Determinants of Laplacians on Surfaces of Finite Volume

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Determinants of the Laplace and other elliptic operators on compact manifolds have been an object of study for many years (see [MP, RS, Vor]). Up until now, however, the theory of determinants has not been extended to non-compact situations, since these typically involve a mixture of discrete and continuous spectra. Recent advances in this theory, which are partially motivated by developments in mathematical physics, have led to a connection, in the compact Riemann surface case, between determinants of Laplacians on spinors and the Selberg zeta function of the underlying surface (see [DP, Kie, Sar, Vor]).

Our purpose in this paper is to introduce a notion of determinants on non-compact (finite volume) Riemann surfaces. These will be associated to the Laplacian Δ shifted by a parameter $s(1-s)$, and will be defined in terms of a Dirichlet series $\zeta(w, s)$ which is a sum that represents the discrete as well as the continuous spectrum. It will be seen to be regular at $w = 0$, and our main theorem (see Sect. 1) will express $\exp\left(-\dfrac{\partial}{\partial w}\zeta(w, s)\Big|_{w=0} \right)$ as the Selberg zeta function of the surface times the appropriate Γ-factor.

1.

Let $M = \Gamma \backslash \mathbf{H}$ be a non-compact, finite volume surface obtained as the quotient of the upper half plane $\mathbf{H}$ by a discrete subgroup Γ of $\mathrm{PSL}_2(\mathbf{R})$. For simplicity we assume that Γ has no fixed points. Let χ be a unitary character of Γ. We consider the spectral problem

$$\Delta f + \lambda f = 0, \quad f(\gamma z) = \chi(\gamma)f(z) \quad (\gamma \in \Gamma, z \in \mathbf{H}), \quad \int_M |f(z)|^2 dz < \infty. \quad (1.1)$$

Here $\Delta = y^2\left(\dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial x^2} \right)$ is the Laplacian of $\mathbf{H}$. In addition to a discrete spectrum $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$, this set-up gives rise to a continuous spectrum as well, as we now

briefly describe (see [Hej, Sel, Ven] for details). The conditions on M mean that it consists of a relatively compact part together with finitely many regions that stretch out to infinity. These are the *cusps*, which we parametrize by their parabolic endpoints $\kappa_1, \ldots, \kappa_h$. They are ordered so that the first h_1 are singular, i.e., if Γ_i is the stabilizer of κ_i in Γ then $\chi(\Gamma_i) = 1$ for $1 \leq i \leq h_1$. To such a cusp κ_i we associate an *Eisenstein series*, defined for $z \in \mathbf{H}$, $\mathrm{Re}(s) > 1$ by

$$E_i(z, s, \chi) = \sum_{\gamma \in \Gamma_i \backslash \Gamma} \chi(\gamma) y^{(i)}(\gamma z)^s.$$

Here $z^{(i)} = (y^{(i)}, x^{(i)})$ is the local parameter at κ_i. Roughly speaking, these Eisenstein series span the subspace of $L^2(\Gamma \backslash \mathbf{H}, \chi)$ which is orthogonal to the one spanned by the discrete spectrum.

Each $E_i(z, s, \chi)$ admits a Fourier expansion at each cusp κ_j, whose zero coefficient is of the form

$$\delta_{ij} y^{(j)s} + \phi_{ij}(s, \chi) y^{(j)1-s}.$$

Let $\Phi(s, \chi) = (\phi_{ij}(s, \chi))_{i, j = 1, \ldots, h_1}$. Then the function $\phi(s) = \phi(s, \chi) = \det \Phi(s, \chi)$ which can be meromorphically continued to all of $\mathbf{C}$, and satisfies $\phi(s, \chi)\phi(1 - s, \chi) = 1$, carries much of the information derived from the continuous spectrum. More specifically, we have three sequences associated with our spectral problem (1.1):

1. The set S_1 of $s_n \in \mathbf{C}$ such that $s_n(1 - s_n) = \lambda_n$, where λ_n is in the discrete spectrum of (1.1).

