Commun. Math. Phys. 115, 607–629 (1988)

Communications in
Mathematical
Physics

On the Linearized Relativistic Boltzmann Equation

I. Existence of Solutions

Marek Dudyński[1] and Maria L. Ekiel-Jeżewska[2]

[1] Institute for Fundamental Technological Research, Polish Academy of Sciences, ul. Swietokrzyska 21, PL-00-049 Warszawa, Poland
[2] Institute for Theoretical Physics, Polish Academy of Sciences, Al. Lotnikow 32/46, PL-02-668 Warszawa, Poland

Abstract. The linearized relativistic Boltzmann equation in $L^2(\mathbf{r}, \mathbf{p})$ is investigated. The detailed analysis of the collision operator L is carried out for a wide class of scattering cross sections. L is proved to have a form of the multiplication operator $\nu(\mathbf{p})$ plus the compact in $L^2(\mathbf{p})$ perturbation K. The collisional frequency $\nu(\mathbf{p})$ is analysed to discriminate between relativistic soft and hard interactions. Finally, the existence and uniqueness of the solution to the linearized relativistic Boltzmann equation is proved.

1. Introduction

In the standard approach to the relativistic kinetic theory one assumes the Boltzmann equation as an evolution equation for the one-particle distribution function [1, 2]. One of the main mathematical problems one faces in that approach is to prove, under physically reasonable assumptions, existence of a unique solution of the Cauchy problem. Such a proof depends crucially on a specific choice of a function space one uses to describe a physical system. Several interesting results concerning solutions of the Boltzmann equation in general relativity have already been obtained [3–5]. For functions of a compact support bounded by $\exp[-\beta_\alpha(x)p^\alpha]$ Bichteler [3] proved local existence of a solution to the Boltzmann equation under the assumption that the total scattering cross section is finite. A similar result, but in Sobolev spaces and with additional assumptions on a form of cross section, was obtained by Bancel [4]. The Sobolev spaces are also appropriate for analysis of the coupled Boltzmann and Einstein-Maxwell equations. The Cauchy problem for such a system of equations has been solved by Bancel and Choquet-Bruhat [5].

It is still not known whether solutions in those spaces allow for any hydrodynamical approximation. On the other hand, the detailed analysis of the nonrelativistic Boltzmann equation, including rigorous justification of the hydrodynamic approximation, has been given by Grad [6, 7], Ellis and Pinsky [8], Nishida [9], and Kawashima et al. [10] for a different space of functions corresponding to systems close to a global equilibrium. Our aim is to perform a similar analysis for the relativistic Boltzmann equation.

In this paper we shall prove that with some quite weak bounds on a possible form of scattering cross section there exists a nonincreasing in norm, global in time, unique solution to the linearized relativistic Boltzmann equation in a flat space-time.

Physical relevance of our results is discussed in [27].

The paper is organized as follows.

In Sect. 2 preliminaries on the system and the linearized relativistic Boltzmann equation are given. For a suitable regular cross section the collisional operator L is decomposed as $L=-\nu(\mathbf{p})+K$, where $\nu(\mathbf{p})$ is a function of $\mathbf{p}$ called collision frequency.

In Sect. 3 the compactness of the K operator in $L^2(\mathbf{p})$ is proved.

In Sect. 4 the dependence of the collision frequency $\nu(\mathbf{p})$ on the scattering cross section is analysed and the notion of relativistic hard and soft interactions is introduced.

In Sect. 5 the existence and uniqueness of the solution to the linearized relativistic Boltzmann equation is proved.

2. The Linearized Relativistic Boltzmann Equation

We consider a one-component classical relativistic gas of particles with rest mass $m\neq 0$ in the flat space-time and in the absence of all external forces. We assume that the system is close to a global equilibrium and that in order to determine its state it suffices to know the one-particle distribution function.

It is convenient to introduce dimensionless variables x^μ and p^μ, which are defined by relations: $x^\mu:=y^\mu/c\tau$ and $p^\mu:=q^\mu c/kT$, where y^μ and q^μ represent the usual, dimensional four-vectors of position and momentum respectively. The dimensionless mass is $M=mc^2/kT$. It is convenient to interpret T as the temperature in the global equilibrium state: τ is a time scale to be specified.

We write down formulas in a covariant manner, using an arbitrarily chosen frame of reference. In this frame we decompose x^μ and p^μ as: $x^\mu=(t,\mathbf{r})$ and $p^\mu=(p_0,\mathbf{p})$. The signature is $(+\,-\,-\,-)$.

