

The Action Functional in Non-Commutative Geometry

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Abstract. We establish the equality between the restriction of the Adler-Manin-Wodzicki residue or non-commutative residue to pseudodifferential operators of order $-n$ on an n -dimensional compact manifold M , with the trace which J. Dixmier constructed on the Macaeu ideal. We then use the latter trace to recover the Yang Mills interaction in the context of non-commutative differential geometry.

Introduction

The non-commutative residue was discovered in the special case of one dimensional symbols by Adler [1] and Manin [8] in the context of completely integrable systems. In a quite remarkable work [13], Wodzicki proved that it could still be defined in arbitrary dimension and gave the only non-trivial trace, noted Res , for the algebra of pseudodifferential operators of arbitrary order. Given such an operator P on the manifold M , $\text{Res} P$ is the coefficient of $\text{Log} t$ in the asymptotic expansion of $\text{Trace}(P e^{-t\Delta})$, where Δ is a Laplacian. Equivalently it is the residue at $s=0$ of the ζ function $\zeta(s) = \text{Trace}(P \Delta^{-s})$. It is *not* the usual regularisation $\zeta(0)$ of the trace, and it vanishes on any P of order strictly less than $-\dim M$, and on any differential operator. In general this trace: Res , has no positivity property, i.e. one does not have $\text{Res}(P^*P) \geq 0$. However its restriction to operators of order $-n$, $n = \dim M$ is positive. This restriction of Res to pseudodifferential operators of order $-n$ was discovered and studied by Guillemin [14]. Even though it is easier to handle than the general residue, it will be of great help for our purpose which is to show how conformal geometry fits with [3], the case of Riemannian geometry being treated in [5].

Our first result is the equality between Res and a trace on the dual Macaeu ideal, introduced by Dixmier in [6] in order to show that the von Neumann algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators in Hilbert space possessed non-trivial tracial weights. I am grateful to J. Dixmier for explaining his result to me and to D. Voiculescu for helpful conversations on the subject of Macaeu ideals. Thus we recall that, given a Hilbert space \mathcal{H} , the Macaeu ideal $\mathcal{L}^\omega(\mathcal{H})$ is the ideal of

compact operators T , whose characteristic values satisfy: [7]

$$\sum_1^\infty \frac{1}{n} \mu_n(T) < \infty .$$

It contains all the Schatten classes $\mathcal{L}^p(\mathcal{H})$ for finite p , and the dual ideal, which we denote \mathcal{L}^{1+} consists of all compact operators T , whose characteristic values satisfy:

$$\text{Sup}_{N>1} \frac{1}{\text{Log} N} \sum_1^N \mu_n(T) < \infty .$$

Gifted with the obvious norm it is a non-separable Banach space containing strictly the ideal \mathcal{L}^1 as well as the closure of finite rank operators (thus \mathcal{L}^1 is *not* norm dense in \mathcal{L}^{1+} for the natural norm of the latter).

Now in [6], J. Dixmier showed that for any mean ω on the amenable group of upper triangular two by two matrices, one gets a trace on \mathcal{L}^{1+} , given by the formula:

$$\text{Tr}_\omega(T) = \lim_\omega \frac{1}{\text{Log} N} \sum_1^N \lambda_n(T)$$

when T is a positive operator, $T \in \mathcal{L}^{1+}$, with eigenvalues $\lambda_n(T)$ in decreasing order, and \lim_ω is the linear form on bounded sequences defined in [6] using ω .

We shall prove in Sect. 1 that when T is pseudodifferential of order $-\text{dim}(M)$, the value of $\text{Tr}_\omega(T)$ does not depend upon ω and is equal to $\text{Res}(T)$. In Sect. 2 we shall apply the above result to show how one can deduce ordinary differential forms and the natural conformal invariant norm on them from the quantized forms which we introduced in [3]. The key point is that we do not need to take a “classical limit” to achieve this goal but only to use the Dixmier trace appropriately. In particular we obtain a simple formula for the conformal structure in terms of the operator F , $F^2 = 1$, given by the polar decomposition of the Dirac operator.

In Sects. 3 and 4 after discussing the analogue of the Yang Mills action in the context of non-commutative differential geometry and showing, as expected, that 4 is the critical dimension, we exploit the above construction to show that if $d = 4$ the leading divergency of the action is the usual local Yang Mills action. The latter result was announced on several occasions.

1. The Main Equality

Theorem 1. *Let M be a compact n -dimensional manifold, E a complex vector bundle on M , and P a pseudodifferential operator of order $-n$ acting on sections of E . Then the corresponding operator P in $\mathcal{H} = L^2(M, E)$ belongs to the Macaev ideal $\mathcal{L}^{1+}(\mathcal{H})$ and one has:*

$$\text{Trace}_\omega(P) = \frac{1}{n} \text{Res}(P)$$

for any invariant mean ω .

Note first that both $\mathcal{L}^{1+}(\mathcal{H})$ and Trace_ω are invariant under similarities T . T^{-1} with T and T^{-1} bounded, so that the choice of inner product in the space of L^2 sections of E is irrelevant.

