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Comment

Global Anomalies on Orbifolds

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In this brief addendum we correct an error in our paper and report a solution to some open questions.

Unfortunately, Proposition 7.6 is incorrect. The mistake comes in our definition of the map $\phi_{\mathbb{R}}$. Although it is well-defined, it is not a homomorphism, hence not a representation. Thus we cannot see the characteristic classes of representations of the space group G as preimages of characteristic classes of representations of the point group P.

A simple example illustrates the problem. Let G be the infinite dihedral group. It is an extension of $\mathbb{Z}/2\mathbb{Z}$ by the integers, with the nontrivial element in $\mathbb{Z}/2\mathbb{Z}$ acting by reflection $x \mapsto -x$. Now construct a 2 dimensional complex representation by letting the generator of the integers act by the matrix $\begin{pmatrix} \exp(it) & 0 \\ 0 & \exp(-it) \end{pmatrix}$ and the reflection act by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then if t is irrational this representation does not factor through any finite group.

On the other hand, in any specific example it is usually easy to reduce the calculation of w_2 and λ to a representation of a finite group, in view of Proposition 7.8.

The representation "ϕ_P" in Corollary 7.7 should be changed to "ϕ".

We also warn the reader that although Proposition 7.3 is correct, it is possible for $v(\varrho_P)$ to be nonzero and $v(\varrho)=j^*v(\varrho_P)$ to vanish.

Similarly, the argument in Sect. 6, which also purports to reduce space group anomalies to point group anomalies, is correct only when the Wilson lines can be continuously turned off. This does not happen in general, as the previous example illustrates. However, if we can continuously change the gauge group imbedding of the space group to an imbedding in a finite group, then again the vanishing of the first Pontrjagin class in the group cohomology of the finite group will be sufficient to guarantee the absence of anomalies to all loops. In all the examples we know this can be done, and is equivalent to level matching condition for each element of the space group in the case of an abelian point group. Examples of this type can be found in L. E. Ibanez, H. P. Nilles, and F. Quevedo: Phys. Lett. **187B**, 25 (1987). (We would like to thank L. Dixon for bringing these examples to our attention.)

S. Morita has kindly written us a letter answering some questions in Sect. 8 of our paper. (He credits T. Mizutani with the main argument.) We summarize his arguments in

Proposition. *Let M be a smooth manifold. Then any element of $H_3(M)$ can be represented by a smooth map $e: K \to M$, where K is a closed oriented 3-manifold K which fibers over the circle.*

Proof. It is a standard fact that any 3-dimensional homology class is represented by some closed oriented manifold. Hence we may as well replace M by a closed oriented 3-manifold, and we are left to construct a degree 1 map $K \to M$. By a theorem of Alexander [A, R] we can find a link L (i.e., a disjoint union of embedded circles) in M such that $M \backslash L$ fibers over a circle. The typical fiber F is a Riemann surface with boundary L. Note that all of the fibers share a common boundary. This presents M as an "open book" [W]. (In fact, a refinement due to Harer [H] states that L can be chosen to a knot, i.e., a single circle embedded in M.) Let N be a tubular neighborhood of L in M. Then $\partial(M \backslash N) = S^1 \times L$ is a product. Thus we can glue the trivial bundle $S^1 \times F$ to $M \backslash N$ along their common boundary. The result is a closed oriented 3-manifold K which fibers over S^1 with typical fiber the double of F. Now construct a map $K \to M$ as follows. Over $M \backslash N$ use the given embedding. Collapse the S^1 in the remaining $S^1 \times F$ to a point, thereby stretching the given embedding over $N \backslash L = (D^2 \backslash pt) \times L$. It remains to map in the collapsed $S^1 \times F$, which is simply a copy of F. But since F bounds L inside M, this is possible–simply map F to a fiber of $M \backslash L \to S^1$. It is clear that the resulting map $K \to M$ has degree 1, since it is surjective and 1:1 almost everywhere.

It follows that any anomaly in the setup of Sect. 5 can be detected by some Riemann surface bundle over a circle. This applies to any spacetime. Hence the sufficient conditions for anomaly cancellation set out in [W′, F] are also necessary conditions. (Remark 2 on p. 508 of [F] should be appropriately modified, as should the last paragraph in Sect. 8 of our paper. Also, we implicitly assume that all manifolds and bundles are spin, so that the anomaly involves only the integral characteristic λ.) Our example of the quaternionic group in Sect. 8 shows that some anomalies are not detected at 1-loop (by a torus), but are detected at higher loops.

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