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Non-Linear Multi-Plane Wave Solutions of Self-Dual Yang-Mills Theory

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Abstract. New solutions of self-dual Yang-Mills (SDYM) equations are constructed in Minkowski space-time for the gauge group $SL(2, \mathbb{C})$. After proposing a Lorentz covariant formulation of Yang's equations, a set of Ansätze for exact non-linear multiplane wave solutions are proposed. The gauge fields are rational functions of $e^{x \cdot k_i}$ $(k_i^2 = 0, 1 \leqq i \leqq N)$ for these Ansätze. At least, three families of multisoliton type solutions are derived explicitly. Their asymptotic behaviour shows that non-linear waves scatter non-trivially in Minkowski SDYM.

1. Introduction

Integrable theories in $1+1$ dimensions have been developed very successfully in the last years. It is then a natural problem to investigate their multidimensional analogues. Namely four- (or n-) dimensional field theories having an associate linear differential system with a spectral parameter(s). Such linear systems are known for self-dual (and antiself-dual) Yang-Mills equations (SDYM) [1, 2] and SUSY Yang-Mills [3], as well as for non-Lorentz invariant equations like Kadomtsev-Petviashvili (KP), or three-wave equations in $2+1$ dimensions [4, 6]. Actually, the dynamics of KP is known much better than that of SDYM in Minkowski space-time.

The construction of multi-soliton (non-linear multi-plane wave) solutions of SDYM in $3+1$ dimensions is the purpose of this paper. The SDYM reads there

$$F_{\mu\nu} = \frac{i}{2}\, \varepsilon_{\mu\nu\lambda\sigma} F^{\lambda\sigma} \tag{1.1}$$

(here $\varepsilon_{0123} = +1$). Since an explicit factor (i) appears, these equations describe complex solutions for $SU(N)$ gauge fields or equivalently real solutions (real gauge

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potentials) for a $SL(N, \mathbb{C})$ gauge theory. The last point of view will be preferred from now on. For a non-compact gauge group the energy density $T_{00}(x)$ is not positive definite. Actually, it is easy to show that the energy-momentum tensor $T_{\mu\nu}(x)$ of the Yang-Mills theory identically vanishes in Minkowskian space-time when the self-dual equations (1.1) are imposed.

Yang's equations [7] are a particularly elegant form of $SU(2)$ Euclidean SDYM equations. Although the SDYM equations are $SO(4)$ (proper Lorentz) invariant in Euclidean (Minkowski) space-time, Yang's equations are not fully $SO(4)$ (Lorentz) invariant. In Sect. 2, we find a fully $SO(4)$ (proper Lorentz) covariant version of Yang's equations [Eqs. (2.11) and (2.20) respectively]. These covariant equations derive from a Lagrangian density of a four-dimensional (generalized) sigma model

$$\mathscr{L} = \frac{1}{2} g^{\mu\nu} \frac{\mathrm{Tr}\,[P_+ \partial_\mu Q \partial_\nu Q]}{\mathrm{Tr}\,[P_+ Q P_+ Q]} \,. \tag{1.2}$$

Here

$$Q(x) = \begin{pmatrix} \phi(x) & -\bar{\varrho}(x) \\ -\varrho(x) & -\phi(x) \end{pmatrix}$$

contain the fields, $P_+ = (1+\sigma_3)/2$ and the space-time metric is

$$g^{\mu\nu} = \eta^{\mu\nu} + \mathbb{C}^{\mu\nu} \,, \tag{1.3}$$

where $\eta^{\mu\nu}$ is the Minkowski metric and $\mathbb{C}^{\mu\nu}$ a constant antisymmetric and antiself-dual tensor [Eq. (2.22)]. The fields $\phi, \varrho, \bar{\varrho}$ are assumed to be Lorentz scalars ($\bar{\varrho}$ *is not* the complex conjugate of ϱ). The Lagrangian (1.2) defines a translationally invariant but non-rotationally invariant field theory. The $\mathbb{C}_{\mu\nu}$ coefficients in (1.2) can be considered as non-Lorentz scalar couplings spoiling rotational invariance.

In Sect. 3, a first Ansatz describing non-linear multi-plane wave solutions of the covariant version of Yang's equations is proposed. They can be considered as $3+1$ dimensional solitons. Actually, the form of the Ansatz is inspired by Hirota's method. Namely

$$Q(x) = q_0 + \sum_{i=1}^{N} q_i e^{\eta_i} \,, \tag{1.4}$$

where $q_\alpha (0 \leqq \alpha \leqq N)$ are two-by-two constant matrices with $(q_\alpha)_{11} = (q_\alpha)_{22}$ and $\eta_i = k_i \cdot x$. Here $k_i (1 \leqq i \leqq N)$ are arbitrary constant null vectors: $k_i^2 = 0$.

In Sect. 3, an explicit formula [Eq. (3.14)] for the coefficients q_α is derived from the self-duality equations. The solution depends on $N+3$ free parameters besides the null vectors $k_i (1 \leqq i \leqq N)$. Actually the SDYM equations are fulfilled, thanks to a trilinear identity [Eq. (3.12)] satisfied by the metric tensor (1.3).

The analysis of the asymptotic behaviour of these solutions shows that they describe the scattering of N non-linear single plane-wave solutions.

In Sect. 4, more general solutions are constructed. We start from a fractional Ansatz where ϕ, ϱ, $\bar{\varrho}$ are ratios of polynomials in e^{η_i} of the form

$$\mathbb{C}^{(0)} + \sum_{i=1}^{N} \mathbb{C}_i^{(1)} e^{\eta_i} + \sum_{1 \leqq i < j \leqq N} \mathbb{C}_{ij}^{(2)} e^{\eta_i + \eta_j} + \mathbb{C}^{(N)} e^{\eta_1 + \eta_2 + \ldots + \eta_N} \,. \tag{1.5}$$

Notice that all η's are different in each exponent. For the case $N=2$ the algebraic equations for the coefficient are written down explicitly (Appendix A). The analysis of the asymptotic behaviour suggests constraints on the coefficients such that there is a single ingoing and outgoing plane wave in directions k_1 and k_2. These constraints are compatible with the algebraic equations (A.1)–(A.15) and simplify them enormously. The family of solutions (4.13)–(4.15) depend on six free parameters besides k_1 and k_2, and it describes the scattering of two non-linear plane waves. The scattering produces zero phase-shift and only a $SL(2, C)$ rotation of the outgoing field with respect to the ingoing field [Eq. (4.18)].

A second family of solutions with a fractional structure follows by applying the Bäcklund transformations γ of [8] to a solution of type (1.4). This family of self-dual fields depends for $N=2$ on eight free parameters besides k_1 and k_2. The interpretation of this family as interacting plane waves is straightforward after introducing two constraints in the parameters [Eq. (4.23)]. Then, we find an ingoing pure ϱ-plane wave (A-soliton) plus a $\bar{\varrho}$-plane wave ($\bar{A}$-soliton) yielding for $x^0 \to +\infty$ two non-linear plane waves of type B. The $\bar{\varrho}$ field is constant for type B solitons (see the Table 1).

