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Perturbations of Gibbs Semigroups

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Abstract. We present the analytic perturbation theory for Gibbs semigroups in the case when perturbations of generators are relatively bounded. Analyticity with respect to perturbation and semigroup parameters in the Tr-norm topology is proved and the corresponding domains are described.

1. Introduction

Let $\mathfrak{C}_p(\mathfrak{H})$ be the Banach space of compact operators on a separable Hilbert space $\mathfrak{H}$ which have the finite $\|\cdot\|_p$-norm

$$\|A\|_p=\left(\sum_{n=1}^{\infty}\lambda_n^p\right)^{1/p}, \qquad 1\leqq p<\infty .$$

Here $\lambda_n=\mu_n(|A|)$, where $\mu_n(|A|)$ is a n^{th} (taking into account degeneracy) eigenvalue of the operator $|A|=\sqrt{AA^*}$. The Banach spaces $\{\mathfrak{C}_p(\mathfrak{H})\}_{p=1}^{\infty}$ are $*$-ideals in the space of all bounded operators $\mathscr{L}(\mathfrak{H})$ and $\mathfrak{C}_1(\mathfrak{H})$ *(trace-class)* $\subset\mathfrak{C}_2(\mathfrak{H})$ *(Hilbert-Schmidt operators)* $\subset\ldots\subset\mathfrak{C}_\infty(\mathfrak{H})$ *(compact operators)* $\subset\mathscr{L}(\mathfrak{H})$, see e.g. [1] or [2, VI.6].

In quantum statistical mechanics one faces strongly continuous one-parametric semigroups of self-adjoint operators $G:\mathbb{R}^1_+\cup\{0\}\to\mathscr{L}(\mathfrak{H})$ which have the property that $G:\mathbb{R}^1_+\to\mathfrak{C}_1(\mathfrak{H})$. They are naturally created by the *density matrix* $\exp(-\beta H)$ for a finite system with the Hamiltonian H and temperature $\beta^{-1}\in\mathbb{R}^1_+$ and got the name of the *Gibbs semigroups* [3–5]. But if we want to make an analytic continuation in the "interaction constant" then the operator H becomes non-self-adjoint. Its numerical range $\theta(H)=\{(H\psi,\psi)\colon \psi\in D(H),\ \|\psi\|=1\}$ belongs to the sector $\mathscr{S}_\gamma(\Omega)=\{z\in\mathbb{C}\colon|\arg(z-\gamma)|\leqq\Omega<\pi/2\}$ and $G(t)=\exp(-tH)\in\mathfrak{C}_1(\mathfrak{H})$ for $t\in\mathbb{R}^1_+$, see e.g. [6, 7]. This was the reason for the following general definition.

Definition 1.1 [5]. A strongly continuous semigroup $G(t)$ in a separable Hilbert space $\mathfrak{H}$ is called a *Gibbs semigroup* if $G:\mathbb{R}^1_+\to\mathfrak{C}_1(\mathfrak{H})$.

Remark 1.1. From the continuity of multiplication $\left(A_nB_n \xrightarrow{\|\cdot\|_p} AB \text{ if } A_n \xrightarrow{s} A\right.$ (strongly) and $\left.B_n \xrightarrow{\|\cdot\|_p} B \text{ for } 1 \leqq p < \infty \text{ [8, 9]}\right)$ and the fact that $\mathfrak{C}_1(\mathfrak{H})$ is $*$-ideal in $\mathscr{L}(\mathfrak{H})$ one gets the $\|\cdot\|_1$-continuity of the Gibbs semigroup $G(t)$ on $\mathbb{R}^1_+$.

By the Hille-Yosida-Phillips theorem [9, X.8] each strongly continuous semigroup $V(t)$ is related to a closed operator T (semigroup generator) with a dense domain $D(T)$, i.e., $V(t)=V_T(t)=\exp(-tT)$. The corresponding properties of the generator T can be formulated using the resolvent $R_\zeta(T)=(\zeta-T)^{-1}$ and the resolvent set $P(T)=\{\zeta\in\mathbb{C}: R_\zeta(T)\in\mathscr{L}(\mathfrak{H})\}$. Therefore, the study of semigroups is mainly connected with properties of their generators, the central question being the problem of stability and perturbation theory.

Let $G_T(t)$ be a Gibbs semigroup with the generator T. The aim of the present paper is to develop a perturbation theory for the Gibbs semigroups. More precisely, we shall describe perturbations U for which $G_\lambda(t)=G_{T+\lambda U}(t)$ is still the Gibbs semigroup $\|\cdot\|_1$-analytic with respect to the parameters $\{\lambda, t\}$.

In Sect. 2 we study U from the Hille-Phillips perturbation class $\mathscr{P}_0$. This case has been earlier considered in refs. 3, 4. Here we simplify their proofs considerably and show that $\mathscr{P}_0$ consists of the operators U which are T-bounded with the relative T-bound $b=0$. In this case the semigroup $G_\lambda(t)$ is $\|\cdot\|_1$-analytic on the parameter λ in the whole complex plane $\mathbb{C}$.