2. The set S_2 of poles $\varrho_m = \beta_m + i\gamma_m$ of $\phi(s, \chi)$ with $\beta_m < 1/2$.

3. The set $S_3 = \{\eta_1, \ldots, \eta_N\}$ of exceptional poles of $\phi(s, \chi)$ in $(1/2, 1]$.

We then have the asymptotic relation

$$\#\{n \mid |\mathrm{Im}(s_n)| \leq T\} + \#\{m \mid |\mathrm{Im}(\varrho_m)| \leq T\} \sim \frac{|M|}{2\pi} T^2 \quad \text{as} \quad T \to \infty. \tag{1.2}$$

Here $|M|$ denoted the area of the surface M. In general these two terms cannot be estimated separately and it is conjectured that the main contribution can come from either one. It is therefore natural to put the numbers $\lambda_n = s_n(1 - s_n)$ and $\varrho_m(1 - \varrho_m)$ together and define for $s, w \gg 0$,

$$\zeta(w, s) = \sum_{\sigma \in S} (\sigma(1 - \sigma) - s(1 - s))^{-w}, \tag{1.3}$$

where

$$S = S_1 \cup S_2 - S_3.$$

We shall see in Sect. 2 that $\zeta(w, s)$ can be continued meromorphically in w and is regular at $w = 0$. In view of the formal identity

$$-\frac{\partial}{\partial w} \zeta(w, s)\bigg|_{w = 0} = \sum_{\sigma \in S} \log(\sigma(1 - \sigma) - s(1 - s)),$$

we define our *determinant* associated to M (and χ) to be

$$\det{}^2(\Delta - s(1 - s)) = e^{-\frac{\partial}{\partial w} \zeta(w, s)|_{w = 0}}. \tag{1.4}$$

We have chosen this notation since the discrete eigenvalues (apart from those that come from exceptional poles) occur twice in the sum (1.3).

Recall now the *Selberg zeta function* ([Hej, Sel, Ven])

$$Z(s) = Z(s, \Gamma, \chi) = \prod_{\{P\}_{hyp}} \prod_{k=0}^{\infty} (1 - \chi(P) NP^{-s-k}), \tag{1.5}$$

where $\{P\}_{hyp}$ run through the primitive Γ-conjugacy classes of hyperbolic elements in Γ and NP is its norm, so that $\log NP$ is the length of the closed geodesic on M which arises from P.

$Z(s)$ is meromorphic in $\mathbf{C}$ and its zeros are at the points in S (with the same multiplicity) as well as at the nonpositive integers $-j$ with multiplicity $(j + \frac{1}{2})|M|/\pi$. It has a pole at $s = \frac{1}{2}$ of multiplicity $\frac{1}{2}(h_1 - \operatorname{tr}\Phi(\frac{1}{2}, \chi))$ and trivial poles at $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$ of multiplicity h_1. $Z(s)$ is also known to admit a functional equation which involves the Γ-factor

$$Z_{\infty}(s) = \left((2\pi)^s \frac{\Gamma_2(s)^2}{\Gamma(s)} \right)^{\frac{|M|}{2\pi}}, \tag{1.6}$$

where $\Gamma_2(s)$ is the double Gamma function (see [Bar, Var, Vig]). $Z_{\infty}(s)$ gives rise to poles which exactly cancel out the trivial zeros of $Z(s)$. We can now state our main theorem:

Theorem. *We have the identity*

$$\det{}^2(\Delta - s(1-s)) = \phi(s) Z(s)^2 Z_{\infty}(s)^2 \Gamma(s + \tfrac{1}{2})^{-2h_1} (2s-1)^A e^{B(2s-1)^2 + C(2s-1) + D}, \tag{1.7}$$

where

$$A = h_1 - \operatorname{tr}\Phi(\tfrac{1}{2}, \chi),$$

$$B = -\frac{|M|}{4\pi},$$

$$C = - \sum_{h_1 < i \leq h} \log|1 - \chi(\gamma_i)| - h_1 \log 2 \quad (\gamma_i \ a \ generator \ of \ \Gamma_i),$$

$$D = \frac{|M|}{\pi}(2\zeta'(-1) - \log\sqrt{2\pi}) + 2h_1 \log\sqrt{2\pi} - A \log 2$$

$$(\zeta(s) = Riemann's \ zeta \ function).$$

It follows from the above that $\det^2(\Delta - s(1-s))$ can be extended to an entire function whose zeros are s_n with doubled multiplicity as well as ϱ_m and $1 - \varrho_m$ with their multiplicity as poles and zeros of $\phi(s)$ respectively. We obtain precisely the zeros of the formal product

$$\det{}^2(\Delta - s(1-s)) = \prod_{\sigma \in S} (\sigma(1-\sigma) - s(1-s)).$$

Thus, $\phi(s)^{-1} \det^2(\Delta - s(1-s))$ is the square of an entire function whose zeros are the s_n and ϱ_m.