Table 1

	ϕ	ϱ	$\bar{\varrho}$
$\eta_1 \to -\infty$ $\eta = \eta_2 = \text{fixed}$ $\bar{A}$-soliton	$F_0/\Delta_0 = \text{const}$	$N_0/\Delta_0 = \text{const}$	$(\bar{N}_0 + \bar{N}_2 e^{\eta})/(\Delta_0 + \Delta_2 e^{\eta})$
$\eta_2 \to -\infty$ $\eta = \eta_1 = \text{fixed}$ A-soliton	$F_0/\Delta_0 = \text{const}$	$(N_0 + N_1 e^{\eta})/(\Delta_0 + \Delta_1 e^{\eta})$	$\bar{N}_0/\Delta_0 = \text{const}$
$\eta_1 \to +\infty$ $\eta = \eta_2 = \text{fixed}$ B-soliton	$F_1/(\Delta_1 + \Delta_{12} e^{\eta})$	$(N_1 + N_{12} e^{\eta})/(\Delta_1 + \Delta_{12} e^{\eta})$	$\bar{N}_1/\Delta_1 = \text{const}$
$\eta_2 \to +\infty$ $\eta = \eta_1 = \text{fixed}$ B-soliton	$F_2/(\Delta_2 + \Delta_{12} e^{\eta})$	$(N_2 + N_{12} e^{\eta})/(\Delta_2 + \Delta_1 e^{\eta})$	$\bar{N}_2/\Delta_2 = \text{const}$

The explicit solutions found here show that Minkowski SDYM have a rich dynamics that can be uncovered by generalizing appropriately $1+1$ or $2+1$ dimensional soliton theory. Although SDYM possess an infinite number of conserved currents, a non-trivial S-matrix is found here for the interaction of non-linear plane waves. The Coleman-Mandula theorem is supposed to hold even classically and it would imply a trivial S-matrix. The reason why this theorem is bypassed is probably linked to the infinite spatial extension of the wave fronts. An analogous phenomenon appears in KP where lumps have trivial scattering, whereas plane waves exhibit non-zero phase shifts [4, 5]. In conclusion, the study of scattering of non-linear plane waves seems to be the right multi-dimensional generalization of $1+1$ dimensional multi-soliton dynamics. Actually, in the paper we shall use the terms soliton and non-linear plane wave as synonyms.

The development of a τ-function like formalism for SDYM in $3+1$ would be very interesting. The absence of terms with powers of e^{η_i} higher than one in all our solutions gives a hint about a possible fermionic character underlying the solutions obtained in this paper.

2. Lagrangian Theory of Self-Dual Yang-Mills Field: Lorentz Covariant Formulation

The Euclidean self-dual Yang-Mills equations read simpler in complex coordinates $(y\bar{y}z\bar{z})$:

$$F_{yz}=F_{\bar{y}\bar{z}}=0 \ , \quad F_{y\bar{y}}+F_{z\bar{z}}=0 \ . \tag{2.1}$$

Here

$$\begin{aligned} \sqrt{2}y=x_1+ix_2 \ , \quad \sqrt{2}z=x_3-ix_4 \ , \\ \sqrt{2}\bar{y}=x_1-ix_2 \ , \quad \sqrt{2}\bar{z}=x_3+ix_4 \ . \end{aligned} \tag{2.2}$$

Equation (2.1) tells us that self-dual fields are pure gauge in the yz plane for fixed $\bar{y}$ and $\bar{z}$ and in the $\bar{y}\bar{z}$ plane for fixed y and z. So, we can write [7]

$$\begin{aligned} A_y=D^{-1}\partial_y D \ , \quad A_z=D^{-1}\partial_z D \ , \\ A_{\bar{y}}=E^{-1}\partial_{\bar{y}} E \ , \quad A_{\bar{z}}=E^{-1}\partial_{\bar{z}} E \ , \end{aligned} \tag{2.3}$$

where the matrices E and D take values in the gauge group. Since we will be interested in Minkowski space fields, the gauge group will be taken to be $SL(N, C)$. In this case one can gauge transform Eq. (2.3) yielding

$$\begin{aligned} &A_{\bar{y}}=A_{\bar{z}}=0 \ , \\ &A_y=\chi^{-1}\partial_y\chi \ , \quad A_z=\chi^{-1}\partial_z\chi \ . \end{aligned} \tag{2.4}$$

Here $\chi \in SL(N, \mathbb{C})$ so $\det\chi=1$. For $N=2$ we can parametrize χ as

$$\chi=\frac{1}{\phi}\begin{pmatrix} 1 \ , & \bar{\varrho} \\ \varrho \ , & \phi^2+\varrho\bar{\varrho} \end{pmatrix} . \tag{2.5}$$

It will be sometimes convenient to work in the so-called R gauge [7], where for $SL(2, \mathbb{C})$ gauge fields

$$A_\mu^R=\frac{1}{2\phi}\begin{pmatrix} -\partial_\mu\phi \ , & 0 \\ 2\partial_\mu\varrho \ , & \partial_\mu\phi \end{pmatrix} ; \quad \mu=y,z \ , \tag{2.6}$$

$$A_{\bar{\nu}}^R=\frac{1}{2\phi}\begin{pmatrix} \partial_{\bar{\nu}}\phi & -2\partial_{\bar{\nu}}\bar{\varrho} \\ 0 & -\partial_{\bar{\nu}}\phi \end{pmatrix} ; \quad \bar{\nu}=\bar{y},\bar{z} \ . \tag{2.7}$$

We would like to stress that $\bar{\varrho}$ is not the complex conjugate of ϱ.

Inserting Eqs. (2.4) and (2.5) in the self-duality Eqs. (2.1) yields

$$\begin{aligned}
&\phi(\partial_{y\bar{y}}+\partial_{z\bar{z}})\phi-\partial_y\phi\partial_{\bar{y}}\phi-\partial_z\phi\partial_{\bar{z}}\phi+\partial_y\varrho\partial_{\bar{y}}\bar{\varrho}+\partial_z\varrho\partial_{\bar{z}}\bar{\varrho}=0\ ,\\
&\phi(\partial_{y\bar{y}}+\partial_{z\bar{z}})\varrho-2\partial_y\varrho\partial_{\bar{y}}\phi-2\partial_z\varrho\partial_{\bar{z}}\phi=0\ ,\\
&\phi(\partial_{y\bar{y}}+\partial_{z\bar{z}})\bar{\varrho}-2\partial_{\bar{y}}\bar{\varrho}\partial_y\phi-2\partial_{\bar{z}}\bar{\varrho}\partial_z\phi=0\ .
\end{aligned}\tag{2.8}$$

These equations are not $SO(4)$ invariant, whereas Eqs. (2.1) are clearly rotationally invariant. It is convenient to use quaternions to describe four-dimensional rotations. Let us define the 2×2 matrix

$$x=\begin{pmatrix} y & z \\ \bar{z} & -\bar{y}\end{pmatrix}\ .\tag{2.9}$$

A $SO(4)$ transformation yields

$$x\to x'=sxt^{\dagger}\ ,\tag{2.10}$$

where $s, t\in SU(2)$. This is an explicit realization of $SO(4)$ as $SU(2)\otimes SU(2)$. We call s and t left and right $SU(2)$ respectively. Equations (2.8) are translationally invariant and right-$SU(2)$ invariant. They are not left-$SU(2)$ invariant. Applying left-$SU(2)$ transformations leads to the more general equations

$$\begin{aligned}
&\phi\partial^2\phi-\partial_\mu\phi\partial^\mu\phi+\partial_\mu\varrho\partial^\mu\bar{\varrho}+\partial_\mu\varrho\Sigma^{\mu\nu}\partial_\nu\bar{\varrho}=0\ ,\\
&\phi\partial^2\varrho-2\partial_\mu\varrho\partial^\mu\phi-2\partial_\mu\varrho\Sigma^{\mu\nu}\partial_\nu\phi=0\ ,\\
&\phi\partial^2\bar{\varrho}-2\partial_\mu\bar{\varrho}\partial^\mu\phi-2\partial_\mu\phi\Sigma^{\mu\nu}\partial_\nu\bar{\varrho}=0\ .
\end{aligned}\tag{2.11}$$

