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Symanzik's Improved Actions from the Viewpoint of the Renormalization Group

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Abstract. We investigate Symanzik's improvement program in a four-dimensional Euclidean scalar field theory with smooth momentum space cutoff. We use Wilson's renormalization group transformation to define the improved actions as a sequence of initial data for the effective action at the fundamental cutoff. This leads to a sequence of solutions to the renormalization group equation. We define the parameters of the improved actions implicitly by conditions on the effective action at a renormalization scale. The improved actions are close approximations to the continuum effective action. We prove their existence to every order of improvement and to every order of renormalized perturbation theory.

1. Introduction

Wilson's effective actions [Wil 74] have become an important tool in quantum field theory. They have proved to be useful in establishing perturbative renormalizability without relying on detailed estimates on Feynman diagrams [Pol 84]. Work in constructive field theory also demonstrates that the investigation of effective actions provides a natural tool to understand the full theory, at least in the case of asymptotically free models [GK 84].

In this paper we address the question of the locality properties of effective actions. Generally effective actions are infinite sums of nonlocal terms. The question arises how to find good, i.e. local, approximations. This important problem has received little attention except within Monte Carlo renormalization group studies [Gup 85].

Symanzik's improved actions [Sym 83] are candidates for local approximations to effective actions. They were found by Symanzik in a different context and he did not investigate their relationship with the renormalization group. The parameters of Symanzik's improved actions are determined by demanding a

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certain rate of convergence for a set of observables in a cutoff theory in the limit when the cutoff is removed [Lüs 84], [Par 85]. In principle the full interacting theory has to be solved in order to determine the improvement parameters. In this paper we show that Symanzik's improved actions lead to a sequence of close approximations to the effective actions. We define the improvement parameters implicitly by conditions on the effective action at a renormalization scale.

We consider a four-dimensional Euclidean real scalar field φ with an ultraviolet cutoff Λ_0 in momentum space. The action $S(\varphi|\Lambda_0)$ is parameterized by a set of bare coupling constants g_a^0 with dimension $d_a \geqq 0$ in units of mass. The renormalization group maps the fundamental theory exactly to an effective theory with ultraviolet cutoff $\Lambda \in [\Lambda_R, \Lambda_0]$ and an effective action $S(\varphi|\Lambda, \Lambda_0, (g_a^0)_{d_a \geqq 0})$. The idea is to integrate out degrees of freedom which correspond to momenta $\Lambda \leqq |p| \leqq \Lambda_0$. Here Λ_R denotes a renormalization scale which we choose of the order of the physical mass scale. The complete map includes a transformation of observables in the fundamental theory to effective observables which preserves expectation values. This is the renormalization of composite operators. Observables which measure the behavior of the theory at large length scales transform trivially. We restrict our attention to such observables, i.e., products of blockspins for instance, and to the effective action itself.

Let the bare coupling constants $(g_a^0)_{d_a \geqq 0}$ depend on Λ_0, Λ_R, and a set of renormalized coupling constants $(g_b^R)_{d_b \geqq 0}$. The bare coupling constants are chosen so that the renormalization conditions

$$g_b(\Lambda, \Lambda_0, (g_a^0)_{d_a \geqq 0})|_{\Lambda = \Lambda_R} = g_b^R \tag{1.1}$$

on the running coupling constants with dimension $d_b \geqq 0$ at the renormalization scale are satisfied in order to take the continuum limit. Polchinski [Pol 84] and Gallavotti [Gal 85] have proved that the limit

$$\lim_{\Lambda_0 \to \infty} S(\varphi|\Lambda, \Lambda_0, (g_a^0(\Lambda_0, \Lambda_R, (g_b^R)_{d_b \geqq 0}))_{d_a \geqq 0}) = S^{\mathrm{cont}}(\varphi|\Lambda, \Lambda_R, (g_b^R)_{d_b \geqq 0}) \tag{1.2}$$

exists to every order in perturbation theory in the renormalized φ^4-coupling constant for $\Lambda \geqq \Lambda_R$. This is the theorem of perturbative renormalizability. (We work within the framework of perturbation theory.) The rate of convergence is estimated by

$$\|S(\cdot|\Lambda, \Lambda_0, (g_a^0(\Lambda_0, \Lambda_R, (g_b^R)_{d_b \geqq 0}))_{d_a \geqq 0}) - S^{\mathrm{cont}}(\cdot|\Lambda, \Lambda_R, (g_b^R)_{d_b \geqq 0})\| \,|_{\Lambda = \Lambda_R} = O\left(\frac{\Lambda_R}{\Lambda_0}\right)^{2-\kappa} \tag{1.3}$$

with a suitable norm $\|\cdot\|$, where $0 < \kappa \ll 1$ controls logarithmic corrections which depend on the order of perturbation theory. Improved renormalization schemes should improve upon this rate of convergence.

Following Symanzik we make an ansatz for the fundamental action $S(\varphi|\Lambda_0)$ of a more general form. It includes nonrenormalizable coupling constants g_a^0 of negative dimension $-2s \leqq d_a < 0$. Here $s = 0, 1, 2, \ldots$ is an improvement index, and $s = 0$ corresponds to standard renormalization theory. The additional bare coupling constants are called improvement parameters. We let them depend on Λ_0, Λ_R, and the renormalized coupling constants $(g_b^R)_{d_b \geqq 0}$ such that the improvement conditions

$$g_c(\Lambda, \Lambda_0, (g_a^0)_{d_a \geqq -2s})|_{\Lambda = \Lambda_R} = g_c^{\mathrm{cont}}(\Lambda, \Lambda_R, (g_b^R)_{d_b \geqq 0})|_{\Lambda = \Lambda_R} \tag{1.4}$$

on the running coupling constants with dimension $-2s \leqq d_c < 0$ at the renormalization scale are satisfied in addition to the renormalization conditions (1.1). It is essential that they be fixed to their continuum values. We will prove that the effective action converges to its continuum limit within this scheme. The crucial observation is that the rate of convergence is speeded up to

$$\begin{aligned}&\|S(\cdot\,|\Lambda, \Lambda_0, (g_a^0(\Lambda_0, \Lambda_R, (g_b^R)_{d_b \geqq 0}))_{d_a \geqq 0})\\&\quad - S^{\text{cont}}(\cdot\,|\Lambda, \Lambda_R, (g_b^R)_{d_b \geqq 0})\| \,|_{\Lambda=\Lambda_R} = O\left(\frac{\Lambda_R}{\Lambda_0}\right)^{2(s+1)-\kappa}.\end{aligned} \tag{1.5}$$

We prove this to every order of improvement and to every order of perturbation theory. The improvement is universal in the sense that it affects all observables which transform trivially under the renormalization group.

The improvement parameters g_a^0 with dimension $-2s \leqq d_a < 0$ depend on Λ_0, Λ_R, and $(g_b^R)_{d_b \leqq 0}$. To achieve improved convergence it suffices to take the leading order terms in an asymptotic expansion in inverse powers of Λ_0 as was proven in [Wie 87]. The expansion coefficients are Symanzik's improvement constants. To compute them with the method which we describe here it is not necessary to have control over the renormalization group flow down to the physical mass scale. The renormalization scale can be chosen to be arbitrarily large. The improvement conditions are then imposed at a large scale in units of the physical mass.

The technical framework of our paper is a perturbative analysis of the renormalization group differential equation along the lines of Polchinski's proof of the perturbative renormalizability of φ_4^4-theory. Differing from Polchinski's work we expand the effective action in terms of normal ordered products of the field φ with respect to the free covariance. The expansion kernels satisfy a Riccati type system of differential equations *without linear terms*. This system is used to iterate estimates on the order of perturbation theory. Infinitesimal renormalization group transformations were first considered by Wilson [Wil 74]. Brydges and Kennedy have proved the existence of nonperturbative solutions to the renormalization group equation for sufficiently small changes of the cutoff [Bry 86].

The paper is organized as follows. In Chap. 2 we establish the renormalization group equation for the effective action and illustrate it graphically. Chapter 3 is devoted to the description of the boundary value problems which we investigate here. We perform the perturbative analysis in Chap. 4 under the assumption that the improvement conditions have well behaved solutions. Theorem 4.12 of the s-improved convergence is proven in Chap. 4. In Chap. 5 we establish the Corollary 5.10 of the uniqueness of the continuum limit. Finally we prove Theorem 6.5 of the existence of solutions to the improvement conditions in Chap. 6. A generalized perturbation theory, where we expand the effective potential into a power series with respect to the bare nonrenormalizable coupling constants, allows us to put sufficient bounds on the improvement parameters.

2. The Renormalization Group Differential Equation

Let φ denote a real scalar field in four Euclidean dimensions with a smooth momentum space ultraviolet (UV)-cutoff Λ_0. The free propagator $v(\Lambda_0)$ is defined by the Fourier transform of

$$v(p|\Lambda_0)=\frac{K(p^2/\Lambda_0^2)}{p^2+m^2}. \tag{2.1}$$

By K we denote a smooth, positive, and monotone cutoff function which takes the value $K(t)=1$ for $0\leqq t\leqq 1-\varepsilon$ and $K(t)=0$ for $t\geqq 1$. Without loss of generality we can assume that $0<|m|\leqq\Lambda_0$. Let the action $S(\varphi|\Lambda_0)$ consist of a quadratic term $S_0(\varphi|\Lambda_0)=(1/2)(\varphi, v(\Lambda_0)^{-1}\varphi)$ and a not necessarily local interaction term $V(\varphi|\Lambda_0)$,

$$S(\varphi|\Lambda_0)=S_0(\varphi|\Lambda_0)+V(\varphi|\Lambda_0). \tag{2.2}$$

By $d\mu_{v(\Lambda_0)}(\varphi)$ we denote the Gaussian measure with covariance $v(\Lambda_0)$. The generating function $Z[J|\Lambda_0]$ of the cutoff Green's functions is an expectation value with respect to $d\mu_{v(\Lambda_0)}(\varphi)$,

$$Z[J|\Lambda_0]=\int d\mu_{v(\Lambda_0)}(\varphi)e^{-V(\varphi|\Lambda_0)+(J,\varphi)}. \tag{2.3}$$

The field independent term in $V(\varphi|\Lambda_0)$ is fixed by the normalization condition $Z[0|\Lambda_0]=1$.

Let us investigate the theory by integrating out the degrees of freedom which correspond to fluctuations with momenta $\Lambda\leqq|p|\leqq\Lambda_0$ in (2.3). For this purpose we split the propagator $v(\Lambda_0)$ into a low momentum part $v(\Lambda)$ and a high momentum part $\Gamma(\Lambda,\Lambda_0)$,

$$v(\Lambda_0)=v(\Lambda)+\Gamma(\Lambda,\Lambda_0). \tag{2.4}$$

This induces a splitting $\varphi=\psi+\zeta$ of the field φ into a block spin ψ and a fluctuation field ζ. The block spin needs not to be rescaled, as no anomalous dimension is generated in perturbation theory. An effective action $S(\psi|\Lambda)$ in the sense of Wilson is defined by the quadratic term $S_0(\psi|\Lambda)=1/2(\psi, v(\Lambda)^{-1}\psi)$ and an effective potential $V(\psi|\Lambda)$,

$$e^{-V(\psi|\Lambda)}=\int d\mu_{\Gamma(\Lambda,\Lambda_0)}(\zeta)e^{-V(\psi+\zeta|\Lambda_0)}. \tag{2.5}$$

Let us rename the block spin and write $V(\varphi|\Lambda)$, where φ has an UV-cutoff Λ. The dependence of $V(\varphi|\Lambda)$ on Λ is described by the renormalization group (RG) functional differential equation [Wil 74, Pol 84]. Let $\dot v(\Lambda)\equiv dv(\Lambda)/d\Lambda$ and

$$\Delta_{\dot v(\Lambda)}\equiv\left(\frac{\delta}{\delta\varphi},\dot v(\Lambda)\frac{\delta}{\delta\varphi}\right)=(2\pi)^4\int d^4p\,\dot v(p|\Lambda)\frac{\delta^2}{\delta\varphi(-p)\delta\varphi(p)}. \tag{2.6}$$

Proposition 2.1. *The effective potential* $V(\varphi|\Lambda)$ *satisfies the* RG *equation*

$$\frac{\partial}{\partial\Lambda}V(\varphi|\Lambda)=\frac{1}{2}\left(\left(\frac{\delta}{\delta\varphi}V(\varphi|\Lambda),\dot v(\Lambda)\frac{\delta}{\delta\varphi}\right)-\Delta_{\dot v(\Lambda)}\right)V(\varphi|\Lambda). \tag{2.7}$$

Proof. Let $\langle\cdot\rangle_{\Gamma(\Lambda,\Lambda_0)}$ denote the expectation value with respect to $d\mu_{\Gamma(\Lambda,\Lambda_0)}(\zeta)$. By the change of covariance formula [GJ 87],

$$\begin{aligned}\frac{\partial}{\partial\Lambda}e^{-V(\psi|\Lambda)}&=\frac{1}{2}\langle\Delta_{\dot\Gamma(\Lambda,\Lambda_0)}e^{-V(\psi+\zeta|\Lambda_0)}\rangle_{\Gamma(\Lambda,\Lambda_0)}\\&=\frac{1}{2}\left\langle\left(\left(\frac{\delta}{\delta\zeta}V(\psi+\zeta|\Lambda_0),\dot\Gamma(\Lambda,\Lambda_0)\frac{\delta}{\delta\zeta}V(\psi+\zeta|\Lambda_0)\right)\right.\right.\\&\qquad\left.\left.-\Delta_{\dot\Gamma(\Lambda,\Lambda_0)}V(\psi+\zeta|\Lambda_0)\right)e^{-V(\psi+\zeta|\Lambda_0)}\right\rangle_{\Gamma(\Lambda,\Lambda_0)},\end{aligned} \tag{2.8}$$

where $\dot{\Gamma}(\Lambda, \Lambda_0) \equiv \partial\Gamma(\Lambda, \Lambda_0)/\partial\Lambda = -\dot{v}(\Lambda)$. In the limit $\Lambda \to \Lambda_0$ the Gaussian measure $d\mu_{\Gamma(\Lambda,\Lambda_0)}(\zeta)$ becomes a Dirac measure. This establishes the RG equation at $\Lambda = \Lambda_0$. $\square$

In the momentum space representation the RG equation takes the form

$$\frac{\partial V(\varphi|\Lambda)}{\partial \Lambda} = \frac{(2\pi)^4}{2} \int d^4p \dot{v}(p|\Lambda) \left(\frac{\delta V(\varphi|\Lambda)}{\delta\varphi(-p)} \frac{\delta V(\varphi|\Lambda)}{\delta\varphi(p)} - \frac{\delta^2 V(\varphi|\Lambda)}{\delta\varphi(-p)\delta\varphi(p)} \right). \tag{2.9}$$

With our choice of an UV-cutoff the right-hand side of the RG equation is invariant under the full Euclidean group, and $\varphi \to -\varphi$. We will assume these symmetries to hold for $V(\varphi|\Lambda)$ by imposing them on the initial data $V(\varphi|\Lambda_0) \equiv V^0(\varphi)$.

