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Cycling, Twisting, and Sewing in the Group Theoretic Approach to Strings

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Abstract. The general theory for cycling transformations in the group theoretic approach to strings is presented. It leads to a simple physical interpretation of the method which is discussed. We also demonstrate that twisting and sewing, i.e. factorization are an inevitable consequence of the method. We show that there exists a particularly simple choice of cycling transformations that leads to very great simplifications in the results for excited string scattering.

1. Introduction

In spite of its spectacular developments, string theory remains ill-understood in many respects. The existence of several approaches, each with its own starting point and advantages is a sure sign that we have not yet reached the deepest level of understanding. Of these approaches, light-cone [1] and covariant string field theory [2] is the most complete and ambitious, since at least the latter can address in principle non-perturbative effects; although formally elegant, it lacks practical efficiency even to recover simple perturbation theory results which had been derived 15 years ago by most primitive methods. The Polyakov approach stresses the two-dimensional world-sheet conformal field theory features of the string. It has led to deeper understanding of the critical dimension and of the rôle of the Faddeev-Popov ghosts of two-dimensional reparametrization invariance, but remains clumsy when one asks questions about excited state scattering and does not include non-perturbative effects.

Two more approaches have been developed more recently; one, based on Grassmanians [3] emphasizes elegant mathematical structures connected with Riemann surfaces. The other one, which we have developed in preceding publications [4–7], makes extensive use of the specific features of string scattering. Basically the geometrical fact underlying the original idea of duality, that strings interact by joining and splitting, from which overlap conditions can be derived, and the realization that the multistring vertices transform simply under the two-

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dimensional conformal group. The multistring vertices are defined by these transformation properties which involve cycling transformations that take conformal operators from one external line of the graph to another, or around an internal loop. In turn, these cycling transformations determine in a rather straightforward way the vertex. The actual scattering amplitude is given by integrating the vertex with an appropriate measure over the Koba-Nielsen variables and internal loop moduli. The measure is determined by first order partial differential equations obtained by imposing decoupling of zero norm physical states. This decoupling is required for unitarity and is essentially a statement of gauge invariance. We thus recovered very efficiently many known facts of perturbation theory: tree level scattering of arbitrary excited states for example [4]. We could also derive new results: the multiloop vertex with arbitrary excited physical external states [6], tree level arbitrary excited Neveu-Schwarz string scattering [7], and a one-loop L_2 anomaly of the open bosonic string [5]. The last result was also independently found in [8] by other methods. We have not yet applied our method to the general case of Ramond strings, and shall comment on this problem at the end of this paper.

In the papers quoted above, the cycling transformations used were partly guessed. Partly derived, and it was not clear what their admissible class is. We realized [4] that given one choice which worked, then one could also take any other choice related by the action of L_n, $n \geqq 0$ which was still cyclic in the Koba-Nielsen variables. This fact was used to obtain particularly simple cycling transformations [5, 6]. As we will see, this is essentially the largest class, however, it is rather useful to spell out the complete theory behind cycling transformations. To make this paper self-contained, we shall explain again our approach, and then proceed to the general theory of the cycling transformations. This addition makes our approach completely systematic. In particular, out of this will come, for the bosonic and Neveu-Schwarz strings, a specific choice of cycling transformation much simpler than previous ones, which makes the tree measure trivial, clarifies the geometrical meaning of the decoupling of zero norm states, and leads to a radical simplification of the oscillator vertex. It also allows us to give a simple physical interpretation of the overlap conditions.

In Sect. 4 we find what transformations implement the complete reversal of all the external legs of a graph and demonstrate that they show that physical scattering amplitudes are invariant under such a reversal. This generalizes the previous result [9]. We also demonstrate how to interchange any two legs of the vertex and show that apart from the integration limits on the Koba-Nielsen points the open string is invariant under such permutations. As a consequence, the extension of the approach to closed strings is trivial.

In Sect. 5 we demonstrate that the approach automatically leads to amplitudes that factorize correctly in accord with unitarity. This is demonstrated in a few lines by studying the overlap identities as a conformal operator, such as Q^{μ}, passes through the composite vertex.

In Sect. 6, we obtain simple results for the one-loop planar diagrams using this method. Section 7 is the conclusion.

2. Review of the Group Theoretic Approach

The essence behind this method [4–7] was the realization that multiloop multistring vertices in string theory behave as group theoretic objects under the conformal group. Here we review the method, but the reader is referred to the above references for a more detailed discussion and to [10] for a review. We will in this section concentrate on the group theoretic aspect of the approach, however, the reader is referred to the next section for a simple physical interpretation of the assumptions.

Given a vertex V which depends on the Koba-Nielsen variables z_k, $k=1,\dots,N$ and loop parameters, i.e., moduli v_r, $r=1,\dots,M$, where N is the number of external legs and M the number of loops, we specify its behaviour under the conformal group by specifying the way it cycles. Namely there exist cycling conformal transformations associated with cycling the external legs, T_k and going around each loop P^i_n. The cycling of the external legs is achieved by

$$V(z_1,\dots,z_N;v_r)\prod_{k=1}^{N}[T^{(k)}_{k+1}]^{-1}=V(z_2,\dots,z_N,z_1,v_r), \tag{2.1}$$

where the upper index on T_{k+1} indicates which leg of the vertex it acts on. For each loop the vertex satisfies a relation of the form:

$$V(z_j,v_r)P^{(i)}_n=V(z_j,v_r), \qquad n=1,\dots,M, \tag{2.2}$$

where again the upper index refers to the external leg of V being acted on.

One need not specify the actual form of V, but to be concrete the most useful representation, at least for perturbation theory is the usual oscillator representation of multistring vertices.

The cycling transformations are taken to factorize, meaning that they can be written in the form

$$T_j=(V^j)^{-1}(V^{j-1}), \tag{2.3}$$

and V^j is obtained from V^{j-1} by cycling its dependence on z_k. This guarantees that $T_N T_{N-1}\dots T_2 T_1=1$ must hold as a result of applying Eq. (2.1) N times. This factorization of T_j in the above equation will be motivated in the next section. The loop cycling transformations P^j_n are arbitrary elements of $SL(2,R)$, $SL(2,C)$ for the open and closed strings respectively. For Neveu-Schwarz strings, we use the analogous graded groups. What are the most general external leg cycling transformations is one of the subjects of this paper and will be discussed in the next section. By cycling the external legs of Eq. (2.2) we may deduce that

$$P^j_n=(V^j)^{-1}P_n V^j \quad \forall_j, \tag{2.4}$$

and hence there is in effect only one cycling transformation for each loop.

We also assume that the vertex obeys some overlap identities. These are equations which relate the action of a conformal operator on one external line of the vertex to its action on other external lines. From the existence of the generic overlap we may use Eqs. (2.1) and (2.2) to deduce their specific form to be

$$V\{Q^{\mu(i)}((V^i)^{-1}z)-Q^{\mu(j)}((V^j)^{-1}z)\}=0, \quad \forall i,j, \tag{2.5}$$

$$V\{Q^{\mu(k)}(z)-Q^{\mu(k)}(P^k_n z)\}=0, \quad \forall_{n,k}. \tag{2.6}$$

We refer the reader to the previous papers of the authors for this derivation and the overlap identities for other conformal operators.

Given the conformal transformations of the vertex of Eqs. (2.1) and (2.2), we may consider a general conformal transformation $\mathscr{A}^i$ which may change the parameters z_i and v_r, namely

$$V(\hat{z}_i, \hat{v}_r) = V(z_i, v_r) \prod_{i=1}^{N} \mathscr{A}^{(i)} . \tag{2.7}$$

In order to maintain the cyclic property of the vertex, $\mathscr{A}^{i+1}$ is obtained from $\mathscr{A}^i$ by cycling its dependence on z_j. We find from Eqs. (2.1) and (2.2) that $\mathscr{A}^i$ must satisfy

$$T_{i+1}^{-1} \mathscr{A}^{i+1} \hat{T}_{i+1} = \mathscr{A}^i \quad \forall i , \tag{2.8}$$

and

$$(P_n^i)^{-1} \mathscr{A}^i \hat{P}_n^i = \mathscr{A}^i \quad \forall i . \tag{2.9}$$

The first of these equations is solved by taking

$$\mathscr{A}^i = (V^i)^{-1} \mathscr{A} \hat{V}^i , \tag{2.10}$$

where

$$\hat{V}^i = V^i(\hat{z}_k) , \tag{2.11}$$

and the second equation now reads

$$P_n^{-1} \mathscr{A} \hat{P}_n = \mathscr{A} . \tag{2.12}$$

Taking $\mathscr{A}$ to be an infinitesimal conformal transformation

$$\mathscr{A}(z) = z + \varepsilon f(z) , \tag{2.13}$$

where $f(z)$ is single valued, we find that Eq. (2.12) becomes

$$P_n(z) + \varepsilon f(z) \frac{dP_n}{dz} = \hat{P}_n(z) + \varepsilon f(P_n z) . \tag{2.14}$$