Remarks

1. We note that with this normalization the determinant is finitely multiplicative, that is, for $S_0 \subset S$ finite,

$$\det^2(\varDelta - s(1-s)) = \prod_{S-S_0}(\sigma(1-\sigma) - s(1-s))\prod_{S_0}(\sigma(1-\sigma) - s(1-s)).$$

2. As an immediate corollary to the theorem we obtain a symmetric functional equation for $Z(s)$ (see [Vig] for the case of $SL_2(\mathbf{Z})$).

3. More generally, one can proceed in a similar fashion to define and compute determinants of Laplacians corresponding to Dirac operators on spinors. The relevant automorphic forms then transform via a multiplier system, and one applies a trace formula of the type given in [Hej, Chap. 9].

Our proof, which we now turn to, is a non-compact analog of that of [Sar].

2.

The analysis of our $\zeta(w,s)$ will be carried out via that of the kernel function

$$\theta(t) = \sum_{\sigma \in S} e^{-\sigma(1-\sigma)t} \qquad t > 0,$$

for which we have

$$\zeta(w,s) = \frac{1}{\Gamma(w)} \int_0^\infty \theta(t)e^{s(1-s)t}t^w\frac{dt}{t}.$$

The behavior of $\zeta(w,s)$ at $w=0$ is thus related to the asymptotics of $\theta(t)$ as $t\to 0$. These will be determined using the *Selberg trace formula* of Γ and χ which we now recall (see [Hej, p. 314]).

Let $h(r)$ be an even function, analytic in $|\mathrm{Im}(r)| < \frac{1}{2} + \delta$, for some $\delta > 0$, and $O(1+|r|)^{-2-\delta}$ there, and put $g(u) = \frac{1}{2\pi}\int_{-\infty}^\infty h(r)e^{-iru}dr$. Then

$$\sum_{1/4+r_n^2 = \lambda_n} h(r_n) + \frac{1}{2\pi}\int_{-\infty}^\infty -\frac{\phi'}{\phi}\left(\frac{1}{2}+ir\right)h(r)dr \qquad (2.1)+(2.2)$$

$$= \frac{|\mathbf{M}|}{2\pi}\int_{-\infty}^\infty r\tanh(\pi r)h(r)dr \qquad (2.3)$$

$$+ 2\sum_{\{P\}_{\mathrm{hyp}}}\sum_{k\geq 1}\frac{\mathrm{tr}\,\chi^k(P)\log NP}{NP^{k/2}-NP^{-k/2}}g(k\log NP) \qquad (2.4)$$

$$-\frac{h_1}{\pi}\int_{-\infty}^\infty h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr + \frac{A}{2}h(0) + 2Cg(0) \qquad (2.5)+(2.6)+(2.7)$$

(with A and C as above). We apply this formula to the pair

$$h(r) = e^{-(\frac{1}{4}+r^2)t}, \qquad g(u) = \frac{1}{\sqrt{2\pi t}}\cdot e^{-\frac{u^2}{4t}-\frac{t}{4}}.$$

The term (2.1) clearly gives $2\sum e^{-\lambda_n t}$. To compute (2.2) we expand (see [Hej, p. 438])

$$-\frac{\phi'}{\phi}\left(\tfrac{1}{2}+ir\right)=\sum_{\varrho=\beta+i\gamma}\frac{1-2\beta}{(r-\gamma)^2+(\tfrac{1}{2}-\beta)^2}-\sum_{i=1}^{N}\frac{2\eta_i-1}{r^2+(\eta_i-\tfrac{1}{2})^2}+c_0.$$

The corresponding integral is computed with the residue theorem as the limit as $T\to\infty$ of the integrals over the rectangles in the lower half plane connecting $-T$, T, $T-iT$, and $-T-iT$. It gives

$$\frac{1-2\beta}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-(\tfrac{1}{4}+r^2)t}}{(r-\gamma)^2+(\tfrac{1}{2}-\beta)^2}\,dr=e^{-\bar\varrho(1-\bar\varrho)t},$$

and we note that $\bar\varrho$ is a pole of $\phi(s)$ if and only if ϱ is. Similarly we have

$$\frac{2\eta_i-1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-(\tfrac{1}{4}+r^2)t}}{r^2+(\eta_i-\tfrac{1}{2})^2}\,dr=e^{-\eta_i(1-\eta_i)t}.$$

These cancel out one of the two terms (2.1) corresponding to $\lambda=\eta_i(1-\eta_i)$, and so the left-hand side yields $\theta(t)$.