Here

$$a_\mu b^\mu=\tfrac{1}{2}\,(a_y b_{\bar{y}}+a_{\bar{y}}b_y+a_z b_{\bar{z}}+a_{\bar{z}}b_z)\ ,$$

and Σ is a constant antisymmetric, Hermitian and antiself-dual tensor transforming under the $(1,0)$ irreducible representation of $SO(4)$ [2]. Σ reads in the real basis (x_i)

$$\Sigma_{\mu\nu}=i\begin{pmatrix} 0 & \varepsilon & -\gamma & \delta \\ -\varepsilon & 0 & -\delta & -\gamma \\ \gamma & \delta & 0 & -\varepsilon \\ -\delta & \gamma & \varepsilon & 0\end{pmatrix},\tag{2.12}$$

where $\varepsilon, \delta, \gamma\in R$ and $\varepsilon^2+\gamma^2+\delta^2=1$. Yang's equations (2.8)[7] are recovered in the particular frame where $\varepsilon=1, \gamma=\delta=0$. Yang's equations can be derived from a local Lagrangian [9]. This is also true for the covariant system (2.11). They are the Euler-Lagrange equation of the invariant Lagrangian

$$\mathscr{L}=\frac{1}{2\phi^2}\,(\delta^{\mu\nu}+\Sigma^{\mu\nu})\,(\partial_\mu\phi\partial_\nu\phi+\partial_\mu\varrho\partial_\nu\bar{\varrho})\ .\tag{2.13}$$

This can be recast as a sigma model introducing a three-component complex field

$$\Psi=\left(\phi, \frac{\varrho+\bar{\varrho}}{2}, \frac{\varrho-\bar{\varrho}}{2i}\right),\tag{2.14}$$

and the unit vector $\check{n}=(1,0,0)$. One finds

$$\mathscr{L}=\tfrac{1}{2}\, g^{\mu\nu} G_{ab}(\psi)\partial_\mu\psi_a\partial_\nu\psi_b \ , \tag{2.15}$$

where $g^{\mu\nu}=\delta^{\mu\nu}+\Sigma^{\mu\nu}$ is a non-symmetric metric tensor in the four-dimensional space and $G_{ab}(\psi)$ is the metric in the internal space

$$G_{ab}(\psi)=\frac{1}{(\check{n}\cdot\Psi)^2}\,[\delta_{ab}-i\varepsilon_{abc}\check{n}_c] \ . \tag{2.16}$$

An alternative matrix form of Eq. (2.15) is

$$\mathscr{L}=\frac{1}{2}\, g^{\mu\nu}\,\frac{\mathrm{Tr}\,(P_+\partial_\mu Q\partial_\nu Q)}{\mathrm{Tr}\,(P_+QP_+Q)} \ , \tag{2.17}$$

where

$$Q=\begin{pmatrix}\phi & -\bar{\varrho}\\ -\varrho & -\phi\end{pmatrix}, \qquad P_+=\frac{1+\sigma_3}{2} \ .$$

All the derivations up to now hold both for $SL(2,\mathbb{C})$ and $SU(2)$ gauge fields. In the latter case, one must impose $\bar{\varrho}=\varrho^*$ (where * means complex conjugate) thoroughly. This constraint can be consistently imposed since the tensor $\Sigma_{\mu\nu}$ is Hermitian.

Let us now consider the Minkowskian version of the self-dual Yang-Mills theory. We choose the time co-ordinate x_0 as

$$x_0=ix_4 \ , \qquad x_0\in\mathbb{R} \ . \tag{2.18}$$

Then

$$\partial_z=\partial_3-\partial_0 \ , \qquad \partial_{\bar{z}}=\partial_3+\partial_0 \ . \tag{2.19}$$

The field equations (2.11) now read

$$\begin{aligned}
&\phi\,\Box\phi-\partial\phi\cdot\partial\phi+\partial\varrho\cdot\partial\bar{\varrho}+\partial\varrho\cdot C\cdot\partial\bar{\varrho}=0 \ ,\\
&\phi\,\Box\varrho-2\partial\varrho\cdot\partial\phi-2\partial\varrho\cdot C\cdot\partial\phi=0 \ ,\\
&\phi\,\Box\bar{\varrho}-2\partial\bar{\varrho}\cdot\partial\phi-2\partial\phi\cdot C\cdot\partial\bar{\varrho}=0 \ .
\end{aligned} \tag{2.20}$$

The dot means here Lorentzian scalar product:

$$a\cdot b=\eta^{\mu\nu}a_\mu b_\nu=a_0b_0-\sum_{k=1}^{3} a_kb_k \ , \qquad \Box=\partial_0^2-\sum_{k=1}^{3}\partial_k^2 \ , \tag{2.21}$$

and

$$\begin{aligned}
&\mathbb{C}_{\mu\nu}=-\mathbb{C}_{\nu\mu} \ ,\\
&\mathbb{C}_{ik}=i\varepsilon_{ikj}\mathbb{C}_{0j} \ , \qquad 1\leqq i,k,j\leqq 3 \ ,\\
&\mathbb{C}_{0i}=\check{n}_i\in\mathbb{R} \ , \qquad \check{n}=(\delta,\gamma,\varepsilon) \ , \qquad \check{n}^2=1 \ .
\end{aligned} \tag{2.22}$$

In Minkowski space-time the Lagrange density (2.13) is written

$$\mathscr{L}=\frac{1}{2\phi^2}\,[\langle\partial\phi|\partial\phi\rangle+\langle\partial\varrho|\partial\bar{\varrho}\rangle] \ , \tag{2.23}$$

where

$$\langle a|b\rangle \equiv (\eta^{\mu\nu}+\mathbb{C}^{\mu\nu})a_\mu b_\nu \ .$$

A conserved energy momentum tensor follows from this Lagrangian,

$$T_{\mu\nu}=\frac{1}{\phi^2}\left[\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}(\partial_\mu\varrho\partial_\nu\bar{\varrho}+\partial_\mu\bar{\varrho}\partial_\nu\varrho)+\mathbb{C}_{\mu\lambda}(\partial_{\nu\varrho}\partial^\lambda\bar{\varrho}-\partial_\nu\bar{\varrho}\partial^\lambda\varrho)\right]-\eta_{\mu\nu}\mathscr{L} \ ,$$

$$\partial^\mu T_{\mu\nu}=0 \ . \tag{2.24}$$

However, this tensor $T_{\mu\nu}$ is not symmetric due to the presence of the tensor $C_{\mu\nu}$,

$$T_{\mu\nu}-T_{\nu\mu}=\frac{\mathbb{C}_{\mu\lambda}}{\phi^2}(\partial_\nu\varrho\partial^\lambda\bar{\varrho}-\partial_\nu\bar{\varrho}\partial^\lambda\varrho)-(\mu\leftrightarrow\nu)\neq 0 \ .$$

This difference cannot be recast in general as the total divergence of an antisymmetric tensor. That is

$$T_{\mu\nu}-T_{\nu\mu}\neq\partial_\varrho\theta^\varrho_{\mu\nu} \quad \text{with} \quad \theta^{\mu\varrho\nu}+\theta^{\varrho\mu\nu}=0 \ .$$

This indicates that there is no conserved angular momentum in the theory of fields $(\phi,\varrho,\bar{\varrho})$ with Lagrangian (2.23). The presence of the constant tensor $\mathbb{C}_{\mu\nu}$ in this Lagrangian makes it manifestly frame dependent. One can think of the unit vector $\mathbb{C}_{0i}=(\delta,\gamma,\varepsilon)$ as a "vectorial coupling constant", which clearly spoils rotational invariance.