In particular, we can consider that set of transformations that leave the vertex inert, i.e., $P_n = \hat{P}_n$ and $V_i = \hat{V}_i$. This occurs when

$$\mathscr{A}^j = (V^j)^{-1} V^i \mathscr{A}^i (V^i)^{-1} V^j , \tag{2.15}$$

and $P_n = \mathscr{A}^{-1} P_n \mathscr{A}^{-1}$ or infinitesimally

$$f(z) \frac{dP_n}{dz} = f(P_n z) . \tag{2.16}$$

The latter equation states that f is an automorphic form of degree -1. More concretely

$$f(z) \frac{\partial}{\partial z} \tag{2.17}$$

is invariant under the action of P_n. Any non-analytic behaviour of f must be located at the Koba-Nielsen points. For a tree graph, there are no loop cycling transformations, and one finds that the solution of Eq. (2.15) implies that f must have a pole of degree one or more. For a M loop graph, Eq. (2.16) means that $f(z)$ is defined on the Riemann surface itself and by the Riemann-Roch theorem, if it is analytic everywhere except at one point, it must have a pole of at least degree $3M-3+1$ there. The exception to this is when the number of loops M is one, in which case it must have a pole of at least order two. In this case, we recognized that the solution of (2.15) and of the associated reality condition on f is a doubly periodic vector field which can be constructed from the Weierstrass $\mathscr{P}$ function and its derivatives. For a more extensive discussion of this point see [10]. Here we make contact with the coset construction of moduli space of [11] which has also been exploited by the Grassmannian approach [3].

It is instructive to write the dependence of the vertex on z_n and v_r directly in terms of the corresponding group elements:

$$V(z_k, v_r) = V(V_i, P_n). \tag{2.18}$$

We may then write the conformal transformations of V as

$$V(V_i, P_n) \prod_{i=1}^{N} \mathscr{A}^i = V(V^i \mathscr{A}^i, \mathscr{A}^{-1} P_n \mathscr{A}), \tag{2.19}$$

where $\mathscr{A}^i = (V^i)^{-1} \mathscr{A} \hat{V}^i$. We could have started from this equation and deduced the particular cases of Eqs. (2.1) and (2.2). There is much here in common with the theory of induced representations and it is similar to the theory for coherent states of groups (see [12]).

The actual scattering is of the form

$$W = \int \prod_i dz_i \prod_r dv_r \bar{f}(z, v_r) V. \tag{2.20}$$

We may deduce the measure $\bar{f}$ by demanding that zero-norm physical states decouple. This is necessary for unitarity, and is a statement about the gauge symmetries of the theory.

Any of the above discussions could be implemented to deduce a vertex which involves anticommuting as well as the usual α_n^μ oscillators. For example, the ghost contribution to the three-string scattering vertex was first given, using the present method, in [13]. However the techniques discussed here can be used to find this contribution to all vertices and this will be given shortly.

The demand of decoupling of zero-norm physical states inevitably leads to a first order differential equation which determines the measure $\bar{f}$. This comes about as follows, zero-norm physical states involve $L_{-n}|\Omega\rangle$, or in BRST language are of the form $Q|\Lambda\rangle$. For such states to vanish, we are required to move L_{-n} or Q through the vertex in such a way that we obtain L_m, $m \geqq 0$ or Q on the other legs. Studying the above equations, one realizes that this can only be achieved by using moduli changing transformations, i.e., derivatives of the vertex with respect to the moduli (i.e., v_r's or z_i's). Such derivatives can only vanish, after integration by parts, if certain first order differential equations for the measure are satisfied. We refer to [1–4] for specific examples of this very general procedure.

Given a choice of cycling transformation we can compute the vertex using the overlap equations (2.5) and (2.6), or more conveniently the integrated form of the P^μ overlap, namely

$$V\left\{\sum_{j=1}^{N} \oint_{\xi_j=0} \frac{d\xi_j}{\xi_j} P^{n(j)}(\xi_j)\phi(\xi_i)\right\}=0, \tag{2.21}$$

where $\xi_j=(V^j)^{-1}(V^i)\xi_i$. The function $\phi(\xi_i)$ must be analytic everywhere except at the Koba-Nielsen points ($\xi_j=0, \forall_j$) where it may have poles. In the case of the multiloop vertex, one must also satisfy certain automorphic relations and reality conditions (see [7]). For trees there are no further restrictions, and taking $\phi(\xi_i) =(\xi_i)^{-n}$ we find the vertex to be

$$\begin{aligned} V=&\left\{\prod_{i=1}^{N}\langle 0|\right\}\exp-\left\{\sum_{1\leqq i<j<N}\left\{(a^i|\Gamma(V^i)^{-1}V^j|a^j)\right.\right.\\ &+(a^i|\Gamma(V^i)^{-1}V^j(0))a_0^j\\ &+a^{0i}(\Gamma(V^j)^{-1}V^i(0)|a^j)\\ &\left.\left.-\frac{a^{0i}a^{0j}}{2}\ln\left[\left\{\frac{d}{dz}(\Gamma(V^i)^{-1}V^jz)\right\}\bigg|_{z=0}\right]\right\}\right\}. \end{aligned} \tag{2.22}$$

The reader is referred to [4–7] for more details, in particular the determination of the $a_0^i a_0^j$ term from the Q^μ overlap and the multiloop vertices. This concludes our review of previous work.

3. Cycling Transformations

In the previous section, we saw how multistring vertices were determined by their conformal properties which required cycling transformations for loops and external legs. For the former we are required to specify conformal transformations $(V^j)^{-1}$. These transformations $(V^j)^{-1}$ must be of the form

$$(V^j)^{-1}(z)=(z-z_j)f^j(z), \tag{3.1}$$

where $f^j(z)$ is any function that is analytic and non-vanishing at $z=z_j$. As mentioned above V^{j+1} is obtained from V^j by cycling its dependence on the Koba-Nielsen variables $z_k \to z_{k+1}$. We may write

$$f^j(z)=\sum_{n=0}^{\infty} a_n^j(z-z_j)^n, \qquad a_0^j \neq 0, \tag{3.2}$$

where a_n^j may depend on z_k and satisfies

$$a_n^{j+1}(z_1,\ldots,z_N)=a_n^j(z_2,\ldots,z_N,z_1). \tag{3.3}$$

In terms of the L_n's, $(V^j)^{-1}$ has the form

$$(V^j)^{-1}=\exp\left\{\sum_{n=1}^{\infty}\bar{a}_n^j L_n^{(j)}\right\}e^{\ln a_0^j L_0^j}e^{-z_j L_{-1}^j}, \tag{3.4}$$

where $\bar{a}_n^j$ are functions of the a_n^j. The form of $(V^j)^{-1}$ of Eq. (3.1) has a simple interpretation; $(V^j)^{-1}$ is the most general analytic transformation which maps the point z_j to zero. The object $Q^{\mu(j)}$ is naturally associated with its Koba-Nielsen point $z=z_j$, and so $Q^{\mu j}((V^j)^{-1}z)$ is naturally associated with the point zero. The Q^μ overlap of Eq. (2.5) then states that $Q^{\mu(j)}$ when conformally mapped to the point zero is the same as $Q^{\mu(i)}$ when also conformally mapped to zero. Thinking about a string emitted from a Riemann surface, this is of course most natural; see Fig. 1. The factorization, Eq. (2.3), of the cycling transformations is particularly apparent from this viewpoint. For the loops we recall that Eq. (2.6) simply states that when we take a Q^μ around a hole in the surface, and compare it with the Q^μ already there, it is the same (see Fig. 2). What is remarkable is that this seemingly timid requirement determines the scattering amplitude completely when one also demands the decoupling of zero-norm physical states.

Of course one's choice of reference point is arbitrary and can be any point on the surface. Below we give an explicit discussion of this arbitrariness for the case of trees. One can also consider other conformal operators such as $P^\mu(z)$ which, having conformal weight one, is naturally written as $P^\mu(z)dz/z$, and one interprets their overlaps in a similar way.

From the point of view of two-dimensional field theory, we can interpret the overlap identity as propagating Q^μ from z_j to 0 using $(V^j)^{-1}$ and on to z_i using (V^i) to find Q^μ at that point z_i. [At first sight, the order of the factors (V^i) and $(V^j)^{-1}$ is puzzling, but the order becomes inverted when we consider what change we must make to a physical state on the right which induces this change on the vertex, and so on the overlap.] The large freedom of choice (3.1) for V_j can be interpreted as reflecting the arbitrariness in the choice of a local evolution operator (Hamiltonian) in the reparametrization invariant two-dimensional world-sheet.