In Sects. 3 and 4 we consider perturbations with $b>0$. The $\|\cdot\|_1$-analyticity in the parameter λ requires that $b<1$ (class $\mathscr{P}_1$). Then the domain of $\|\cdot\|_1$-analyticity of the Gibbs semigroup $G_\lambda(z)$ has the form $C_b\times S_{\lambda,b}$. Here $C_b=\{z\in\mathbb{C}: |z|<(2b)^{-1}\}$ and $S_{\lambda,b}$ is sector in the right half-plane $\mathbb{C}_+=\{z\in\mathbb{C}: \operatorname{Re} z>0\}$:

$$S_{\lambda,b}=\left\{z\in\mathbb{C}_+ : |\arg z|<\omega=\operatorname{arctg}\frac{\sqrt{1-2|\lambda|b}}{|\lambda|b}\right\}.$$

For perturbations from class $\mathscr{P}_0$ one correspondingly gets that a domain of the $\|\cdot\|_1$-analyticity of $G_\lambda(z)$ has the form $\mathbb{C}\times\mathbb{C}_+$.

Section 5 contains the concluding remarks and formulations of some other results of the Gibbs semigroup theory.

2. Perturbations from Class $\mathscr{P}_0$

The standard perturbation theory for strongly continuous semigroups has been developed by Hills and Phillips [11, XIII] for perturbations from class $\mathscr{P}_0$.

Definition 2.1. A closed operator U belongs to the *class* $\mathscr{P}_0$ of perturbations for the generator T of the semigroup $G_T(t)$ if the domain $D(U)\supseteq \bigcup_{t>0} G_T(t)\mathfrak{H}$ and

$$\int_0^1 dt\, \|UG_T(t)\| < \infty\,. \tag{2.1}$$

We show that the Gibbs semigroups are stable with respect to perturbations $U\in\mathscr{P}_0$, the corresponding perturbation series converges in the $\|\cdot\|_1$-topology and defines a Gibbs semigroup.

Theorem 2.1. *Let $G_T(t)$ be a self-adjoint Gibbs semigroup with the generator $T \geqq -\alpha$. If the operator $U \in \mathscr{P}_0$, then the series*

$$\mathscr{F}_\lambda(t) = \sum_{n=0}^{\infty} \lambda^n S_n(t) \tag{2.2}$$

with

$$S_{n=0}(t) = G_T(t), \quad S_{n\geqq 1}(t) = -\int_0^t d\tau G_T(t-\tau) U S_{n-1}(\tau), \tag{2.3}$$

(i) *converges uniformly on t from any compact $K \subset \mathbb{R}^1_+$ and on λ from any disc $C \subset \mathbb{C}$ in the $\|\cdot\|_1$-topology;*

(ii) *defines the Gibbs semigroup $G_\lambda(t)$ with the generator $H_\lambda = T + \lambda U$ which is the $\|\cdot\|_1$-holomorphic function of the parameter λ for any fixed $t > 0$.*

Proof. (i) The operator U is T-bounded and $\mathfrak{C}_1(\mathfrak{H})$ is $*$-ideal in $\mathscr{L}(\mathfrak{H})$. Then, $UG_T(t) = [UR_\zeta(T)][(\zeta - T)G_T(t)] \in \mathfrak{C}_1(\mathfrak{H})$ for $t > 0$ and $\zeta \in P(T)$. Together with condition (2.1) this implies that for $t > 0$ the operator $S_n(t)$ can be represented as the n-fold $\|\cdot\|_1$-convergent Bochner integral:

$$\begin{aligned} S_n(t) = \int_0^t d\tau_1 \int_0^t d\tau_2 \ldots \int_0^t d\tau_n \chi_n^t(\tau_0, \tau_1, \ldots, \tau_n) \\ \times (-1)^n G_T(\tau_0) U G_T(\tau_1) \ldots U G_T(\tau_n). \end{aligned}$$

Here $\chi_n^t(\tau_0, \tau_1, \ldots, \tau_n)$ is the characteristic function of the set $\left\{\tau_i \geqq 0;\ i = 0, 1, \ldots, n: \sum_{i=0}^{n} \tau_i = t\right\}$. From the above representation one gets

$$\begin{aligned} &\|S_n(t)\|_1 \\ &\leqq \int_0^t d\tau_1 \ldots \int_0^t d\tau_n \chi_n^t(\tau_0, \tau_1, \ldots, \tau_n) \|G_T(\tau_0) U G_T(\tau_1) \ldots U G_T(\tau_n)\|_1. \end{aligned} \tag{2.4}$$

Now we can use the Ginibre-Gruber inequality [12]:

$$\left\|\prod_{i=0}^{n} A_i V(x_i)\right\|_1 \leqq \left(\prod_{i=0}^{n} \|A_i\|\right) \operatorname{Tr}\left[V\left(\sum_{i=0}^{n} x_i\right)\right], \tag{2.5}$$

where $V(x)$ is an arbitrary Gibbs semigroup and $A_i \in \mathscr{L}(\mathfrak{H})$, $i = 0, 1, \ldots, n$. To this end we introduce $A_0 = G_T(\tau_0/2)$, $A_i = UG_T(\tau_i/2)$ for $i \geqq 1$ and the functions $q(t) = \|G_T(t)\| = \exp \alpha t$ and $p(t) = \|UG_T(t)\|$. Then, using (2.4) and (2.5) one gets

$$\|S_n(t)\|_1 \leqq 2^n (q * \underbrace{p * \ldots * p}_{n})(t/2) \operatorname{Tr} G_T(t/2),$$

where $*$ denotes convolution operation. From condition (2.1) it follows that for any $R > 0$ there is $\varepsilon > 0$ small enough, that $\int_0^\varepsilon dt \|UG_T(t)\| = \gamma_\varepsilon < (2R)^{-1}$. Then, for $0 < t \leqq 2\varepsilon$ series (2.2) converges in the $\|\cdot\|_1$-topology uniformly on λ from disc $C_R = \{z \in \mathbb{C} : |z| < R\}$ and on t from any compact $K \subset (0, 2\varepsilon]$. In addition, from (2.2) one gets that

$$\|\mathscr{F}_\lambda(t)\|_1 \leqq \frac{(\exp\alpha t/2-1)}{\alpha(1-2|\lambda|\gamma_\varepsilon)} \operatorname{Tr} G_T(t/2) \tag{2.7}$$

and that the operator-valued function $\mathscr{F}_\lambda(t)$ satisfies the equation *(Duhamel formula* [10, X.9]):

$$\mathscr{F}_\lambda(t) = G_T(t) - \lambda \int_0^t d\tau G_T(t-\tau) U \mathscr{F}_\lambda(\tau). \tag{2.8}$$