On the contrary, the original self-dual equations (2.1) and the gauge fields A_μ are fully Poincaré covariant. A_μ follows from the reduced fields $(\varrho,\bar{\varrho},\phi)$ through

$$A_\mu=\tfrac{1}{2}(\delta_\mu^\nu+\mathbb{C}_\mu^\nu)\chi^{-1}\partial_\nu\chi \ , \tag{2.25}$$

where χ is given by Eq. (2.5). Therefore, in this formulation of the self-dual Yang-Mills fields in terms of the auxiliary field χ, we find that χ is a Lorentz scalar but not a gauge invariant one. However, it is easy to express gauge invariant quantities in terms of the field χ. One finds for example for $(\det F_{\mu\bar{\nu}})$ with fixed μ and $\bar{\nu}$,

$$\det[F_{\mu\bar{\nu}}]=-\frac{1}{\phi^4}[(\phi\partial_{\mu\bar{\nu}}\phi+\partial_\mu\varrho\partial_{\bar{\nu}}\bar{\varrho}-\partial_\mu\phi\partial_{\bar{\nu}}\phi)^2$$
$$+(\phi\partial_{\mu\bar{\nu}}\varrho-2\partial_\mu\varrho\partial_{\bar{\nu}}\phi)(\phi\partial_{\mu\bar{\nu}}\bar{\varrho}-2\partial_\mu\phi\partial_{\bar{\nu}}\bar{\varrho})] \ . \tag{2.26}$$

3. Self Dual Soliton Solutions in 3+1 Minkowski Space-Time

We construct in this section Ansätze that solve the covariant version of Yang's equations presented in the previous section [e.g., (2.20)]. By solitons in 3+1 dimensions, we mean a *non-linear* superposition of plane waves. That is a field like

$$A_\mu=A_\mu(x\cdot k_1,x\cdot k_2,\dots,x\cdot k_N) \ , \quad 0\leqq\mu\leqq 3 \tag{3.1}$$

for an N-soliton solution. Here the constant vectors $k_1,\dots,k_N$ are taken null in accordance with the massless character of classical gauge theories

$$k_i\cdot k_i=0 \ , \quad 1\leqq i\leqq N \ . \tag{3.2}$$

Equations (2.20) are bilinear in ϕ, ϱ and $\bar{\varrho}$. Therefore, one can hope that Hirota's method can be applied [4]. This suggests to propose the following Ansatz:

$$\begin{aligned}\phi(x)&=K_0+\sum_{i=1}^{N} K_i e^{\eta_i} ,\\ \varrho(x)&=\mathbb{C}_0+\sum_{i=1}^{N} \mathbb{C}_i e^{\eta_i} ,\\ \bar{\varrho}(x)&=\bar{\mathbb{C}}_0+\sum_{i=1}^{N} \bar{\mathbb{C}}_i e^{\eta_i} .\end{aligned} \tag{3.3}$$

Here the K_α, $\mathbb{C}_\alpha$ and $\bar{\mathbb{C}}_\alpha$ $(0\leqq\alpha\leqq N)$ are constants and

$$\eta_i\equiv k_i\cdot x=\omega_i t-\mathbf{k}_i\cdot\mathbf{x} , \quad |\omega_i|=|\mathbf{k}_i| ; \quad 1\leqq i\leqq N . \tag{3.4}$$

It is trivial to check that the fields (3.3) fulfill Eqs. (2.20) fir $N=1$.

Inserting the Ansatz (3.3) in Eqs. (2.20) and equating to zero the coefficient of $e^{\eta_i+\eta_j}$ $(1\leqq i<j\leqq N, N\geqq 2)$ yields the algebraic equations

$$\begin{aligned}&(1+\Lambda_{ij})\mathbb{C}_i\bar{\mathbb{C}}_j+(1-\Lambda_{ij})\mathbb{C}_j\bar{\mathbb{C}}_i=2K_iK_j ,\\ &(1+\Lambda_{ij})\mathbb{C}_iK_j+(1-\Lambda_{ij})\mathbb{C}_jK_i=0 ,\\ &(1-\Lambda_{ij})\bar{\mathbb{C}}_iK_j+(1+\Lambda_{ij})\bar{\mathbb{C}}_jK_i=0 , \quad 1\leqq i<j\leqq N .\end{aligned} \tag{3.5}$$

Here

$$\Lambda_{ij}\equiv\frac{k_i\cdot\mathbb{C}\cdot k_j}{k_i\cdot k_j} , \tag{3.6}$$

and $\mathbb{C}^{\mu\nu}$ is the antiself-dual tensor introduced in Sect. 2 [Eq. (2.22)].

The system (3.5) looks heavily overdetermined for $N>2$ since it contains $3N(N-1)/2$ equations and only $2N$ unknown (we can absorb then constants K_i in a constant shift of the variables η_i). However, as we shall see below, there exist non-trivial solutions of (3.5) for any N. For $N=2$, Eqs. (3.5) admit the following solution:

$$\begin{aligned}\phi&=K_0+K_1e^{\eta_1}+K_2e^{\eta_2} ,\\ \varrho&=\mathbb{C}_0+A\left(K_1e^{\eta_1}+\frac{\Lambda+1}{\Lambda-1}K_2e^{\eta_2}\right) ,\\ \bar{\varrho}&=\bar{\mathbb{C}}_0-\frac{1}{A}\left(K_1e^{\eta_1}+\frac{\Lambda-1}{\Lambda+1}K_2e^{\eta_2}\right) ,\end{aligned} \tag{3.7}$$

where A is an arbitrary constant and $\Lambda\equiv\Lambda_{12}$. We find, therefore, a two-soliton solution (two non-linear plane waves) that depends on six arbitrary parameters $(K_0, K_1, K_2, A, C_0, \bar{C}_0)$ besides the two light-like vectors k_1 and k_2.

It must be noticed that Eqs. (3.5) for a given couple (i,j) and arbitrary N are identical to the equations for $N=2$, where only the couple (1,2) is present. Therefore, we find for a fixed pair (i,j) $(1\leqq i<j\leqq N)$,

$$\begin{aligned}&\mathbb{C}_i=A_{ij}(\Lambda_{ij}-1)K_i , \quad \bar{\mathbb{C}}_j=-K_j^2/\mathbb{C}_j ,\\ &\mathbb{C}_j=A_{ij}(\Lambda_{ij}+1)K_j , \quad \bar{\mathbb{C}}_i=-K_i^2/\mathbb{C}_i .\end{aligned} \tag{3.8}$$

Now, we must show that there exist coefficients A_{ij} such that Eqs. (3.5) hold for all pairs (i,j) $(1 \leqq i < j \leqq N)$.