In the past, for the bosonic string we took

$$(V^j)=\begin{pmatrix} \infty & 0 & 1 \\ z_{j-1} & z_j & z_{j+1} \end{pmatrix}, \tag{3.5}$$

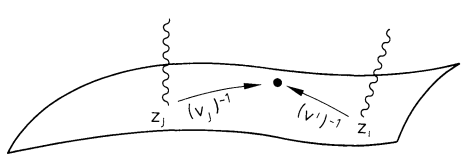

Fig. 1. Interpretation of an external leg cycling transformation

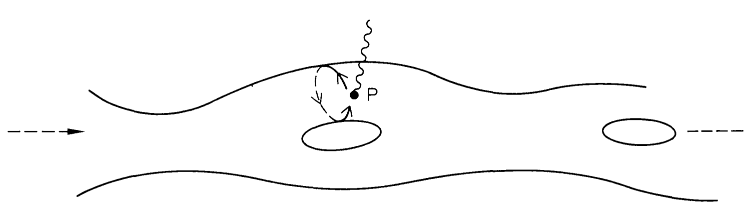

Fig. 2. Interpretation of a loop cycling transformation

as it was this choice that led to the Caneschi-Schwimmer-Veneziano vertex [9], when twisted, and the Lovelace-Olive vertices [14]. This corresponds to taking

$$(V^j)^{-1}(z) = \frac{(z-z_j)(z_{j+1}-z_{j-1})}{(z-z_{j-1})(z_{j+1}-z_j)} \tag{3.6}$$

or

$$a_0^j = \frac{(z_{j+1}-z_{j-1})}{(z_{j+1}-z_j)} \frac{(-1)}{(z_{j-1}-z_j)}, \qquad \bar{a}_1^j = \frac{(z_{j+1}-z_j)}{(z_{j-1}-z_{j+1})}, \qquad \bar{a}_n^j = 0,\ n \geqq 2. \tag{3.7}$$

There is, however, a more obvious and considerably simpler choice, namely

$$(V^j)^{-1}(z) = (z-z_j) \quad \text{or} \quad f^j = 1. \tag{3.8}$$

We now demonstrate some of the theory given in the previous section for this choice. Using Eq. (2.10), we find that

$$\begin{aligned} V(\hat{z}_k) &= V(z_k) \prod_{j=1}^{N} \left\{ [V^j(z_k)]^{-1} [V^j(\hat{z}_k)] \right\} \\ &= V(z_k) \prod_{j=1}^{N} e^{-(z_j-\hat{z}_j)L_{-1}^j}. \end{aligned} \tag{3.9}$$

Infinitesimally this equation becomes

$$\frac{\partial V}{\partial z_k} = + V L_{-1}^{(k)}. \tag{3.10}$$

This has the very simple geometrical interpretation that $L_{-1}^k = +\overleftarrow{\partial}/\partial z_k$ generates a translation of the Koba-Nielson point z_k. Of course, such a geometrical interpretation exists for arbitrary V_k's, but is obscured by gauge transformations.

Computing the measure is now trivial. From (3.9), we immediately deduce that zero-norm physical states of the form $L_{-1}^{(k)}|\Omega\rangle$ decouple from the physical vertex W:

$$W L_{-1}^{(k)}|\Omega\rangle = 0 \tag{3.11}$$

for

$$W = \int \prod_{c} dz_i V, \tag{3.12}$$

that is, the measure is a constant.

For the previous choice of cycling transformation, Eq. (3.5), one has:

$$[V^j(gz_k)] = g[V^j(z_k)], \tag{3.13}$$

if g is a member of $SL(2,R)$, and hence the vertex [see Eq. (2.21)] itself is $SL(2,R)$ inert. In this case, the measure [4] is also found by inspection to be $SL(2,R)$ inert. Demanding separate invariance of the vertex and measure is, of course, not necessary as it is only W which should be invariant. It is this over-concern to incorporate $SL(2,R)$ which was probably responsible for the appearance of the V^j of Eq. (3.5) in previous string vertices.

We now discuss the $SL(2,R)$ invariance for the new choice. We will require a knowledge of the transformations which leave the vertex inert and according to Eq. (2.15) are given by

$$\mathscr{A}^j = e^{(z_i - z_j)L_{-1}} \mathscr{A}^i e^{-(z_i - z_j)L_{-1}} . \tag{3.14}$$

Under a translation of the Koba-Nielsen variables $z_k \to z_k + a\ \forall k$, we have

$$\begin{aligned} V(z_{k+a}) &= V(z_k) \prod_{j=1}^{N} \{[V^j(z_j)]^{-1} V^j(z_{j+a})\} \\ &= V(z_k) \prod_{j=1}^{N} e^{-aL^{(j)}_{-1}} = V(z_k) . \end{aligned} \tag{3.15}$$

For the last step we used Eq. (3.14) with $\mathscr{A}^i = \exp(-aL^{(i)}_{-1})$. Under an infinitesimal dilation we have

$$\begin{aligned} V(z_k + \lambda z_k) &= V(z_k) \prod_{j=1}^{N} \{[V^j(z_j)]^{-1} V^j(z_j + \lambda z_j)\} \\ &= V(z_k) \prod_{j=1}^{N} e^{\lambda z_j L^j_{-1}} . \end{aligned} \tag{3.16}$$

Taking $\mathscr{A}^i = \exp \lambda (L^{(i)}_0 + z_i L^{(i)}_{-1})$ in Eq. (3.13), we find that a transformation which leaves the vertex inert is

$$\mathscr{A}^j = \exp \lambda \{L^{(j)}_0 + z_j L^{(j)}_{-1}\} . \tag{3.17}$$

Consequently, for infinitesimal λ we find that

$$V(z_k + \lambda z_k) = V(z_k) \prod_{j=1}^{N} (1 - \lambda L^{(j)}_0) \tag{3.18}$$

which on-shell becomes

$$V(z_k + \lambda z_k) = V(z_k)(1-\lambda)^N . \tag{3.19}$$

For the action of L_1 that is $z_k \to z_k + \varepsilon z_k^2$, one can show straightforwardly that, for on-shell states

$$V(z_k + \varepsilon z_k^2) = V(z_k) \prod_{k=1}^{N} (1 - 2\varepsilon z_k) . \tag{3.20}$$

Iterating these infinitesimal results, we conclude that for finite $SL(2,R)$ transformations

$$V(\hat{z}_k) = V(z_k) \prod_{k=1}^{N} \left\{ \frac{dz_k}{d\hat{z}_k} \right\} , \tag{3.21}$$

where $\hat{z}_n = g(z_n)$ for $g \in SL(2,R)$. However, this is precisely what is required to cancel the change in $\Pi_k dz_k$, and so on-shell we find that the results of physical states scattering is the same:

$$\hat{W} \equiv \int \prod_k d\hat{z}_k \hat{V}(\hat{z}_k) = W = \int \prod_k dz_k V(z_k) . \tag{3.22}$$

Let us now evaluate the vertex for N string scattering at tree level which was given for a general cycling transformation in Eq. (2.21). Substituting $(V^j)^{-1}(z) = (z - z_j)$, we find that

$$V = \left\{ \prod_{i=1}^{N} {}_i\langle 0| \right\} \exp - \left\{ \sum_{1 \leq i < j \leq N} \left\{ \sum_{n,m=1}^{\infty} \frac{a_n^{\mu(i)}}{\sqrt{n}} \frac{(n+m-1)!(-1)^m}{(m-1)!(n-1)!(z_j - z_i)^{n+m}} \right. \right.$$
$$\times \frac{a_m^{\mu(j)}}{\sqrt{m}} + \sum_{n=1}^{\infty} \left\{ \frac{a_n^{\mu(i)}}{\sqrt{n}} \frac{a_0^{j\mu}}{(z_j - z_i)^n} + \frac{a_n^{(j)\mu}}{\sqrt{n}} \frac{a_0^{(i)\mu}}{(z_i - z_j)} \right\}$$
$$\left. \left. + a_0^{(i)\mu} a_0^{\mu(j)} \ln (z_j - z_i) \right\} \right\} . \tag{3.23}$$

It is obvious that one recovers the well-known scattering formula [15] for N tachyonic states from this very simple formula which is much simpler than the previous ones. We observe that V is invariant under the exchange of $a_n^{i\mu} \leftrightarrow a_m^{j\mu}$, $z^i \leftrightarrow z^j$; that is the interchange of two legs. Such an invariance had not been established beyond tachyonic external states for the other vertices in particular the Lovelace-Olive vertex [14] which one finds from the cycling choice of Eq. (3.5).