From Eq. (2.8) according to the standard arguments (see e.g. [13, XI]) it follows that for $0 \leqq t \leqq 2\varepsilon$ the function $\mathscr{F}_\lambda(t)$ is a strongly continuous semigroup with the generator $H_\lambda = T + \lambda U$. Now we can extend the definition of $\mathscr{F}_\lambda(t)$ for $t \geqq 2\varepsilon$ by putting

$$\mathscr{F}_\lambda(t) = (\mathscr{F}_\lambda(\varepsilon))^n \mathscr{F}_\lambda(t-n\varepsilon) \tag{2.9}$$

if $n\varepsilon \leqq t \leqq (n+1)\varepsilon$. From the representation (2.9) and estimates (2.6), (2.7) one gets the uniform $\|\cdot\|_1$-convergence of series (2.2) for t from any compact $K \subset \mathbb{R}^1_+$ and for $\lambda \in C_R$ for a disc of an arbitrary radius R.

(ii) The $\|\cdot\|_1$-convergence of (2.2) implies that the semigroup $\mathscr{F}_\lambda(t)$ for $t>0$ and $\lambda \in \mathbb{C}$ is a Gibbs semigroup, i.e. $\mathscr{F}_\lambda(t) = G_\lambda(t) \in \mathfrak{C}_1(\mathfrak{H})$, which is $\|\cdot\|_1$-continuous on $\mathbb{R}^1_+$ by Remark 1.1. Moreover, by construction of the series (2.2) the Gibbs semigroup $G_\lambda(t)$ is $\|\cdot\|_1$-holomorphic for $\lambda \in \mathbb{C}$ if $t>0$. □

Corollary 2.1. *Let the operator U be symmetric. Then, the semigroup $G_\lambda(t)$ is the self-adjoint Gibbs semigroup for $t>0$ and $\lambda \in \mathbb{R}^1$. On the other hand, as it follows from the proof of Theorem 2.1 the statements* (i) *and* (ii) *hold if one substitutes the self-adjointness of the semigroup $G_T(t)$ with the generator $T \geqq -\alpha$ by a more general condition of its quasi-boundedness: $\|G_T(t)\| \leqq M \exp\alpha t$, i.e. $G_T(t) \in \mathscr{B}(M,\alpha)$.*

Corollary 2.2. *Let $\mathscr{D} \subset \mathbb{C}$ and for any $\lambda \in \mathscr{D}$ the operator $U(\lambda)$ from the family $\{U(\lambda)\}_{\lambda \in \mathscr{D}}$ has the following properties:*

(a) *The operator $U(\lambda)$ is T-bounded: $\|U(\lambda)\psi\| \leqq a\|\psi\| + b\|T\psi\|$, $\psi \in D(T)$, and the function $\lambda \to U(\lambda)G_T(t)$ is $\|\cdot\|$-analytic for $t>0$;*

(b) $\int_0^1 dt \sup_{\lambda \in \mathscr{D}} \|U(\lambda)G_T(t)\| < \infty$.

Then, the operator-valued function $G_{T+U(\lambda)}(t)$ constructed by iterations of Eq. (2.8) *is a Gibbs semigroup which is $\|\cdot\|_1$-analytic for $\lambda \in \mathscr{D}(t>0)$.*

Note that conditions defining the perturbation class $\mathscr{P}_0$ are too severe. The next statement says that this class does not cover even perturbations arising in the quantum statistical mechanics [14, 15].

Theorem 2.2. *Let $G_T(t)$ be a strongly continuous quasi-bounded semigroup with the generator T. If the operator $U \in \mathscr{P}_0$, then it is T-bounded with the relative bound $b=0$.*

Proof. By the closedness of the operator U and the property: $D(U) \supseteq \bigcup_{t>0} G_T(t)\mathfrak{H}$ we get that $UG_T(t) \in \mathscr{L}(\mathfrak{H})$ for $t>0$ and $\|UG_T(t)\| \leqq \|UG_T(t=1)\| M \exp[\alpha(t-1)]$ for

$t>1$. Consequently, for an arbitrary small $\varepsilon>0$ one can find $\xi>\alpha$ large enough that

$$\left\| \int_0^\infty dt \exp(-\xi t) U G_T(t)\psi \right\| \leqq \varepsilon \|\psi\|, \qquad \psi \in \mathfrak{H}. \tag{2.10}$$

Simultaneously for the semigroup $G_T(t)$ and for any ζ from the resolvent set $P(-T)$ one gets:

$$\int_0^\infty dt \exp(-\zeta t) G_T(t)\psi = (\zeta + T)^{-1}\psi, \qquad \psi \in \mathfrak{H}.$$

This relation together with estimate (2.10) and the closedness of U gives that $(\zeta+T)^{-1}\psi \in D(U)$ for $\psi \in \mathfrak{H}$ and $\|U\varphi\| \leqq \varepsilon\xi\|\varphi\| + \varepsilon\|T\varphi\|$ for $\varphi \in D(T)$, i.e., $b=0$. □

The paper [7] is an attempt to develop a perturbation theory for the Gibbs semigroups when $b>0$. But it contains an error in the proof of the $\|\cdot\|_1$-analyticity of the semigroup $G_\lambda(t)$ on the parameter λ and in the estimate of the domain of its $\|\cdot\|_1$-analyticity on the parameter t in the continuation into the half-plane $\mathbb{C}_+$. Below, in Sects. 3 and 4, these errors will be corrected.