Proof. Since $\mathcal{L}^{1+}(\mathcal{H})$ contains $\mathcal{L}^1(\mathcal{H})$, and any element of the latter is in the kernel of Tr_ω , it follows that we can neglect smoothing operators and we just need to prove the statements locally. Thus to show that $P \in \mathcal{L}^{1+}(\mathcal{H})$ we may assume that M is the standard n torus \mathbb{T}^n and E the trivial line bundle. Then $P = T(1 + \Delta)^{-n/2}$, where T is bounded and Δ is the Laplacian of the (flat) torus. Thus as \mathcal{L}^{1+} is an ideal it is enough to check that $(1 + \Delta)^{-n/2} \in \mathcal{L}^{1+}$, which is obvious. In fact the characteristic values of $(1 + \Delta)^{-n/2}$ are the $(1 + l^2)^{-n/2}$, where the l 's are the lengths of elements in the lattice $\Gamma = \mathbb{Z}^n$. Thus we see that the limit of $\frac{1}{\text{Log } N} \sum_1^N \lambda_j$, when N goes to ∞ , does exist for this operator so that, for any ω :

$$\text{Trace}_\omega((1 + \Delta)^{-n/2}) = \frac{1}{n} \int_{S^{n-1}} d\sigma = \frac{1}{n} 2\pi^{\frac{n-1}{2}} / \Gamma\left(\frac{n-1}{2}\right).$$

Let us now prove the main equality. We may assume that M is the standard n -sphere S^n . Since Trace_ω is positive and vanishes on $\mathcal{L}^1(\mathcal{H})$ it defines a positive linear form on *symbols of order $-n$* , because it only depends upon the principal symbol $\sigma_{-n}(P)$ for P of order $-n$. Since a positive distribution is a measure, we get a measure on the unit sphere cotangent bundle of S^n . But as Tr_ω is a trace, the latter measure is invariant under the action of any isometry of S^n , and hence is proportional to the volume form on $(T^*S^n)_1 = \{(x, \xi) \in T^*S^n; \|\xi\| = 1\}$. By the above computation the constant of proportionality is $\frac{1}{n}(2\pi)^{-n}$, thus:

$$\text{Trace}_\omega(P) = \frac{1}{n}(2\pi)^{-n} \int_{(T^*S^n)_1} \sigma_{-n}(P) dv$$

for any P of order $-n$ and any ω . As the right-hand side is the formula for $\frac{1}{n} \text{Res}(P)$, we get the conclusion. \square

Corollary 2. *All the traces Tr_ω agree on pseudodifferential operators of order $-\dim M$, on a manifold M .*

One can then conclude that suitable averages of the sequence $\frac{1}{\text{Log } N} \sum_1^N \lambda_j(P)$ do converge, when $N \rightarrow \infty$, to this common value.

2. Conformal Geometry

Let M be a compact Riemannian manifold of dimension n , and $A^1 = C^\infty(M, T^*M)$ be the space of smooth 1-forms on M . There is a natural norm on A^1 which depends only upon the *conformal* structure of M . If $\dim M = 2$, it is the ordinary Dirichlet integral: $\int \|\omega\|^2 dv = \int \omega \wedge * \omega$. If $\dim M = n$, it is the L^n norm, given by the (n^{th} root of) following integral:

$$\|\omega\|^n = \int \|\omega(x)\|^n d^n x.$$

In [3] we introduced (assuming that M is Spin^c) the quantized differential forms on M , obtained as operators of the form $\sum adb$; $a, b \in C^\infty(M)$, in the Hilbert space \mathcal{H} of L^2 spinors on M . Here db is given by the commutator $i[F, b]$, where the operator F , $F^2 = 1$, is the sign $D|D|^{-1}$ of the Dirac operator. (We can ignore the non-invertibility of D , since it only modifies F by a finite rank operator.)

The next result shows how to pass from quantized 1-forms to ordinary forms, not by a classical limit, but by a direct application of the Dixmier trace.

Theorem 3. *Let M be a Spin^c Riemannian manifold of dimension $n > 1$, $\mathcal{H} = L^2(M, S)$ the Hilbert space of L^2 spinors, $F = D|D|^{-1}$ the sign of the Dirac operator. Let $\mathcal{A} = C^\infty(M)$ be the algebra of smooth functions on M and $\Omega^1 = \{\Sigma a[F, b]; a, b \in \mathcal{A}\}$ be the \mathcal{A} -bimodule of quantized forms of degree 1.*

1) *For any $\alpha \in \Omega^1$ one has $|\alpha|^n \in \mathcal{L}^{1+}(\mathcal{H})$.*

2) *There exists a unique bimodule linear map $\Omega^1 \xrightarrow{c} A^1$ such that $c(i[F, a]) = da \forall a \in C^\infty(M)$. This map is surjective and the image of the self adjoint elements of Ω^1 are the real forms.*

3) *For any $\alpha = \alpha^* \in \Omega^1$ one has $\text{Trace}_\omega(|\alpha|^n) = \lambda_n \int \|c(\alpha)\|^n$ with $\lambda_n = 2(2\pi)^{-n/2} \Gamma\left(n - \frac{1}{2}\right) \Gamma\left(\frac{n-1}{2}\right)^{-1} \Gamma(n+1)^{-1}$.*

Proof. 1. By construction α is a pseudodifferential operator of order -1 , so that $|\alpha|^n$ is also a pseudodifferential operator and is of order $-n$. The conclusion follows from Theorem 1.