It is instructive to consider the relation between the simple vertex denoted V above and vertices denoted $\hat{V}$, obtained with other choices of cycling transformations (i.e., $f^{(j)}$'s). The relation is given by Eq. (2.10) to be

$$\begin{aligned} \hat{V} &= V \prod_{j=1}^{N} \{ [V^j(z_n)]^{-1} \hat{V}_j(z_n) \} \\ &= V \prod_{j=1}^{N} \exp \left\{ \sum_{n=1}^{\infty} \bar{a}_n^j L_n^{(j)} \right\} \exp \{ \ln a_0^j L^j \} . \end{aligned} \tag{3.24}$$

Hence for on-shell external physical states, we find that

$$\hat{V} = V \prod_{j=1}^{N} \exp \{ - \ln a_0^j \} . \tag{3.25}$$

Consequently, in order for W and $\hat{W}$ to agree on-shell, the measure $\hat{f}$ associated with $\hat{V}$ must be given by

$$\hat{f} = \prod_j a_0^j . \tag{3.26}$$

For the choice of Eq. (3.8) we find the measure is

$$\hat{f} = \prod_{j=1}^{N} \frac{(z_{j-1} - z_{j+1})}{(z_{j-1} - z_j)(z_j - z_{j+1})} \tag{3.27}$$

in agreement with [4]. It is clear that as $\hat{W}$ and W agree on-shell, then $\hat{W}$ will decouple zero-norm physical states as W did. Since the measure is determined uniquely by decoupling, it follows that the measure must be given by the above equation. However, one can compute from Eq. (3.24) the analogue of Eq. (3.10) by gauge transforming the latter and then explicitly compute the measure from decoupling to find the above result of Eq. (3.26).

One can also consider whether the cycling transformation of Eq. (3.1) is the most general one allowed. As the above argument shows, all the allowed choices of

f lead to the same scattering amplitude, as they must. Given a vertex $\hat{V}$ related to V by $\hat{V} = VG$, then any factors such as $e^{L_{-1}}$ in G which do not annihilate on physical states, unlike $L_n - \delta_{n,0}$, $n \geqq 0$ must be in effect not there, due to some identity satisfied by V. This can only happen if such factors correspond to transformations that leave the vertex V inert, that is, one of the form of Eq. (3.13). Given any G, one can easily verify whether this is the case or not. When it is the case, these factors can in effect be eliminated, so that indeed the cycling transformations of Eq. (3.1) form the most general class leading to the same S matrix.

The extension of these ideas to the Neveu-Schwarz sector is completely straightforward. The tree scattering of excited Neveu-Schwarz strings has already been considered by the authors [7] within the framework of the graded extension of $SL(2, R)$ whose generators are $L_0, L_{\pm 1}, G_{\pm 1/2}$. We used for $(V^j)^{-1}$ the graded extension of the invariant cross-ratio of Eq. (3.6). But we see now that there is a much simpler choice, which generalizes Eq. (3.8) to include the anticommuting variables θ and θ_j:

$$(V^j)^{-1}(z) \equiv Z_j = z - z_j - \theta\theta_j, \tag{3.28}$$

$$(V^j)^{-1}(\theta) \equiv \Theta_j = \theta - \theta_j. \tag{3.29}$$

The differential equations (3.10) obeyed by the vertex are unchanged, and there is a new one, related to an infinitesimal change in θ_j. Using Eqs. (3.28) and (3.29), it reads:

$$\frac{\partial V}{\partial \theta_j} + V[G^{(j)}_{-1/2} + \theta_j L^j_{-1}] = 0. \tag{3.30}$$

Combining with Eq. (3.10), this can be written

$$Q_j V \equiv \frac{\partial V}{\partial \theta_j} + \theta_j \frac{\partial}{\partial z_j} V = -VG^{(j)}_{-1/2}. \tag{3.31}$$

This has the natural geometric meaning that $G^{(j)}_{-1/2}$ generates a supertranslation of the Koba-Nielsen variable (z_j, θ_j).

The vertex V can be written in completely explicit form. First, one expands in powers of θ_i:

$$V = V_0 + \sum_i \theta_i V_i + \sum_{i<j} \theta_i\theta_j V_{ij} + \sum_{i<j<k} \theta_i\theta_j\theta_k V_{ijk} + \dots \tag{3.32}$$

From the overlap of the Neveu-Schwarz field when all θ_i are taken to be zero, one finds that V_0 is obtained by multiplying the orbital part of Eq. (3.23) by the following analogous contribution of the anticommuting oscillators $b^{(j)}_r$:

$$\langle 0| \exp \sum_{1 \leqq i<j \leqq N} \sum_{\substack{r=1/2 \\ s=1/2}}^{\infty} b^{(i)}_r b^{(j)}_s (z_j - z_i)^{-r-s} \frac{(s+r-1)!}{(r-\frac{1}{2})!(s-\frac{1}{2})!} (-1)^{s-1/2}. \tag{3.33}$$

We note that the coefficient of $b^{(i)}_r b^{(j)}_s$ in this expression is antisymmetric in the interchange $(i, r) \leftrightarrow (j, s)$, as it should be. The V_i, V_{ij}, ..., vertices are obtained trivially from solving Eq. (3.29); one finds

$$\begin{aligned} V_i &= -V_0 G^{(i)}_{-1/2}, \qquad V_{ij} = V_i G^{(j)}_{-1/2} = -V_0 G^{(i)}_{-1/2} G^{(j)}_{-1/2}, \\ V_{ijk} &= -V_{ij} G^{(k)}_{-1/2} = V_0 G^{(i)}_{-1/2} G^{(j)}_{-1/2} G^{(k)}_{-1/2}, \dots \text{etc.} \end{aligned} \tag{3.34}$$

The physical S matrix element, from which zero-norm physical states decouple, is obtained by integrating over the θ_i variables, as well as the z_i variables. Here, we have two choices: for the N point function, one may integrate over all $N\theta_i$ variables, or, using the graded part of super $SL(2,R)$, only over $N-2$ of them, dividing out the super $SL(2,R)$ invariant measure $(z_{i_1}-z_{i_2})d\theta_{i_1}d\theta_{i_2}$, for some i_1 and i_2. In the first case, the S-matrix element is

$$\int \prod_i dz_i V_0 G^{(1)}_{-1/2} G^{(2)}_{-1/2} \ldots G^{(N)}_{-1/2}\,, \tag{3.35}$$

while in the second case, it is (up to a sign)

$$\int \prod_i \frac{dz_i}{(z_{i_1}-z_{i_2})} V_0 G^{(1)}_{-1/2} G^{(2)}_{-1/2} \ldots G^{(i_1-1)}_{-1/2} G^{(i_1+1)}_{-1/2} \ldots G^{(i_2-1)}_{-1/2} G^{(i_2+1)}_{-1/2} \ldots G^{(N)}_{-1/2}\,. \tag{3.36}$$

These two possibilities correspond to the old F_1 and F_2 formulations of the Neveu-Schwarz sector, and lead to identical results on-shell. This can be shown by using repeatedly the overlaps for G gauges together with Eq. (3.10) and the proof is left to the reader.

Of course, as in the bosonic case, three of the z_i integrations must be removed by $SU(1,1)$ invariance from either forms (3.35) or (3.36). For (3.36), z_{i_1} and z_{i_2} may or may not be part of this set of three, without affecting the resulting answer.

4. Twisting

One feature of open string scattering is the ability to reverse all the variables of an amplitude $A(1,2,\ldots,N)$ and obtain the same result. That is

$$A(1,2,\ldots,N)=A(N,N-1,\ldots,2,1)\,. \tag{4.1}$$

We recall that the actual scattering amplitude was made up of the sum, with unit weight, of all such contributions A which were not related by cycling or reversing the order. This gives $(N-1)/2!$ terms for an N leg graph. For example, the scattering of N tachyons is left the same if we change $z_i \to z_{N-i+1}$ and at the same time change $p_i \to p_{N-i+1}$.