Let us expand $V(\varphi|\Lambda)$ in terms of normal ordered products of the field φ with respect to the free propagator $v(\Lambda)$. Normal ordered products are defined by $:P(\varphi):_{v(\Lambda)} \equiv \exp(-1/2\Delta_{v(\Lambda)})P(\varphi)$ for polynomials P of φ,

$$V(\varphi|\Lambda_0) = g_0(\Lambda) + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \left(\prod_{i=1}^{2n} \frac{d^4p_i}{(2\pi)^4} \right) (2\pi)^4 \delta\left(\sum_{i=1}^{2n} p_i \right)$$
$$\times V_{2n}(p_1, \ldots, p_{2n}|\Lambda) : \left(\prod_{i=1}^{2n} \varphi(p_i) \right) :_{v(\Lambda)} . \tag{2.10}$$

The dependence of the V-kernels $V_{2n}(\cdot|\Lambda)$ on Λ is described by a Riccati type coupled system of differential equations. Let $\mathscr{G}$ denote the index set

$$\{(m, l, k) : 1 \leqq m,\ 1 \leqq l \leqq m,\ 0 \leqq k \leqq 2\min\{l, m-l+1\} - 1\}$$

of integer three tupels, and $F_{m,l,k}$ the combinatorial factor

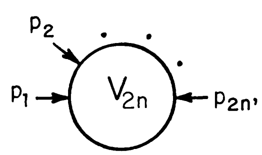
$$(2(k+1)!(2l-k-1)!(2(m-l)-k+1)!)^{-1} .$$

Proposition 2.2. *The V-kernels satisfy the* RG *equation*

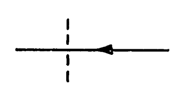
$$\frac{\partial}{\partial\Lambda} V_{2n}(p_1, \ldots, p_{2n}|\Lambda) = \sum_{\substack{(m,l,k)\in\mathscr{G} \\ n=m-k}} F_{m,l,k} \int \left(\prod_{j=1}^{k+1} \frac{d^4q_j}{(2\pi)^4} \right)$$
$$\times (2\pi)^4 \delta\left(\sum_{i=1}^{2l-k-1} p_{\pi(i)} + \sum_{j=1}^{k+1} q_j \right) \left(\frac{\partial}{\partial\Lambda} \prod_{j=1}^{k+1} v(q_j|\Lambda) \right)$$
$$\times V_{2l}(p_{\pi(1)}, \ldots, p_{\pi(2l-k-1)}, q_1, \ldots, q_{k+1}|\Lambda)$$
$$\times V_{2(m-l+1)}(p_{\pi(2l-k)}, \ldots, p_{\pi(2n)}, -q_1, \ldots, -q_{k+1}|\Lambda) . \tag{2.11}$$

Here the sum $\sum_{(m,l,k)\in\mathscr{G}}$ includes a summation over all permutations $\pi \in \mathscr{S}_{2(m-k)}$. The Proof of Proposition 2.2 is deferred to the appendix.

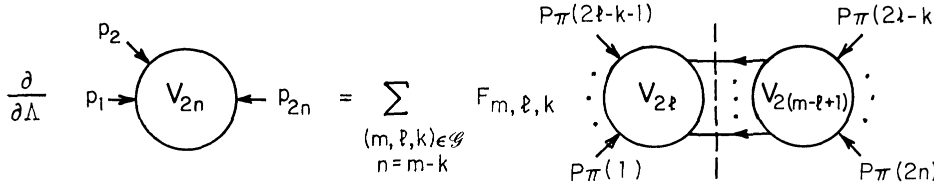
Let us introduce the following graphical notation

$$(2\pi)^4 \delta\left(\sum_{i=1}^{2n} p_i \right) V_{2n}(p_1, \ldots, p_{2n}|\Lambda) \equiv$$ 

$v(q|\Lambda) \equiv$,

$\dot{v}(q|\Lambda) \equiv$  . (2.12)

Reversing arrow means multiplication with -1. Integration over internal momenta in connected diagrams is understood. A cut means differentiation with respect to Λ. The RG equation for the V-kernels takes the form



(2.13)

The graphical representation illustrates how the RG builds up the effective potential $V(\varphi|\Lambda)$ from its initial value $V(\varphi|\Lambda_0)$ when the cutoff is lowered. The right-hand side of (2.13) is an infinite sum of connected diagrams with a fixed number of external legs. Each of these is well defined due to the UV- and IR-cutoff in the free propagator.

3. The Renormalization and Improvement Conditions

Let us define a sequence of initial value problems for the effective potential $V(\varphi|\Lambda)$ over a scale interval $[\Lambda_R, \Lambda_0] \ni \Lambda$, where Λ_R is the *renormalized* and Λ_0 the *bare* scale.

A sequence of bare potentials $V^s(\varphi|\Lambda_0) \equiv V^{0s}(\varphi)$, which is labelled by the *improvement index* $s \in \{0, 1, 2, \ldots\}$, is defined by

$$V^{0s}(\varphi) = \sum_{d_a \geqq -2s} g_a^{0s} \int d^4x \, \mathcal{O}_a(x, \varphi|\Lambda_0). \tag{3.1}$$

Here $\mathcal{O}_a(x, \varphi|\Lambda_0)$ denotes a local composite field which is normal ordered with respect to $v(\Lambda_0)$. The corresponding bare coupling constant g_a^{0s} has the classical dimension d_a in units of mass. We sum over a complete set of linearly independent $\mathcal{O}_a(x, \varphi|\Lambda_0)$ with $d_a \geqq -2s$ which transform like scalars under the group $O(4)$ in Euclidean space. The initial value $V^{0s}(\varphi)$ is parameterized in terms of V-kernels $V_{2n}^{0s}(p_1, \ldots, p_{2n})$ with dimension $[4-2n]$ in units of mass and $O(4)$ invariant Taylor expansions of order $4-2n+2s$,

$$V_{2n}^{0s}(p_1, \ldots, p_{2n}) = \sum_{\substack{d_a \geqq -2s \\ \{a\} = n}} g_a^{0s} T_a(p_1, \ldots, p_{2n}). \tag{3.2}$$

Here $\{a\}$ denotes the degree of the V-kernel whose Taylor coefficient is g_a^{0s}. In other words $g_a^{0s} = (\mathbf{D}_a V_{2\{a\}}^{0s})(0, \ldots, 0)$, where $\mathbf{D}_a$ is a differential operator of order $4-2\{a\}-d_a$. By $T_a(p_1, \ldots, p_{2n})$ we denote the corresponding invariant homogeneous polynomials of degree $4-2n-d_a$.

Consider for instance $s=1$. The bare V-kernels $V_{2n}^{01}(p_1, \ldots, p_{2n})$ take the form

$$\begin{aligned} V_{2n}^{01}(p_1, \ldots, p_{2n}) = {} & \delta_{n,1}(g_1^{01} + g_2^{01} p_1^2 + g_4^{01}(p_1^2)^2) \\ & + \delta_{n,2}\left(g_3^{01} + g_5^{01} \sum_{i=1}^{3} p_i^2 + g_6^{01} \sum_{\substack{i,j=1 \\ i \neq j}}^{3} p_i p_j\right) + \delta_{n,3} g_7^{01}. \end{aligned} \tag{3.3}$$

The parameters $(g_1^{01}, g_1^{01}, g_3^{01}) \equiv (g_a^{01})_{d_a \geqq 0}$ are the mass counterterm, the wave function renormalization, and the bare φ^4-coupling constant. The additional parameters $(g_4^{01}, \ldots, g_7^{01}) \equiv (g_a^{01})_{d_a = -2}$ are the improvement counterterms of first order [Sym 84].

The integration of the RG equation (2.7) with the initial condition (3.1) yields a trajectory

$$[\Lambda_R, \Lambda_0]_3 \mapsto V(\varphi|\Lambda, \Lambda_0, (g_a^{0s})_{d_a \geqq -2s}) \equiv V^s(\varphi|\Lambda),$$

where the dependence on the initial data is shown explicitly. Its boundary value $V^s(\varphi|\Lambda_R) \equiv V^{Rs}(\varphi)$ is called renormalized effective potential. It has the form

$$V^{Rs}(\varphi) = \sum_{d_a \geqq 2s} g_a^{Rs} \int d^4x \mathcal{O}_a(x, \varphi|\Lambda_R) + R^{Rs}(\varphi), \tag{3.4}$$

where R^{Rs} denotes a nonlocal remainder term.

Let us impose the following *renormalization conditions* on the boundary values $g_a^{Rs} \equiv g_a^s(\Lambda_R)$ of the running coupling constants $g_a^s(\Lambda) \equiv g_a(\Lambda, \Lambda_0, (g_b^{0s})_{d_b \geqq -2s})$ with dimension $d_a \geqq 0$,

$$g_a(\Lambda_R, \Lambda_0, (g_b^{0s})_{d_b \geqq -2s}) = \delta_{a,3} g_3^R, \tag{3.5}$$

where g_3^R denotes the renormalized φ^4-coupling constant.

In the unimproved case, which corresponds to $s=0$, the renormalization conditions determine implicitly the bare parameters $(g_a^{00})_{d_a \geqq 0}$ as functions of Λ_0, Λ_R, and g_3^R. The *continuum limit* of the effective action is defined by

$$V^{\text{cont}}(\varphi|\Lambda, \Lambda_R, g_3^R) = \lim_{\Lambda_0 \to \infty} V(\varphi|\Lambda, \Lambda_0, (g_a^{00}(\Lambda_0, \Lambda_R, g_3^R))_{d_a \geqq 0}) \tag{3.6}$$

for $\Lambda \geqq \Lambda_R$, where the parameters $(g_a^{00})_{d_a \geqq 0}$ are tuned according to the renormalization conditions. The RG maps the continuum theory exactly to an effective theory with UV-cutoff Λ. The price is the nonlocality of $V^{\text{cont}}(\varphi|\Lambda, \Lambda_R, g_3^R) \equiv V^{\text{cont}}(\varphi|\Lambda)$. There is much evidence that φ_4^4-theory possesses a trivial continuum limit beyond perturbation theory [Wil 74]. The implicit equations for $(g_a^{00})_{d_a \geqq 0}$ do not admit solutions for arbitrary Λ_0, once g_3^R is fixed. However in perturbation theory the continuum limit exists to every order of g_3^R [Pol 84, Gal 85]. The speed of convergence can be estimated by $\|V^0(\cdot|\Lambda_R) - V^{\text{cont}}(\cdot|\Lambda_R)\| = O(\Lambda_R/\Lambda_0)^{2-\kappa}$, where $0 < \kappa \ll 1$ takes into account logarithmic corrections which depend on the order of perturbation theory. We will later define the norm $\|\cdot\|$ precisely.

For a given index $s \geqq 1$, let us impose the following *improvement conditions* on the coupling constants g_a^{Rs} with $-2s \leqq d_a < 0$:

$$g_a(\Lambda_R, \Lambda_0, (g_b^{0s})_{d_b \geqq -2s}) = g_a^{R\,\text{cont}}. \tag{3.7}$$

Here $g_a^{R\,\text{cont}} = g_a^{\text{cont}}(\Lambda_R)$ are the boundary values of the continuum running coupling constants $g_a^{\text{cont}}(\Lambda) = g_a^{\text{cont}}(\Lambda, \Lambda_R, g_3^R)$. The renormalization and improvement conditions together determine the parameters $(g_a^{0s})_{d_a \geqq -2s}$ as functions of Λ_0, Λ_R, and g_3^R. We will prove to every order of perturbation theory in g_3^R that $V^{Rs}(\varphi)$ converges to its unique limit $V^{R\,\text{cont}}(\varphi)$, when the cutoff Λ_0 is removed according to this scheme. Here the crucial observation is

$$\|V^s(\cdot|\Lambda_R) - V^{\text{cont}}(\cdot|\Lambda_R)\| = O(\Lambda_R/\Lambda_0)^{2(s+1)-\kappa},$$

which defines Symanzik's improvement concept in the asymptotic regime $\Lambda_R/\Lambda_0 \ll 1$.

4. Renormalization Group s-Improved Perturbation Theory

Let the bare parameters $(g_b^{0s})_{d_b \geqq -2s}$ be tuned as functions of Λ_0, Λ_R, and g_3^R such that the renormalization and improvement conditions are satisfied. To investigate the V-kernels in s-improved perturbation theory we expand

$$V_{2n}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, (g_a^{0s}(\Lambda_0, \Lambda_R, g_3^R))_{d_a \geqq -2s}) = \sum_{\alpha=0}^{\infty} V_{2n}^{s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, \Lambda_R)(g_3^R)^\alpha. \tag{4.1}$$

The expansion coefficients $V_{2n}^{s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, \Lambda_R) \equiv V_{2n}^{s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda)$ describe multiparticle interactions whose strength can be measured with the norm

$$\| V_{2n}^{s(\alpha)}(\cdot|\Lambda)\| \equiv \sup_{|p_1|, \ldots, |p_{2n}| \leqq \Lambda} |V_{2n}^{s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda)|, \tag{4.2}$$

which exists as $V_{2n}^{s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda)$ has the compact support $|p_i| \leqq \Lambda$ for $1 \leqq i \leqq 2n$. Let $P(n) \equiv P^{(n)}[\ln \Lambda_0/\Lambda_R]$ denote a polynomial in $\ln(\Lambda_0/\Lambda_R)$ of degree n with positive coefficients. We define $P(n) \equiv 0$ for $n < 0$.