For $N=3$, it follows from Eqs. (3.5) and (3.8) that the A_{ij} must fulfill the linear system:

$$\begin{aligned} A_{12}(A_{12}-1) &= A_{13}(A_{13}-1) \ , \\ A_{12}(A_{12}+1) &= A_{23}(A_{23}-1) \ , \\ A_{13}(A_{13}+1) &= A_{23}(A_{23}+1) \ . \end{aligned} \tag{3.9}$$

The determinant of this homogeneous system reads

$$\Delta = 1 - A_{12}A_{13} - A_{13}A_{23} + A_{12}A_{23} \ . \tag{3.10}$$

After some calculations, one finds using (3.6) that Δ is identically zero. Hence, a non-trivial solution exists for $N=3$. For $N \geqq 4$ the coefficients A_{ij} $(1 \leqq i < j \leqq N)$ are constrained by N equations instead of three for $N=3$ [Eqs. (3.9)]. However, their structure is the same and we find that they have nontrivial solutions since the corresponding determinants vanish as Δ does in Eq. (3.10).

The key equation $\Delta = 0$ can be written with the help of the non-symmetric metric tensor of Sect. 2,

$$g^{\mu\nu} = \eta^{\mu\nu} + \mathbb{C}^{\mu\nu} \ . \tag{3.11}$$

(The self-dual tensor $\mathbf{C}^{\mu\nu}$ is given in Eq. (2.22)). We find

$$\langle k_i|k_j\rangle \langle k_j|k_l\rangle \langle k_l|k_i\rangle + \langle k_i|k_l\rangle \langle k_l|k_j\rangle \langle k_j|k_i\rangle = 0 \ , \tag{3.12}$$

where we use the scalar product defined in Sect. 3,

$$\langle a|b\rangle = g^{\mu\nu} a_\mu b_\nu \ . \tag{3.13}$$

Actually, Eq. (3.12) corresponds to an algebraic property of the tensor $\mathbf{C}_{\mu\nu}$ since it holds for the arbitrary null vector k_i, k_j, k_l. It is tempting to make a parallel with classical two-dimensional theories where integrability is linked to a bilinear algebraic equation in the r-matrices: the classical Yang-Baxter equations [10]. Here, in $3+1$ dimensions the multisoliton solutions exist thanks to the trilinear identity (3.12). Moreover, the structure of (3.12) somehow resembles a zero curvature (path-independence) condition.

Finally, collecting all equations we find the general expression for C_i and $\mathbb{C}_i$ $(1 \leqq i \leqq N)$ in terms of K_i and k_i:

$$\begin{aligned} \mathbb{C}_i &= A_{(N)} K_i \prod_{n=1}^{N-1} [A_{n,n+1} + \operatorname{sign}(i-n-\tfrac{1}{2})] \ , \\ \bar{\mathbb{C}}_i &= -K_i^2/\mathbb{C}_i \ , \qquad 1 \leqq i \leqq N \ . \end{aligned} \tag{3.14}$$

Therefore, this N-soliton solution depends on $N+4$ free parameters

$$\bar{\mathbb{C}}_0 \ , \quad \bar{\mathbb{C}}_0 \ , \quad A_{(N)} \ , \quad K_\alpha (0 \leqq \alpha \leqq N) \tag{3.15}$$

besides the null vectors $k_1, \ldots, k_N$. Indeed the parameters K_i $(1 \leqq i \leqq N)$ can be considered as initial phases of the η_i. In addition, one can always rescale $(\phi, \varrho, \bar{\varrho})$ as

$$\phi(x) \to K\phi(x) \ , \quad \varrho(x) \to K^2 \varrho(x) \ , \quad \bar{\varrho}(x) \to \bar{\varrho}(x) \ , \tag{3.16}$$

leaving the gauge fields invariant [Eq. (2.6)–(2.7)]. So we have actually $(N+3)$ parameters. We are not taking into account the $SL(2, \mathbb{C})$ symmetries in the counting of parameters.

Let us now analyze these multisoliton solutions by studying their asymptotic behaviour. In the R-gauge [Eqs. (2.6)–(2.7)] the gauge field reads for the two-soliton solution (3.7):

$$A_\mu^{(2)}(x) = \frac{1}{2(K_0 + K_1 e^{\eta_1} + K_2 e^{\eta_2})} \cdot \begin{pmatrix} -K_1 k_\mu^1 e^{\eta_1} - K_2 k_\mu^2 e^{\eta_2} \,, & 0 \\ 2A_{(2)}[(\Lambda-1)K_1 k_\mu^1 e^{\eta_1} + (\Lambda+1) k_\mu^2 K_2 e^{\eta_2}] \,, & K_1 k_\mu^1 e^{\eta_1} + K_2 k_\mu^2 e^{\eta_2} \end{pmatrix} \tag{3.17}$$

When $\eta_1 \to -\infty$ at fixed η_2 one gets

$$A_\mu^{(2)}(x)_{\eta_1 \to -\infty} = a_\mu(k, A_{(2)}(\Lambda+1), K_2/K_0) \,, \tag{3.18}$$

and a similar expression for $\bar{A}_\mu^{(2)}$. Here $a_\mu(k, B, C)$ may be considered as a one-soliton solution

$$a_\mu(k, B, \mathbb{C}) \equiv \frac{k_\mu}{2\left(1 + \dfrac{e^{-k \cdot x}}{\mathbb{C}}\right)} \begin{pmatrix} -1 & 0 \\ 2B & 1 \end{pmatrix} = k_\mu a(k, B, \mathbb{C}) \,.$$

More generally, we can study the N-soliton solution (3.13) when $\eta_k \to -\infty$ $(1 \leqq k \leqq N, k \neq i)$ and η_i is kept fixed. In this regime

$$A_\mu^{(N)}(x)_{\eta_k \to -\infty} = a_\mu(k_i, A_{(N)}\sigma_i, K_i/K_0) \,, \tag{3.19}$$

where

$$\sigma_i = \prod_{k=1}^{N-1} [\Lambda_{k,k+1} + \operatorname{sign}(i - k - \tfrac{1}{2})] \,.$$

An analogous expression holds for $\bar{A}_\mu^{(N)}(x)$.

When all the frequencies ω_k $(1 \leqq k \leqq N)$ are positive, the limit $\eta_k \to -\infty$ for $k \neq i$ $(1 \leqq k \leqq N)$ and $\eta_i = \text{fixed}$ corresponds to $t \to -\infty$ and

$$\omega_i t - \mathbf{k}_i \cdot \mathbf{x} = \text{fixed} \,. \tag{3.20}$$

That amounts to stay on the wave front orthogonal to the vector $\mathbf{k}_i$. So Eq. (3.19) indicates that the solution $A_\mu^{(N)}$ when $\omega_k > 0$ $(1 \leqq k \leqq N)$ describes the collision of N plane waves. Equation (3.16) describes each of them when $t \to -\infty$. It is reasonable to call one soliton to the solution $a_\mu(k, B, \mathbb{C})$ [Eq. (3.18)] since $A_\mu(x)$ tends exponentially to a constant when one goes away from the wave front $k \cdot x = \text{cte}$. In fact, $a_\mu(k, B, \mathbb{C})$ is almost a zero field solution since the local gauge invariants associated to a_μ vanish. That is $F_{\mu\nu}(a)$ is non-zero, but its determinant for a given $\mu\nu$ in the internal space is identically zero for all x. However, a_μ is not a pure gauge. When some $\eta_k \to +\infty$ and the rest is kept fixed, $A_\mu^{(N)}$ and $\bar{A}_\mu^{(N)}$ give just a constant connection. A non-trivial behaviour appears when several $\eta_k \to +\infty$, keeping fixed their differences. For example, let η_1 and $\eta_2 \to +\infty$ in $A_\mu^{(2)}$ with $\eta_3 \equiv \eta_2 - \eta_1$ fixed. We find

$$A_\mu^{(2)}(x)_{\eta_{1,2}\to+\infty} = k_\mu^1 a(-k_3, A_2(\Lambda-1), K_1/K_2) + k_\mu^2 a(k_3, A_2(\Lambda+1), K_2/K_1)$$
$$\eta_3 = \text{fixed} \; . \tag{3.21}$$

Here $k_3 \equiv k_2 - k_1$. The gauge field here tends exponentially to a constant when one goes away from the plane $\eta_3 = \text{cte}$. Since

$$k_3^2 = (k_1 - k_2)^2 = -2\omega_1\omega_2[1 - \cos(\check{k}_1 \cdot \check{k}_2)] < 0 \tag{3.22}$$

if $\omega_1\omega_2 > 0$, the resulting outgoing waves are not like the ingoing ones (3.19). For $t \to +\infty$, the wave speed is less than one. So, we can conclude that the scattering of the two massless plane waves leads to a pair of massive lumps. The fact that the two terms in Eq. (3.21) peak at the same plane $\eta_3 = \ln(K_1/K_2)$ is probably a feature of the simple Ansatz (3.3).