2. For $x \in M$ let $C_x = \text{Cliff}_{\mathbb{C}}(T_x^*M)$ be the complexified Clifford algebra of the cotangent space T_x^*M of M at x . One has $C_x = \text{End}(S_x)$, where S is the Spinor bundle. For each $\xi \in T_x^*M$ we let $\gamma(\xi) \in C_x$ be the corresponding γ matrix, $\gamma(\xi) = \gamma(\xi)^*$, $\gamma(\xi)^2 = -\|\xi\|^2$, and we extend γ to a linear map of $T_{x,\mathbb{C}}^*(M)$ to C_x . Given $a \in \mathcal{A} = C^\infty(M)$, the symbol of order -1 of $[F, a]$ is the Poisson bracket $\{\sigma, a\}$, where $\sigma(x, \xi) = \gamma(\xi)/\|\xi\|$, and thus its restriction to the unit sphere is the transverse part $\varrho(x, \xi) = \gamma(da - \langle da, \xi \rangle \xi)$ of $\gamma(da)$. It is a homogeneous function of degree -1 on T_x^*M with values in C_x . Now provided $n > 1$, a vector $\eta \in T_x^*$ is uniquely determined by the transverse part $\xi \rightarrow \eta - \langle \eta, \xi \rangle \xi$, as a function of $\xi \in S_x^*$, and this still holds for $\eta \in T_{x,\mathbb{C}}^*$. Thus the map c exists and is characterized by the equality:

$$\sigma_{-1}(x)(\alpha, \xi) = \gamma(c(\alpha)(x) - \langle c(\alpha)(x), \xi \rangle \xi) \quad \forall (x, \xi) \in S^*M.$$

The image of $\sum ai[F, b] \in \Omega^1$ is $\sum adb \in A^1$ so the surjectivity of c is clear. The image of $ai[F, b] + (ai[F, b])^*$ is $adb + (db^*)a^*$ which is a real form, so 2. follows.

3. The absolute value of $\gamma(\eta)$ for $\eta \in T_x^*(M)$ (but not its complexification) is $\|\eta\|$, where 1 is the unit of C_x . Thus by Theorem 1 we have:

$$\text{Trace}_\omega(|\alpha|^n) = \frac{(2\pi)^{-n}}{n} \int_{S^*M} \|\alpha_x - \langle \alpha_x, \xi \rangle \xi\|^n \text{trace}(1) d^n x d^{n-1} \xi.$$

Here $\text{trace}(1) = \dim(S_x) = 2^{n/2}$. Thus we just need to show that for any $\eta \in \mathbb{R}^n$ one has $\int_{S^{n-1}} \|\eta - \langle \eta, \xi \rangle \xi\|^n (d^{n-1} \xi) = 2^{-n/2} \lambda_n \|\eta\|^n$. By homogeneity and invariance under rotations we are reduced to the computation of an integral, which is obviously > 0 for $n > 1$. \square

As an immediate corollary of the theorem we see that the Fredholm module (\mathcal{H}, F) allows us to recover both the bimodule of 1-forms A^1 with the ordinary differentiation: $\mathcal{A} \xrightarrow{d} A^1$ (given by $a \rightarrow \text{Class of } i[F, a]$), and also the conformal structure of M since the L^n norm on A^1 uniquely determines it.

Another equivalent way to formulate the result is to consider for each n the ideal \mathcal{L}^{n+} , n^{th} root of \mathcal{L}^{1+} , in $\mathcal{L}(\mathcal{H})$,

$$\mathcal{L}^{n+} = \left\{ T \in \mathcal{L}(\mathcal{H}), T \text{ compact, } \text{Sup}_N \left(\frac{1}{\text{Log } N} \sum_1^N \mu_j(T)^n \right) < \infty \right\},$$

and the ideal \mathcal{L}_0^{n+} which is the norm closure, for the norm of \mathcal{L}^{n+} , of operators of finite rank (cf. [7]). Then on an n -dimensional manifold M as above the quantized 1-forms are all in \mathcal{L}^{n+} , and the ordinary forms are obtained by moding out $\mathcal{L}_0^{n+} \subset \mathcal{L}^{n+}$. The ordinary differential is obtained in the same way from the quantized differential $a \rightarrow i[F, a] \in \Omega^1$.

For forms of arbitrary degree there are two more points which we have to clarify before we can handle the Yang Mills action. Given an n -dimensional Euclidean space E , we let Π_E be the homomorphism of the tensor algebra $T(E)$ in $C^\infty(S_E, \text{Cliff}(E))$, (the algebra of smooth maps from the unit sphere $S_E = \{\xi \in E, \|\xi\| = 1\}$ to the Clifford algebra of E) obtained from the linear map $\eta \rightarrow \varrho(\eta)$, $\varrho(\eta)(\xi) = \gamma(\eta - \langle \eta, \xi \rangle \xi) \forall \xi \in S_E$.

We let $J(E)$ be the kernel of Π_E .