3. Perturbations from Class $\mathscr{P}_1$

As it follows from the above discussion a class of Gibbs semigroups which are stable with respect to perturbations from the class $\mathscr{P}_0$ contains all *quasi-bounded Gibbs semigroups* $\mathscr{B}(M,\alpha)$. We show that extension of the perturbations to the *class* $\mathscr{P}_b$ (operators which are relatively bounded by the generator with $b>0$) implies reduction of stable semigroups to the *holomorphic Gibbs semigroups* $\mathscr{H}(\omega,\alpha)$ with generators having the following property: $R_\zeta(T) \in \mathfrak{C}_p(\mathfrak{H})$ for $\zeta \in P(T)$ and some $p<\infty$. It is the class of semigroups that one encounters in quantum statistical mechanics [14–17].

Theorem 3.1. *Let $G_T(t)$ be a self-adjoint Gibbs semigroup with the generator $T \geqq -\alpha$. Then, it admits an analytic continuation to the $\|\cdot\|_1$-holomorphic in the half-plane $\mathbb{C}_+$ Gibbs semigroup $G_T(z) \in \mathscr{H}(\pi/2,\alpha)$.*

Proof. Using the spectral representation $G_T(t) = \int_{-\alpha}^\infty dE_\xi(T)\exp(-t\xi)$ one can check that $G_T(t>0)\mathfrak{H} \subseteq D(T)$. Therefore, the operator $TG_T(t) \in \mathscr{L}(\mathfrak{H})$ for $t>0$. Consequently, the semigroup $G_T(t>0)$ is strongly differentiable. By the Remark 1.1 this property can be lifted up to $\|\cdot\|_1$-differentiability

$$\begin{aligned} \|\cdot\|_1 - \partial_t G_T(t) &= \|\cdot\|_1 - \lim_{\Delta\to 0} \left\{ \frac{1}{\Delta}[G_T(t/2+\Delta) - G_T(t/2)] \right\} G_T(t/2) \\ &= (-T)G_T(t) \in \mathfrak{C}_1(\mathfrak{H}). \end{aligned}$$

In the same way one gets for $t>0$ that

$$\|\cdot\|_1 - \partial_t^n G_T(t) = (-T)^n G_T(t) \in \mathfrak{C}_1(\mathfrak{H}), \qquad n \geqq 1.$$

Then, due to the estimate $\|TG_T(t)\| \leqq \max\{t^{-1}, \alpha \exp \alpha t\}$, we obtain

$$\|\partial_t^n G_T(t)\|_1 \leqq \|TG_T(t/2n)\|^n \|G_T(t/2)\|_1 \leqq (2n)^n t^{-n} \|G_T(t/2)\|_1 .$$

Hence, the semigroup $G_T(t)$ can be continued from $\mathbb{R}_+^1$ to the disc $C_t = \{z \in \mathbb{C}_+ : |z-t| < t/2e\}$ via the $\|\cdot\|_1$-convergent series

$$G_T(z) = \sum_{n=0}^{\infty} \frac{(z-t)^n}{n!} \partial_t^n G_T(t) = \exp(-zT) .$$

The union of these discs $\bigcup_{t>0} C_t$ is the sector

$$\mathscr{S}_0(\Omega) = \{z \in \mathbb{C}_+ : |\arg z| < \Omega = \arcsin(2e)^{-1}\}$$

in which one can continue the Gibbs semigroup $G_T(t)$ from $\mathbb{R}_+^1$ by the $\|\cdot\|_1$-convergent power series. For a further $\|\cdot\|_1$-analytic continuation from the sector $\mathscr{S}_0(\Omega)$ to the half-plane $\mathbb{C}_+$ one has to take into account that $G_T(z=t+i\tau) = G_T(t) G_T(i\tau) \in \mathfrak{C}_1(\mathfrak{H})$ for $z \in \mathbb{C}$ and $\|\cdot\|_1 - \partial_z G_T(z) = (-T) G_T(z) \in \mathfrak{C}_1(\mathfrak{H})$. □

Definition 3.1. The closed operator U belongs to the $\mathscr{P}_1$*-class* perturbations of the generator T for a strongly continuous semigroup $G_T(t)$ if $D(U) \supset D(T)$ and U is a T-bounded operator with a relative boundary $b < 1$:

$$\|U\psi\| \leqq a\|\psi\| + b\|T\psi\| , \qquad \psi \in D(T) . \tag{3.1}$$

Remark 3.1. If, in addition, the operator T is self-adjoint and semibounded from below and the operator U is symmetric, then by the Kato-Rellich theorem [10, X.2], [13, V, Sect. 4] the algebraic sum $H = T + U$ with $D(H) = D(T)$ is the self-adjoint operator and $H \geqq -\alpha' = -\alpha - \max(a/(1-b);\ a + b|\alpha|)$.