The initial conditions for the perturbative coupling constants with negative dimension are obtained from the expansion

$$g_a^{0s}(\Lambda_0, \Lambda_R, g_3^R) = \sum_{\alpha=0}^{\infty} g_a^{0s(\alpha)}(\Lambda_0, \Lambda_R)(g_3^R)^\alpha.$$

Let us state the following *assumptions* upon the initial values $g_a^{0s(\alpha)}(\Lambda_0, \Lambda_R)$ with $-2s \leqq d_a < 0$.

(1) To zeroth order and to all orders $\alpha \geqq 1$ for $\{a\} > \alpha + 1$,

$$g_a^{0s(\alpha)}(\Lambda_0, \Lambda_R) = 0. \tag{4.3}$$

(2) To all orders $\alpha \geqq 1$ for $\{a\} \leqq \alpha + 1$,

$$|g_a^{0s(\alpha)}(\Lambda_0, \Lambda_R)| \leqq \Lambda_0^{d_a} P(2\alpha - \{a\}). \tag{4.4}$$

We will later prove these assumptions to be satisfied.

The boundary conditions for the perturbative coupling constants $g_a^{s(\alpha)}(\Lambda) = g_a^{s(\alpha)}(\Lambda, \Lambda_0, \Lambda_R)$ with $d_a \geqq 0$ are the renormalization conditions

$$g_a^{s(\alpha)}(\Lambda_R) = \delta_{a,3}\delta_{\alpha,1}. \tag{4.5}$$

The expansion coefficients $V_{2n}^s(p_1, \ldots, p_{2n}|\Lambda)$ satisfy the RG equation with the above boundary conditions on the running coupling constants $g_a^{s(\alpha)}(\Lambda)$ with dimension $d_a \geqq -2s$. All V-kernels vanish to zeroth order and only finitely many V-kernels do not vanish to order α, namely those with degrees $1 \leqq n \leqq \alpha + 1$. This implies that $(\partial V_{2n}^{s(\alpha)}/\partial \Lambda)(p_1, \ldots, p_{2n}|\Lambda)$ is to order α a finite sum of terms which are determined by V-kernels of lower orders $1 \leqq \beta \leqq \alpha - 1$. We prove estimates on the nonvanishing V-kernels by induction on the order α, where the boundary conditions are taken care of by Taylor expansion. As a result $\| V_{2n}^{s(\alpha)}(\cdot|\Lambda)\|$ is bounded by $\Lambda^{4-2n}P(2\alpha - n)$. Here $P(\cdot)$ denotes a polynomial of the above form and Λ^{4-2n} is a dimension factor. The proofs are written out explicitly, as these will serve for reference later.

The effective potential $V(\psi|\Lambda)$ as defined in (2.5) is a generating function, where the block spin plays the role of a source. It generates the free propagator amputated connected Greens functions in the theory with UV-cutoff Λ_0, which is defined by the free propagator $\Gamma(\Lambda, \Lambda_0)$ and the interaction $V(\zeta|\Lambda_0)$ [Mac 85]. As a consequence the V-kernels have perturbation expansions in terms of *connected* diagrams. The following Proposition 4.1 expresses the fact that with a given number of vertices of a specific type one can only build connected diagrams with a certain maximal number of external legs. We restrict our attention to solutions of the RG equation with this property [Pol 84]. In Proposition 4.1 we prove that there *exists* a solution of the RG equation (2.11) with the appropriate vanishing properties.

Proposition 4.1. *To order $\alpha=0$ for all degrees $n\geqq 1$ and to all orders $\alpha\geqq 1$ for all degrees $n>\alpha+1$,*

$$V_{2n}^{s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda)=0\,. \tag{4.6}$$

Proof. The vanishing is consistent with the above boundary conditions. It also satisfies the RG equation. $(\partial V_{2n}^{s(\alpha)}/\partial\Lambda)(\cdot)$ is an infinite sum of terms proportional to $V_{2l}^{s(\beta)}(\cdot)V_{2(m-l+1)}^{s(\alpha-\beta)}(\cdot)$ with $n=m-k$ and $0\leqq\beta\leqq\alpha$. Suppose that $n>\alpha+1$ and $l\leqq\beta+1$, then $m-l+1>\alpha-\beta+k+1$. Therefore each term contains a vanishing kernel. $\square$

A dimensional analysis yields the following estimates for the free propagator $v(p|\Lambda)$ with UV-cutoff Λ.

Lemma 4.2. *There exist positive constants A_0, A_1, and B_n for all $n\geqq 0$, such that*

$$\begin{gathered}\int\frac{d^4p}{(2\pi)^4}|v(p|\Lambda)|\leqq A_0\Lambda^2\,,\\ \int\frac{d^4p}{(2\pi)^4}|\dot v(p|\Lambda)|\leqq A_1\Lambda\,,\\ \sum_{\mu_1,\ldots,\mu_n=1}^{4}\sup_{p\in\mathbb{R}^4}\left|\frac{\partial^n\dot v(p|\Lambda)}{\partial p^{\mu_1}\ldots\partial p^{\mu_n}}\right|\leqq\frac{B_n}{\Lambda^{n+3}}\,.\end{gathered} \tag{4.7}$$

Let us introduce the following notation for $i,j\in\{1,\ldots,2n\}$, and

$$\mu\in\{1,\ldots,4\}:\partial_\mu^{[i,j]}\equiv\partial/\partial p_i^\mu-\partial/\partial p_j^\mu\,.$$

Given $N=\{(i_l,j_l,\mu_l):1\leqq l\leqq|N|\}$ as above, we define

$$\partial_N\equiv\prod_{(i,j,\mu)\in N}\partial_\mu^{[i,j]},\quad\text{and}\quad p_N\equiv\prod_{(i,j,\mu)\in N}p_i^\mu\,.$$

Let $\partial_N\equiv 1$ for $|N|=0$.

Proposition 4.3. *To every order $\alpha\geqq 1$ for all degrees $1\leqq n\leqq\alpha+1$ the norm of the V-kernels is bounded as follows:*

$$\|(\partial_N V_{2n}^{s(\alpha)})(\cdot|\Lambda)\|\leqq\Lambda^{4-2n-|N|}P(2\alpha-n)\,. \tag{4.8}$$

Proof. The estimate is proved by induction on the order of perturbation theory. All V-kernels are independent of Λ to first order, as Proposition 4.1 implies $(\partial V_{2n}^{s(1)}/\partial\Lambda)(p_1,\ldots,p_{2n}|\Lambda)=0$. By the initial conditions:

$$V_{2n}^{s(1)}(p_1,\ldots,p_{2n}|\Lambda)=\delta_{n,2}+\sum_{\substack{-2s\leqq d_a<0\\ \{a\}=n}}g_a^{0s(1)}T_a(p_1,\ldots,p_{2n})\,. \tag{4.9}$$

The assumption (4.4) implies

$$\|V_{2n}^{s(1)}(\cdot|\Lambda)\| \leqq \delta_{n,2} + \sum_{\substack{-2s \leqq d_a < 0 \\ \{a\}=n}} \Lambda_0^{d_a} P(2-\{a\}) O(1) \cdot \Lambda^{4-2n-d},$$
$$\leqq \Lambda^{4-2n} P(2-n), \tag{4.10}$$

which proves the estimate (4.8) to first order in the case $|N|=0$. The general case follows by application of ∂_N to Eq. (4.9).

Let us suppose that the estimate is true to order $\alpha-1 \geqq 1$. The RG equation together with Proposition 4.1 and Lemma 4.2 yields the inequality

$$\left\|\left(\frac{\partial V_{2n}^{s(\alpha)}}{\partial \Lambda}\right)(\cdot|\Lambda)\right\| \leqq \sum_{\beta=1}^{\alpha-1} \sum_{\substack{(m,l,k)\in\mathscr{G} \\ m-k=n \\ m-\alpha+\beta \leqq l \leqq \beta+1}} O(1) \cdot \Lambda^{2k-3}$$
$$\times \|V_{2l}^{s(\beta)}(\cdot|\Lambda)\| \cdot \|V_{2(m-l+1)}^{s(\alpha-\beta)}(\cdot|\Lambda)\| \tag{4.11}$$

to order α. Here the right-hand side is a finite sum of terms which are bounded by the induction hypothesis. For all $1 \leqq n \leqq \alpha+1$ simultaneously we have the estimate

$$\left\|\left(\frac{\partial V_{2n}^{s(\alpha)}}{\partial \Lambda}\right)(\cdot|\Lambda)\right\| \leqq \sum_{\beta=1}^{\alpha-1} \sum_{\substack{(m,l,k)\in\mathscr{G} \\ m-k=n \\ m-\alpha+\beta \leqq l \leqq \beta+1}} O(1) \cdot \Lambda^{2k-3}$$
$$\times \Lambda^{4-2l} P(2\beta-l) \cdot \Lambda^{4-2(m-l+1)} P(2(\alpha-\beta)-(m-l+1)),$$
$$\leqq \Lambda^{3-2n} P(2\alpha-n-1). \tag{4.12}$$

When ∂_N is applied to the RG equation (2.11), each differentiation with respect to the momentum variables acts on either of the factors $\dot{v}(\cdot)$, $V_{2l}^{s(\beta)}(\cdot)$, or $V_{2(m-l+1)}^{s(\alpha-\beta)}(\cdot)$. We suppose that the q_{k+1}-integral has been performed. Taking norms all of these factors are bounded by Lemma 4.2 and the induction hypothesis. The above argument leads to the estimate

$$\left\|\left(\frac{\partial \partial_N V_{2n}^{s(\alpha)}}{\partial \Lambda}\right)(\cdot|\Lambda)\right\| \leqq \Lambda^{3-2n-|N|} P(2\alpha-n-1). \tag{4.13}$$

We integrate this inequality from Λ_0 to Λ for $4-2n-|N| < -2s$ with the initial condition $(\partial_N V_{2n}^{0s(\alpha)})(\cdot)=0$ which completes the induction in this case,

$$\|(\partial_N V_{2n}^{s(\alpha)})(\cdot|\Lambda)\| \leqq \int_\Lambda^{\Lambda_0} d\Lambda' \left\|\left(\frac{\partial \partial_N V_{2n}^{s(\alpha)}}{\partial \Lambda'}\right)(\cdot|\Lambda')\right\|$$
$$\leqq \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \Lambda'^{4-2n-|N|} P(2\alpha-n-1)$$
$$\leqq \Lambda^{4-2n-|N|} P(2\alpha-n). \tag{4.14}$$

For the running coupling constants $g_a^{s(\alpha)}(\Lambda)$ formula (4.13) leads to the estimate

$$\left|\left(\frac{\partial g_a^{s(\alpha)}}{\partial \Lambda}\right)(\Lambda)\right| \leqq \Lambda^{d_a-1} P(2\alpha-\{a\}-1), \tag{4.15}$$

which is integrated from Λ_0 to Λ for $-2s \leqq d_a < 0$. Taking into account the assumption (4.4) we have

$$\begin{aligned}|g_a^{s(\alpha)}(\Lambda)| &\leqq |g_a^{0s(\alpha)}| + \int_{\Lambda}^{\Lambda_0} d\Lambda' \left|\left(\frac{\partial g_a^{s(\alpha)}}{\partial \Lambda'}\right)(\Lambda')\right| \\ &\leqq \Lambda_0^{d_a} P(2\alpha - \{a\}) + \int_{\Lambda}^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \Lambda'^{d_a} P(2\alpha - \{a\} - 1) \\ &\leqq \Lambda^{d_a} P(2\alpha - \{a\}). \end{aligned} \tag{4.16}$$

For $d_a \geqq 0$ we integrate (4.15) from Λ_R to Λ, the initial values vanishing by the renormalization conditions to order $\alpha \geqq 2$,

$$\begin{aligned}|g_a^{s(\alpha)}(\Lambda)| &\leqq \int_{\Lambda_R}^{\Lambda} d\Lambda' \left|\left(\frac{\partial g_a^{s(\alpha)}}{\partial \Lambda'}\right)(\Lambda')\right|, \\ &\leqq \int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \Lambda'^{d_a} P(2\alpha - \{a\} - 1) \\ &\leqq \Lambda^{d_a} P(2\alpha - \{a\}). \end{aligned} \tag{4.17}$$

Finally we reconstruct the kernels with $4-2n \geqq -2s$ by Taylor expansion,

$$\begin{aligned} V_{2n}^{s(\alpha)}(p_1, \dots, p_{2n}|\Lambda) = &\sum_{\substack{d_a \geqq -2s \\ \{a\}=n}} g_a^{s(\alpha)}(\Lambda) T_a(p_1, \dots, p_{2n}) \\ &+ \frac{1}{(2n)^{4-2n+2s}} \sum_{|N|=4-2n+2s} \int_0^1 dt \frac{(1-t)^{5-2n+2s}}{(5-2n+2s)!} \\ &\times (\partial_N V_{2n}^{s(\alpha)})(tp_1, \dots, tp_{2n}|\Lambda) p_N, \end{aligned} \tag{4.18}$$

which yields the estimate

$$\begin{aligned} \| V_{2n}^{s(\alpha)}(p_1, \dots, p_{2n}|\Lambda) \| \leqq &\sum_{\substack{d_a \geqq -2s \\ \{a\}=n}} \Lambda^{d_a} P(2\alpha - \{\alpha\}) \cdot O(1) \Lambda^{4-2n-d_a} \\ &+ \sum_{|N|=4-2n+2s} \Lambda^{4-2n-|N|} P(2\alpha - n) \cdot O(1) \Lambda^{|N|} \\ \leqq &\Lambda^{4-2n} P(2\alpha - n). \end{aligned} \tag{4.19}$$

Here we used the bounds (4.16) and (4.17) on the running coupling constants and the bound (4.14) for the remainder term. An analogous argument for the derivatives of the V-kernels completes the induction. □

We will now investigate a set of auxiliary quantities which we call W-kernels. At Λ_R they are derivatives of the V-kernels with respect to g_a^{Rs}, the renormalized coupling constants with $d_a \geqq -2s$. The bare coupling constants depend on the latter implicitly through the renormalization and improvement conditions.