Lemma 4. *With the notations of Theorem 3, let Ω^k be the the \mathcal{A} -bimodule of quantized forms of degree k .*

1. *For $1 \leq k \leq n$ one has $\Omega^k \subset \mathcal{L}^{n/k+}(\mathcal{H})$ and the direct sum $\bigoplus_0^n \Omega_0^k$, with $\Omega_0^k = \mathcal{L}^{n/k+} \cap \Omega^k$ is a two sided ideal in the algebra $\bigoplus_0^n \Omega^k = \Omega^*$.*

2. *The principal symbol map gives a canonical isomorphism c of graded algebras, from Ω^*/Ω_0^* to the graded algebra of smooth sections of the vector bundle $\bigoplus_0^n E_k$, where E_k is obtained from the cotangent bundle by applying the functor:*

$$E \rightarrow T^k(E)/J(E) \cap T^k(E) = f_k(E).$$

Proof. 1. Any element of Ω^k is a pseudodifferential operator P of order $-k$; thus $|P|^{n/k}$ is of order $-n$ and Theorem 1 applies. The Holder inequality also holds for the ideals \mathcal{L}^{p+} and shows that $\mathcal{L}^{p_1+} \times \mathcal{L}^{p_2+} \subset \mathcal{L}^{p_3+}$, $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and also that $\mathcal{L}_0^{p_1+} \times \mathcal{L}_0^{p_2+} \subset \mathcal{L}_0^{p_3+}$, $\mathcal{L}^{p_1+} \times \mathcal{L}_0^{p_2+} \subset \mathcal{L}_0^{p_3+}$ (cf. [7]).

2. First, by Theorem 1, an element P of Ω^k belongs to $\mathcal{L}_0^{n/k+}$ if and only if its principal symbol vanishes. (If it does then the operator is of order $< -k$ and hence even belongs to $\mathcal{L}^{n/k}$; if it does not then the Dixmier trace of $|P|^{n/k}$ does not vanish.) The quotient Ω^k/Ω_0^k is a commutative bimodule over $\mathcal{A} = C^\infty(M)$, and since any element of Ω^k is a finite sum of products of k elements of Ω^1 , the symbols $\sigma_{-k}(P)$, $P \in \Omega^k$ are exactly the smooth sections of $f_k(T^*M)$. \square

For our purpose we only need to determine f_1 and f_2 . For $n > 1$ we have seen that $f_1(E) = E$. For $n > 2$ let us show that $J(E) \cap T^2(E) = \{0\}$, i.e. that the map Π_E is injective on tensors of rank 2. Since $J(E)$ is invariant under the action of the orthogonal group $O(E)$, it is enough to check that Π_E is non-zero on the three irreducible subspaces of $T^2(E)$, namely a) antisymmetric tensors b) symmetric traceless tensors c) the inner product (viewed as a symmetric tensor). Since $n > 2$ we

can take $\eta_1, \eta_2 \in E$ linearly independent, and $\xi, \|\xi\| = 1$, orthogonal to both, to get that $\Pi_E(\eta_1 \otimes \eta_2 - \eta_2 \otimes \eta_1) \neq 0$. The image by Π_E of the symmetric tensor $\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1 (\eta_i \in E)$ is the scalar valued function on $S_E: \Pi_E(\eta_1 \otimes \eta_2 + \eta_2 \otimes \eta_1)(\xi) = \langle \eta_1, \eta_2 \rangle - \langle \eta_1, \xi \rangle \langle \eta_2, \xi \rangle$. This is enough to show that Π_E is non-zero and hence injective on tensors of type a) b) or c). Thus we get:

Lemma 5. *If $\dim E > 2, f_2(E) = T^2(E)$.*

The next point that we need to clarify is that even though $f = \Omega^*/\Omega_0^*$ is a graded algebra of tensors on the manifold M , and c is a homomorphism from the graded algebra Ω^* to Ω^*/Ω_0^* , we do not have a natural differential in f . The point is that the ideal Ω_0^* is not in general stable under the map:

$$\alpha \in \Omega^k \rightarrow d\alpha = i(F\alpha - (-1)^k \alpha F) \in \Omega^{k+1}.$$

However since $d^2 = 0$, this is easily cured:

Lemma 6. 1. *The direct sum $\Omega_{00}^* = \bigoplus_0^n \Omega_{00}^k$ with $\Omega_{00}^k = \{\alpha \in \Omega_0^k, d\alpha \in \Omega_0^{k+1}\}$ is a graded differential two sided ideal in the graded differential algebra Ω^* .*

2. The map $\tilde{c}, \tilde{c}(\alpha) = (c(\alpha), c(d\alpha))$ is a linear injection of the quotient Ω^k/Ω_{00}^k in the space of sections of the bundle $f_k(T^) \oplus f_{k+1}(T^*)$.*

Proof. 1. We just have to check that it is a two sided ideal, which follows from Lemma 4 1) and the equality $d(\alpha_1 \alpha_2) = (d\alpha_1)\alpha_2 + (-1)^{\rho_1} \alpha_1 d\alpha_2$.

2. Apply Lemma 4 2). \square

Assuming $n > 2$ let us determine the image $\tilde{c}(\Omega^1)$, i.e. the pairs $(c(\alpha), c(d\alpha))$ when α varies in Ω^1 .

Lemma 7. *For $n > 2, \tilde{c}(\Omega^1)$ consists of all smooth tensors (ω, β) , where ω is of rank 1, β of rank 2 and one has:*

$$A\beta = d\omega,$$

where A is the projection on antisymmetric tensors of rank 2.