Therefore, the operator H is a generator of a strongly continuous (differentiable for $t > 0$) self-adjoint semigroup $V_H(t)$. From the same arguments as in Theorem 3.1 it can be extended to the $\|\cdot\|$-analytic in $\mathbb{C}_+$ semigroup $V_H(z)$. If the operator T is a generator of a Gibbs semigroup, then from Remark 3.1 and the Weyl min-max principle (see e.g. [18, XIII.1]) it follows that the spectrum of the operator H is a pure point and $\mu_n(H) \geqq a + (1-b)\mu_n(T)$ for $n \geqq 1$ large enough. Hence $V_H(t)$ is a Gibbs semigroup: $V_H(t) = G_H(t)$, and by Theorem 3.1 one gets that $G_H(z) \in \mathscr{H}(\pi/2, \alpha')$. Therefore, we have proved the following statement:

Theorem 3.2. *Let $G_T(t)$ be a self-adjoint quasi-bounded Gibbs semigroup. Let perturbation $U \in \mathscr{P}_1$ and be a symmetric operator. Then the operator $H = T + U$ is a generator of the quasi-bounded self-adjoint Gibbs semigroup $G_H(t)$ which can be extended to the $\|\cdot\|_1$-holomorphic semigroup $G_H(z) \in \mathscr{H}(\pi/2, \alpha')$.*

Remark 3.2. Thus, self-adjoint Gibbs semigroups are stable with respect to perturbations from the class $\mathscr{P}_1$. Moreover, the conditions $D(U) \supset D(T)$ and $b < 1$ can be relaxed. Indeed, one can require that the perturbation $U = U^*$ be semibounded from below and the algebraic sum $H = T + U$ be essentially self-adjoint on the domain $D(T) \cap D(U)$. These conditions allow us to use the *Golden-*

Thompson inequality [18, XIII.17]

$$\|\exp(-\tilde{H})\|_1 \leqq \|\exp(-T/2)\exp(-U)\exp(-T/2)\|_1,$$

which says that the closure $(T+U)^{\sim}=\tilde{H}$ is a generator of the self-adjoint, semi-bounded Gibbs semigroup $G_{\tilde{H}}(t)$.

But as above (see Sect. 2), we would like in addition to stability of the self-adjoint Gibbs semigroups to study the analytic properties of the semigroup $G_\lambda(z)$ on the parameter $\lambda \in \mathbb{C}$ for perturbations $\lambda U \in \mathscr{P}_1$. Therefore, instead of the spectral representation for constructing the semigroup $G_\lambda(z)$ one has to exploit the *Dunford-Taylor formula* (see e.g. [2, VII] or [13, XI, Sect. 1.6]):

$$G_\lambda(z)=\frac{1}{2\pi i}\int_\Gamma d\zeta \exp(-\zeta z) R_\zeta(H_\lambda). \tag{3.2}$$

The right-hand side of (3.2) is the $\|\cdot\|$-convergent Bochner integral with a positively oriented contour $\Gamma \subset P(H_\lambda)$ with the spectrum of H_λ contained within Γ. Formula (3.2) connects the properties of the semigroup $G_T(z)$ with that for a nonperturbed semigroup $G_T(z)$ by means of the resolvent $R_\zeta(H_\lambda)=(\zeta-H_\lambda)^{-1}$, $H_\lambda=T+\lambda U$.

Theorem 3.3. *Let $T=T^* \geqq -\alpha$. Then for any $\lambda \in C_b=\{z\in\mathbb{C}: |z|<(2b)^{-1}\}$ formula* (3.2) *defines the semigroup $G_\lambda(z)$ which is strongly analytic on the parameter z in the sector $S_{\lambda,b}=\{z\in\mathbb{C}_+ : |\arg z|<\omega=\operatorname{arctg}(\sqrt{1-2|\lambda|b}/|\lambda|b)\}$.*

Proof. First of all we have to note that for $\lambda=0$ formula (3.2) gives the same result as Theorem 3.1. The line of reasoning is the following. For the operator T the resolvent set is $P(T)=\{\mathbb{C}\backslash[-\alpha,+\infty)\}$. Therefore, for any $z\in\mathbb{C}_+$ the contour $\Gamma=\Gamma_0$ can be chosen in such a way that $\operatorname{Re}z\operatorname{Re}\zeta>\operatorname{Im}z\operatorname{Im}\zeta$ for $\operatorname{Re}\zeta\to\infty$ (Fig. 1), i.e. the right-hand side of (3.2) is the $\|\cdot\|$-convergent Bochner integral. To verify that this integral defines a strongly holomorphic semigroup with the generator T

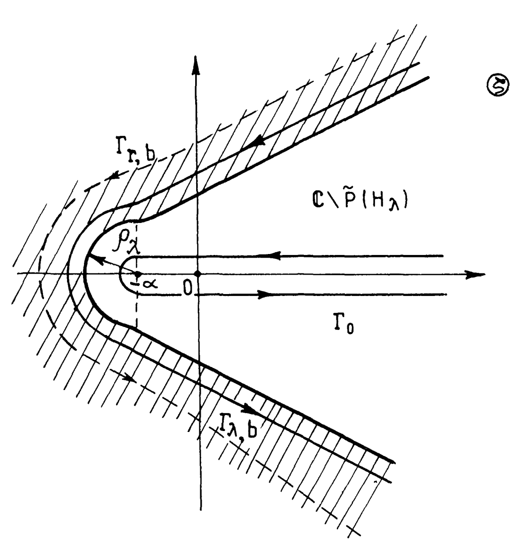

Fig. 1

in the half-plane $\mathbb{C}_+$, one has to use the Cauchy theorem and differentiation under the integral (3.2). Now let perturbation $U \in \mathscr{P}_1$. Then, the operator $H_\lambda = T + \lambda U$ is well-defined as an algebraic sum and is closed on the domain $D(H_\lambda) = D(T)$ if $|\lambda| b < 1$. The appropriate resolvent set $P(H_\lambda)$ is defined by the condition $|\lambda| \, \|UR_\zeta(T)\| < 1$ corresponding to the convergence of the Neumann series for $R_\zeta(H_\lambda)$:

$$R_\zeta(H_\lambda) = R_\zeta(T) \sum_{n=0}^{\infty} (\lambda U R_\zeta(T))^n . \tag{3.3}$$