Definition 4.4. Let us define quantities

$$W_{2n,b}(p_1, \dots, p_{2n}|\Lambda, \Lambda_0, (g_c^{0s})_{d_c \geqq -2s}) \equiv W_{2n,b}^s(p_1, \dots, p_{2n}|\Lambda)$$

by

$$W_{2n,b}(p_1,\dots,p_{2n}|\Lambda,\Lambda_0,(g_c^{0s})_{d_c\geqq -2s})\equiv(\mathrm{W}_b^s[V_{2n}])(p_1,\dots,p_{2n}|\Lambda,\Lambda_0,(g_c^{0s})_{d_c\geqq -2s})$$
$$\equiv\sum_{d_c\geqq -2s}\left(\frac{\partial V_{2n}}{\partial g_a^{0s}}\right)(p_1,\dots,p_{2n}|\Lambda,\Lambda_0,(g_c^{0s})_{d_c\geqq -2s})$$
$$\times\left[\left(\frac{\partial g}{\partial g^{0s}}\right)(\Lambda,\Lambda_0,(g_c^{0s})_{d_c\geqq -2s})\right]_{a,b}^{-1}. \tag{4.20}$$

Here $(\partial g/\partial g^{0s})^{-1}$ denotes an inverse matrix, and W_b^s a linear differential operator with $d_b\geqq -2s$.

The W-kernels are completely determined by the V-kernels. For the further analysis however, their dependence on Λ is better characterized by the following RG equation.

Lemma 4.5. *The W-kernels satisfy the following system of differential equations,*

$$\left(\frac{\partial W_{2n,b}^s}{\partial\Lambda}\right)(p_1,\dots,p_{2n}|\Lambda)=\sum_{(m,l,k)\in\mathscr{G}}F_{m,l,k}\int\left(\prod_{j=1}^{k+1}\frac{d^4q_j}{(2\pi)^4}\right)$$
$$\times\left[\delta_{n,m-k}\mathbf{1}|_{\mathbf{p}'=\mathbf{p}}-\sum_{d_a\geqq -2s}\delta_{\{a\},m-k}W_{2n,a}^s(p_1,\dots,p_{2n}|\Lambda)\cdot D_a|_{\mathbf{p}'=0}\right]$$
$$\times(2\pi)^4\delta\left(\sum_{i=1}^{2l-k-1}p'_{\pi(i)}+\sum_{j=1}^{k+1}q_j\right)\left(\frac{\partial}{\partial\Lambda}\prod_{j=1}^{k+1}v(q_j|\Lambda)\right)$$
$$\times(W_{2l,b}^s(p'_{\pi(1)},\dots,p'_{\pi(2l-k-1)},q_1,\dots,q_{k+1}|\Lambda)$$
$$\times V_{2(m-l+1)}^s(p'_{\pi(2l-k)},\dots,p'_{\pi(2(m-k))},-q_1,\dots,-q_{k+1}|\Lambda)+W\leftrightarrow V). \tag{4.21}$$

Here $\mathbf{1}|_{x'=x}f(x')\equiv f(x)$, *and the sum runs over all a such that* $d_a\geqq -2s$. *The initial conditions at* Λ_0 *for the W-kernels are*

$$W_{2n,b}^{0s}(p_1,\dots,p_{2n})=\sum_{\substack{d_a\geqq -2s\\ \{a\}=n}}\delta_{a,b}T_a(p_1,\dots,p_{2n}). \tag{4.22}$$

The proof is deferred to the appendix.

The kernel $W_{2n,b}^s(p_1,\dots,p_{2n}|\Lambda)$ is expanded exactly like the kernel $V_{2n}^s(p_1,\dots,p_{2n}|\Lambda)$, i.e. first in a Taylor series of order $4-2n+2s$ with remainder term and second in an s-improved perturbation series. The Taylor coefficients are constants.

Proposition 4.6. *The zeroth order W-kernels are independent of* Λ *and given by*

$$W_{2n,b}^{s(0)}(p_1,\dots,p_{2n}|\Lambda)=\sum_{\substack{d_a\geqq -2s\\ \{a\}=n}}\delta_{a,b}T_a(p_1,\dots,p_{2n}). \tag{4.23}$$

To every order $\alpha\geqq 0$ *and for all degrees* $n>\alpha+s+2$,

$$W_{2n,b}^{s(\alpha)}(p_1,\dots,p_{2n}|\Lambda)=0. \tag{4.24}$$

Proof. As the zeroth order V-kernels vanish, we have $(\partial W_{2n,b}^{s(0)}/\partial\Lambda)(\cdot)=0$ by (4.21). The initial conditions (4.22) imply (4.23). They are compatible with (4.24) as the highest degree W-kernel with nonvanishing zeroth order has $n=s+2$. In the RG equation $(\partial W^{s(\alpha)}_{2n,b}/\partial \Lambda)(\cdot)$ is an infinite sum of terms which are either proportional to $W^{s(\beta)}_{2l,b}(\cdot)V^{s(\alpha-\beta)}_{2(m-l+1)}(\cdot)$ with $n=m-k$ or

$$W^{s(\gamma)}_{2n,a}(\cdot)W^{s(\beta)}_{2l,b}(\cdot)V^{s(\alpha-\beta-\gamma)}_{(m-l+1)}(\cdot)$$

with $\{a\}=m-k$. Both types contain a vanishing kernel for $n>\alpha+s+2$. Consider for instance the first type with $l\leqq\beta+s+2$, then $m-l+1>\alpha-\beta+k+1$ and Proposition 4.1 applies. □

Proposition 4.7. *The norm of the W-kernels is bounded to every order $\alpha\geqq 0$ and for all degrees $1\leqq n\leqq\alpha+s+2$ as follows:*

$$\|(\partial_N W^{s(\alpha)}_{2n,b})(\cdot|\Lambda)\|\leqq \Lambda^{4-2n-d_b-|N|}P(2\alpha-n+\{b\}). \tag{4.25}$$

Proof. To order zero $W^{s(0)}_{2n,b}(p_1,\ldots,p_{2n}|\Lambda)=\delta_{n,\{b\}}T_b(p_1,\ldots,p_{2n})$. Here $T_b(p_1,\ldots,p_{2n})$ is a homogeneous polynomial of degree $4-2n-d_b$ with $d_b\geqq -2s$. The above estimate is satisfied to order zero.

Let us suppose that it holds to order $\alpha-1\geqq 0$. By Lemma 4.5, Proposition 4.6, and Lemma 4.2 the W-kernels satisfy the inequality

$$\begin{aligned}\left\|\left(\frac{\partial W^{s(\alpha)}_{2n,b}}{\partial\Lambda}\right)(\cdot|\Lambda)\right\| &\leqq \sum_{\beta=1}^{\alpha-1}\sum_{\substack{(m,l,k)\in\mathcal{G}\\ m-k=n\\ m-\alpha+\beta\leqq l\leqq\beta+s+2}} O(1)\cdot\Lambda^{2k-3}\cdot\|W^{s(\beta)}_{2l,b}(\cdot|\Lambda)\|\;\|V^{s(\alpha-\beta)}_{(2m-l+1)}(\cdot|\Lambda)\|\\ &+\sum_{\beta+\gamma+\delta=\alpha}\sum_{d_a\geqq -2s}\sum_{\substack{(m,l,k)\in\mathcal{G}\\ m-k=\{\alpha\}\\ m-\delta\leqq l\leqq\gamma+s+2}}\sum_{|M_1|+|M_2|\leqq 4-2\{a\}-d_a} O(1)\cdot\Lambda^{2k+2\{a\}+d_a+|M_1|+|M_2|-7}\\ &\times\|W^{s(\beta)}_{2n,a}(\cdot\,\|\Lambda)\|\cdot\|(\partial_{M_1}W^{s(\gamma)}_{2l,b})(\cdot|\Lambda)\|\cdot\|(\partial_{M_2}V^{s(\delta)}_{2(m-l+1)})(\cdot|\Lambda)\|+W\leftrightarrow V\end{aligned}\tag{4.26}$$

to order α. Here M_1 and M_2 denote index sets of triples (i,j,μ) with $1\leqq i,j\leqq 2l$ and $1\leqq i,j\leqq 2(m-l+1)$ respectively (and $1\leqq\mu\leqq 4$). By $W\leftrightarrow V$ we denote the terms, where W and V are interchanged under the sums over $(m,l,k)\in\mathcal{G}$. The right-hand side is bounded by the induction hypothesis and Proposition 4.3. It is a finite sum of terms in which no order α W-kernels appear. The inequality (4.26) yields the estimate

$$\begin{aligned}\left\|\left(\frac{\partial W^{s(\alpha)}_{2n,b}}{\partial\Lambda}\right)(\cdot|\Lambda)\right\| &\leqq \sum_{\beta=1}^{\alpha-1}\sum_{\substack{(m,l,k)\in\mathcal{G}\\ m-k=n\\ m-\alpha+\beta\leqq l\leqq\beta+s+2}} O(1)\cdot\Lambda^{2k-3}\Lambda^{4-2l-d_b}P(2\beta-l+\{b\})\\ &\quad\times\Lambda^{4-2(m-l+1)}P(2(\alpha-\beta)-(m-l+1))\\ &\quad+\sum_{\beta+\gamma+\delta=\alpha}\sum_{d_a\geqq -2s}\sum_{\substack{(m,l,k)\in\mathcal{G}\\ m-k=\{a\}\\ m-\delta\leqq l\leqq\gamma+s+2}}\sum_{|M_1|+|M_2|\leqq 4-2\{a\}-d_a} O(1)\\ &\quad\times\Lambda^{2k+2\{a\}+d_a+|M_1|+|M_2|-7}\cdot\Lambda^{4-2n-d_a}P(2\beta-n+\{a\})\\ &\quad\times\Lambda^{4-2l-d_b-|M_1|}P(2\gamma-l+\{b\})\cdot\Lambda^{4-2(m-l+1)-|M_2|}\\ &\quad\times P(2\delta-(m-l+1))\\ &\leqq\Lambda^{3-2n-d_b}P(2\alpha-n+\{b\}-1).\end{aligned}\tag{4.27}$$

By a similar argument we conclude that

$$\left\|\left(\frac{\partial \partial_N W_{2n,b}^{s(\alpha)}}{\partial \Lambda}\right)(\cdot|\Lambda)\right\| \leqq \Lambda^{3-2n-d_b-|N|} P(2\alpha-n+\{b\}-1) \tag{4.28}$$

for all ∂_N with $|N| \geqq 1$ to order α.

The inequality (4.28) is integrated from Λ_0 to Λ for $4-2n-|N| < -2s$ with the initial condition $(\partial_N W_{2n,b}^{0s(\alpha)})(\cdot|\Lambda)=0$:

$$\begin{aligned} \|(\partial_N W_{2n,b}^{s(\alpha)})(\cdot|\Lambda)\| &\leqq \int_\Lambda^{\Lambda_0} d\Lambda' \left\|\left(\frac{\partial \partial_N W_{2n,b}^{s(\alpha)}}{\partial \Lambda'}\right)(\cdot|\Lambda')\right\| \\ &\leqq \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \Lambda'^{4-2n-d_b-|N|} P(2\alpha-n-1) \\ &\leqq \Lambda^{4-2n-d_b-|N|} P(2\alpha-n). \end{aligned} \tag{4.29}$$

The remaining W-kernels with $4-2n-|N| \geqq -2s$ are reconstructed by Taylor expansion,

$$\begin{aligned} W_{2n,b}^{s(\alpha)}(p_1,\ldots,p_{2n}|\Lambda) &= \delta_{n,\{b\}} \cdot T_b(p_1,\ldots,p_{2n}) \\ &+ \frac{1}{(2n)^{4-2n+2s}} \sum_{|N|=4-2n+2s} \int_0^1 dt \frac{(1-t)^{5-2n+2s}}{(5-2n+2s)!} \\ &\times (\partial_N W_{2n,b}^{s(\alpha)})(tp_1,\ldots,tp_{2n}|\Lambda) \cdot p_N. \end{aligned} \tag{4.30}$$

Here the Taylor coefficients are independent of Λ. By taking norms we have

$$\begin{aligned} \|W_{2n,b}^{s(\alpha)}(\cdot|\Lambda)\| &\leqq \delta_{n,\{b\}} \cdot O(1)\Lambda^{4-2n-d_b} + \sum_{|N|=4-2n+2s} O(1)\cdot \Lambda^{|N|} \\ &\times \Lambda^{4-2n-d_b-|N|} P(2\alpha-n+\{b\}) \\ &\leqq \Lambda^{4-2n-d_b} P(2\alpha-n+\{b\}), \end{aligned} \tag{4.31}$$

which completes the induction step in this case. In the same way we obtain

$$\|(\partial_N W_{2n,b}^{s(\alpha)})(\cdot|\Lambda)\| \leqq \Lambda^{4-2n-d_b-|N|} P(2\alpha-n+\{b\})$$

for $4-2n-|N| \geqq -2s$. $\square$

We will now introduce another set of auxiliary quantities which we call X-kernels. At Λ_R they are total derivatives of the V-kernels with respect to Λ_0, where the bare coupling constants $(g_a^{0s})_{d_a \geqq -2s}$ are tuned according to the renormalization and improvement conditions.

Definition 4.8. Let us define quantities $X_{2n}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_a^{0s})_{d_a \geqq -2s}) \equiv X_{2n}^s(p_1,\ldots,p_{2n}|\Lambda)$, the X-kernels, by

$$\begin{aligned} &X_{2n}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_a^{0s})_{d_a \geqq -2s}) \equiv (\mathbf{X}^s[V_{2n}])(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_a^{0s})_{d_a \geqq -2s}) \\ &\equiv \left[\frac{\partial}{\partial \Lambda_0} - \sum_{d_b \geqq -2s} \left(\frac{\partial g_b}{\partial \Lambda_0}\right)(\Lambda,\Lambda_0,(g_a^{0s})_{d_a \geqq -2s}) \mathbf{W}_b^s\right] \\ &\quad \times V_{2n}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_a^{0s})_{d_a \geqq -2s}), \end{aligned} \tag{4.32}$$

where $\mathbf{X}^s$ denotes a differential operator and $\mathbf{W}_b^s$ is introduced in Definition 4.4.

The dependence on Λ of the X-kernels is described by a RG equation which is similar to that of the W-kernels.