Everybody familiar with Hirota's method would probably have noticed that terms of the form $e^{\eta_i+\eta_j}$ are absent in Eq. (3.3). The equations of motion (2.20) do not admit them through the Ansatz (3.3). They will appear in the solutions of the next section.

4. Multi-Soliton Ansatz and Soliton Scattering

The Ansätze found in Sect. 3 are not the more general self-dual fields with ingoing multi-plane wave asymptotics in Minkowski space-time. The first possible generalization is a fractional Ansatz

$$\phi = \frac{F}{\Delta} \; , \qquad \varrho = \frac{N}{\Delta} \; , \qquad \bar{\varrho} = \frac{\bar{N}}{\Delta} \; , \tag{4.1}$$

where Δ, F, N, and $\bar{N}$ are polynomials in e^{η_i} $(1 \leqq i \leqq n)$ of the form

$$\mathbb{C}^{(0)} + \sum_{i=1}^{n} C_i^{(1)} e^{\eta_i} + \sum_{1 \leqq i < j \leqq n} \mathbb{C}_{ij}^{(2)} e^{\eta_i+\eta_j} + \ldots + \mathbb{C}^{(n)} e^{\eta_1+\eta_2+\ldots+\eta_n} \; . \tag{4.2}$$

The coefficients $\mathbb{C}_{i_1\ldots i_k}^{(k)}$ $(0 \leqq k \leqq n)$ are constants to be determined by imposing the field equations (2.20). The Ansatz (4.1) is clearly inspired by the structure of the known solutions in other integrable equations [4–6].

Equations (2.20) and (4.1) yields the following set of homogeneous equations for Δ, F, N and $\bar{N}$:

$$\Delta F[\Delta\partial^2 F - F\partial^2\Delta - \Delta^2(\partial F)^2 + \Delta^2\langle\partial N|\partial\bar{N}\rangle]$$
$$-\Delta[N\langle\partial\Delta|\partial\bar{N}\rangle + \bar{N}\langle\partial N|\partial\Delta\rangle] + (F^2 + N\bar{N})(\partial\Delta)^2 = 0 \; , \tag{4.3}$$

$$F(\Delta\partial^2 N - N\partial^2\Delta) + 2F[\langle\partial N|\partial\Delta\rangle - \partial N \cdot \partial\Delta] + 2N\langle\partial\Delta|\partial F\rangle - 2\Delta\langle\partial N|\partial F\rangle = 0 \; , \tag{4.4}$$

$$F(\Delta\partial^2\bar{N} - \bar{N}\partial^2\Delta) - 2F[\langle\partial\bar{N}|\partial\Delta\rangle - \partial\bar{N} \cdot \partial\Delta] + 2\bar{N}\langle\partial F|\partial\Delta\rangle - 2\Delta\langle\partial F|\partial\bar{N}\rangle = 0 \; . \tag{4.5}$$

Here we use the scalar product $\langle a|b\rangle$ [Eq. (2.23)]. It must be noticed that

$$\langle a|a\rangle = a\cdot a\ . \tag{4.6}$$

Let us first restrict to the two-soliton case ($n=2$). That is

$$\begin{aligned}
\Delta &= \Delta_0 + \Delta_1 e^{\eta_1} + \Delta_2 e^{\eta_2} + \Delta_{12} e^{\eta_1+\eta_2}\ ,\\
F &= F_0 + F_1 e^{\eta_1} + F_2 e^{\eta_2} + F_{12} e^{\eta_1+\eta_2}\ ,\\
N &= N_0 + N_1 e^{\eta_1} + N_2 e^{\eta_2} + N_{12} e^{\eta_1+\eta_2}\ ,\\
\bar{N} &= \bar{N}_0 + \bar{N}_1 e^{\eta_1} + \bar{N}_2 e^{\eta_2} + \bar{N}_{12} e^{\eta_1+\eta_2}\ .
\end{aligned} \tag{4.7}$$

Here $\eta_i = x\cdot k_i$ and $k_1^2 = k_2^2 = 0$ as in Sect. 3. Inserting Eqs. (4.7) in Eqs. (4.3)–(4.5) and equating to zero the coefficient of each independent exponential: e^{η_1}, e^{η_2}, $e^{\eta_1+\eta_2}$, $e^{2\eta_1+\eta_2}$, etc., leads to the set of 15 algebraic equations listed in the Appendix after remarkable simplifications. Since they contain 16 unknowns [the constant coefficients in Eq. (4.7)], a non-trivial solution may exist as we explicitly show below.

Before describing the explicit solutions, let us study the asymptotic behaviour of the Ansatz (4.7) and the invariants associated to it.

When η_1 or η_2 tends to $\pm\infty$ keeping η_2 or η_1 fixed respectively, we find

$$\phi = \frac{f_0 + f e^{\eta}}{\delta_0 + \delta e^{\eta}}\ ,\qquad \varrho = \frac{n_0 + n e^{\eta}}{\delta_0 + \delta e^{\eta}}\ ,\qquad \bar{\varrho} = \frac{\bar{n}_0 + \bar{n} e^{\eta}}{\delta_0 + \delta e^{\eta}}\ . \tag{4.8}$$

Here $\eta = k\cdot x$ stands for η_2 or η_1 respectively and $\delta_0, \delta, f_0, f, \eta_0, \eta, \bar{\eta}_0$, and $\bar{\eta}$ are constants.

In order to analyze these fields, it is useful to look at the invariant physical quantities associated to them. One could consider the energy-momentum tensors of the Yang-Mills theory since any self-dual field is a solution of the Yang-Mills equations. For the gauge group $SL(2,\mathbb{C})$, T_μ^ν depends on a free parameter [11]. It is easy to show that

$$T_{\mu\nu} = \mathrm{Tr}\left(F_{\alpha\mu}F_\nu^\alpha - \frac{\eta_{\mu\nu}}{4}\, F_{\alpha\beta}F^{\alpha\beta}\right) \tag{4.9}$$

identically vanishes using the self-dual Minkowski equations

$$F_{\mu\nu} = \frac{i}{2}\,\varepsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}\ . \tag{4.10}$$

It should be recalled that the energy density $T_{00}(x)$ is not positive definite for a non-compact gauge group as $SL(2,\mathbb{C})$.