We now find the transformations which implement such changes on the vertex and verify that they vanish on-shell. As with all computations in the new approach, it is an exercise in conformal group theory. Changing the external momenta $p_i \to p_{N-i+1}$ and the oscillators $\alpha_n^{\mu(N)} \to \alpha_n^{\mu(N-i+1)}$ is implemented by relabelling the legs by $i \to N-i+1$ of the vertex. We must also exchange $z_i \to \hat{z}_i = z_{N-i+1}$. We could carry out this change directly on the vertex using its conformal definitions, but a change is as good as a rest, so we perform these manoeuvres on the Q^μ overlaps. We start with a vertex V which satisfies

$$V\{Q^i((V^i)^{-1}z)-Q^j((V^j)^{-1}z)\}=0\,, \tag{4.2}$$

and we would like to obtain a vertex $\hat{V}$ which satisfies

$$\hat{V}\{Q^{N-i+1}((\hat{V}^i)^{-1}z)-Q^{N-j+1}((\hat{V}^j)^{-1}z)\}=0\,, \tag{4.3}$$

where

$$\hat{V}^i \equiv V^i(\hat{z}_k) \equiv V^i(z_{N-k+1}). \tag{4.4}$$

The last equation can be written as

$$\hat{V}\{Q^i((\hat{V}^{N-i+1})^{-1}z) - Q^j((\hat{V}^{N-j+1})^{-1}(z))\} = 0, \tag{4.5}$$

since it holds for all i and j and in particular $i \to N-i+1$ and $j \to N-j+1$. Taking V and $\hat{V}$ to be related by a conformal transformation $\mathscr{A}^i$, i.e.,

$$\hat{V} = V \prod_{i=1}^{N} \mathscr{A}^i, \tag{4.6}$$

we find that $\hat{V}$ satisfies

$$0 = \hat{V}\{Q^i((\mathscr{A}^i)^{-1}(V^i)^{-1}z) - Q^j(\mathscr{A}^j)^{-1}(V^j)^{-1}z)\}. \tag{4.7}$$

Comparing this last equation with Eq. (4.2), we find that

$$\mathscr{A}^i = (V^i)^{-1}(\hat{V}^{N-i+1}). \tag{4.8}$$

Now for the choice of cycling transformations of Eq. (3.5), we find that

$$\mathscr{A}^i = \begin{pmatrix} \infty & 0 & 1 \\ 1 & 0 & \infty \end{pmatrix}, \tag{4.9}$$

namely

$$\mathscr{A}^i = (-1)^{L_0} e^{-L_1}, \tag{4.10}$$

while for the choice of cycling transformation of Eq. (3.8) we find

$$\mathscr{A}^i = 1. \tag{4.11}$$

The above results look unfamiliar due to the appearance of $(-1)^{L_0}$ rather than $\exp i\pi(L_0 - p^2/2)$ as for example discussed in [16]. This somewhat subtle point is related to the fact that in our vertex we have $p^i p^j \ln(z^i - z^j)$, while in previous discussions one used $p^i p^j \ln|z^i - z^j|$. For the usual evaluation of the amplitude, there is no distinction due to the ordering of the z_i's, however, for twisting we reverse the order of the legs and z_i's and so must take account of the distinction. Explicitly examining the vertex, one can readily confirm that the above twists are correct. We could have adopted a vertex with $p^i p^j \ln|z^i - z^j|$. Then we would find a factor of $\exp i\pi(p^i)^2/2$ when twisting as compared to before. Hence the twist for this alternative prescription for the vertex would be $(-1)^N e^{-L_1}$ and $(-1)^N$, where $N = L_0 - \frac{1}{2}p^2$ to replace Eqs. (4.10) and (4.11) respectively. For the case of Eq. (4.1), one can see the entire result by inspection of the vertex of Eq. (3.23). The twist for the new vertex of Eq. (4.11) is, however, simpler than that of Eq. (4.12). We can also consider the interchange of only two legs and their corresponding variables. It is straightforward to find the conformal transformations one must apply to the vertex to induce such a change. One can show that on-shell these conformal transformations yield factors which are precisely cancelled by the corresponding change in the measure. For the vertex of Eq. (3.23), as we have already mentioned, this is obvious. For the open string, of course, the limits of integration and the

Chan-Paton factors mean the final result is not invariant under an interchange. However, for the closed string, we have no such obstruction and so the generalization of our approach to closed strings is obvious and will be discussed elsewhere.

For the case of Neveu-Schwarz strings, the theory above applies and for the cycling transformations of [7] we find that the twist is given by

$$\mathscr{A}^j=(e^{-\Omega_j G_{1/2}^{(j)}}(-1)^{G/2}e^{-L_1}(-1)^{L_0}), \tag{4.12}$$

where $\Omega^{(j)}$ is the super $SL(2,R)$ invariant discussed in [7] and is given by

$$\begin{aligned}\Omega_j=\{&-\theta_{j+1}(z_j-z_{j-1}-\theta_j\theta_{j-1})+\theta_{j-1}(z_j-z_{j+1}-\theta_j\theta_{j+1})\\&-\theta_j(z_{j-1}-z_{j+1}-\theta_{j-1}\theta_{j+1})+\theta_{j-1}\theta_j\theta_{j+1}\}\\&\times\{(z_j-z_{j-1}-\theta_j\theta_{j-1})^{-1/2}(z_{j-1}-z_{j+1}-\theta_{j-1}\theta_{j+1})^{-1/2}\\&\times(z_j-z_{j+1}-\theta_{j-1}\theta_{j+1})^{-1/2}\}.\end{aligned} \tag{4.13}$$

5. Sewing

In a quantum theory based on a Feynman path integral or in an S-matrix theory unitarity and in particular factorizability is guaranteed [17]. However, in the approach advocated by the authors, this is not the case and must be verified explicitly. We will now demonstrate that the amplitudes do factorize correctly and this is a simple consequence of the method. The problem one faces has much in common with the old dual approach where the vertices were shown [15, 18] to factorize by direct calculation using oscillator algebra. What was actually shown was that one could sew two vertices together in such a way so as to yield a third vertex which was of the same type. One then realized that factorization corresponded to reversing these steps.

Let us first consider sewing together two tree vertices to yield a third. In the group theoretic approach, the vertices are determined by their overlap relations, and it therefore suffices to show that the final vertex has the correct overlap identities if the original two do. In sewing we will take the adjoint of a vertex, and since

$$Q^+(z)=-Q\left(\frac{1}{z}\right)\equiv-Q(\Gamma z), \tag{5.1}$$

we find that V^+ has the overlap

$$V^+\{Q^i(\Gamma(V^i)^{-1}z)-Q^j(\Gamma(V^j)^{-1}z)\}=0. \tag{5.2}$$

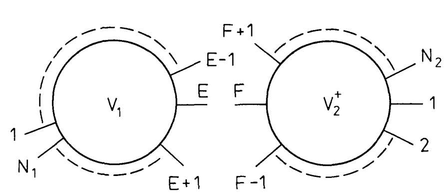

Fig. 3. Sewing two vertices to form a third

Let us sew legs E and F together of vertices with N_1 and N_2 external legs as in Fig. 3. We join V_1 and V_2^+ with a propagation which when written in parametric form is an integration which we discuss later, times a conformal factor $\mathscr{P}$. Thus the composite vertex V_c is of the generic form $V_c = V_1 \mathscr{P}^{(E)} V_2^+$. In what follows, we will often take the adjoint with respect to the remaining legs of V_2 to get the final vertex in standard form.

Let us assume that $(V^j)^{-1}$ depends only on Koba-Nielsen variables z_{j-1}, z_j, and z_{j+1} as does the cycling choice of Eq. (3.5). The choice of Eq. (3.8) only depends on z_j and so is included as a special case of this assumption.

Consider first the overlap identities on the remaining legs of the first vertex; all identities are correct for the new vertex except for those involving legs $E-1$ and $E+1$. However, we may choose z_E in two ways, either

a)
$$z_E = z_{F-1} \tag{5.3}$$

or

b)
$$z_E = z_{F+1} . \tag{5.4}$$

For these choices we find that identities involving legs $E+1$ and legs $E-1$ respectively are now correct for the composite vertex.

With choice a) we must perform a conformal transformation on leg $E-1$; the identity

$$V_1\{Q^i((V_0^i)^{-1}z) - Q^{E-1}((V_0^{E-1})^{-1}z)\} = 0, \qquad i \neq E, E-1, \tag{5.5}$$

must become

$$V_c\{Q^i((V^i)^{-1}z) - Q^{E-1}((V^{E-1})^{-1}z)\} = 0, \qquad i \neq E, E-1, \tag{5.6}$$

where (V_0^i) are for the original vertex V_1, and V^i are for the composite vertex V_c. Clearly $V_0^i = V^i$, and so the conformal transformation one must apply to V_1 on leg E is

$$G^{E-1} = (V_c^{E-1})^{-1} V_0^{E-1} . \tag{5.7}$$

Cycling transformations of Eq. (3.5), we have

$$G^{E-1} = \begin{pmatrix} z_{E-2} & z_{E-1} & z_{F-1} \\ \infty & 0 & 1 \end{pmatrix} \begin{pmatrix} \infty & 0 & 1 \\ z_{E-2} & z_{E-1} & z_{F+1} \end{pmatrix} = r^{L_0}, \tag{5.8}$$

where

$$r = \frac{(z_{F-1} - z_{E-2})\,(z_{E-1} - z_{F+1})}{(z_{F-1} - z_{E-1})\,(z_{E-2} - z_{F+1})} . \tag{5.9}$$

For choice b) we must make the gauge transformation $(V_c^{E+1})^{-1} V_0^{E+1}$ on leg $E+1$. However, for the choice of cycling transformation of Eq. (3.8) [i.e., $(V^j)^{-1}(z) = z - z_j$], no gauge transformations are required.