Lemma 4.9. *The X-kernels satisfy the following coupled system of linear differential equations:*

$$\begin{aligned}\left(\frac{\partial X_{2n}^s}{\partial \Lambda}\right)(p_1, \dots, p_{2n}|\Lambda) = & \sum_{(m,l,k)\in\mathcal{G}} F_{m,k,l} \int \left(\prod_{j=1}^{k+1} \frac{d^4 q_j}{(2\pi)^4}\right) \\ & \times \left[\delta_{n,m-k}\mathbf{1}\Big|_{\mathbf{p}'=\mathbf{p}} - \sum_{d_b \geqq -2s} \delta_{\{b\},m-k} W_{2n,b}^s(p_1, \dots, p_{2n}|\Lambda)\mathbf{D}_b\Big|_{\mathbf{p}'=0}\right] \\ & \times (2\pi)^4 \delta\left(\sum_{i=1}^{2l-k-1} p_{\pi(i)} + \sum_{j=1}^{k+1} q_j\right)\left(\frac{\partial}{\partial\Lambda}\prod_{j=1}^{k+1} v(q_j|\Lambda)\right) \\ & \times (X_{2n}^s(p'_{\pi(i)}, \dots, p'_{\pi(2l-k-1)}, q_1, \dots, q_{k+1}|\Lambda) \\ & \times V_{2(m-l+1)}^s(p'_{\pi(2l-k)}, \dots, p'_{\pi(2(m-k))}, -q_1, \dots, -q_{k+1}|\Lambda) + X \leftrightarrow V).\end{aligned} \tag{4.33}$$

The proof of Lemma 4.9 is deferred to the appendix.

The X-kernels with $4-2n \geqq -2s$ are expanded in Taylor series of order $4-2n+2s$ with remainder terms. By Definition 4.8 the Taylor coefficients vanish identically, and the X-kernels are nonlocal in this sense. We can restrict the discussion to $(\partial_N X_{2n}^s)(p_1, \dots, p_{2n}|\Lambda)$ with $4-2n-|N| < -2s$. The initial values at Λ_0 are bounded with the identity

$$(\partial_N X_{2n}^{0s})(p_1, \dots, p_{2n}) = -\left(\frac{\partial \partial_N V_{2n}^s}{\partial \Lambda}\right)(p_1, \dots, p_{2n}|\Lambda)\Big|_{\Lambda=\Lambda_0} \tag{4.34}$$

for these X-kernels and Proposition 4.3. Finally, the kernels $X_{2n}^s(p_1, \dots, p_{2n}|\Lambda)$ are expanded into perturbation series whose coefficients $X_{2n}^{s(\alpha)}(p_1, \dots, p_{2n}|\Lambda)$ we are now concerned about.

Proposition 4.10. *The X-kernels vanish to zeroth order. To every order $\alpha \geqq 1$ and for all degrees $n > \alpha + s + 1$,*

$$X_{2n}^{s(\alpha)}(p_1, \dots, p_{2n}|\Lambda) = 0. \tag{4.35}$$

The proof is identical to that of Proposition 4.6. The vanishing is consistent with the initial conditions which are imposed by (4.34), and with the RG equation (4.33).

Proposition 4.11. *The norm of the X-kernels is bounded to every order $\alpha \geqq 1$, and for all degrees $1 \leqq n \leqq \alpha + s + 1$ as follows:*

$$\|(\partial_N X_{2n}^{s(\alpha)})(\cdot|\Lambda)\| \leqq \frac{\Lambda^{4-2n-|N|}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{2(s+1)} P(2\alpha - n - 1). \tag{4.36}$$

Proof. The estimate is proven by induction on the order α. To first order the X-kernels are independent of Λ, and determined by the initial conditions at Λ_0. For $4-2n-|N| < -2s$ the identity

$$(\partial_N X_{2n}^{0s(1)})(\cdot) = -(\partial \partial_N V_{2n}^{s(1)}/\partial \Lambda)(\cdot|\Lambda_0)$$

implies

$$\|(\partial_N X^{s(1)}_{2n})(\cdot|\Lambda)\| \leqq \frac{\Lambda_0^{4-2n-|N|}}{\Lambda_0} P(2n-1),$$
$$= \frac{\Lambda^{4-2n-|N|}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{-(4-2n-|N|)} P(2n-1)$$
$$\leqq \frac{\Lambda^{4-2n-|N|}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{2(s+1)} P(2n-1). \tag{4.37}$$

Let us suppose that the estimate holds to order $\alpha-1\geqq 1$. With Lemma 4.9, Proposition 4.10, and Lemma 4.2 we infer the inequality

$$\left\|\left(\frac{\partial X^{s(\alpha)}_{2n}}{\partial\Lambda}\right)(\cdot|\Lambda)\right\| \leqq \sum_{\beta=1}^{\alpha-1} \sum_{\substack{(m,l,k)\in\mathscr{G}\\ m-k=n\\ m-\alpha+\beta\leqq l\leqq\beta+s+1}} O(1)\cdot\Lambda^{2k-3}\|X^{s(\beta)}_{2l}(\cdot|\Lambda)\|\cdot\|V^{s(\alpha-\beta)}_{2(m-l+1)}(\cdot|\Lambda)\|$$
$$+\sum_{\beta+\gamma+\delta=\alpha}\sum_{d_b\geqq -2s}\sum_{\substack{(m,l,k)\in\mathscr{G}\\ m-k=\{b\}\\ m-\delta<l\leqq\gamma+s+1}}\sum_{|M_1|+|M_2|\leqq 4-2\{b\}-d_b}$$
$$\times O(1)\cdot\Lambda^{2k+2\{b\}+d_b+|M_1|+|M_2|-7}\|W^{s(\beta)}_{2n,b}(\cdot|\Lambda)\|$$
$$\times\|(\partial_{M_1}X^{s(\gamma)}_{2l})(\cdot|\Lambda)\|\cdot\|(\partial_{M_2}V^{s(\delta)}_{2(m-l+1)}(\cdot|\Lambda)\| + X\leftrightarrow V. \tag{4.38}$$

The right-hand side of (4.38) is a finite sum of terms which are bounded by the induction hypothesis, Proposition 4.3, and Proposition 4.7. It is linear in the X-kernels, and no order α X-kernels appear. Thus

$$\left\|\left(\frac{\partial X^{s(\alpha)}_{2n}}{\partial\Lambda}\right)(\cdot|\Lambda)\right\| \leqq \sum_{\beta=1}^{\alpha-1} \sum_{\substack{(m,l,k)\in\mathscr{G}\\ m-k=n\\ m-\alpha+\beta\leqq l\leqq\beta+s+1}} O(1)\cdot\Lambda^{2k-3}\frac{\Lambda^{4-2l}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{2(s+1)} P(2\beta-l-1)$$
$$\times\Lambda^{4-2(m-l+1)}P(2(\alpha-\beta)-(m-l+1)$$
$$+\sum_{\beta+\gamma+\delta=\alpha}\sum_{d_b\geqq -2s}\sum_{\substack{(m,l,k)\in\mathscr{G}\\ m-k=\{b\}\\ m-\delta\leqq l\leqq\gamma+s+1}}\sum_{|M_2|+|M_2|\leqq 4-2\{b\}-d_b}$$
$$\times O(1)\cdot\Lambda^{2k+2\{b\}+d_b+|M_1|+|M_2|-7}\cdot\Lambda^{4-2n-d_b}P(2\beta-n+\{b\})$$
$$\times\frac{\Lambda^{4-2l-|M_1|}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{2(s+1)}P(2\gamma-l-1)$$
$$\times\Lambda^{4-2(m-l+1)-|M_2|}P(2\delta-(m-l+1)),$$
$$\leqq\frac{\Lambda^{3-2n}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{2(s+1)}P(2\alpha-n-2). \tag{4.39}$$

By an analogous argument we infer the estimate

$$\left\|\left(\frac{\partial\partial_N X^{s(\alpha)}_{2n}}{\partial\Lambda}\right)(\cdot|\Lambda)\right\| \leqq \frac{\Lambda^{3-2n-|N|}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{2(s+1)} P(2\alpha-n-2). \tag{4.40}$$

The X-kernel on the right-hand side of the RG-equation preserves the convergence factor $(\Lambda/\Lambda_0)^{2(s+1)}$ in the iteration. The inequality (4.40) is integrated from Λ_0 to Λ. With $4-2n-|N|<-2s$, and the above estimate on the initial values, we conclude that

$$\begin{aligned}
\|(\partial_N X_{2n}^{s(\alpha)})(\cdot|\Lambda)\| &\leqq \left\|\left(\frac{\partial\partial_N V_{2n}^{s(\alpha)}}{\partial\Lambda}\right)(\cdot|\Lambda_0)\right\| + \int_\Lambda^{\Lambda_0} d\Lambda' \left\|\left(\frac{\partial\partial_N X_{2n}^{s(\alpha)}}{\partial\Lambda'}(\cdot|\Lambda')\right\|\right. \\
&\leqq \Lambda_0^{3-2n-|N|} P(2\alpha-n-1) \\
&\quad + \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \frac{\Lambda'^{4-2n-|N|}}{\Lambda_0}\left(\frac{\Lambda'}{\Lambda_0}\right)^{2(s+1)} P(2\alpha-n-2) \\
&\leqq \frac{\Lambda^{4-2n-|N|}}{\Lambda_0}\left(\frac{\Lambda}{\Lambda_0}\right)^{2(s+1)} P(2\alpha-n-1),
\end{aligned} \tag{4.41}$$

which completes the induction step. Note that Λ appears to a negative power on the right side of the estimate. □

We are ready now to show that the renormalized V-kernels approach their continuum limit values in s-improved perturbation theory with an s-increased rate of convergence.

Theorem of s-Improved Convergence 4.12. *Let the bare parameters g_a^{0s} with dimension $d_a \geqq -2s$ depend on Λ_0, Λ_R, and g_3^R such that the renormalization and improvement conditions hold. Let us assume that $g_a^{0s(\alpha)}(\Lambda_0,\Lambda_R)=0$ for $\alpha\geqq 1$ and $\{\alpha\}>\alpha+1$ and that $|g_a^{0s}(\Lambda_0,\Lambda_R)|\leqq \Lambda_0^{d_a}P(2\alpha-\{a\})$ for $\alpha\geqq 1$ and $\{a\}\leqq\alpha+1$, where α denotes the order of perturbation theory in g_3^R. Then the continuum limit*

$$\lim_{\Lambda_0\to\infty} V_{2n}^{s(\alpha)}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,\Lambda_R)|_{\Lambda=\Lambda_R} \equiv V_{2n}^{s\,\mathrm{cont}(\alpha)}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_R)|_{\Lambda=\Lambda_R} \tag{4.42}$$

of the renormalized V-kernels exists to every order $\alpha\geqq 1$ of s-improved perturbation theory for the degrees $1\leqq n\leqq \alpha+1$. The remaining V-kernels vanish identically. The rate of convergence in (4.42) *is bounded as follows:*

$$\|V_{2n}^{s(\alpha)}(\cdot|\Lambda,\Lambda_0,\Lambda_R)-V_{2n}^{s\,\mathrm{cont}(\alpha)}(\cdot|\Lambda,\Lambda_R)\|\,|_{\Lambda=\Lambda_R} \leqq \Lambda_R^{4-2n}\left(\frac{\Lambda_R}{\Lambda_0}\right)^{2(s+1)} P(2\alpha-n). \tag{4.43}$$

Proof. Let $\alpha\geqq 1$, and $1\leqq n\leqq\alpha+1$. The X-kernels were defined, such that their renormalized boundary values are the total derivatives of the V-kernels with respect to Λ_0 at Λ_R, where the parameters $(g_a^{0s})_{d_a\geqq -2s}$ are tuned according to the renormalization and improvement conditions. Thus for $\Lambda_{00}\geqq\Lambda_0$,

$$\begin{aligned}
&\|V_{2n}^{s(\alpha)}(\cdot|\Lambda,\Lambda_0,\Lambda_R)-V_{2n}^{s(\alpha)}(\cdot|\Lambda,\Lambda_{00},\Lambda_R)\|\,|_{\Lambda=\Lambda_R} \\
&\leqq \int_{\Lambda_0}^{\Lambda_{00}} d\Lambda_0' \|X_{2n}^{s(\alpha)}(\cdot|\Lambda,\Lambda_0',\Lambda_R)\|\,|_{\Lambda=\Lambda_R} \\
&\leqq \int_{\Lambda_0}^{\Lambda_{00}} \frac{d\Lambda_0'}{\Lambda_0'} \Lambda_R^{4-2n}\left(\frac{\Lambda_R}{\Lambda_0'}\right)^{2(s+1)} P^{(2\alpha-n-1)}\left[\log\frac{\Lambda_0'}{\Lambda_R}\right] \\
&\leqq \Lambda_R^{4-2n}\left[\left(\frac{\Lambda_R}{\Lambda_0'}\right)^{2(s+1)} P^{(2\alpha-n)}\left[\log\frac{\Lambda_0'}{\Lambda_R}\right]\right]_{\Lambda_{00}}^{\Lambda_0}.
\end{aligned} \tag{4.44}$$

Cauchy's criterion implies the existence of (4.42) and the estimate (4.43). □

5. Uniqueness of the Continuum Limit

The limit values $V_{2n}^{s\,\mathrm{cont}(\alpha)}(\cdot\,|\Lambda)$ in Theorem 4.12 do not depend on the improvement index $s \geqq 0$. The continuum limit values $V_{2n}^{\mathrm{cont}(\alpha)}(\cdot\,|\Lambda)$ are uniquely defined within the index $s=0$ scheme. We prove this to every order of perturbation theory by means of a homotopy to the index $s \geqq 1$ scheme.

Let us introduce auxiliary quantities which we call Y-kernels. They are closely related to the index zero W-kernels and we refer to these for the proofs of Lemma 5.2, Propositions 5.3, and 5.4.

Definition 5.1. Let us define kernels

$$Y_{2n,b}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_c^{0s})_{d_c \geqq -2s}) \equiv Y_{2n,b}^s(p_1,\ldots,p_{2n}|\Lambda)$$

by

$$\begin{aligned} Y_{2n,b}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_c^{0s})_{d_c \geqq -2s}) &\equiv (Y_b^s[V_{2n}])(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_c^{0s})_{d_c \geqq -2s}) \\ &\equiv \sum_{a=1}^{3}\left(\frac{\partial V_{2n}}{\partial g_a^{0s}}\right)(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_c^{0s})_{d_c \geqq -2s}) \\ &\quad \times \left[\left(\frac{\partial g}{\partial g^{0s}}\right)(\Lambda,\Lambda_0,(g_c^{0s})_{d_c \geqq -2s})\right]_{a,b}^{-1}, \end{aligned} \tag{5.1}$$

where $d_a, d_b \geqq 0$ and $1 \leqq a,b \leqq 3$.