We begin the estimation of the right-hand side with the term

$$-\frac{h_1}{\pi}\int_{-\infty}^{\infty}e^{-(\tfrac{1}{4}+r^2)t}\frac{\Gamma'}{\Gamma}(1+ir)dr,$$

which by integration by parts is

$$e^{-t/4}t\frac{2ih_1}{\pi}\int_{-\infty}^{\infty}re^{-r^2t}\log\Gamma(1+ir)dr.$$

Recall Stirling's formula

$$\log\Gamma(s+1)=\left(s+\frac{1}{2}\right)\log(s+1)-(s+1)+\log\sqrt{2\pi}+O\left(\frac{1}{s}\right)\quad\text{as}\quad s\to\infty.$$

The first term in the integral is then

$$\int_{-\infty}^{\infty}re^{-r^2t}\left(\frac{1}{2}+ir\right)\left(\frac{1}{2}\log(1+r^2)+i\arg(1+ir)\right)dr$$

$$=\frac{i}{2}\left(\int_{-\infty}^{\infty}r^2e^{-r^2t}\log(1+r^2)dr+\int_{-\infty}^{\infty}re^{-r^2t}\arctan(r)dr\right)=\frac{i}{2}(I+II).$$

Some calculation leads to

$$I=\int_{0}^{\infty}e^{-ut}\log(1+u)\sqrt{u}\,du=t^{-\tfrac{3}{2}}e^t\int_{t}^{\infty}e^{-u}\sqrt{u-t}(\log u-\log t)du$$

$$=t^{-\tfrac{3}{2}}(c_1\log t+c_2e^t+O(t))\quad\text{as}\quad t\to0,$$

and

$$II=c_3\frac{e^t}{t}+c_4\frac{1}{\sqrt{t}}+O(\sqrt{t})\quad\text{[GR, 3.466]}.$$

The linear term contributes $c_5 t^{-\frac{3}{2}}$ and the remainder is $O(1)$.

It is not difficult to see that the contribution from (2.1) is

$$\frac{|M|}{2\pi}\frac{1}{t}+c_6+O(t) \quad \text{as} \quad t\to 0,$$

and that of (2.2) is of exponential decay. Finally the term $g(0)$ gives $c_7 e^{-t/4}\dfrac{1}{\sqrt{t}}$.

Summarizing we have

Proposition 1. *For some constants* $\alpha, \beta, \gamma, \delta$ *(given explicitly in Sect. 3)*

$$\theta(t)=\frac{\alpha}{t}+\beta\frac{\log t}{\sqrt{t}}+\frac{\gamma}{\sqrt{t}}+\delta+O(\sqrt{t}\log t) \quad \text{as} \quad t\to 0.$$

We apply Proposition 1 to

$$\zeta(w,s)=\frac{1}{\Gamma(w)}\int\limits_0^\infty \theta(t)e^{s(1-s)t}t^w\frac{dt}{t}.$$

For the first four terms we need

$$\frac{1}{\Gamma(w)}\int\limits_0^\infty t^{-\varepsilon}e^{s(1-s)t}t^{w-1}dt=\frac{1}{\Gamma(w)}(s(s-1))^{\varepsilon-w}\Gamma(w-\varepsilon) \tag{2.8}$$

with $\varepsilon=0,\frac{1}{2},1$, and

$$\frac{1}{\Gamma(w)}\int\limits_0^\infty \frac{\log t}{\sqrt{t}}e^{s(1-s)t}t^{w-1}dt=\frac{\Gamma\left(w-\frac{1}{2}\right)}{\Gamma(w)}(s(s-1))^{\frac{1}{2}-w}\left(\frac{\Gamma'}{\Gamma}\left(w-\frac{1}{2}\right)-\log(s(s-1))\right) \tag{2.9}$$

(see [GR, 4.352]). The remainder is

$$\frac{1}{\Gamma(w)}\int\limits_0^\infty \psi(t)e^{s(1-s)t}t^{w-1}dt \tag{2.10}$$

with $\psi(t)=O(\sqrt{t}\log t)$ as $t\to 0$ and $O(1)$ as $t\to\infty$. Since $\dfrac{1}{\Gamma(w)}$ vanishes at $w=0$ we have

Proposition 2. *For a fixed* $s\gg 0$, $\zeta(w,s)$ *is regular at* $w=0$.