To ensure the overlap relations between the legs of the second vertex are correct, we must carry out an equivalent discussion which is the same as above provided we make the replacements $E \leftrightarrow F$ everywhere.

To find the propagator we consider the overlap identities between legs on the composite vertex which originated from the vertices V_1 and V_2 before sewing. We have

$$V_1\{Q^i((V^i)^{-1}z)-Q^E((V^E)^{-1}z)\}=0\,, \tag{5.10}$$

and so

$$V_1\mathscr{P}\{Q^i((V^i)^{-1}z)-Q^E(\mathscr{P}^{-1}(V^E)^{-1}z)\}=0\,. \tag{5.11}$$

The latter factor now faces V_2 on leg F, and so can be considered as a factor

$$V_2Q^F(\Gamma\mathscr{P}^{-1}(V^E)^{-1}z)\,, \tag{5.12}$$

after taking the Hermitian conjugate. From Eq. (2.5), we have that

$$0=V_2\{Q^F(\Gamma\mathscr{P}^{-1}(V^E)^{-1}z)-Q^k((V^k)^{-1}V_F\Gamma\mathscr{P}^{-1}(V^E)_z^{-1})\}\,. \tag{5.13}$$

However, the composite vertex must satisfy

$$V_c\{Q^i((V^i)^{-1}z)-Q^k((V^k)^{-1}z)\}=0\,, \tag{5.14}$$

and so we must conclude that

$$V_F\Gamma\mathscr{P}^{-1}(V^E)^{-1}=1 \quad \text{or} \quad \mathscr{P}=V_E^{-1}V_F\Gamma\,. \tag{5.15}$$

What one finds for the propagator $\mathscr{P}$ depends in general on the identifications made. For the choice of cycling transformation of Eq. (3.5), we find for choice a) that

$$\mathscr{P}=\left[\frac{c}{c-1}\right]^{L_0}, \tag{5.16}$$

where

$$c\equiv\frac{(z_{E-1}-z_{E+1})\,(z_{F+1}-z_{F-1})}{(z_{E-1}-z_{F-1})\,(z_{F+1}-z_{E+1})}, \tag{5.17}$$

while for choice b)

$$\mathscr{P}=e^{-L_1}\left(\frac{c}{c-1}\right)^{L_0}e^{-L_{-1}}\,. \tag{5.18}$$

For the cycling choice $(V^j)^{-1}z=z-z_j$, we find

$$\mathscr{P}=e^{\frac{L_1}{z_E-z_F}}(-1)^{L_0}e^{2\ln(z_E-z_F)L_0}e^{\frac{L_{-1}}{z_F-z_E}}\,. \tag{5.19}$$

We note that also in this case we are obliged to make one of the identifications a) or b) above. For Eqs. (5.16) and (5.17) the propagator is obtained by integrating over dc and that of Eq. (5.19) over ds where $s=z_E-z_F$. Here we have glossed over any additional factors of c and s respectively which may be needed in order to obtain the correct measure. Such factors follow from ensuring that zero-norm physical states decouple from the composite vertex.

As mentioned above, what we must really guarantee is factorization. Here we look at the residue of the pole in

$$(p_{E+1}+\ldots+p_{N1}+p_1+\ldots+p_{E-1})^2=(q_{F+1}+\ldots+q_{N2}+q_1+\ldots+q_{F-1})^2\,,$$

where p and q label the momenta of the vertices V_1 and V_2 respectively. Such a pole occurs when all the remaining Koba-Nielsen variables on a leg coalesce. On V_1, say, this means

$$(z_i-z_j)=\varepsilon(\tilde{z}_i-\tilde{z}_j)\quad \forall i,j;\ i,j\neq E\,,$$

where $\varepsilon\to 0$. Due to the ordering of the z's this is enforced by taking $z_{E-1}\to z_{E+1}$. Reversing the above steps, we may rewrite V_c in terms of V_1, V_2, and $\mathscr{P}$. However, $\mathscr{P}$ contains a u^{L_0} factor which for $u\to 0$ implies the poles discussed above in the momenta squared. The residue is then recognizable as V_1 times V_2 augmented by conformal transformations which vanish on-shell.

It is in order to make a comment on how $SL(2,R)$ has been fixed. The above procedure has implicitly assumed that on V_1 we used $SL(2,R)$ to fix z_{E-1}, z_E and z_{E+1} and similarly on V_2. We also inserted one integration over a combination of three variables. After sewing, only four of these legs are left as well as the one integration. Hence in effect only three legs on the composite vertex have had their Koba-Nielsen variables fixed by $SL(2,R)$ invariance.

We now consider sewing two legs so as to form a loop. To treat the most general situation, it suffices to consider an N string vertex V with M loops and we sew two external legs to form an $N-2$ string vertex V with $M+1$ loops (see Fig. 4). We recall Eq. (2.5)

$$V\{Q^i((V^i)^{-1}z)-Q^j((V^j)^{-1}z)\}=0\,, \tag{5.20}$$

which is the same overlap as for trees. On legs $E-1, E+1$ and $F-1, F+1$, the new transformations V^j do not coincide with those of the old vertex V_0^j, and hence we make gauge transformations $\mathscr{A}^{E-1}=(V_0)^{E-1}V^{E-1}$, etc. Unlike for the tree case, we cannot identify z_E to be z_{E-1}, since although it makes $V^{E+1}=V_0^{E+1}$, it reduces V^{E-1} to be non-invertible.

To find the loop cycling transformation, we transport Q^μ from leg i through the sewn legs E and F and back to leg i. We have

$$V\mathscr{P}\{Q^i((V^i)^{-1}z)-Q^E(\mathscr{P}^{-1}(V^E)^{-1}z)\}=0\,, \tag{5.21}$$

which becomes a factor

$$Q^F(\Gamma\mathscr{P}^{-1}(V^E)^{-1}z) \tag{5.22}$$

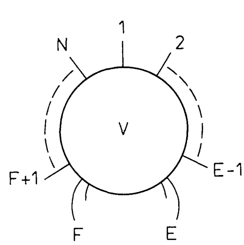

Fig. 4. Sewing to create an extra loop

on line F. However, using the original overlap we have

$$V\{Q^F(\Gamma\mathscr{P}^{-1}(V^E)^{-1}z)-Q^i((V^i)^{-1}V^F\Gamma\mathscr{P}^{-1}(V^E)^{-1})\}=0\,, \tag{5.23}$$

and so

$$V\{Q^i((V^i)^{-1}z)-Q^i((V^i)^{-1}V^F\Gamma\mathscr{P}^{-1}(V^E)^{-1}z)\}=0\,. \tag{5.24}$$

Comparing with the result of Eq. (2.6) we find

$$P^i_n=(V^i)^{-1}V^F\Gamma\mathscr{P}^{-1}(V^E)^{-1}V^i$$

or

$$P_n=V^F\Gamma\mathscr{P}^{-1}(V^E)^{-1}\,.$$

Factorization implies that we must sew the legs above with the same propagator corrections which were used to sew the three graphs. Since z_E and z_F are unconstrained and we gain one new variable from the propagator, we have a P_n which depends on three variables as should be for an arbitrary element of $SL(2,R)$.

To find the correct measure for the composite vertex one must in general sew with vertices which include their ghost contributions. However, this is ensured by demanding that the corresponding anticommuting conformal operators are correctly transmitted around the graph. This discussion, at least in outline, is the same as above and will be considered elsewhere. We note here that the BRST invariant propagators are easily obtained from those given above by letting $L_0, L_{\pm1}$ include their ghost extensions and inserting appropriate factors of $\beta_0, \beta_{\pm1}$ next to the corresponding $L_0, L_{\pm1}$. For example, $\mathscr{P}$ of Eq. (5.16) becomes

$$\mathscr{P}=\beta_0\left(\frac{c}{c-1}\right)^{L_0}.$$

The propagator for Neveu-Schwarz strings is computed as above and for the cycling transformation of [7] and the identification (a) is the super $SL(2,R)$ invariant extension of Eq. (5.16).

6. One Loop Amplitude

The one loop amplitudes for the open bosonic string were computed using the group theoretic approach for arbitrary number of external strings in [5, 6]. Here we repeat this computation using the particular simple choice of cycling transformation discussed above and for the torus when mapped onto a rectangle rather than in the Schottky representation which was used before. We consider the rectangle (see Fig. 5) to have length $\ln w$ and width $2i\pi$. The section of the rectangle

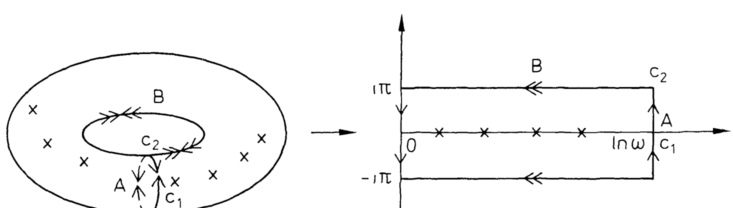

Fig. 5. Mapping the torus to the rectangle

above the real axis is mapped onto the open string, and the section below is mapped onto its double [19].