The dependence on Λ of the Y-kernels is described by the following RG equation:

Lemma 5.2. *The Y-kernels satisfy the following system of differential equations:*

$$\begin{aligned} \left(\frac{\partial Y_{2n,b}^s}{\partial \Lambda}\right)(p_1,\ldots,p_{2n}|\Lambda) &= \sum_{(m,l,k)\in\mathscr{G}} F_{m,l,k}\int\left(\prod_{j=1}^{k+1}\frac{d^4q_j}{(2\pi)^4}\right) \\ &\times [\delta_{n,m-k}\mathbf{1}|_{p'=p} - \delta_{1,m-k}Y_{2n,1}^s(p_1,\ldots,p_{2n}|\Lambda)\mathbf{1}|_{\mathbf{p}'=0} \\ &- \delta_{1,m-k}Y_{2n,2}^s(p_1,\ldots,p_{2n}|\Lambda)\tfrac{1}{8}\Box^{[1,2]'}|_{\mathbf{p}'=0} - \delta_{2,m-k}Y_{2n,3}^s(p_1,\ldots,p_{2n}|\Lambda)\mathbf{1}|_{\mathbf{p}'=0}] \\ &\times (2\pi)^4\delta\left(\sum_{i=1}^{2l-k-1}p_{\pi(i)} + \sum_{j=1}^{k+1}q_j\right)\left(\frac{\partial}{\partial\Lambda}\prod_{j=1}^{k+1}v(q_j|\Lambda)\right) \\ &\times (Y_{2l,b}^s(p'_{\pi(1)},\ldots,p'_{\pi(2l-k-1)},q_1,\ldots,q_{k+1}|\Lambda) \\ &\times V_{2(m-l+1)}^s(p'_{\pi(2l-k)},\ldots,p'_{\pi(2(m-k))},-q_1,\ldots,-q_{k+1}|\Lambda) + Y\leftrightarrow V) \end{aligned} \tag{5.2}$$

with the initial values

$$Y_{2n,b}^{0s}(p_1,\ldots,p_{2n}) = \delta_{n,1}(\delta_{b,1}+\delta_{b,2}p_1^2) + \delta_{n,2}\delta_{b,3} \tag{5.3}$$

at Λ_0.

$Y_{2n,b}^s(p_1,\ldots,p_{2n}|\Lambda)$ is expanded in an s-improved perturbation series with coefficients $Y_{2n,b}^{s(\alpha)}(p_1,\ldots,p_{2n}|\Lambda)$.

Proposition 5.3. *To zeroth order the Y-kernels are exactly given by the Λ-independent expression*

$$Y_{2n,b}^{s(0)}(p_1,\ldots,p_{2n}|\Lambda) = \delta_{n,1}(\delta_{b,1}+\delta_{b,2}p_1^2) + \delta_{n,2}\delta_{b,3}\,. \tag{5.4}$$

To every order $\alpha \geqq 0$ *and for all degrees* $n > \alpha + 2$,

$$Y^{s(\alpha)}_{2n,b}(p_1, \ldots, p_{2n}|\Lambda) = 0 . \tag{5.5}$$

The nonvanishing perturbative Y-kernels are bounded in norm as follows:

Proposition 5.4. *To every order* $\alpha \geqq 0$ *and for all degrees* $1 \leqq n \leqq \alpha + 2$,

$$\|(\partial_N Y^{s(\alpha)}_{2n,b})(\cdot|\Lambda)\| \leqq \Lambda^{4-2n-d_b-|N|} P(2\alpha - n + \{b\}) . \tag{5.6}$$

Let us finally introduce another set of auxiliary quantities, the Z-kernels. At Λ_R they are the total derivatives with respect to the bare coupling constants g^{0s}_a with dimension $-2s \leqq d_a < 0$, where the renormalization conditions are kept fixed. We omit the proofs of Lemma 5.6, Propositions 5.7 and 5.8.

Definition 5.5. Let us define kernels

$$Z_{2n,b}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, (g^{0s}_c)_{d_c \geqq -2s}) \equiv Z^s_{2n,b}(p_1, \ldots, p_{2n}|\Lambda)$$

by

$$\begin{aligned} Z^s_{2n,b}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, (g^{0s}_c)_{d_c \geqq -2s}) &\equiv (\mathbf{Z}^s_b[V_{2n}])\,(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, (g^{0s}_c)_{d_c \geqq -2s}) \\ &\equiv \left[\frac{\partial}{\partial g^{0s}_b} - \sum_{a=1}^{3} \left(\frac{\partial g_a}{\partial g^{0s}_b}\right)(\Lambda, \Lambda_0, (g^{0s}_c)_{d_c \geqq -2s}) \mathbf{Y}^s_a\right] \\ &\quad \times V_{2n}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, (g^{0s}_c)_{d_c \geqq -2s}) , \end{aligned} \tag{5.7}$$

where $-2s \leqq d_b < 0$.

For the perturbative analysis we characterize the Z-kernels by a RG equation which is similar to that of the X-kernels.

Lemma 5.6. *The dependence on* Λ *of the* Z*-kernels is described by the following system of linear differential equations:*

$$\begin{aligned} \left(\frac{\partial Z^s_{2n,b}}{\partial \Lambda}\right)(p_1, \ldots, p_{2n}|\Lambda) = &\sum_{(m,l,k) \in \mathscr{G}} F_{m,l,k} \int \left(\prod_{j=1}^{k+1} \frac{d^4 q_j}{(2\pi)^4}\right) \\ &\times [\delta_{n,m-k} \mathbf{1}|_{\mathbf{p}'=\mathbf{p}} - \delta_{1,m-k} Y^s_{2n,1}(p_1, \ldots, p_{2n}|\Lambda) \mathbf{1}|_{\mathbf{p}'=0} \\ &- \delta_{1,m-k} Y^s_{2n,2}(p_1, \ldots, p_{2n}|\Lambda) \tfrac{1}{8} \square^{[1,2]'}|_{\mathbf{p}'=0} - \delta_{2,m-k} Y^s_{2n,3}(p_1, \ldots, p_{2n}|\Lambda) \mathbf{1}|_{\mathbf{p}'=0}] \\ &\times (2\pi)^4 \delta\left(\sum_{i=1}^{2l-k-1} p_{\pi(i)} + \sum_{j=1}^{k+1} q_j\right) \left(\frac{\partial}{\partial \Lambda} \prod_{j=1}^{k+1} v(q_j|\Lambda)\right) \\ &\times (Z^s_{2l,b}(p'_{\pi(1)}, \ldots, p'_{\pi(2l-k-1)}, q_1, \ldots, q_{k+1}|\Lambda) \\ &\times' V^s_{2(m-l+1)}(p'_{\pi(2l-k)}, \ldots, p'_{\pi(2(m-k))}, -q_1, \ldots, -q_{k+1}|\Lambda) + Z \leftrightarrow V) . \end{aligned} \tag{5.8}$$

Their initial values at Λ_0 *are given by:*

$$Z^{0s}_{2n,b}(p_1, \ldots, p_{2n}) = \sum_{\substack{-2s \leqq d_a < 0 \\ \{a\} = n}} \delta_{a,b} T_a(p_1, \ldots, p_{2n}) . \tag{5.9}$$

Let the parameters g_c^{0s} with $d_c \geqq -2s$ be functions of Λ_0, Λ_R, and g_3^R, such that the renormalization and improvement conditions are fulfilled. $Z_{2n}^s(\cdot|\Lambda)$ is expanded in powers of g_3^R, the coefficients being denoted $Z_{2n}^{s(\alpha)}(\cdot|\Lambda)$.

Proposition 5.7. *To order zero the Z-kernels are independent of Λ and given by*

$$Z_{2n,b}^{s(0)}(p_1, \ldots, p_{2n}|\Lambda) = \sum_{\substack{-2s \leqq d_a < 0 \\ \{a\} = n}} \delta_{a,b} T_a(p_1, \ldots, p_{2n}). \tag{5.10}$$

To every order $\alpha \geqq 0$ and for all degrees $n > \alpha + s + 1$,

$$Z_{2n,b}^{s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda) = 0. \tag{5.11}$$

The nonvanishing Z-kernels can be bounded in norm to every order of perturbation theory by means of an induction on the order with the RG equation (5.8).

Proposition 5.8. *To every order $\alpha \geqq 0$ of perturbation theory and for all degrees $1 \leqq n \leqq \alpha + s + 1$,*

$$\|(\partial_N Z_{2n,b}^{s(\alpha)})(\cdot|\Lambda)\| \leqq \Lambda^{4-2n-d_b-|N|} P(2\alpha - n + \{b\}). \tag{5.12}$$

Let us implement the improvement conditions in two steps. First we take $(g_a^{0s})_{-2s \leqq d_a < 0}$ as independent parameters and impose the renormalization conditions only. They determine $(g_b^{0s})_{d_b \geqq 0}$ implicitly as functions of Λ_0, Λ_R, g_3^R, and $(g_a^{0s})_{-2s \leqq d_a < 0}$. Second we choose $(g_a^{0s})_{-2s \leqq d_a < 0}$ as functions of Λ_0, Λ_R, and g_3^R, such that the improvement conditions are satisfied. Let us define a homotopy

$$\begin{aligned} &V_{2n}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, (g_b^{0s}(\Lambda_0, \Lambda_R, g_3^R, t(g_a^{0s}(\Lambda_0, \Lambda_R, g_3^R))_{-2s \leqq d_a < 0})_{d_b \geqq 0}, \\ &\quad t(g_a^{0s}(\Lambda_0, \Lambda_R, g_3^R))_{-2s \leqq d_a < 0}) \equiv V_{2n}^{t \cdot s}(p_1, \ldots, p_{2n}|\Lambda) \end{aligned} \tag{5.13}$$

with $t \in [0, 1]$ between $V_{2n}^0(p_1, \ldots, p_{2n}|\Lambda)$ and $V_{2n}^s(p_1, \ldots, p_{2n}|\Lambda)$. Let

$$V_{2n}^{t \cdot s(\alpha)}(p_1, \ldots, p_{2n}|\Lambda, \Lambda_0, \Lambda_R)$$

denote the V-kernel of order α of $t \cdot s$-improved perturbation theory within the above homotopy.

Theorem 5.9. *To every order $\alpha \geqq 1$ and for all degrees $1 \leqq n \leqq \alpha + 1$, the norm of the difference of the V-kernels in s-improved and unimproved perturbation theory is bounded as follows:*

$$\|V_{2n}^{s(\alpha)}(\cdot|\Lambda, \Lambda_0, \Lambda_R) - V_{2n}^{0(\alpha)}(\cdot|\Lambda, \Lambda_0, \Lambda_R)\| |_{\Lambda = \Lambda_R} \leqq \Lambda_R^{4-2n} \left(\frac{\Lambda_R}{\Lambda_0}\right)^2 P^{(2\alpha - n)}\left[\ln \frac{\Lambda_0}{\Lambda_R}\right]. \tag{5.14}$$

Corollary 5.10. *The continuum limit of the V-kernels is unique in the sense that for all values $s \geqq 0$ of the improvement index, to every order $\alpha \geqq 0$ of s-improved perturbation theory, and for all degrees $1 \leqq n \leqq \alpha + 1$*

$$V_{2n}^{s\,\mathrm{cont}(\alpha)}(\cdot|\Lambda, \Lambda_R)|_{\Lambda = \Lambda_R} = V_{2n}^{\mathrm{cont}(\alpha)}(\cdot|\Lambda_R). \tag{5.15}$$

Proof of Theorem 5.9. By the above homotopy we have

$$\begin{aligned}
&\| V_{2n}^{s(\alpha)}(\cdot\,|\Lambda,\Lambda_0,\Lambda_R)-V_{2n}^{0(\alpha)}(\cdot\,|\Lambda,\Lambda_0,\Lambda_R)\|\,|_{\Lambda=\Lambda_R}\\
&\leqq \int_0^1 dt \sum_{\beta=0}^{\alpha}\ \sum_{-2s\leqq d_a\leqq -2} \|Z_{2n,a}^{t\cdot s(\beta)}(\cdot\,|\Lambda,\Lambda_0,\Lambda_R)\|\,|_{\Lambda=\Lambda_R}|g_a^{0s(\alpha-\beta)}(\Lambda_0,\Lambda_R)|\\
&\leqq \sum_{\beta=0}^{\alpha}\ \sum_{-2s\leqq d_a\leqq -2} \Lambda_R^{4-2n-d_a}P(2\beta-n+\{a\})\Lambda_0^{d_a}P(2(\alpha-\beta)-\{a\})\\
&\leqq \Lambda_R^{4-2n}\left(\frac{\Lambda_R}{\Lambda_0}\right)^2 P(2\alpha-n)\,.
\end{aligned}\tag{5.16}$$

Proposition 5.8 applies also for the improvement index $t\cdot s$, as none of the above estimates becomes invalid when the bare parameters $g_a^{0s(\alpha)}(\Lambda_0,\Lambda_R)$ with $-2s\leqq d_a<0$ and $\{a\}\leqq\alpha+1$ are scaled by a factor t. □

6. Generalized Perturbation Theory

By induction on the order s of improvement we conclude that the bare parameters g_a^{0s} with dimension $d_a\geqq -2s$ can be chosen as functions of Λ_0,Λ_R, and the renormalized φ^4-coupling constant g_3^R such that the renormalization and the improvement conditions hold to every order of perturbation theory. We prove the estimates on the nonvanishing orders which we assumed above. For this purpose we represent them as iterative solutions of a set of nonlinear equations.

Let us consider the initial value problem which corresponds to the improvement index $s\geqq 1$. Suppose that the bare coupling constants g_a^{0s} with dimension $d_a\geqq 2(s-1)$ depend on Λ_0, Λ_R, g_3^R, and the bare coupling constants g_b^{0s} with dimensions $d_b=-2s$. The V-kernels are expanded to generalized perturbation series as follows:

$$\begin{aligned}
&V_{2n}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,(g_a^{0s}(\Lambda_0,\Lambda_R,g_3^R,(g_b^{0s})_{d_b=-2s})_{d_a\geqq -2(s-1)},(g_a^{0s})_{d_a=-2s})\\
&\quad=\sum_{|\underset{\sim}{\alpha}|\geqq 0} V_{2n}^{s(\alpha)}(p_1,\ldots,p_{2n}|\Lambda,\Lambda_0,\Lambda_R)(g_3^R)^{\alpha_3}\prod_{d_a=-2s}(g_a^{0s})^{\alpha_a}\,.
\end{aligned}\tag{6.1}$$

Here the sum runs over integer multi-indices $\underset{\sim}{\alpha}\equiv(\alpha_3,(\alpha_a)_{d_a=-2s})$ whose order is defined by $|\underset{\sim}{\alpha}|\equiv\alpha_3+\sum_{d_a=-2s}\alpha_a$.