Lastly we need the asymptotics of $\dfrac{\partial}{\partial w}\zeta(w,s)\Big|_{w=0}$ as $s\to\infty$. Going back to the formulae above we find using $-\dfrac{\Gamma'}{\Gamma^2}(0)=1$ that $\dfrac{\partial}{\partial w}\Big|_{w=0}$ of (2.8) with $\varepsilon=0,\frac{1}{2},1$ is, respectively

$$-\log(s(s-1)); \quad -2\sqrt{\pi}(s(s-1))^{1/2}; \quad s(s-1)(\log(s(s-1))-1);$$

that of (2.9) is

$$2\sqrt{\pi}(\mathbf{c}+\log 4-2)(s(s-1))^{1/2}+2\sqrt{\pi}(s(s-1))^{1/2}\log(s(s-1))$$
$$(\mathbf{c}=\text{Euler's constant}),$$

and that of (2.10) goes to 0 as $s\to\infty$. Hence

Proposition 3.

$$\frac{\partial}{\partial w}\zeta(w,s)|_{w=0}\sim\alpha s(s-1)(\log(s(s-1))-1)$$
$$+2\sqrt{\pi}\beta(s(s-1))^{1/2}(\log(s(s-1))+(\mathbf{c}+\log 4-2))$$
$$-2\sqrt{\pi}\gamma(s(s-1))^{1/2}-\delta\log(s(s-1))\quad as\quad s\to\infty.$$

3.

The Selberg zeta function arises from the trace formula with the choice

$$h(r)=((s-\tfrac{1}{2})^2+r^2)^{-1}-((a-\tfrac{1}{2})^2+r^2)^{-1},$$

$$g(u)=\frac{1}{2s-1}e^{-|u|(s-\frac{1}{2})}-\frac{1}{2a-1}e^{-|u|(a-\frac{1}{2})},$$

where $a>1$ is fixed and $1<\mathrm{Re}(s)<a$. The hyperbolic term (2.4) becomes

$$\frac{1}{s-\frac{1}{2}}\frac{Z'}{Z}(s)-\frac{1}{a-\frac{1}{2}}\frac{Z'}{Z}(a).$$

To compute the integral in (2.2) we follow [Ven, p. 84] and integrate $\frac{\phi'}{\phi}(\tau)h(\tau)$ along the contour consisting of the interval $\mathrm{Re}(\tau)=\frac{1}{2}$, $-T\le\mathrm{Im}(\tau)\le T$ and the semicircle connecting the two endpoints. One obtains

$$\frac{1}{2s-1}\frac{\phi'}{\phi}(s)-\sum_{\varrho}\left(\left(s-\frac{1}{2}\right)^2-\left(\varrho-\frac{1}{2}\right)^2\right)^{-1}-\left(\left(a-\frac{1}{2}\right)^2-\left(\varrho-\frac{1}{2}\right)^2\right)^{-1}$$
$$+\sum_{i=1}^{N}\left(\left(s-\frac{1}{2}\right)^2-\left(\eta_i-\frac{1}{2}\right)^2\right)^{-1}-\left(\left(a-\frac{1}{2}\right)^2-\left(\eta_i-\frac{1}{2}\right)^2\right)^{-1}+c_8.$$

For (2.5) we use ([Hej, p. 435])

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(r^2+\left(s-\frac{1}{2}\right)^2\right)^{-1}\frac{\Gamma'}{\Gamma}(1+ir)dr=\frac{1}{s-\frac{1}{2}}\frac{\Gamma'}{\Gamma}\left(s+\frac{1}{2}\right),$$

and for (2.3) we need

$$\int_{-\infty}^{\infty}r\tanh(\pi r)h(r)dr=2\sum_{k=0}^{\infty}\left(\frac{1}{s+k}-\frac{1}{a+k}\right).$$