Note that taking into account inequality (3.1) and the structure of the set $P(T)$ one gets

$$\begin{aligned} \|UR_\zeta(T)\| &\leqq a \|R_\zeta(T)\| + b \|TR_\zeta(T)\| \\ &\leqq \begin{cases} \dfrac{a}{|\zeta+\alpha|} + b, & \operatorname{Re}\zeta < -\alpha \\ \dfrac{a}{\operatorname{Im}(\zeta+\alpha)} + b\left[1 + \dfrac{|\zeta+\alpha|}{\operatorname{Im}(\zeta+\alpha)}\right], & \operatorname{Re}\zeta \geqq -\alpha . \end{cases} \end{aligned} \tag{3.4}$$

Then, from inequality (3.4) we obtain that the resolvent set $P(H_\lambda)$ contains as a subset

$$\tilde{P}(H_\lambda) = \begin{cases} |\zeta+\alpha| > \varrho_\lambda = \dfrac{a|\lambda|}{1-b|\lambda|}, & \operatorname{Re}\zeta < -\alpha \\ \operatorname{Im}(\zeta+\alpha) > \varrho_\lambda + \dfrac{|\lambda| b}{\sqrt{1-2|\lambda| b}} \operatorname{Re}(\zeta+\alpha), & \operatorname{Re}\zeta \geqq -\alpha . \end{cases} \tag{3.5}$$

Therefore, the contour Γ in the integral (3.2) can be chosen in $\tilde{P}(H_\lambda)$, e.g. $\Gamma = \Gamma_{\lambda, b}$, see Fig. 1. Then, the integral convergence condition ($\operatorname{Re} z \operatorname{Re}\zeta > \operatorname{Im} z \operatorname{Im}\zeta$ for $\zeta \to \infty$) defines a sector $S_{\lambda, b}$, where the semigroup $G_\lambda(z)$ is strongly analytic in z, see Fig. 2:

$$S_{\lambda, b} = \left\{ z \in \mathbb{C}_+ : \left| \frac{\operatorname{Im} z}{\operatorname{Re} z} \right| < \frac{\sqrt{1-2|\lambda| b}}{|\lambda| b} \right\}. \tag{3.6}$$

The semigroup and analytic properties of the function $z \to G_\lambda(z)$ can be verified similarly to the case $\lambda = 0$. From (3.5) and (3.6) one gets that $G_\lambda(z) \in \mathscr{H}(\omega, \alpha'(\lambda))$,

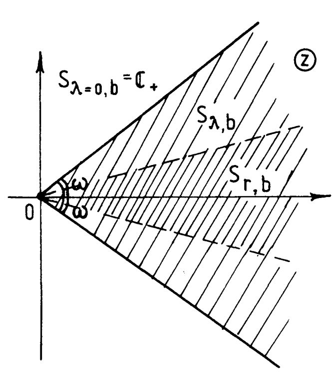

Fig. 2

where $\alpha'(\lambda) \geqq \alpha + a|\lambda|/(1-b|\lambda|)$ and $\omega = \operatorname{arctg}(\sqrt{1-2|\lambda|b}/|\lambda|b)$, see Fig. 2. $\square$

Remark 3.3. If the operator T is a generator of the Gibbs semigroup $G_T(t)$ and perturbation $U \in \mathscr{P}_1$ then the perturbed semigroup $G_T(t)$ for $\lambda \in C_b$ should not necessarily be Gibbsian. One needs supplementary conditions either on the operator λU (see Theorem 3.2 and Remark 3.2) or on the generator T. The latter will be considered in the next section.

4. Gibbs Semigroups with *p*-Generators

Let an unperturbed semigroup $G_T(t)$ be the Gibbs semigroup. For perturbations from the class $\mathscr{P}_0$ this guarantees that the semigroup $G_\lambda(t)$ for $\lambda \in \mathbb{C}$ will also be Gibbsian. Theorem 3.2 ensures the same for the semigroup $G_{T+U}(t)$ if the operator $U \in \mathscr{P}_1$ and is symmetric. Remark 3.2 makes clear the price one needs to pay for getting rid of the condition: $U \in \mathscr{P}_1$. But as before the symmetricity of the operator U is essential. Since for perturbation λU with complex parameter λ this property is violated, we start with conditions ensuring the stability of Gibbs semigroups with respect to these perturbations, see Remark 3.3.

Definition 4.1. We say that a self-adjoint Gibbs semigroup has *p-generator* T if for some $\zeta_0 \in P(T)$ the resolvent $R_{\zeta_0}(T) \in \mathfrak{C}_p(\mathfrak{H})$ for some finite $p \geqq 1$, see [7].

Remark 4.1. Let $T_\sigma(\Lambda^N) = \left[\sum_{1 \leqq i \leqq N} (-\Delta_i/2m)\right]_\sigma$ be the kinetic-energy operator of N particles enclosed in a bounded vessel $\Lambda \subset \mathbb{R}^d$. Here σ fixes a boundary condition on $\partial\Lambda$ corresponding to a self-adjoint extension of symmetric operator $T(\Lambda^N)$. Then one gets: $R_{\zeta_0}(T_\sigma(\Lambda^N)) \in \mathfrak{C}_p(\mathfrak{H})$ for $p \geqq \frac{1}{2} \cdot N \cdot d$ and arbitrary $\zeta_0 \in P(T_\sigma(\Lambda^N))$.