Proof. It is enough to check the equation for the pair $\omega = c(\alpha), \beta = c(d\alpha)$ with $\alpha = adb; a, b \in C^\infty(M)$. Then by Theorem 3 2), $c(\alpha)$ is the 1-form adb and since $d\alpha = da db$, we see that $A\beta$ is the antisymmetric tensor $\frac{1}{2}(da \otimes db - db \otimes da)$, thus the equality $A\beta = d\omega$. It remains to show that $\tilde{c}(\Omega^1)$ contains all the smooth symmetric tensors of rank 2. Now with $\alpha = adb$ as above and $x \in C^\infty(M)$ we have $c(x\alpha - \alpha x) = 0$ and $c(d(x\alpha - \alpha x)) = c((dx)\alpha + \alpha(dx))$. Thus $\tilde{c}(x\alpha - \alpha x)$ is the smooth symmetric two tensor $(dx)\alpha + \alpha(dx)$. As every smooth symmetric two tensor is a finite sum of such terms we get the conclusion. \square

3. The Action Functional in Non-Commutative Differential Geometry

We begin this section by a very simple example, the case of the circle S^1 , where we show that using our quantized differential forms, the quantized flat connections correspond exactly to the Grassmannian which plays a fundamental role in the theory of totally integrable systems [9].

Thus we let $\mathcal{A} = C^\infty(S^1)$ be the algebra of smooth functions on S^1 and let (\mathcal{H}, F) be the Fredholm module over \mathcal{A} given by $\mathcal{H} = L^2(S^1)$ and $F = 2P - 1$, where P is the Toeplitz projection. In other words the operator F multiplies the n^{th} Fourier component of $\xi \in L^2(S^1)$ by 1 if $n \geq 0$ and -1 otherwise.

Lemma 8. *The space $\Omega^1 = \{\sum a[F, b]; a, b \in \mathcal{A}\}$ of 1-forms is dense in the space $\mathcal{L}^2(\mathcal{H})$ of Hilbert Schmidt operators.*

Proof. Let $u \in \mathcal{A}$ be the function $u(\theta) = \exp i\theta, \theta \in S^1$. The operator $\frac{1}{2}u^{-1}[F, u]$ is the rank one projection on the subspace $\mathbb{C}e_0$, where $(e_n)_{n \in \mathbb{Z}}$ is the canonical basis of $\mathcal{H} = L^2(S^1)$, $e_n(\theta) = \exp(in\theta), \forall \theta \in S^1$. Thus the quantized forms $\omega_{n,m} = u^n(\frac{1}{2}u^{-1}[F, u])u^m$ form the natural orthonormal basis of $\mathcal{L}^2(\mathcal{H})$. \square

We cannot entirely justify the choice of the Hilbert Schmidt norm in the above lemma, since it happens in dimension 1, that 1-forms are traceable. (As we saw above, by Theorem 1, it is not true that 1-forms belong to \mathcal{L}^n for an n -dimensional manifold, $n > 1$.) The only sensible justification is that the definition of the character of the Fredholm module only requires that 1-forms be of Hilbert Schmidt class, and is continuous in this norm (cf. [3]). Next consider the trivial line bundle, with fiber \mathbb{C} , on S^1 , or equivalently the finite projective module $\mathcal{E} = C^\infty(S^1)$ over \mathcal{A} . Then as in [3, Definition 18, p. 110] a connection ∇ on \mathcal{E} is given by a linear map $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$ such that

$$\nabla(\xi \cdot x) = (\nabla\xi)x + \xi \otimes dx,$$

where here $dx = i[F, x]$, according to our definition of the quantized differential. We endow the above line bundle with its obvious metric, i.e. we view \mathcal{E} as a C^* module over \mathcal{A} , with $\langle \xi, \eta \rangle(\theta) = \bar{\xi}(\theta)\eta(\theta), \forall \theta \in S^1, \forall \xi, \eta \in \mathcal{E}$. Obviously a connection on \mathcal{E} is specified by the 1-form $\alpha = \nabla 1$, and the latter is an arbitrary element of Ω^1 . Moreover the connection associated to α is compatible with the metric (cf. [4]), (i.e. such that $\langle \nabla\xi, \eta \rangle + \langle \xi, \nabla\eta \rangle = d\langle \xi, \eta \rangle \forall \xi, \eta \in \mathcal{E}$) iff $\alpha + \alpha^* = 0$.

We thus get the elementary but significant result:

Theorem 9. *The map $\nabla \rightarrow \frac{1}{2}(1 + F) - \frac{1}{2}i\nabla(1)$ is a one-to-one bijection from flat compatible and square integrable connections on \mathcal{E} with the restricted Grassmannian. It is equivariant with respect to the natural action of $C^\infty(S^1, U(1))$.*

Proof. First ∇ is characterized by $\alpha = \nabla(1)$ and is compatible iff $\alpha^* = -\alpha$, and square integrable iff $\alpha \in \mathcal{L}^2$; thus by Lemma 8, without the flatness condition the allowed α 's are the skew adjoint elements of $\mathcal{L}^2(\mathcal{H})$. Now (cf. [9]) the restricted Grassmannian consists exactly of the idempotents $Q, Q = Q^*$ such that $Q - P \in \mathcal{L}^2$. Thus if we set $Q = \frac{1}{2}(1 + F) - \frac{1}{2}i\alpha$, we just need to check that $Q^2 = Q$ iff ∇_α is flat, i.e. iff one has $i(F\alpha + \alpha F) + \alpha^2 = 0$, which is obvious. The unitary group $\mathcal{U} = C^\infty(S^1, U(1))$ of $\text{End}_{\mathcal{A}}(\mathcal{E})$ acts by gauge transformations on compatible connections (cf. [4]) with $\gamma_u(\nabla) = u\nabla u^{-1}$ for $u \in \mathcal{U}$, or equivalently $\gamma_u(\alpha) = ui[F, u^{-1}] + \alpha u u^{-1}$. Thus the corresponding Q_α is replaced by $uQ_\alpha u^{-1}$. \square

A similar statement holds for the bundle with fiber \mathbb{C}^n , with \mathcal{U} replaced by $C^\infty(S^1, U(n))$.