The evolution of the distribution function $F(\mathbf{r},\mathbf{p},t)$ is governed by the relativistic Boltzmann equation [1, 2]:

$$\frac{\partial F}{\partial t}+\frac{\mathbf{p}}{p_0}\cdot\frac{\partial F}{\partial \mathbf{r}}=\frac{1}{2}\int d^3p_1 d\Omega g\frac{s^{1/2}}{p_{10}p_0}\sigma(g,\theta)\,[F'F'_1-FF_1]\,, \tag{2.1}$$

where

$s^{1/2}:=|p_1+p|$ is the total energy;

$2g:=|p_1-p|$ is the value of the relative momentum;

$\cos\theta:=1-2(p_\mu-p_{1\mu})(p^\mu-p'^\mu)(4M^2-s)^{-1}$ defines the angle of scattering; $d\Omega=\sin\theta\, d\theta\, d\varphi$;

$\sigma(g,\theta)$ is the differential scattering cross section. All the above variables refer to the centre of mass (c.m.) frame.

Since the system is close to the global equilibrium, its distribution function $F(\mathbf{r},\mathbf{p},t)$ can be written as:

$$F(\mathbf{r},\mathbf{p},t)=f_0(\mathbf{p})+[f_0(\mathbf{p})]^{1/2}f(\mathbf{r},\mathbf{p},t)\,, \tag{2.2}$$

where

$$f \ll f_0^{1/2}, \tag{2.3}$$

and the relativistic equilibrium distribution function $f_0(\mathbf{p})$ has the Jüttner form [11]:

$$f_0(\mathbf{p}) = \frac{1}{4\pi M^2 K_2(M)} \exp(-U^\mu p_\mu), \tag{2.4}$$

where U^μ has the meaning of a dimensionless hydrodynamic velocity; $K_2(M)$ is the Bessel function of the second kind of index two [12].

Now linearizing Eq. (2.1) we find that the evolution of f is governed by the linearized relativistic Boltzmann equation (LRBE) [13, 14]:

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{p_0} \cdot \frac{\partial f}{\partial \mathbf{r}} = \frac{f_0^{1/2}}{2p_0} \int d^3 p_1 d\Omega \frac{g s^{1/2}}{p_{10}} \sigma(g, \theta) f_{10} \times \left[\frac{f_1'}{(f_{10}')^{1/2}} + \frac{f'}{(f_0')^{1/2}} - \frac{f_1}{(f_{10})^{1/2}} - \frac{f}{(f_0)^{1/2}} \right]. \tag{2.5}$$

Equation (2.5) is our basic equation. We shall consider this equation in the Hilbert space $L^2(\mathbf{r}, \mathbf{p})$ of real valued functions with the scalar product given by

$$(f|g) := \int d^3 r\, d^3 p\, fg. \tag{2.6}$$

Functions from this space have a simple physical interpretation as describing systems with a finite entropy[1].

To analyse properties of the LRBE (2.5) we need bounds on a form of the scattering cross section $\sigma(g, \theta)$. Specifically, we assume that there exist constants α, β, γ, B, B' such that σ obeys:

$$\sigma(g, \theta) \leqq (B g^\beta + B' g^{-\alpha}) \sin^\gamma \theta, \tag{2.7}$$

where $\gamma > -2$, $0 < \alpha < \min(4, 4+\gamma)$ and $0 \leqq \beta < \gamma + 2$ (see Fig. 1).

This restriction is justified on physical grounds, as will be discussed in a separate work [27].

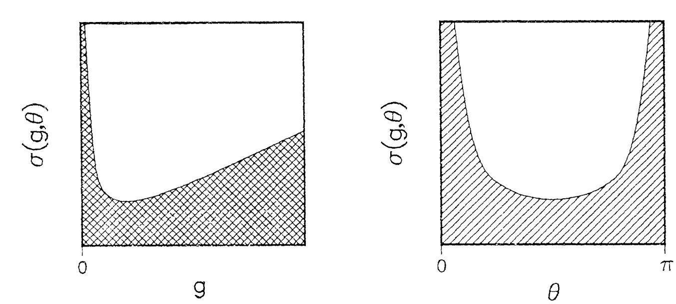

Fig. 1. The momentum g and angle θ dependence of the scattering cross section σ [see Eq. (2.7)]. All σ lying in the shaded region are admissible

[1] The entropy for the linearized Boltzmann equation is to be understood as $\int d^3 p d^3 r f^2$ [17]

The condition (2.7) is our main assumption. It represents a non-trivial generalization of the restriction used in the mathematical analysis of the non-relativistic Boltzmann equation [6, 16].