Let us impose the following boundary conditions on the running coupling constants $g_a^{s(\underset{\sim}{\alpha})}(\Lambda)=g_a^{s(\underset{\sim}{\alpha})}(\Lambda,\Lambda_0,\Lambda_R)$ with dimensions $d_a\geqq -2s$.

The coupling constants with dimension $d_a\geqq -2(s-1)$ are fixed at the renormalization scale Λ_R. The renormalization conditions on the coupling constants with dimension $d_a\geqq 0$ take the form $g_a^{s(\underset{\sim}{\alpha})}(\Lambda_R)=\delta_{\alpha_3,1}\delta_{a,3}$. The coupling constants with dimension $-2(s-1)\leqq d_a<0$ are fixed by the improvement conditions $g_a^{s(\underset{\sim}{\alpha})}(\Lambda_R)=g_a^{\mathrm{cont}(\alpha_3)}(\Lambda_R)$ of order $s-1$. Here we assume that $|\underset{\sim}{\alpha}|=\alpha_3$, i.e., $\underset{\sim}{\alpha}=(\alpha_3,0,\ldots,0)$. The coefficients with $|\underset{\sim}{\alpha}|>\alpha_3$ have vanishing boundary values.

The coupling constants with dimension $d_a=-2s$ are fixed at the bare scale Λ_0. Only the coefficients of order $|\underset{\sim}{\alpha}|=1$ with $\alpha_a=\delta_{a,b}$, where $d_b=-2s$, have boundary values which do not vanish, namely $g_a^{s(\underset{\sim}{\alpha})}(\Lambda_0)=\delta_{a,b}$.

Let us assume that we can satisfy the improvement conditions of order $s-1$ by an appropriate choice of bare coupling constants $g_a^{s(\underline{\alpha})}(\Lambda_0)$ with dimension $d_a \geqq -2(s-1)$ and $\underline{\alpha}=(\alpha_3,0,\ldots,0)$ with $g_a^{s(\underline{\alpha})}(\Lambda_0)=0$ for $\alpha=0$ and for $\{a\}>\alpha_3+1$, and

$$|g_a^{s(\underline{\alpha})}(\Lambda_0)| \leqq \Lambda_0^{d_a} P(2\alpha_3-\{a\}) \quad \text{for} \quad \{a\} \leqq \alpha_3+1 .$$

The expansion coefficients, which correspond to multi-indices of the form $\underline{\alpha}=(\alpha_3,0,\ldots,0)$, can be identified as expansion coefficients in $(s-1)$-improved perturbation theory by $g_a^{s(\underline{\alpha})}(\Lambda)=g_a^{s-1(\alpha_3)}(\Lambda)$. The above assumptions constitute the hypothesis in the induction on the order s of improvement.

Let us define $L(\underline{\alpha}) \equiv \alpha_3 + \sum_{d_a=-2s} \alpha_a(\{a\}-1)+1$, i.e., $2L(\underline{\alpha})$ is the maximal number of external legs a connected diagram can have, which consists of α_3 vertices with four legs and α_a vertices with $2\{a\}$ legs respectively where $d_a=-2s$. We also introduce the symbol $\{\underline{\alpha}\} \equiv 2\alpha_3 + \sum_{d_a=-2s} \alpha_a\{a\}$. It is used to estimate logarithmic corrections which depend on the order of generalized perturbation theory. Finally, let $\theta(a,b)\equiv 1$ for $a>b$ and $\theta(a,b)\leqq 0$ for $a<b$.

Proposition 6.1. *To order* $|\underline{\alpha}|=0$ *for all degrees* $n \geqq 1$, *and to every order* $|\underline{\alpha}| \geqq 1$ *for all degrees* $n > L(\underline{\alpha})$,

$$V_{2n}^{s(\underline{\alpha})}(p_1,\ldots,p_{2n}|\Lambda)=0 . \tag{6.2}$$

Proof. For $|\underline{\alpha}|=\alpha_3$ Proposition 6.1 reduces to Proposition 4.1, where the improvement order is $s-1$. For $|\underline{\alpha}|>\alpha_3$ it is consistent with the above boundary conditions. $(\partial V_{2n}^{s(\underline{\alpha})}/\partial\Lambda)(\cdot|\Lambda)$ has an infinite expansion with terms which are proportional to $V_{2l}^{s(\underline{\beta})}(\cdot\,\Lambda)V_{2(m-l+1)}^{s(\underline{\gamma})}$, where $\underline{\beta}+\underline{\gamma}=\underline{\alpha}$ and $m-k=n$. Suppose that $l \leqq L(\underline{\beta})$, then $m-l+1>L(\underline{\gamma})$. Therefore every term contains a vanishing V-kernel. □

Proposition 6.2. *To every order* $|\underline{\alpha}| \geqq 1$, *for all degrees* $1 \leqq n \leqq L(\underline{\alpha})$ *the norm of the* V*-kernels is bounded as follows,*

$$\|(\partial_N V_{2n}^{s(\underline{\alpha})})(\cdot|\Lambda)\| \leqq \Lambda^{4-2n-|N|}\Lambda_0^{2s(|\underline{\alpha}|-\alpha_3)}\left(\frac{\Lambda}{\Lambda_0}\right)^{2s\theta(|\underline{\alpha}|,\alpha_3)} P(\{\underline{\alpha}\}-n) . \tag{6.3}$$

Proof. The case $|\underline{\alpha}|=\alpha_3$ is treated in Proposition 4.3, where the improvement order is $s-1$. We assume that $|\underline{\alpha}|>\alpha_3$ in the following.

The estimate is proven by induction on the order $|\underline{\alpha}|$. To first order the V-kernels are independent of Λ. By the initial conditions at Λ_0,

$$V_{2n}^{s(\underline{\alpha})}(p_1,\ldots,p_{2n}|\Lambda)=\delta_{n,\{b\}}T_b(p_1,\ldots,p_{2n}) , \tag{6.4}$$

for $\alpha_a=\delta_{a,b}$ where $d_b=-2s$. As $T_b(p_1,\ldots,p_{2n})$ is a homogeneous polynomial of degree $4-2n-d_b$, we conclude that

$$\|V_{2n}^{s(\underline{\alpha})}(\cdot|\Lambda)\| \leqq O(1)\cdot\Lambda^{4-2n+2s} . \tag{6.5}$$

Therefore the estimate holds to order $|\underline{\alpha}|=1$.

Let us suppose that it is true to order $|\underline{\alpha}|-1\geqq 1$. The induction step is performed with the inequality

$$\left\|\left(\frac{\partial V_{2n}^{s(\alpha)}}{\partial \Lambda}\right)(\cdot|\Lambda)\right\| \leqq \sum_{\substack{\beta+\gamma=\alpha \\ 1\leqq|\underline{\beta}|,|\underline{\gamma}|\leqq|\underline{\alpha}|-1}} \sum_{\substack{(m,l,k)\in\mathscr{G} \\ m-k=n \\ m-L(\gamma)+1\leqq l\leqq L(\beta)}} O(1)\cdot\Lambda^{2k-3}\|V_{2l}^{s(\beta)}(\cdot|\Lambda)\|\,\|V_{2(m-l+1)}^{s(\gamma)}(\cdot|\Lambda)\|\,. \tag{6.6}$$

Due to Proposition 6.1 the sum is finite. Using the induction hypothesis we infer the estimate

$$\begin{aligned}\left\|\left(\frac{\partial V_{2n}^{s(\alpha)}}{\partial \Lambda}\right)(\cdot|\Lambda)\right\| &\leqq \sum_{\substack{\beta+\gamma=\alpha \\ 1\leqq|\underline{\beta}|,|\underline{\gamma}|\leqq|\underline{\alpha}|-1}} \sum_{\substack{(m,l,k)\in\mathscr{G} \\ m-k=n \\ m-L(\gamma)+1\leqq l\leqq L(\beta)}} O(1)\cdot\Lambda^{2k-3} \\ &\quad\times\Lambda^{4-2l}\Lambda_0^{2s(|\underline{\beta}|-\beta_3)}\left(\frac{\Lambda}{\Lambda_0}\right)^{2s\theta(|\underline{\beta}|,\beta_3)} P(\{\underline{\beta}\}-l) \\ &\quad\times\Lambda^{4-2(m-l+1)}\cdot\Lambda_0^{2s(|\underline{\gamma}|-\gamma_3)}\left(\frac{\Lambda}{\Lambda_0}\right)^{2s\theta(|\underline{\gamma}|,\gamma_3)} P(\{\underline{\gamma}\}-(m-l+1)) \\ &\leqq\Lambda^{3-2n}\cdot\Lambda_0^{2s(|\underline{\alpha}|-\alpha_3)}\left(\frac{\Lambda}{\Lambda_0}\right)^{2s\theta(|\underline{\alpha}|,\alpha_3)} P(\{\underline{\alpha}\}-n-1)\,.\end{aligned} \tag{6.7}$$

Note that $\theta(|\underline{\beta}|,\beta_3)+\theta(|\underline{\gamma}|,\gamma_3)\geqq\theta(|\underline{\alpha}|,\alpha_3)$ for $\underline{\beta}+\underline{\gamma}=\underline{\alpha}$. By a similar argument we obtain the estimate for $\|(\partial\partial_N V_{2n}^{s(\alpha)}/\partial\Lambda)(\cdot|\Lambda)\|$ with an additional factor $\Lambda^{-|N|}$. It is integrated from Λ_0 to Λ for $4-2n-|N|<-2s$. Here Λ appears to negative power in the estimate. The initial values vanish. Thus

$$\begin{aligned}\|(\partial_N V_{2n}^{s(\alpha)})(\cdot|\Lambda)\| &\leqq \int_\Lambda^{\Lambda_0} d\Lambda'\left\|\left(\frac{\partial\partial_N V_{2n}^{s(\alpha)}}{\partial\Lambda'}\right)(\cdot|\Lambda')\right\| \\ &\leqq \int_\Lambda^{\Lambda_0}\frac{d\Lambda'}{\Lambda'}\Lambda'^{4-2n-|N|}\Lambda_0^{2s(|\underline{\alpha}|-\alpha_3)}\left(\frac{\Lambda'}{\Lambda_0}\right)^{2s\theta(|\underline{\alpha}|,\alpha_3)} P(\{\underline{\alpha}\}-n-1) \\ &\leqq\Lambda^{4-2n-|N|}\Lambda_0^{2s(|\underline{\alpha}|-\alpha_3)}\left(\frac{\Lambda}{\Lambda_0}\right)^{2s\theta(|\underline{\alpha}|,\alpha_3)} P(\{\underline{\alpha}\}-n)\,.\end{aligned} \tag{6.8}$$

The running coupling constants satisfy the estimate

$$\left|\left(\frac{\partial g_a^{s(\alpha)}}{\partial\Lambda}\right)(\Lambda)\right|\leqq\Lambda^{d_a-1}\Lambda_0^{2s(|\underline{\alpha}|-\alpha_3)}\left(\frac{\Lambda}{\Lambda_0}\right)^{2s\theta(|\underline{\alpha}|,\alpha_3)} P(\{\underline{\alpha}\}-\{a\}-1)\,. \tag{6.9}$$

For $d_a\geqq-2(s-1)$ it is integrated from Λ_R to Λ. Here Λ appears to a positive power in the estimate because $|\underline{\alpha}|>\alpha_3$. For $d_a=-2s$ it is integrated from Λ_0 to Λ. In both cases the initial values vanish as $|\underline{\alpha}|\geqq 2$. Thus

$$|g_a^{s(\alpha)}(\Lambda)|\leqq\Lambda^{d_a}\Lambda_0^{2s(|\underline{\alpha}|-\alpha_3)}\left(\frac{\Lambda}{\Lambda_0}\right)^{2s\theta(|\underline{\alpha}|,\alpha_3)} P(\{\underline{\alpha}\}-\{a\}) \tag{6.10}$$

holds for all coupling constants with $d_a\geqq-2s$.

Finally the V-kernels with $4-2n-|N|\geqq -2s$ are reconstructed by Taylor expansion which completes the induction step. $\square$

Corollary 6.3. *The coupling constants $g_a^{s(\underline{\alpha})}(\Lambda)$ with dimension $d_a=-2s$ satisfy to all orders $|\underline{\alpha}|\geqq 1$ for $\sum_{d_a=-2s}\alpha_a\geqq 1$ and $\{a\}\leqq L(\underline{\alpha})$ the estimate*

$$|g_a^{s(\underline{\alpha})}(\Lambda)|\leqq \Lambda_0^{2s\left(\sum_{d_a=-2s}\alpha_a-1\right)}P(\{\underline{\alpha}\}-\{\alpha\}). \tag{6.11}$$

Next we let the bare coupling constants g_a^{0s} with dimension $d_a=-2s$ depend on Λ_0, Λ_R, and g_3^R. A perturbation expansion yields

$$g_a^{0s}(\Lambda_0,\Lambda_R,g_3^R)=\sum_{\alpha=0}^{\infty} g_a^{0s(\alpha)}(\Lambda_0,\Lambda_R)\cdot(g_3^R)^\alpha .$$

Here the expansion coefficients $g_a^{0s(\alpha)}$ to vanish to zeroth order and to all orders $\alpha\geqq 1$ for $\{a\}>\alpha+1$.