In relation with [2] and [12] we also want to point out that on the space of all compatible connections (i.e. all $\alpha = -\alpha^*$ in $\mathcal{L}^2(\mathcal{H})$) one has a natural Chern-

Simons action given by

$$I(\alpha) = \int (\alpha d\alpha + \frac{2}{3}\alpha^3),$$

where the integral is the *trace* and as usual $d\alpha$ is the graded commutator $d\alpha = i(F\alpha + \alpha F)$.

But let us now pass to the analogue of the Yang Mills action. The set up is, as in [3] and as above, fixed by a_* algebra \mathcal{A} and a Fredholm module (\mathcal{H}, F) over \mathcal{A} which is p -summable, i.e. $[F, x] \in \mathcal{L}^p(\mathcal{H})$ for some finite p , which as explained in [3] has to do with dimension. We are also given the analogue of a Hermitian bundle, i.e. a finite projective module \mathcal{E} over \mathcal{A} , with an \mathcal{A} valued inner product (cf. [4]). This latter data can be ignored for a first reading and specialized to $\mathcal{E} = \mathcal{A}$ with $\langle a, b \rangle = a^*b \in \mathcal{A}$.

Then using the differential algebra of quantized differential forms, $\Omega^k = \{\sum a^0 da^1 \dots da^k; a^j \in \mathcal{A}, da = i[F, a]\}$ (cf. [3]) we get the notions of connection, compatible connection, curvature relative to \mathcal{E} . For $\mathcal{E} = \mathcal{A}$ a connection is just an element α of Ω^1 , it is compatible iff $\alpha^* = -\alpha$ and its curvature is $\theta = d\alpha + \alpha^2 = i(F\alpha + \alpha F) + \alpha^2$. (cf. [3, p. 110] and [4]). Using [3, Lemma 1, p. 56], we get:

Theorem 10. 1. *The action $I_+(\alpha) = \|\theta\|_{HS}^2$ is finite if $p \leq 4$.*

2. *When $p \leq 4$, the action I_+ is a quartic positive function of α invariant under the action of the gauge group of second kind*

$$\mathcal{U} = \{u \in \text{End}(\mathcal{E}); uu^* = u^*u = 1\}.$$

Proof. For the sake of clarity we take $\mathcal{E} = \mathcal{A}$. By construction $\theta = d\alpha + \alpha^2 \in \Omega^2$, and by [3, Lemma 1, p. 56] one has $\Omega^k \subset \mathcal{L}^{p/k}$, so that $\Omega^2 \subset \mathcal{L}^{p/2}$. Thus θ is Hilbert Schmidt when $p/2 \leq 2$, i.e. when $p \leq 4$. If we replace α by $\gamma_u(\alpha) = udu^{-1} + \alpha u u^{-1}$, the curvature θ is replaced by $u\theta u^{-1}$ so that the statement 2. is obvious. \square

It is well known that the dimension $n=4$ is the relevant dimension for the classical Yang Mills action since it is only for $n=4$ that it is conformally invariant, but for the action I_+ the situation is slightly different: 1. The action I_+ is *finite* only if the degree of summability p is ≤ 4 , 2. For a 4-dimensional manifold M , the Fredholm module (\mathcal{H}, F) on $C^\infty(M)$ given by Theorem 3 is p summable for any $p=4+\varepsilon, \varepsilon>0$ but not for $p=4$. Thus in this case the action I_+ is divergent. However by Lemma 4 one has $\Omega^2 \subset \mathcal{L}^{2^+}$ so that the divergence of $\|\theta\|_{HS}^2 = \text{Trace}(\theta^*\theta)$ is only logarithmic ($\theta^*\theta \in \mathcal{L}^{1^+}$) and the principal term (i.e. the coefficient of $\text{Log}K$ in terms of a cut off K) is given by the Dixmier trace $\text{Trace}_\omega(\theta^*\theta)$. In the next section we shall fully identify this leading term in I_+ with the classical Yang Mills action.

4. The Leading Term of the Action in 4 Dimensions

Let M be a 4 dimensional compact smooth Riemannian manifold. We assume that M is Spin^c and let (\mathcal{H}, F) be the Fredholm module over $\mathcal{A} = C^\infty(M)$, with \mathcal{H} the Hilbert space of L^2 spinors and $F = D|D|^{-1}$, where D is the Dirac operator. We let (Ω^*, d) be the graded differential algebra of quantized forms, and define as in Sect. 3 the notion of compatible connection for a Hermitian vector bundle E over M . This