Let us consider the gauge invariants $\det(F_{\mu\bar{\nu}})$. One could also use the conserved asymmetric energy momentum tensor (2.24), but the behaviour of all these quantities are qualitatively similar. We obtain from Eqs. (2.26) and (4.8),

$$\begin{aligned}
\det(F_{\mu\bar{\nu}}) = &-\frac{(k_\mu k_{\bar{\nu}})^2\, e^{2\eta}}{(f_0 + f e^{\eta})^4\, (\delta_0 + \delta e^{\eta})^4}\ \{(\delta_0 f - f_0\delta)^2\\
&\cdot(\delta_0 f_0 - e^{2\eta}\delta f)^2 + (\delta_0 n - n_0\delta)(\delta_0\bar{n} - \delta\bar{n}_0)\,[\delta_0^2 f_0^2\\
&+ (\delta f_0 - f\delta_0)(\delta f_0 - f\delta_0 + 2\delta f e^{\eta})e^{2\eta}] + (\delta_0 n - \delta n_0)\\
&\cdot(\delta_0\bar{n} - \delta\bar{n}_0)^2\, e^{2\eta}\}\ .
\end{aligned} \tag{4.12}$$

Since e^{η} varies fast with η, $\det(F_{\mu\bar{v}})$ is peaked around its stationary points. There are up to six stationary points, although some of them may be complex. One would naturally call a soliton a field having only one real maximum. Therefore, a solution with an asymptotic behaviour (4.8), in general, describes several ingoing "one-solitons" propagating collinearly. It is natural to look for particular cases of (4.8) with only one peak. Let us first choose $\phi(x)$ to be asymptotically constant for $\eta_1 \to \pm\infty$ and $\eta_2 \to \pm\infty$. It follows in that case from Eq. (4.1) and (4.7) that

$$\phi(x) = \frac{F_0}{\Delta_0} = \frac{F_1}{\Delta_1} = \frac{F_2}{\Delta_2} = \frac{F_{12}}{\Delta_{12}} , \tag{4.13}$$

and hence $\phi(x)$ is everywhere constant. Now the field equations (4.3)–(4.5) yield

$$\Delta_{12} = \frac{\Delta_1 \Delta_2}{\Delta_0} , \qquad N_{12} = \frac{N_1 \Delta_2 + N_2 \Delta_1}{\Delta_0} - N_0 \frac{\Delta_1 \Delta_2}{\Delta_0^2} , \tag{4.14}$$

and a similar equation for $\bar{N}_{12}$. We find in addition the quartic constraint

$$\langle k_1 | k_2 \rangle (\Delta_0^2 N_1 \bar{N}_2 - \Delta_0 \Delta_1 N_0 \bar{N}_2 - \Delta_0 \Delta_2 \bar{N}_0 N_1 + \Delta_1 \Delta_2 N_0 \bar{N}_0) + (1 \leftrightarrow 2) = 0 . \tag{4.15}$$

We have found therefore a self-dual solution depending on six free parameters between the nine coefficients Δ_0, Δ_1, Δ_2, N_0, N_1, N_2, $\bar{N}_0$, $\bar{N}_1$ and $\bar{N}_2$. [They must fulfill Eq. (4.15), and they are defined up to two general constant factors, see Eq. (3.16.]

Let us analyze the asymptotic behaviour of this solution. For $\eta_1 \to \pm\infty$, $\eta_2 =$ fixed, ϱ and $\bar{\varrho}$ take the form (4.8). More precisely

$$\begin{aligned} \varrho_+(\eta) &= \varrho_-(\eta) + \frac{N_1}{\Delta_1} - \frac{N_0}{\Delta_0} , \\ \bar{\varrho}_+(\eta) &= \bar{\varrho}_-(\eta) + \frac{\bar{N}_1}{\Delta_1} - \frac{\bar{N}_0}{\Delta_0} , \end{aligned} \tag{4.16}$$

where

$$\varrho_\pm(\eta_2) = \lim_{\eta_1 \to \pm\infty} \varrho(\eta_1, \eta_2) ,$$

and similarly for $\bar{\varrho}_\pm(\eta)$. Analogous results follow for $\eta_2 \to \pm\infty$ and fixed η_1. In conclusion, we have two non-linear plane waves both in the initial $(t \to -\infty)$ and final $(t \to +\infty)$ state. Equations (4.16) show that there is no phase-shift due to the interaction in this case. The effect of the interaction is a constant gauge rotation of the field

$$A_\mu^+(\eta) = S A_\mu^-(\eta) S^{-1} , \tag{4.17}$$

where

$$A_\mu^+(\eta_2) = \lim_{\eta_1 \to \pm\infty} A_\mu(\eta_1, \eta_2) ,$$

and S follows from Eq. (4.16):

$$S = \begin{pmatrix} 1 & \bar{N}_1/\Delta_1 - \bar{N}_0/\Delta_0 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{C}) . \tag{4.18}$$

It is possible to obtain fractional Ansatz solutions [Eq. (4.1)–(4.2)] by applying a suitable Bäcklund transformation to the N-soliton polynomial solution (3.3)–

(3.14). Let us apply the Bäcklund transformation (γ) of [8] to the $N=2$ solution (3.7). We find an expression like (4.1) where

$$\begin{aligned} \Delta &= V(c_1, c_2, \delta_1, \delta_2) \ , \\ N &= V(a_1, c_2, \beta_1, \delta_2) \ , \\ \bar{N} &= -V(c_1, a_2, \delta_1, \beta_2) \ , \\ F &= F_0 + F_1 e^{\eta_1} + F_2 e^{\eta_2} \ . \end{aligned} \tag{4.19}$$

Here

$$\begin{aligned} V(\alpha, \beta, \gamma, \delta) &= \gamma\delta - \alpha\beta F_0^2 + F_1 e^{\eta_1}\left(A\alpha\delta - 2F_0\alpha\beta + \frac{\beta\gamma}{A}\right) \\ &\quad + F_2 e^{\eta_2}\left(S\alpha\delta + \frac{\beta\gamma}{S} - 2F_0\alpha\beta\right) \\ &\quad + \frac{4e^{\eta_1+\eta_2}}{\Lambda^2 - 1} F_1 F_2 \alpha\beta \ , \qquad \Lambda = \frac{\langle k_1 | k_2 \rangle}{k_1 \cdot k_2} - 1 \ , \qquad S = A\,\frac{\langle k_1 | k_2 \rangle}{\langle k_2 | k_1 \rangle} \ . \end{aligned} \tag{4.20}$$

The coefficients

$$A, a_1, a_2, c_1, c_2, \beta_1, \beta_2, \delta_1, \delta_2, F_0, F_1, \quad \text{and} \quad F_2 \tag{4.21}$$

are only restricted by the constraints

$$a_1\delta_1 - c_1\beta_1 = 1 = a_2\delta_2 - c_2\beta_2 \ . \tag{4.22}$$

Therefore, the solution (4.19)–(4.22) depends actually on eight free parameters besides k_1 and k_2. Asymptotically, it behaves like Eq. (4.8) setting $f=0$. As discussed below Eq. (4.12), it is simpler to consider particular cases of the solution (4.19)–(4.22) in order to analyze their "soliton" content. Let us for example assume

$$S\delta_2 = c_2 F_0 \quad \text{and} \quad \delta_1 = c_1 F_0 S \ . \tag{4.23}$$

The asymptotic behaviour of the fields in this case (see the Table 1) leads us to interpret the solution as describing the collision of a type A soliton with a type $\bar{A}$ soliton which produces two types of solutions in the final state ($t \to +\infty$). The solitons Type A, $\bar{A}$ and B are particular cases of Eq. (4.8) as defined in the Table 1. Therefore, Eq. (4.19) describes inelastic processes: transformations of solitons. It must be noticed that $\det F_{\mu\bar{\nu}}$ vanishes for solitons of type A, $\bar{A}$ and B as well as it vanishes for the one-soliton described in Sect. 3.