We thus arrive at the key formula

$$\sum_{\sigma \in S} ((\sigma(1-\sigma)-(s(1-s))^{-1}-(\sigma(1-\sigma)-(a(1-a))^{-1})$$

$$= \frac{1}{2s-1}\left[\frac{\phi'}{\phi}(s)+2\frac{Z'}{Z}(s)-2h_1\frac{\Gamma'}{\Gamma}\left(s+\frac{1}{2}\right)\right]$$

$$+ \frac{|M|}{\pi}\sum_{k=0}^{\infty}\left(\frac{1}{s+k}-\frac{1}{a+k}\right)+\frac{2A}{(2s-1)^2}+\frac{2C}{2s-1}+c_9. \qquad (3.1)$$

As in [Sar] we now differentiate both sides with respect to s. The left-hand side becomes

$$-(2s-1)\sum_{\sigma \in S}((\sigma(1-\sigma)-(s(1-s))^{-2},$$

which is precisely $-\dfrac{d}{ds}\left(\dfrac{1}{2s-1}\dfrac{d}{ds}\dfrac{\partial}{\partial w}\zeta(w,s)|_{w=0}\right)$. Furthermore, by the product expansion of $\Gamma_2(s)$ (see [Bar, Var, Vig]),

$$\frac{d}{ds}\frac{1}{2s-1}\frac{Z'_\infty}{Z_\infty}(s)=-\frac{|M|}{2\pi}\sum_{k=0}^{\infty}\frac{1}{(s+k)^2}.$$

Thus we obtain the following relation between $Z(s)$ and our determinant:

$$\frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\log\det{}^2(\Delta-s(1-s))$$

$$= \frac{d}{ds}\frac{1}{2s-1}\frac{d}{ds}\left(\log\phi(s)+2\log Z(s)+2\log Z_\infty(s)-2h_1\log\Gamma\left(s+\frac{1}{2}\right)\right)$$

$$+ \frac{d}{ds}\left(\frac{2A}{(2s-1)^2}+\frac{2C}{2s-1}\right)$$

(A and C appear in the trace formula). Integrating twice we see that for some constants B, D,

$$\det{}^2(\Delta-s(1-s))=\phi(s)Z(s)^2 Z_\infty(s)^2 \Gamma(s+\tfrac{1}{2})^{-2h_1}(2s-1)^A e^{B(2s-1)^2} e^{C(2s-1)} e^D.$$

We compute these constants from the asymptotics at $s+1$ as $s\to\infty$. Recall Stirling's formulae [ibid.]

$$\log\Gamma_2(s+1)=-\tfrac{1}{2}(s^2-\tfrac{1}{6})\log s+\tfrac{3}{4}s^2-s\log\sqrt{2\pi}-\zeta'(-1)+o(1).$$

Therefore, since $Z(s+1)$ and $\phi(s+1)$ go to 1 as $s\to\infty$, the logarithm of the right-hand side is

$$\frac{|M|}{\pi}\left(-s^2\log s+\frac{3}{2}s^2-s\log s+s-\frac{1}{3}\log s+\log\sqrt{2\pi}-2\zeta'(-1)\right)$$

$$-2h_1(s\log s-s+\log s+\log\sqrt{2\pi})$$

$$+A\log(2s+1)+B(2s+1)^2+C(2s+1)+D+o(1).$$

On the other hand, Proposition 3 tells us the logarithm of the left-hand side is

$$-\alpha(2s^2\log s+2s\log s-s^2+\tfrac{1}{2}),$$

$$-2\sqrt{\pi}\beta((2s\log s+\log s+1)+(\mathbf{c}+\log 4-2)(s+\tfrac{1}{2})),$$

$$+2\sqrt{\pi}\gamma(s+\tfrac{1}{2})+2\delta\log s+o(1).$$

A comparison of the two formulae determines all constants:

$$\alpha=\frac{|M|}{2\pi},$$

$$\beta=\frac{h_1}{2\sqrt{\pi}},$$

$$\gamma=\frac{1}{2\sqrt{\pi}}(2C+h_1\mathbf{c}+h_1\log 4),$$

$$\delta=\frac{A}{2}-\frac{h_1}{2}-\frac{|M|}{6\pi},$$

$$B=-\frac{|M|}{4\pi},$$

$$D=\frac{|M|}{\pi}(2\zeta'(-1)-\log\sqrt{2\pi})+2h_1\log\sqrt{2\pi}-A\log 2.$$

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