involves the module $\mathcal{E} = C^\infty(M, E)$ (of sections of E) over \mathcal{A} and the \mathcal{A} -valued inner product given by the metric of E . By construction (cf. [3]) the curvature θ is an element of $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2)$, but since here Ω^2 acts in the Hilbert space \mathcal{H} , we can view θ as an operator in the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$. The inner product of the latter space is given by (cf. [4]) $\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle = \langle \langle \xi, \xi' \rangle \eta, \eta' \rangle$ for $\xi, \xi' \in \mathcal{E}$ and $\eta, \eta' \in \mathcal{H}$. In the simple case where \mathcal{E} is the free module \mathcal{A}^q (i.e. E is the trivial bundle with fiber \mathbb{C}^q), the connection is given by a matrix $\omega = \omega_{ij}$ of elements of Ω^1 , with $i, j \in \{1, \dots, q\}$ and the curvature is the operator in \mathcal{H}^q given by the matrix $d\omega + \omega^2$, with $(d\omega + \omega^2)_{ik} = d(\omega_{ik}) + \sum \omega_{ij}\omega_{jk}$. In general if θ is the curvature, $\theta = \nabla^2 \in \text{Hom}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2)$, of the connection ∇ , there exists elements ξ^i of \mathcal{E} , $i \in \{1, \dots, q\}$ and $\theta_{ij} \in \Omega^2$; $i, j \in \{1, \dots, q\}$ such that $\theta(\xi) = \sum (\xi^i \otimes \theta_{ij}) \langle \xi^j, \xi \rangle$. The corresponding operator in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ is then such that:

$$\theta(\xi \otimes \eta) = \sum \xi^i \otimes \theta_{ij} \langle \xi^j, \xi \rangle \eta \quad \forall \xi \in \mathcal{E}, \eta \in \mathcal{H}.$$

The compatibility of the connection ∇ with the metric implies that θ is a selfadjoint operator in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$: If $\mathcal{E} = \mathcal{A}^q$, then the connection given by $\omega = (\omega_{ij}) \in M_q(\Omega^1)$ is compatible iff $\omega^* = -\omega$ and the curvature $\theta = d\omega + \omega^2$ is then selfadjoint since for $\alpha \in \Omega^1$ one has $d\alpha^* = -(d\alpha)^* \in \Omega^2$. For the sake of clarity, since we are going to relate our notion of connection with the usual notion we shall use the term q -connection for the former and c -connection for the latter.

Lemma 11. a) Every q -connection $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^1$ determines uniquely a classical connection ∇_c by composition with the bimodule map $c: \Omega^1 \rightarrow A^1$ of Theorem 3: $\nabla_c = (1 \otimes c) \circ \nabla$.

b) Let θ be the curvature of the q -connection ∇ , then the curvature θ_c of ∇_c is the antisymmetric part $A c(\theta)$ of $c(\theta)$.

Proof. a) One has $c(axb) = ac(x)b$ for $a, b \in \mathcal{A}$, $\alpha \in \Omega^1$, so $(1 \otimes c) \circ \nabla$ is a linear map of $\mathcal{E} = C^\infty(M, E)$ to $\mathcal{E} \otimes_{\mathcal{A}} A^1 = C^\infty(M, E \otimes T^*)$ such that $\nabla_c(\xi a) = (\nabla_c \xi) a + \xi \otimes da$ for any $\xi \in \mathcal{E}, a \in \mathcal{A}$.

b) Since the ordinary exterior product of two 1-forms is the antisymmetric part of their tensor product, the answer follows from Lemma 7. \square

Corollary 12. The map $\nabla \rightarrow \nabla_c$ maps flat q -connections to ordinary flat connections on \mathcal{E} .

Note that the flatness of the q -connection ∇ means as in Theorem 9 that the operator $F_\nabla = 1 \otimes F - i\nabla$ in the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ satisfies $F_\nabla^2 = 1$, and hence, in the compatible case, yields an element of a suitable Grassmanian. Here F_∇ is defined by: $F_\nabla(\xi \otimes \eta) = \xi \otimes F\eta - i \sum \xi^j \otimes \omega_j \eta$, with $\nabla \xi = \sum \xi^j \otimes \omega_j \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$. One checks that the right-hand side is independent of any choice. Now by Lemma 7 we can associate to every q -connexion a classical tensorial data which is a bit more refined than a classical connexion. Indeed the bimodule $\Omega^1/\Omega_{00}^1 = \sum$ is by Lemma 7 isomorphic to the space of smooth tensors $C^\infty(M, T^1 \oplus T^2)$ which satisfy the equation $d\omega = A\beta$, and the bimodule structure of \sum is given by: $a(\omega, \beta) = (a\omega, da \otimes \omega + a\beta)$; $(\omega, \beta)a = (\omega a, \beta a - \omega \otimes da)$. By the map $(\omega, \beta) \rightarrow (\omega, \beta - d\omega)$, we can identify \sum with the space of all smooth tensors $C^\infty(M, T^1 \oplus S^2 T^1)$ with the bimodule structure given by:

$$a(\omega, \sigma) = (\omega, \sigma)a = (a\omega, a\sigma + \frac{1}{2}(da \otimes \omega + \omega \otimes da))$$

$= (a\omega, a\sigma + da \cdot \omega)$, where $da \cdot \omega$ is the product in the symmetric algebra. Note in particular that the map $(\omega, \sigma) \rightarrow \omega$ is an \mathcal{A} -bimodule map of Σ to A^1 , but that the subspace $\{(\omega, \sigma) \in \Sigma; \sigma = 0\}$ is not a submodule of Σ .