A full and systematic exploration of the two-soliton and N-soliton solutions within Ansätze (4.1)–(4.7) is beyond the scope of the present paper.

We also want to remark that all the Ansätze presented in this paper are specific of a space-time with Lorentzian signature. For Euclidean signature all solutions become just constants due to the requirement $k_i^2=0$ [Eq. (3.2)].

Single non-linear plane wave solutions are known in Yang-Mills theory [12]. If one imposes the self-duality condition in a $SU(2)$ gauge theory, they become dependent on one arbitrary matrix function of $k \cdot x = x^+$ instead of two. Our multiplane wave solutions describe the scattering of such single non-linear plane waves, provided they have a rational form like Eq. (4.8).

Appendix

Inserting the fractional Ansatz (4.1) for $N=2$ in Eqs. (2.20) yields the following fifteen algebraic equations in the sixteen unknowns $[\Delta_A, F_A, N_A, N_{\bar{A}}, A=0,1,2,(12)]$.

$$F_0(N_{12}\Delta_0-\Delta_{12}N_0)+F_2(N_0\Delta_1-\Delta_0N_1)+F_1(N_0\Delta_2-\Delta_0N_2)+\Lambda[F_0(N_1\Delta_2-N_2\Delta_1)$$
$$+F_2(N_0\Delta_1-\Delta_0N_1)+F_1(\Delta_0N_2-N_0\Delta_2)]=0\ , \tag{A.1}$$

$$(1+\Lambda)(N_0\Delta_1-N_1\Delta_0)(F_0F_{12}-\Lambda F_1F_2)+(1-\Lambda)[F_0^2(N_{12}\Delta_1-N_1\Delta_{12})$$
$$+\Lambda F_1^2(\Delta_0N_2-\Delta_2N_0)+(1+\Lambda)F_0F_1(N_1\Delta_2-N_2\Delta_1)]=0\ . \tag{A.2}$$

Equation (A.3) follows by exchanging $(1\leftrightarrow 2)$ and $(\Lambda\to-\Lambda)$ in Eq. (A.2). Equations (A.4)–(A.6) are obtained by exchanging $N_A\to\bar{N}_A$ and $\Lambda\to-\Lambda$ $(A=0,1,2,(12))$ in Eqs. (A.1)–(A.3),

$$F_{12}^2(\Delta_1\Delta_2-\Delta_0\Delta_{12})+\Delta_{12}^2(F_0F_{12}-F_1F_2+N_1\bar{N}_2)+\Delta_1\Delta_2N_{12}\bar{N}_{12}$$
$$-\Delta_1\Delta_{12}(N_{12}\bar{N}_2+\bar{N}_{12}N_2)$$
$$+\Lambda\Delta_{12}[\Delta_{12}\bar{N}_1N_2+\Delta_1(\bar{N}_2N_{12}-\bar{N}_{12}N_2)]+(1\leftrightarrow 2,\Lambda\to-\Lambda)=0\ , \tag{A.7}$$

$$2\Delta_1\Delta_2(2F_0F_{12}+N_1\bar{N}_2)+N_0\bar{N}_0\Delta_{12}^2+N_{12}\bar{N}_{12}\Delta_0^2$$
$$-\Delta_0\Delta_{12}(4F_1F_{12}+N_0\bar{N}_{12}+\bar{N}_0N_{12})-2\Delta_1^2N_2\bar{N}_2$$
$$+2\Lambda[\Delta_{12}\Delta_1(\bar{N}_0N_2-N_0\bar{N}_2)+\Delta_0\Delta_1(\bar{N}_2N_{12}-N_2\bar{N}_{12})]$$
$$+(1\leftrightarrow 2,\Lambda\to-\Lambda)=0\ , \tag{A.8}$$

$$\Delta_0^2(F_0F_{12}-F_1F_2+N_1\bar{N}_2)-F_0^2(\Delta_0\Delta_{12}-\Delta_1\Delta_2)+\Delta_1\Delta_2N_0\bar{N}_0$$
$$-\Delta_0\Delta_1(N_0\bar{N}_2+\bar{N}_0N_2)+\Delta_0\Lambda[\Delta_0N_1\bar{N}_2-\Delta_1(N_0\bar{N}_2-\bar{N}_0N_2)]$$
$$+(1\leftrightarrow 2,\Lambda\to-\Lambda)=0\ , \tag{A.9}$$

$$2F_{12}\Delta_{12}(F_0\Delta_1-F_1\Delta_0)+2F_1\Delta_1(F_{12}\Delta_2-F_2\Delta_{12})+\Delta_{12}N_1(\Delta_1\bar{N}_2-\Delta_2\bar{N}_1)$$
$$+\Delta_{12}N_0(\Delta_{12}\bar{N}_1-\Delta_1\bar{N}_{12})+(\Delta_1\Delta_2-\Delta_0\Delta_{12})N_1\bar{N}_{12}$$
$$+\Delta_0\Delta_1N_{12}\bar{N}_{12}-\Delta_1^2N_2\bar{N}_{12}+\Lambda[\Delta_{12}\bar{N}_0(\Delta_1N_{12}-\Delta_{12}N_1)$$
$$+N_1\bar{N}_{12}(\Delta_0\Delta_{12}+\Delta_1\Delta_2)+\Delta_1N_2(\Delta_{12}\bar{N}_1-\Delta_1\bar{N}_{12})]$$
$$+(N_A\leftrightarrow\bar{N}_A,\Lambda\to-\Lambda)=0\ , \tag{A.10}$$

$$\Delta_1^2(F_0F_{12}-F_1F_2)+F_1^2(\Delta_1\Delta_2-\Delta_0\Delta_{12})$$
$$+(\Delta_0N_1-\Delta_1N_0)(\Delta_1\bar{N}_{12}-\Delta_{12}\bar{N}_1)$$
$$+\Lambda\Delta_1[\bar{N}_0(\Delta_1N_{12}-\Delta_{12}N_1)-\Delta_0N_{12}\bar{N}_1]$$
$$+(N_A\leftrightarrow\bar{N}_A,\Lambda\to-\Lambda)=0\ , \tag{A.11}$$

$$
\begin{aligned}
2\Delta_0\Delta_1(F_{12}F_0-F_1F_2)&+(\Delta_1\Delta_2-\Delta_0\Delta_{12})(2F_0F_1+N_0\bar{N}_1)\\
&+(\Delta_0N_1-\Delta_1N_0)(\Delta_1\bar{N}_2+\Delta_0\bar{N}_{12})+\Delta_2(\Delta_1N_0\bar{N}_0-\Delta_0N_1\bar{N}_1)\\
&+\Lambda[(\Delta_0\bar{N}_{12}+\Delta_1\bar{N}_2)(\Delta_0N_1-\Delta_1N_0)+N_0\bar{N}_1(\Delta_0\Delta_{12}+\Delta_1\Delta_2)]\\
&+(N_A\leftrightarrow\bar{N}_A,\Lambda\to-\Lambda)=0\ . \qquad (A.12)
\end{aligned}
$$

Equations (A.13)–(A.15) follow by exchanging $(1\leftrightarrow 2)$ and $(\Lambda\to-\Lambda)$ in Eqs. (A.10)–(A.12).

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