We may perform the calculation by conformally mapping from the results in the Schottky representation, however, it is quicker to begin at the beginning. For the torus we require two loop cycling transformations rather than just one in the Schottky approach, where going around a B cycle is the condition single valuedness. We can take these to be

$$P_n(z)=z+\ln\omega\,; \qquad R_n(z)=z+2\pi i\,. \tag{6.1}$$

The external cycling transformations are taken to be $(V^j)^{-1}(z)=(z-z_j)$. Using Eq. (2.4) we find that $P_n^j=P_n$ and $R_n^j=R_n$.

It is important to realize that although $P^\mu(z)$ and $L(z)=\Sigma_n L_{-n}z^{-n}$ overlaps exist for both types of cycling transformations, the $Q^\mu(z)$ overlap is only valid for $P_n(z)$ and not for $R_n(z)$. The reason is that $Q^\mu(z)$ contains a $\ln z$ term which measures the distance around a B cycle which is clearly not zero.

Following the discussion on moduli changing [i.e., $\hat{\omega}=\omega(1+\varepsilon)$] in particular Eq. (2.14), we find that this is achieved by the infinitesimal transformation $\mathscr{A}(z)=z+\varepsilon f(z)$, where $f(z)$ obeys the relations

$$f(z+2i\pi)=f(z)\,, \qquad f(z)=1+f(z+\ln\omega)\,. \tag{6.2}$$

The solution to this equation

$$f(z)=\sum_i c_i\bar{\zeta}(z-s_i)\,, \tag{6.3}$$

where c_i and s_i are constants and $\sum_i c_i=1$ and

$$\bar{\zeta}(z)\equiv\zeta(z)-\frac{z\eta}{\pi i}\,; \qquad \eta\equiv\zeta(\pi i)\,, \tag{6.4}$$

and the periods of the Weierstrass ζ function are $2\omega_1=i\pi$, $2\omega_2=\ln\omega$. Taking only one $\bar{\zeta}$ function and adjusting s_i, as we must, such that its pole occurs at the point z_j, where string j is "emitted", we have the equation

$$\omega\frac{\partial V}{\partial\omega}=V\sum_{i=1}^{N}\left\{\oint\frac{dz}{z}\bar{\zeta}(z-z_j+z_i)\frac{L^i(z)}{z}+c\right\}, \tag{6.5}$$

where c is a constant.

We may also shift one of the Koba-Nielsen points, say, $z_j\to z_j+\delta_{jk}\alpha=\hat{z}_j$. Following Eq. (2.10) this is achieved by

$$\begin{aligned} V(z_j+\delta_{jk}\alpha,\omega)&=V(z_j,\omega)\sum_{j=1}^{N}\,[V^j(z_j)]^{-1}[V^j(\hat{z}_j)]\\ &=V(z_j,\omega)\exp(-L_{-1}\alpha)\,. \end{aligned} \tag{6.6}$$

Demanding that the zero-norm physical state of the form $L_{-1}|\Omega\rangle$ decouple from W, we find that the measure $\bar{f}$ is independent of z_j.

Equation (6.5) can be written as

$$\omega\frac{\partial V}{\partial\omega}=V\left\{\sum_{\substack{i=1\\ i\neq j}}^{N}\{\bar\zeta(z_i-z_j)L_{-1}^{(i)}+\bar\zeta'(z_i-z_j)L_0^{(i)}\}\right.$$
$$\left.+\left\{L_{-2}^{(j)}-\frac{\eta}{\pi i}L_0^{(j)}+c\right\}+\{\text{terms with } L_n\ n\geqq 1\}\right\}. \tag{6.7}$$

We now demand that the zero-norm physical state of the form $(L_{-2}+3/2L_{-1}^2)|\Omega'\rangle$ decouple from W. Using the fact that the measure is z_j independent, Eq. (6.6), and that $L_0^{(j)}|\Omega'\rangle_j=-|\Omega'\rangle_j$, we find the differential equation for the measure $\bar f$ to be

$$\omega\frac{\partial}{\partial\omega}\ln\bar f=-c-\frac{\eta}{i\pi}-1. \tag{6.8}$$

The value of c can be found by acting on equation with the vacuum at zero momentum

$$0=c+V\left\{\left(\frac{a_{-1}^{\mu(j)}}{2}\right)^2\,\bigg|\,0,0\right)\right\}. \tag{6.9}$$

From the oscillator form of V given shortly, we find that

$$c=\left(\frac{1}{2\ln\omega}-\frac{\eta}{2\pi i}\right)D, \tag{6.10}$$

and as a result we find that

$$\bar f=\frac{1}{\omega}\cdot\frac{1}{\omega^{\frac{D-2}{24}}}\cdot\frac{1}{\left[\prod_{n=1}^{\infty}(1-\omega^1)\right]^{D-2}}\cdot\frac{1}{(\ln\omega)^{D/2}}, \tag{6.11}$$

which shows the well-known reduction in the powers of the partition function [20].

Let us find V itself by using the equation

$$V\sum_{j=1}^{N}\left\{\oint_{\xi_j=0}\frac{d\xi_j}{\xi_j}\varphi P^\mu(\xi_j)\right\}=0, \tag{6.12}$$

where $\xi_j=\xi_i-z_j+z_i$, and where φ is an arbitrary function which only has poles at the Koba-Nielsen points z_i. Following the discussion of [6], we find the above result holds if

$$\varphi(z+\ln\omega)=\varphi(z), \tag{6.13}$$

and

$$\varphi(z+2\pi i)=\varphi(z)+\text{constant}. \tag{6.14}$$

Such a set of function which have poles at only one point are provided by

$$\begin{aligned}\varphi_n(z)&=\frac{(-1)^n}{n!}\left(\frac{d}{dz}\right)^n\ln\psi(z)\\&=\frac{(-1)^n}{n!}\left(\frac{d}{dz}\right)^n\ln\frac{\psi(z)}{z}+\frac{1}{n}z^{-n}\\&=\frac{z^{-n}}{n}+\sum_{m=1}^{\infty}E_{nm}z^m.\end{aligned} \tag{6.15}$$

In this equation

$$\ln\psi(z)=\ln\sigma(z)+z^2\left\{\frac{1}{2\ln\omega}-\frac{\eta}{\pi i}\right\}, \tag{6.16}$$

where $\sigma(z)$ is the Weierstrass σ function [i.e., $\zeta(z)=d/dz\ln\sigma(z)$]. We note that $\sigma(z)$ only has one zero in the fundamental domain at $z=0$, and so $\ln\psi(z)/z$ is an analytic function. We may write

$$\varphi_n(\xi_i-z_j+z_i)=\frac{1}{n}\frac{1}{(\xi_i-z_j+z_i)^n}+\sum_{\substack{m=0\\ n=1,2,\dots}}^{\infty}E^{ji}_{nm}(\xi_i)^m,$$

and we find that

$$\begin{aligned}E^{ji}_{nm}=&(-1)^n\frac{(n+m-1)!}{n!m!}G_{n+m}(z_i-z_j)-(-1)^nG_n(0)\delta_{m,0}\\&+\delta_{n,1}\left\{\delta_{m,1}\left\{\frac{1}{\ln\omega}-\frac{\eta}{\pi i}+G_2(0)\right\}\right.\\&\left.+\delta_{m,0}(z_i-z_j)\left\{G_2(0)+\frac{1}{\ln\omega}-\frac{\eta}{\pi i}\right\}\right\}\\&n=1,2,3\dots\\&m=0,1,2,\dots,\end{aligned} \tag{6.17}$$

where

$$G_k(z_i-z_j)=\sum_{p,q}{}'\frac{1}{(\Omega_{p,q}-z_i+z_j)}. \tag{6.18}$$

$\Omega_{p,q}=2p\omega_1+2q\omega_2$ and $\sum'_{p,q}$ means sum over all integers p and q except $p=q=0$. We note that $G_k(0)=0$ if k is odd, but if k is even these are the Eisenstein series [21] which are usually labelled slightly differently. Substituting in Eq. (6.12), we find that the vertex V is given by