Lemma 13. 1. *The map $\nabla \rightarrow (1 \otimes \tilde{c}) \circ \nabla$ is a surjection of the space of q -connections on \mathcal{E} to the space $\Gamma_{\mathcal{E}}$ of maps $\chi: \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Sigma$ such that $\chi(\xi a) = \chi(\xi)a + \xi \otimes da \forall \xi \in \mathcal{E}, a \in \mathcal{A}$.*

2. *The map $(\omega, \sigma) \rightarrow \omega$ gives a surjection ϱ of $\Gamma_{\mathcal{E}}$ on the space of classical connections on E , and the fibers of ϱ are affine spaces over the vector space $C^\infty(M, \text{End } E \otimes S^2 T^*)$ of smooth 2-tensors.*

Proof. 1. To prove 1. one can assume, as in [3, Proposition 19], that $\mathcal{E} = \mathcal{A}^n$, so that a q -connection is an element of $M_n(\Omega^1)$ and $\Gamma_{\mathcal{E}} = M_n(\Sigma)$, thus 1. follows from Lemma 7.

2. We view $C^\infty(M, S^2 T^*)$ as a submodule Σ_0 of Σ by the map $\sigma \rightarrow (0, \sigma)$. One has $C^\infty(M, \text{End } E \otimes S^2 T^*) = \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Sigma_0)$. Thus the exact sequence of bimodules:

$$0 \rightarrow \Sigma_0 \rightarrow \Sigma \rightarrow A^1 \rightarrow 0$$

gives the desired answer. \square

Theorem 14. *Let M be a 4-dimensional Spin^c Riemannian compact manifold, $\mathcal{H} = L^2(M, S)$ and $F = D|D|^{-1}$ as above, and E a hermitian vector bundle over M , $\mathcal{E} = C^\infty(M, E)$.*

1. *For every compatible q -connection ∇ on \mathcal{E} , the curvature $\theta \in \mathcal{L}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H})$ belongs to \mathcal{L}^{2+} and the value of the Dixmier trace $\text{Trace}_\omega(\theta^2) = I(\theta)$, is independent of ω and defines a gauge invariant positive functional I .*

2. *The restriction of I to each (affine space) fiber of the map $\nabla \rightarrow \nabla_c$ is Gaussian (i.e. a quadratic form) and one has:*

$$\text{Inf}_{\nabla_c = A} I(\nabla) = (16\pi^2)^{-1} \text{YM}(A),$$

where A is a classical connection and YM the classical Yang Mills action.

In fact we shall prove more since we shall identify the Hilbert space of the Gaussian as $L^2(M, \text{End } E \otimes S^2 T^*)$.

Proof. 1. Follows from the inclusion $\Omega^2 \subset \mathcal{L}^{2+}$, i.e. Lemma 4, 1) and Theorem 1. The gauge invariance (under the unitary group of $\text{End}_{\mathcal{A}}(\mathcal{E})$) follows from the trace property of Trace_ω .

2. The value of $I(\theta)$ depends only upon the element χ of Γ associated to the q -connection ∇ . In order to see that and to compute $I(\theta)$ we shall for simplicity assume that $\mathcal{E} = \mathcal{A}^n$. Then ∇ is given by a matrix (α_{ij}) , $\alpha_{ij} \in \Omega^1$, with $\alpha_{ji} = -\alpha_{ij}^* \forall i, j \in \{1, \dots, n\}$. The curvature θ is given by the matrix (θ_{ij}) , $\theta = d\alpha + \alpha^2$, i.e. $\theta_{ij} = d\alpha_{ij} + \sum_k \alpha_{ik}\alpha_{kj}$. Since $\alpha_{ij} \in \Omega^1$, one has $(d\alpha_{ij})^* = d\alpha_{ji}$ and $\theta_{ij}^* = \theta_{ji}$. Now the value of $\text{Tr}_\omega(\theta^2)$ only depends upon the image of θ in Ω^2/Ω_{00}^2 , and the latter only depends upon the image $\tilde{c}(\alpha_{ij})$ of α_{ij} in Ω^1/Ω_{00}^1 , thus our assertion. Now let us write $\tilde{c}(\alpha_{ij}) = (\omega_{ij}, \beta_{ij})$ with $A\beta_{ij} = d\omega_{ij}$ as in Lemma 7. Then the image $c(\theta_{ij})$ of θ_{ij} in Ω^2/Ω_{00}^2 , considered as a tensor of rank 2, is given by the following formula:

$$c(\theta_{ij}) = \beta_{ij} + \sum_k \omega_{ik}\omega_{kj}.$$

For each ij the antisymmetric part $Ac(\theta_{ij})$ is the i, j component of the curvature of the associated classical connection (cf. 11b)). By 13 2., the symmetric part of the tensors β_{ij} is any smooth symmetric tensor t_{ij} with $t_{ji} = t_{ij}^* \forall i, j$, [where $(\xi \otimes \eta)^* = \eta^* \otimes \xi^*$ for any tensors of rank 1, ξ and η]. By Theorem 1, there exists an $O(4)$ invariant inner product on $T^2\mathbb{R}^4 = A^2\mathbb{R}^4 \oplus S^2\mathbb{R}^4$ such that, with the above notations:

$$I(\mathcal{V}) = \text{Trace}_\omega(\theta^2) = \int_M \|c(\theta_{ij})\|^2.$$

Since in this inner product $A^2\mathbb{R}^4$ is necessarily orthogonal to $S^2\mathbb{R}^4$, it follows that, while $I(\mathcal{V})$ obviously depends quadratically on the symmetric part of β_{ij} , its minimum over each fiber of $\mathcal{V} \rightarrow \mathcal{V}_c$ is reached when the symmetric part of each tensor $c(\theta_{ij})$ is set equal to 0. But then the value of $I(\mathcal{V})$ is, up to a numerical factor, the standard Yang-Mills action. \square

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