Theorem 4.1. *Let a self-adjoint operator $T \geqq -\alpha$ be a p-generator of the strongly continuous semigroup $G_T(t)$. Then, it is a $\|\cdot\|_1$-analytic in $\mathbb{C}_+$ Gibbs semigroup. If the operator $U \in \mathscr{P}_1$, then a perturbed semigroup $G_\lambda(z)$ is also a Gibbs semigroup which is $\|\cdot\|_1$-analytic on z in the sector $S_{\lambda,b}$* (3.6) *for any $\lambda \in C_b = \{\zeta \in \mathbb{C} : |\zeta| < (2b)^{-1}\}$.*

Proof. For each $\zeta \in P(T)$ one gets: $R_\zeta(T) = R_{\zeta_0}(T) + (\zeta_0 - \zeta)\, R_\zeta(T)\, R_{\zeta_0}(T)$. Thus, for an arbitrary $\zeta \in P(T)$ the resolvent $R_\zeta(T) \in \mathfrak{C}_p(\mathfrak{H})$ and the integral (3.2) along the path Γ_0 (Fig. 1) is the $\|\cdot\|_p$-convergent Bochner integral defining a semigroup $G_T(t) \in \mathfrak{C}_p(\mathfrak{H})$ for $t > 0$. Then, the semigroup $G_T(t) = [G_T(t/p)]^p \in \mathfrak{C}_1(\mathfrak{H})$ and its $\|\cdot\|_1$-analyticity: $G_T(z) \in \mathscr{H}(\pi/2, \alpha)$, can be verified in the same way as in Theorem 3.1. To estimate the $\|\cdot\|_p$-norm of the operator $G_\lambda(z)$ we use the expansion (3.3) and formula (3.2) for the case when the contour $\Gamma = \Gamma_{\lambda,b}$ (Fig. 1)

$$\begin{aligned}
\|G_\lambda(z)\|_p &\leqq \frac{1}{2\pi} \int_{\Gamma_{\lambda,b}} |d\zeta| \exp[-\operatorname{Re}(\zeta \cdot z)]\, \|R_\zeta(T)\|_p (1 - |\lambda|\, \|UR_\zeta(T)\|)^{-1} \\
&\leqq \frac{1}{2\pi} \int_{\Gamma_{\lambda,b}} |d\zeta| \exp[-(\operatorname{Re}\zeta \operatorname{Re} z - \operatorname{Im}\zeta \operatorname{Im} z)] \\
&\quad \times \|R_{\zeta_0}(T)\| \frac{1 + |\zeta_0 - \zeta|\, \|R_\zeta(T)\|}{1 - |\lambda|\, \|UR_\zeta(T)\|}.
\end{aligned}$$

Then, by Theorem 3.3 the right-hand side of (4.1) is bounded for $z \in S_{\lambda,b}$ and $|\lambda| < (2b)^{-1}$, see (3.5) and (3.6). Consequently, the semigroup $G_\lambda(z) = [G_\lambda(z/p)]^p \in \mathfrak{C}_1(\mathfrak{H})$ and its $\|\cdot\|_1$-analyticity on z in the sector $S_{\lambda,b}: G_\lambda(z) \in \mathscr{H}(\omega, \alpha'(\lambda))$ can be checked in the similar way as in Theorem 3.3. □

Now we have to prove the $\|\cdot\|_1$-analyticity of the semigroup $G_\lambda(z)$ on the parameter $\lambda \in C_b = \{\zeta \in \mathbb{C} : |\zeta| < (2b)^{-1}\}$ when $z \in S_{\lambda,b}$ (3.6). As a first step we prove the $\|\cdot\|_p$-analyticity.

Lemma 4.1. *Let the operator $U \in \mathscr{P}_1$ and the generator of the semigroup $G_T(z)$ is a p-generator. Then, the Gibbs semigroup $G_\lambda(z)$ is $\|\cdot\|_p$-analytic on the parameter λ in the disc C_b (i.e. for an arbitrary closed domain $\mathscr{D} \subset C_b$) if $z \in \bigcap_{\lambda \in \mathscr{D}} S_{\lambda,b}$.*

Proof. Let $r < (2b)^{-1}$ be a fixed number and $z \in \bigcap_{|\lambda| \leqq r} S_{\lambda,b} \equiv S_b(r)$, see Fig. 2. Then, for all λ such that $|\lambda| \leqq r$ one can pick out a common contour $\Gamma_{r,b} \subset \bigcap_{|\lambda| \leqq r} \tilde{P}(H_\lambda)$ [see (3.5) and Fig. 1] for the family of semigroups $\{G_\lambda(z)\}_{|\lambda| \leqq r}$ which ensures the $\|\cdot\|_p$-convergence of the integral (3.2) which can be estimated by (4.1). From the expansion (3.3) together with inequality (3.4) and estimate

$$\left\| R_\zeta(H_\lambda) - R_\zeta(T) \sum_{n=0}^{N} (\lambda U R_\zeta(T))^n \right\|_p \leqq (1 - |\lambda| \, \|U R_\zeta(T)\|)^{-1} \|R_\zeta(T)\|_p \|\lambda U R_\zeta(T)\|^{N+1},$$

it follows that for $\zeta \in \Gamma_{r,b}$ the series (3.3) is uniformally convergent for $\lambda : |\lambda| \leqq r$, in the $\|\cdot\|_p$-norm. Hence, it defines a $\|\cdot\|_p$-analytic in this disc resolvent $R_\zeta(H_\lambda)$. Therefore, the series (3.3) can be term-by-term integrated by the contour $\Gamma_{r,b}$ for each $z \in \mathscr{D} \subset S_b(r)$:

$$G_\lambda(z) = \sum_{n=0}^{\infty} \lambda^n \frac{1}{2\pi i} \int_{\Gamma_{r,b}} d\zeta \exp(-\zeta z) R_\zeta(T) (U R_\zeta(T))^n. \tag{4.2}$$