Let us recorder the generalized perturbation expansion for the running coupling constants with dimension $d_b=-2s$ as follows:

$$\begin{aligned}
&g_b(\Lambda,\Lambda_0,(g_a^{0s}(\Lambda_0,\Lambda_R,g_3^R))_{d_a\geqq -2s})\\
&=\sum_{\substack{|\underline{\alpha}|\geqq 1\\ \{b\}\leqq L(\underline{\alpha})}} g_b^{s(\underline{\alpha})}(\Lambda,\Lambda_0,\Lambda_R)(g_3^R)^{\alpha_3}\prod_{d_a=-2s}\left(\sum_{\substack{\beta\geqq 1\\ \{a\}\leqq\beta+1}} g_a^{0s(\beta)}(\Lambda_0,\Lambda_R)(g_3^R)^\beta\right),\\
&=\sum_{(\underline{\alpha},\underline{\beta})} g_b^{s(\underline{\alpha})}(\Lambda,\Lambda_0,\Lambda_R)\left(\prod_{d_a=-2s}\prod_{i=1}^{\alpha_a} g_a^{0s(\beta_{a,i})}(\Lambda_0,\Lambda_R)\right)(g_3^R)^{\alpha_3+|\underline{\beta}|}.
\end{aligned}\tag{6.12}$$

Here the sum runs over multi-indices $\underline{\alpha}$ of order $|\underline{\alpha}|\geqq 1$, which satisfy $\{b\}\leqq L(\underline{\alpha})$, and over multi-indices $\underline{\beta}\equiv(\beta_{a,i})_{d_a=-2s,\,1\leqq i\leqq\alpha_a}$ with $\beta_{a,i}\geqq 1$, which satisfy $\{a\}\leqq\beta_{a,i}+1$. Let $|\underline{\beta}|\equiv\sum_{d_a=-2s}\sum_{i=1}^{\alpha_a}\beta_{a,i}$.

To order $|\underline{\alpha}|=1$ with $\alpha_a=\delta_{a,c}$, where $d_c=-2s$, the expansion coefficients take the form $g_b^{s(\underline{\alpha})}(\Lambda,\Lambda_0,\Lambda_R)=\delta_{b,c}$. Furthermore

$$g_b^{s(\gamma,0,\ldots,0)}(\Lambda,\Lambda_0,\Lambda_R)=g_b^{s-1(\gamma)}(\Lambda,\Lambda_0,\Lambda_R)$$

is an expansion coefficient in $(s-1)$-improved perturbation theory. Thus

$$\begin{aligned}
g_b^{0s(\gamma)}(\Lambda_0,\Lambda_R)&=g_b^{s(\gamma)}(\Lambda,\Lambda_0,\Lambda_R)-g_b^{s-1(\gamma)}(\Lambda,\Lambda_0,\Lambda_R)\\
&\quad-\sum_{\substack{(\underline{\alpha},\underline{\beta})\\ \alpha_3\leqq\gamma-1\\ \alpha_3+|\underline{\beta}|=\gamma}} g_b^{s(\underline{\alpha})}(\Lambda,\Lambda_0,\Lambda_R)\prod_{d_a=-2s}\prod_{i=1}^{\alpha_a} g_a^{0s(\beta_{a,i})}(\Lambda_0,\Lambda_R),
\end{aligned}\tag{6.13}$$

which expresses $g_b^{0s(\gamma)}(\Lambda_0,\Lambda_R)$ in terms of lower orders $g_a^{0s(\beta_{a,i})}(\Lambda_0,\Lambda_R)$ with $1\leqq\beta_{a,i}\leqq\gamma-1$.

Proposition 6.4. *To every order $\gamma\geqq 0$ there exist bare parameters $g_b^{0s(\gamma)}(\Lambda_0,\Lambda_R)$ with dimensions $d_b=-2s$ such that the improvement conditions $g_b^{s(\gamma)}(\Lambda,\Lambda_0,\Lambda_R)|_{\Lambda=\Lambda_R}=g_b^{\mathrm{cont}(\gamma)}(\Lambda_R)$ are satisfied. The zeroth order and all orders $\gamma\geqq 1$ for $\{b\}>\gamma+1$*

vanish. The orders $\gamma \geqq 1$ *for* $\{b\} \leqq \gamma+1$ *satisfy the estimate*

$$|g_b^{0s(\gamma)}(\Lambda_0, \Lambda_R)| \leqq \frac{1}{\Lambda_0^{2s}} P(2\gamma - \{b\}). \tag{6.14}$$

Proof. We perform an induction on the order γ in (6.13) with $\Lambda = \Lambda_R$, where $g_b^{\mathrm{cont}(\gamma)}(\Lambda_R)$ is inserted for $g_b^{s(\gamma)}(\Lambda, \Lambda_0, \Lambda_R)|_{\Lambda=\Lambda_R}$. To first order by the induction hypothesis on the order $s-l$ of improvement

$$|g_b^{0s(1)}(\Lambda_0, \Lambda_R)| = |g_b^{s-1(1)}(\Lambda, \Lambda_0, \Lambda_R)|_{\Lambda=\Lambda_R} - g_b^{\mathrm{cont}(1)}(\Lambda_R)| \leqq \frac{1}{\Lambda_R^{2s}} \left(\frac{\Lambda_R}{\Lambda_0}\right)^{2s} P(2 - \{b\}). \tag{6.15}$$

Let us suppose that the estimate holds to order $\gamma - 1 \geqq 1$. Thus,

$$\begin{aligned}
|g_b^{0s(\gamma)}(\Lambda_0, \Lambda_R)| &\leqq |g_b^{s-1(\gamma)}(\Lambda, \Lambda_0, \Lambda_R)|_{\Lambda=\Lambda_R} - g_b^{\mathrm{cont}(\gamma)}(\Lambda_R)| \\
&\quad + \sum_{\substack{(\alpha, \beta) \\ \alpha_3 \leqq \gamma - 1 \\ \alpha_3 + |\beta| = \gamma}} |g_b^{s(\alpha)}(\Lambda, \Lambda_0, \Lambda_R)|_{\Lambda=\Lambda_R} \prod_{d_a = -2s} \prod_{i=1}^{\alpha_a} |g_a^{0s(\beta_{a,i})}(\Lambda_0, \Lambda_R)|, \\
&\leqq \frac{1}{\Lambda_R^{2s}} \left(\frac{\Lambda_R}{\Lambda_0}\right)^{2s} P(2\gamma - \{b\}) + \sum_{\substack{(\alpha, \beta) \\ \alpha_3 \leqq \gamma - 1 \\ \alpha_3 + |\beta| = \gamma}} \Lambda_0^{2\left(\sum_{d_a = -2s} \alpha_a - 1\right)} P(\{\alpha\} - \{b\}) \\
&\quad \times \prod_{d_a = -2s} \prod_{i=1}^{\alpha_a} \frac{1}{\Lambda_0^2} P(2\beta_{a,i} - \{a\}), \\
&\leqq \frac{1}{\Lambda_0^{2s}} P(2\gamma - \{b\}).
\end{aligned} \tag{6.16}$$

Theorem 4.12 provides the estimate on the convergence in the continuum limit for $(s-1)$-improved perturbation theory. The hypothesis in the induction on the order of improvement covers the assumptions of Theorem 4.12. Corollary 6.3 provides the estimate on the expansion coefficients in generalized perturbation theory. This completes the induction step. □

It is essential that the renormalized coupling constants $g_b^{s(\gamma)}(\Lambda_R)$ with dimension $d_b = -2s$ be fixed to their *continuum values* in order to apply the estimates on $(s-1)$-improved perturbation theory.

Proposition 6.4 completes the induction step on the order s of improvement. The first improvement step which corresponds to $s=1$ follows by an analogous line of arguments. There the only ingredient needed is the renormalizability estimate of Theorem 4.12 in the case $s=0$. Thus we conclude:

Theorem 6.5. *To every order* $s \geqq 1$ *of improvement and to every order* $\alpha \geqq 0$ *of renormalized perturbation theory the bare parameters* $g_a^{0s(\alpha)}$ *with dimension* $d_a \geqq -2s$ *can be chosen as functions of* Λ_0 *and* Λ_R *such that the renormalization conditions*

$$g_a^{(\alpha)}(\Lambda_R, \Lambda_0, (g_b^{0s})_{d_b \geqq -2s}) = \delta_{\alpha,1} \delta_{a,3} \tag{6.17}$$

for $d_a \geqq 0$, *and the improvement conditions of order* s

$$g_a^{(\alpha)}(\Lambda_R, \Lambda_0, (g_b^{0s})_{d_b \geqq -2s}) = g_a^{\mathrm{cont}(\alpha)}(\Lambda_R) \tag{6.18}$$

for $-2s \leqq d_a < 0$, *are satisfied. The zeroth order and all orders* $\alpha \geqq 1$ *for* $\{a\} > \alpha + 1$ *of the bare parameters* $g_a^{0s(\alpha)}$ *with dimension* $-2s \leqq d_a < 0$ *vanish. All orders* $\alpha \geqq 1$ *for* $\{a\} \leqq \alpha + 1$ *satisfy the estimate*

$$|g_a^{0s(\alpha)}(\Lambda_0, \Lambda_R)| \leqq \Lambda_0^{d_a} P(2\alpha - a). \tag{6.19}$$

Appendix. RG-Equations

We prove the RG equation (2.11) for the V-kernels by inserting the expansion (2.10) into the RG equation (2.9) for $V(\varphi|\Lambda)$.

The formula for the infinitesimal change of normal ordering [GJ 87] implies

$$\left(\frac{\partial}{\partial \Lambda} + \frac{1}{2} \Delta_{\dot{v}(\Lambda)}\right) V(\varphi|\Lambda) = \dot{g}_0(\Lambda) + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \int \left(\prod_{i=1}^{2n} \frac{d^4 p_i}{(2\pi)^4}\right) (2\pi)^4 \delta\left(\sum_{i=1}^{2n} p_i\right)$$
$$\times \left(\frac{\partial V_{2n}}{\partial \Lambda}\right)(p_1, \ldots, p_{2n}|\Lambda) : \left(\prod_{i=1}^{2n} \varphi(p_i)\right) :_v^{(\Lambda)}. \tag{A1.1}$$

The use of normal ordered products eliminates the linear term on the right-hand side of (2.9). The nonlinear term contributes

$$\frac{1}{2}\left(\frac{\delta V(\varphi|\Lambda)}{\delta \varphi}, \dot{v}(\Lambda) \frac{\delta V(\varphi|\Lambda)}{\delta \varphi}\right) = \sum_{n=1}^{\infty} \sum_{m=1}^{n} (2(2m-1)!(2(n-m)+1)!)^{-1}$$
$$\times \int \left(\prod_{i=1}^{2n} \frac{d^4 p_i}{(2\pi)^4}\right) \frac{d^4 q}{(2\pi)^4} (2\pi)^4 \delta\left(\sum_{i=1}^{2n} p_i\right) (2\pi)^4 \delta\left(\sum_{i=1}^{2m-1} p_i + q\right) \dot{v}(q|\Lambda)$$
$$\times V_{2m}(p_1, \ldots, p_{2m-1}, q|\Lambda) V_{2(n-m+1)}(p_{2m}, \ldots, p_{2n}, -q|\Lambda)$$
$$\times : \left(\prod_{i=1}^{2m-1} \varphi(p_i)\right) :_{v(\Lambda)} : \left(\prod_{i=2m}^{2n} \varphi(p_i)\right) :_{v(\Lambda)}. \tag{A1.2}$$

Let $I \equiv \{1, \ldots, 2m-1\}$ and $J \equiv \{2m, \ldots, 2n\}$ define index sets. By $I = I_1 + I_2$ we denote a partition of I, i.e., $I = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. The product of normal ordered monomials in (A1.2) is reorganized with the identity

$$:\left(\prod_{i \in I} \varphi(p_i)\right) :_{v(\Lambda)} : \left(\prod_{j \in J} \varphi(p_j)\right) :_{v(\Lambda)} = \sum_{k=0}^{\min\{|I|, |J|\}} \sum_{\substack{I = I_1 + I_2, |I_1| = k \\ J = J_1 + J_2, |J_1| = k}} \sum_{\sigma : J_2 \to I_2}$$
$$\times \left(\prod_{j \in J_2} (2\pi)^4 \delta(p_{\sigma(j)} + p_j) v(p_j|\Lambda)\right) : \left(\prod_{i \in I_1} \varphi(p_j)\right) \left(\prod_{j \in J_1} \varphi(p_j)\right) :_{v(\Lambda)}, \tag{A1.3}$$

where we sum over all bijections σ from J_2 to I_2. Formulas (A1.1), (A1.2), and (A1.3) together yield the RG equation (1.11).

By Definition 3.4, the operators $\mathbf{W}_b^s$ satisfy the relations

$$\left[\frac{\partial}{\partial \Lambda}, \mathbf{W}_b^s\right] = -\sum_{d_a \geqq -2s} \left(\mathbf{W}_b^s \left[\frac{\partial g_a^s}{\partial \Lambda}\right]\right) (\Lambda) \mathbf{W}_a^s, \tag{A1.4}$$

and

$$(\mathbf{W}_b^s[g_b^s])(\Lambda) = \delta_{a,b}. \tag{A1.5}$$

The RG equation for the running coupling constants implies

$$\left(\mathbf{W}_b^s\left[\frac{\partial g_a^s}{\partial \Lambda}\right]\right)(\Lambda)=\sum_{\substack{(m,l,k)\in\mathscr{G}\\ m-k=\{a\}}} F_{m,l,k}\int\left(\prod_{j=1}^{k+1}\frac{d^4q_j}{(2\pi)^4}\right)\mathbf{D}_a\Bigg|_{p'=0}$$
$$\times\left[(2\pi)^4\delta\left(\sum_{i=1}^{2l-k-1}p'_{\pi(i)}+\sum_{j=1}^{k+1}q_j\right)\cdot\left(\frac{\partial}{\partial\Lambda}\prod_{j=1}^{k+1}v(q_j|\Lambda)\right)\right.$$
$$\times W_{2l,b}^s(p'_{\pi(1)},\ldots,p'_{\pi(2l-k-1)},q_1,\ldots,q_{k+1}|\Lambda)$$
$$\left.\times V_{2(m-l+1)}^s(p'_{\pi(2l-k)},\ldots,p'_{\pi(2(m-k))},-q_1,\ldots,-q_{k+1}|\Lambda)+W\leftrightarrow V\right]. \quad \text{(A 1.6)}$$

We insert (A 1.6) in (A 1.4). The application to the V-kernels yields the RG equation for the W-kernels. □

The RG *equations for the X-Kernels* follows from the identities

$$\left[\frac{\partial}{\partial\Lambda},\mathbf{X}^s\right]=-\sum_{d_a\geqq-2s}\left(\mathbf{X}^s\left(\frac{\partial g_a^s}{\partial\Lambda}\right]\right)(\Lambda)\mathbf{W}_a, \quad \text{(A 1.7)}$$

and

$$(\mathbf{X}^s[g_a^s])(\Lambda)=0. \quad \text{(A 1.8)}$$

The derivation of the RG equation for the X-kernels is now identical with that for the W-kernels.

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