Commun. Math. Phys. 120, 525–527 (1989)

Communications in
Mathematical
Physics

Erratum

Convergence of Diffusion Waves of Solutions for Viscous Conservation Laws

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Commun. Math. Phys. **110**, 503–517 (1987)

In our paper the convergence rate should be lower due to a nonlinear interaction term we omitted. The nonlinear term (1.22) when rewritten on the last two lines of p. 511, its i^{th} component should be

$$N(a,b)=\frac{1}{2}\sum_{j\neq i} b_{ijj}a_j^2+\frac{1}{2}\sum_{j\neq k} b_{ijk}a_ja_k+\frac{1}{2}\sum_{j,k} b_{ijk}b_jb_k+l_i\cdot H(a,b),$$

$$b_{ijk}\equiv l_i(f''(0)\gamma_j,\gamma_k).$$

The first term on the right-hand side was missing in the original expression. It creates the interaction of i^{th} characteristic mode with other modes. This contributes to a lower rate of convergence of the solution to the diffusion waves. For instance the rate of L_1-convergence is around $t^{-1/4}$ instead of $t^{-1/2}$. The correct expression of (1.24) of the main result, Theorem 1.2 in [1] should be

$$\|D^l(u-\theta^*)\|_{L_p}(t)=0(1)\delta t^{-\left(\frac{l}{2}+\frac{3}{4}-\frac{1}{2p}-\sigma\right)}. \qquad (1.24)'$$

This rate of convergence is in general optimal. We will present a simple example later to illustrate this. The rate of $t^{-\frac{1}{4}}$ for L_1-convergence is consistent with the inviscid theory. The same rate was obtained in [4] for convergence of solutions of hyperbolic conservation laws to N-waves. The L_2-result has also been obtained independently by Kawashima in [2]. The L_1-result for physical systems which are hyperbolic-parabolic has not been obtained.

To obtain this we follow the same technique as before and use the integration form of (1.19) through parametric methods. The missing term yields

$$\xi_i\equiv\int_0^t\int_{-\infty}^{\infty} G_i(x+y,t-s)\frac{1}{2}\sum_{j\neq i} b_{ijj}(\theta_j^2(y,s))_y dyds,$$

where

$$G_i(t,t)\equiv\frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-\lambda_i t)^2}{2t}\right).$$

ξ_i satisfied the inhomogeneous heat equation

$$\xi_{it}+\lambda_i\xi_{ix}+\frac{1}{2}\sum_{j\neq i} b_{ijj}(\theta_j^2)_x=\frac{1}{2}\xi_{ixx}, \qquad \xi_i(x,0)=0.$$

This can be estimated by the technique of hyperbolic waves in [3] as follows. According to Proposition 2.1 of [1],

$$\theta_j(x,t)=\frac{1}{\sqrt{t+1}}\,\theta_j^*\left(\frac{1}{\sqrt{t+1}}\right)+e_j\equiv\tilde{\theta}_j+e_j, \qquad \|e_j\|_{L_p}(t)=O(1)\delta t^{\frac{1}{2p}-\frac{1}{2}-1}.$$

With this we may decompose ξ_i into $\eta_i+\zeta_i$ with

$$\zeta_{it}+\lambda_i\zeta_{ix}+\frac{1}{2}\sum_{j\neq i} b_{ijj}(\tilde{\theta}_j^2)_x=0, \qquad \zeta_i(x,\infty)=0,$$

$$\eta_{it}+\lambda\eta_{ix}=\frac{1}{2}\eta_{ixx}+\frac{1}{2}\zeta_{ixx}-\sum_{j\neq i} b_{ijj}\left(\tilde{\theta}_j e_j+\frac{1}{2}e_j^2\right)_x, \qquad \eta_i(x,0)=-\zeta_i(x,0).$$

The hyperbolic wave ζ_i was estimated in Sect. 7 of [3] by the characteristic method. We have

$$\|D^l\zeta_i\|_{L_p}(t)=O(1)\delta t^{-\left(\frac{l}{2}+1-\frac{1}{2p}\right)}.$$

η_i is estimated by parametric method

$$\eta_i(\cdot,t)=G_i(\cdot,t)*\eta_i(\cdot,0)+\int_0^t G_i(\cdot,t-s)*\left\{\frac{1}{2}\zeta_{iyy}-\sum_{j\neq i} b_{ijj}\left(\tilde{\theta}_j e_j+\frac{1}{2}e_j^2\right)_y\right\}ds.$$

From estimates of ζ_i, e_i, and $\tilde{\theta}_j$, the second term on the right-hand side has the same decay rate as that for ζ_i. Since ξ_i, ζ_i, and η_i all satisfy conservation laws, we have

$$\int_{-\infty}^{\infty}\eta_i(x,0)dx=0,$$

and from the hypothesis in [1] and pointwise estimate of ζ in [3], we have

$$\eta_i(\cdot,O)\in L_1\cap L^\alpha,$$

$$L^\alpha\equiv\left\{v \mid \|v\|_{L^\alpha}\equiv\int_{-\infty}^{\infty}|x|^\alpha|v(x)|dx<\infty\right\}, \qquad 0\leqq\alpha<\frac{1}{2}.$$

With these, a lemma of Kawashima, [2] and Lemma 3.4 of [1], yields

$$\|D^l[G_i(\cdot,t)*\eta_i(\cdot,0)\|_{L_p}(t)=O(1)\delta t^{-\left(\frac{l}{2}+\frac{1}{2}+\frac{\alpha}{2}-\frac{1}{2p}\right)}.$$

This establishes (1.24)′.

We now present a simple example to show that (1.24)′ would be optimal if $\sigma=0$. Consider

$$U_t+(V^2)_x=\tfrac{1}{2}U_{xx}, \qquad V_t+V_x=V_{xx}, \qquad U(x,0)=V(x,0)=\delta(x).$$

The solution V is the heat kernel with speed one and U differs from the heat kernel by

$$d(x,t)=\int_0^t ds\int_{-\infty}^{\infty}(2\pi(t-s))^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{2(t-s)}}(4\pi s)^{-1}\left(e^{-\frac{(y-s)^2}{4s}}\right)_y dy$$
$$=\int_0^t (2\pi t)^{-\frac{1}{2}}(4\pi s)^{-1}\left(e^{-\frac{(x-s)^2}{2t}}\right)_x ds\,.$$

Direct calculations yield

$$\|d\|_{L_\infty}(t)\geqq|d(0,t)|\geqq Ct^{-\frac{3}{4}},\qquad \|d\|_{L_1}(t)\geqq\int_0^{\sqrt{t}}|e(x,t)dx|\geqq Ct^{-\frac{1}{4}},\qquad t\geqq 1\,,$$

for some positive constant C.

References

1. Chern, I.-L., Liu, T.-P.: Convergence to diffusion waves of solutions for viscous conservation laws. Commun. Math. Phys. **110**, 503–517 (1987)
2. Kawashima, S.: Large-time behaviors of solutions to hyperbolic-parabolic systems of conservation laws and applications. Proc. Roy. Soc. Edinburgh **106**A, 169–194 (1987)
3. Liu, T.-P.: Nonlinear stability of shock waves for viscous conservation laws. Mem. Am. Math. Soc. No. **328** (1985)
4. Liu, T.-P.: Pointwise convergence to N-waves for solutions of hyperbolic conservation laws. Bull. Inst. Math. Academia Sinica **15**, 1–17 (1987)

Communicated by A. Jaffe

Received July 21, 1988