$$\begin{aligned}V={}_1\langle 0|\dots{}_N\langle 0|\exp-&\left\{\sum_{i,j=1}^{\infty}\left\{\sum_{\substack{m=1\\n=1}}^{\infty}\left[\frac{1}{2}a^{\mu j}_n\sqrt{n}\frac{(n+m-1)!(-1)^n}{m!n!}G_{n+m}(z_i-z_j)a^{\mu i}_m\sqrt{m}\right]\right.\right.\\&+\frac{1}{2}a^{\mu j}_1\left(\frac{1}{\ln\omega}-\frac{\eta}{\pi i}+G_2(0)\right)a^{\mu i}_1\\&+\left[\sum_{n=1}^{\infty}a^{\mu j}_n\sqrt{n}(-1)^n(G_n(z_i-z_j)-G_n(0))a^{\mu i}_0\right]\\&\left.+a^{\mu j}_1\left(\frac{1}{\ln\omega}-\frac{\eta}{\pi i}+G_2(0)\right)(z_i-z_j)a^{\mu i}_0\right\}\\&+\sum_{\substack{i,j=1\\i\neq j}}^{N}\left\{+\frac{1}{2}\sum_{\substack{n=1\\m=1}}^{\infty}\cdot a^{\mu j}_n\sqrt{n}\frac{(n+m-1)!}{m!n!}(-1)^n\frac{\sqrt{m}}{(z_j-z_i)^{n+m}}a^{\mu i}_m\right.\\&\left.\left.+\sum_{n=1}^{\infty}\frac{a^{\mu j}_n\sqrt{n}(-1)^n}{(z_j-z_i)^n}a^{\mu i}_0\right\}+\sum_{i,j}a^{\mu i}_0\mathscr{A}^{ij}_{00}a^{\mu j}_0\right\}.\end{aligned} \tag{6.19}$$

This result has a simple interpretation in that $\ln\psi$ of Eq. (6.15) is the "energy" between all pairs of Koba-Nielsen points and their images. The Eisenstein coefficients have well-known modular properties, and this should be useful in the analogue formula for the closed string.

7. Conclusion

We have given above an account of the theory of the cycling transformations used in the group theoretic approach of the authors. It allows a simple physical interpretation of the method, namely two conformal operators which act at two points on the world-sheet of the string have the same action after being conformally mapped to the same point. This also applies to mapping a conformal operator around a loop. The corresponding equations, i.e. overlap identities and the decoupling of zero-norm physical states, i.e., gauge invariance completely determine the string s-matrix.

It is now recognized that there is a number of problems in the computation of string scattering that involves Ramond strings. Following the discussion on cycling transformations, we may search for a $(V^j)^{-1}(z,\theta)$ which is a Ramond transformation and vanishes as $z \to z_j$ and $\theta \to \theta_j$. One easily sees that such a transformation is not possible due to the square roots required by it being a Ramond transformation. This result is perhaps to be expected as any point z_j, θ_j is related by a branch cut to some other point where a Ramond string is emitted. Should one not require a knowledge of the part of the transformation that depends on the anticommuting Koba-Nielsen, then there is no problem. For example, one can compute the part of the vertex which is independent of θ_j.

It is straightforward to show that one must perform $I+B-2$ θ_j integrations on a graph with $2I$ fermionic and B bosonic external strings. Hence the vertices with one bosonic and two fermionic lines and four fermionic lines do not require any θ_j integrations and so can be computed without any difficulty from the usual overlap equations. For the former we recover, at least for a certain choice of cycling transformations, the vertex of Corrigan and Olive [22] while the latter generalizes the known four fermion ground state scattering [23]. For higher vertices, we must perform the θ_j integration, and so in effect bring down factors from $V(z_j, \theta_j)$. This in fact amounts to the picture changing operation of [24]. However, here we see that picture changing is the result of quite a number of manoeuvres which begin from a simple starting point. This process is illustrated for the Neveu-Schwarz string scattering in [7]. We will return to these points and to the closed string in a future publication.

References

1. Kaku, M., Kikkawa, K.: Phys. Rev. D **10**, 1110, 1823 (1974)
   Cremmer, E., Gervais, J.-L.: Nucl. Phys. B **90**, 410 (1975)
2. Neveu, A., West, P.: Phys. Lett. **168B**, 192 (1985); Nucl. Phys. B **278**, 601 (1980)
   Witten, E.: Nucl. Phys. B **268**, 23 (1986)
   Hata, H., Itoh, K., Kugo, T., Kunitomo, K., Ogawa, K.: Phys. Lett. **172B**, 186 (1986); Phys. Rev. D **34**, 2369 (1986)

3. Ishibashi, N., Matsuo, Y., Ooguri, H.: University of Tokyo preprint UT-499 (1986)
Alvarez-Gaumé, L., Gomez, C., Reina, C.: Phys. Lett. **190B**, 55 (1987) and CERN preprint TH.4775/87 (1987)
Witten, E.: Conformal field theory, Grassmannians and algebraic curves. Princeton preprint (1987)
4. Neveu, A., West, P.: Phys. Lett. **193B**, 187 (1987)
5. Neveu, A., West, P.: Phys. Lett. **194B**, 200 (1987)
6. Neveu, A., West, P.: Group theoretic approach to the open Bosonic string multiloop s-matrix. Commun. Math. Phys. **114**, 613–643 (1988)
7. Neveu, A., West, P.: Neveu-Schwarz excited string scattering; A superconformal group computation. Phys. Lett. **200B**, 275 (1988)
8. Callan, C., Lovelace, C., Nappi, C., Yost, S.A.: Princeton preprint PUPT (1987)
9. Caneschi, L., Schwimmer, A., Veneziano, G.: Phys. Lett. **30B**, 351 (1969)
10. West, P.: A review of duality, string vertices, overlap identities and the group theoretic approach to string theory. CERN preprint TH.4819/87 (1987)
11. Beilinson, A.A., Manin, Yu.I., Schechtman, V.A.: Localization of the Virasoro and Neveu Schwarz algebra (preprint 1986)
12. Perelomov, A.: Generalized coherent states and their applications. Berlin, Heidelberg, New York: Springer 1986
13. Neveu, A., West, P.: Nucl. Phys. B **278**, 601 (1980)
14. Lovelace, C.: Phys. Lett. **32B**, 496 (1970)
Olive, D.: Nuovo Cimento **3A**, 399 (1971)
15. Bardakci, K., Ruegg, H.: Phys. Lett. **28B**, 342 (1968)
Virasoro, M.A.: Phys. Rev. Lett. **22**, 37 (1969)
Bardakci, K., Ruegg, H.: Phys. Rev. **181**, 1884 (1969)
Chan, H.M., Tsou, S.T.: Phys. Lett. **28B**, 485 (1969)
Goebel, C.J., Sakita, B.: Phys. Rev. Lett. **22**, 257 (1969)
Koba, Z., Nielsen, H.B.: Nucl. Phys. B **10**, 633 (1969)
16. Mandelstam, S.: Phys. Rep. **13**, 259 (1974)
17. Eden, J., Landshoff, P., Olive, D., Polkinghorne, J.: Analytic S matrix. published by CUP and references therein
18. Fubini, S., Veneziano, G.: Nuovo Cimento **64A**, 811 (1969)
Fubini, S., Gordon, D., Veneziano, G.: Phys. Lett. **29B**, 679 (1969)
Fubini, S., Veneziano, G.: Nuovo Cimento **67A**, 29 (1970), Ann. Phys. **63**, 12 (1971)
Bardakci, K., Mandelstam, S.: Phys. Rev. **184**, 1640 (1969)
19. Alessandrini, V.: Nuovo Cimento **2A**, 321 (1971)
Schiffer, M., Spencer, D.: Rieman surfaces. Princeton University Press 1954
20. Brink, L., Olive, D.: Nucl. Phys. B **58**, 237 (1973); B **56**, 256 (1973)
21. Serre, J.P.: A course of arithmetic. Berlin, Heidelberg, New York: Springer 1986
22. Corrigan, E., Olive, D.: Nuovo Cimento **11A**, 749 (1972)
23. Olive, D., Scherk, J.: Nucl. Phys. B **69**, 325 (1974)
Brink, L., Olive, D., Rebbi, C., Scherk, J.: Phys. Lett. **45B**, 379 (1973)
Mandelstam, S.: Phys. Lett. **46B**, 447 (1973)
Schwarz, J.H., Wu, C.C.: Phys. Lett. **47B**, 453 (1973)
Corrigan, E.F., Goddard, P., Smith, R.A., Olive, D.: Nucl. Phys. B **67**, 477 (1973)
Schwarz, J.H., Wu, C.C.: Nucl. Phys. B **73**, 77 (1974)
Bruce, D., Corrigan, E., Olive, D.: Nucl. Phys. B **25**, 427 (1975)
24. Friedan, D., Martinec, E., Shenker, S.: Nucl. Phys. B **271**, 93 (1986)

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