The series in the right-hand side of (4.2) converges uniformly in disc C_b in $\|\cdot\|_p$-topology and defines a $\|\cdot\|_p$-analytic on the $\lambda \in C_b$ function which coincides with the Gibbs semigroup $G_\lambda(z)$. □

Theorem 4.2. *Let all conditions of Lemma 4.1 be satisfied. Then, the Gibbs semigroup $G_\lambda(z)$ is $\|\cdot\|_1$-analytic on λ in the disc C_b for each $z \in \bigcap_{\lambda \in \mathscr{D}} S_{\lambda,b}$.*

Proof. By Lemma 4.1 the semigroup $G_\lambda(z)$ can be represented in the disc C_b as the $\|\cdot\|_p$-convergent Taylor series

$$G_\lambda(z) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \partial_\lambda^n G_\lambda(z) \Big|_{\lambda=0}, \qquad z \in S_{\lambda,b}. \tag{4.3}$$

Here, derivatives on λ have the form

$$\partial_\lambda^n G_\lambda(z)|_{\lambda=0} = \frac{n!}{2\pi i} \int_\gamma d\mu \, \mu^{-(n+1)} G_\mu(z). \tag{4.4}$$

The contour γ belongs to the disc C_b and includes the point $\mu=0$, e.g. $\gamma=\gamma_r$, where γ_r is a circle of the radius $r<(2b)^{-1}$ centered at the origin. Then, by Theorem 4.1 one gets from (4.4) the following inequality:

$$\|\partial_\lambda^n G_\lambda(z)|_{\lambda=0}\|_1 \leqq \frac{n!}{2\pi}\int_0^{2\pi} d\varphi r^{-n} \sup_{|\mu|\leqq r} \|G_\mu(z)\|_1 = n! r^{-n} M_r(z),$$

and, as a consequence, the estimate:

$$\left\|G_\lambda(z) - \sum_{n=0}^{N} \frac{\lambda^n}{n!} \partial_\lambda^n G_\lambda(z)\Big|_{\lambda=0}\right\|_1 \leqq \sum_{n=N+1}^{\infty} |\lambda|^n r^{-n} M_r(z). \tag{4.5}$$

The estimate (4.5) implies for each $\lambda \in C_b$ and $z \in S_{\lambda,b}$ the existence of the contour $\gamma_r \subset C_b$ such that the series (4.3) is $\|\cdot\|_1$-convergent uniformly in $\lambda: |\lambda| \leqq r$. $\square$

5. Concluding Remarks

Summarizing we can conclude that for perturbations of the Gibbs semigroups with the p-generators the mapping

$$G_\lambda(z): C_b \times S_{\lambda,b} \to \mathfrak{C}_1(\mathfrak{H}) \tag{5.1}$$

is $\|\cdot\|_1$-analytic on both the parameters if only perturbations from the class $\mathscr{P}_1$ are involved, see Sects. 3 and 4.

If the relative bound $b\to 0$, then one gets from (5.1) the result of Sect. 2:

$$G_\lambda(z): \mathbb{C} \times \mathbb{C}_+ \to \mathfrak{C}_1(\mathfrak{H}). \tag{5.2}$$

Although in this limit the generators of the unperturbed Gibbs semigroups need not be p-generators.

In quantum statistical mechanics the results (5.1) and (5.2) ensure the analyticity of the partition function

$$Z_\Lambda(\lambda, \beta) = \operatorname{Tr} G_\lambda(\beta); \qquad \lambda \in C_b, \quad \beta \in S_{\lambda,b} \tag{5.3}$$

for a finite volume Λ. This property is a consequence of the $\|\cdot\|_1$-analyticity of the function $\operatorname{Tr}: \mathfrak{C}_1(\mathfrak{H}) \to \mathbb{C}$. It is often exploited in applications, e.g. for the proof of inequalities [15, 16] and of the theorems utilizing the convexity on λ or β [15–17].

The investigation of the Gibbs semigroups has been started in [4] where perturbation theory for the self-adjoint semigroups and bounded operators $U \in \mathscr{L}(\mathfrak{H})$ has been considered. An attempt to develop a consistent perturbation theory for unbounded operators U is the intention of [6]. But it contains inaccuracies which are first of all connected with the incorrect use of the *Duhamel formula* for investigation of the $\|\cdot\|_1$-analytic properties of the semigroup $G_\lambda(t)$ on the parameter λ see also [17, 2.4]. In [3] the perturbation theory for the Gibbs semigroups has been developed for the operators $U \in \mathscr{P}_0$. In the present paper this class of perturbations is extended to $\mathscr{P}_1$ and the corresponding domains of the $\|\cdot\|_1$-analyticity, C_b and $S_{\lambda,b}$, are analyzed in detail.

The $\|\cdot\|_1$-*compactness* of the families of Gibbs semigroups was considered in paper [5]. It is important for statistical mechanics of systems with singular potentials, see e.g. [14]. In the recent paper [19] the $\|\cdot\|_1$-convergent *Trotter-Lie formula* for Gibbs semigroup is discussed.

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