Under the assumption (2.7) the LRBE (2.5) may be rewritten in a form similar to that known from non-relativistic physics [6, 15, 16][2]:

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{p_0}\cdot\frac{\partial f}{\partial \mathbf{r}}=-\nu(\mathbf{p})f+K[f] \tag{2.8}$$

with

$$K=K_2-K_1, \tag{2.9}$$

$$K_i[f(\mathbf{r},\mathbf{p},t)]=\int d^3p_1 k_i(\mathbf{p},\mathbf{p}_1)f(\mathbf{r},\mathbf{p}_1,t), \tag{2.10}$$

$$\nu(\mathbf{p})=\int d^3p_1 k_1(\mathbf{p},\mathbf{p}_1)\exp[(\tau-\tau_1)/2], \tag{2.11}$$

where $\tau:=U^\mu p_\mu$, $\tau_1:=U^\mu p_{1\mu}$.

The integral kernels k_i have the following form:

$$k_1(\mathbf{p},\mathbf{p}_1)=\frac{1}{2M^2K_2(M)}\frac{gs^{1/2}}{p_0p_{10}}\exp[-(\tau+\tau_1)/2]\int_0^\pi d\theta\sin\theta\,\sigma(g,\theta), \tag{2.12}$$

$$k_2(\mathbf{p},\mathbf{p}_1)=\frac{1}{8M^2K_2(M)}\frac{s^{3/2}}{gp_0p_{10}}\int_0^\infty dx\exp[-(1+x^2)^{1/2}(\tau+\tau_1)/2]$$
$$\times\sigma\left[\frac{g}{\sin(\psi/2)},\psi\right]x\frac{1+(1+x^2)^{1/2}}{(1+x^2)^{1/2}}I_0\left[\frac{|\mathbf{p}\wedge\mathbf{p}_1|}{2g}x\right], \tag{2.13}$$

where I_0 is the Bessel function of purely imaginary argument of index zero [12] and ψ is connected with x by the relation:

$$\sin(\psi/2)=2^{1/2}g[g^2-M^2+(g^2+M^2)(1+x^2)^{1/2}]^{-1/2}. \tag{2.14}$$

$\mathbf{p}\wedge\mathbf{p}_1$ is a vector product of $\mathbf{p}$ and $\mathbf{p}_1$ calculated in the rest frame of our gas [in this frame $U^\mu=(1,0,0,0)$], so the explicit expression for $|\mathbf{p}\wedge\mathbf{p}_1|$ has a form:

$$|\mathbf{p}\wedge\mathbf{p}_1|=[4g^2(\tau\tau_1-g^2-M^2)-M^2(\tau-\tau_1)^2]^{1/2}. \tag{2.15}$$

Using results known from the non-linear relativistic Boltzmann equation [1, 2] one checks easily that the collisional operator $L=-\nu+K$ is symmetric and non-positive definite with respect to the scalar product (2.6).

Moreover, $L[\varphi(\mathbf{p})]=0$ iff $\varphi\in N_0$, where N_0 is the subspace spanned by $\{f_0^{1/2}(\mathbf{p}), p_i f_0^{1/2}(\mathbf{p}), p_0 f_0^{1/2}(\mathbf{p}); i=1,2,3\}$. In addition, integral kernels $k_i(\mathbf{p},\mathbf{p}_1)$ are symmetric functions of both arguments.

In order to analyse properties of the LRBE (2.8) we follow the non-relativistic theory [6, 8, 9, 16] and examine first the structure of the K operator and the behaviour of the collision frequency ν (see Sects. 3 and 4).

[2] We emphasize that (2.8) is a different equation than that following from the relativistic Chapman-Enskog approximation and also called the linearized Boltzmann equation by some authors [18]

3. Properties of the K Operator

The main result of this section is the proof of the following theorem:

Theorem 3.1. *Under the assumption* (2.7) *the K operator is compact in* $L^2(\mathbf{p})$.

In the proof we shall use the following lemma, proved by Drange [15]:

Lemma 3.1. *Assume that:*

1) $\int k_i(\mathbf{p}, \mathbf{p}_1) d^3 p_1$ *is bounded in* $\mathbf{p}$ *for* $i=1,2$,
2) $k_i \in L^2(\Omega_n)$ *for* $n=1,2,3,\ldots$, *where* $\Omega_n = A_n \cap B_n$ *and*

$$A_n = \{(\mathbf{p}, \mathbf{p}_1) : |\mathbf{p}-\mathbf{p}_1| \geqq 1/n\}, \tag{3.1}$$

$$B_n = \{(\mathbf{p}, \mathbf{p}_1) : |\mathbf{p}| \leqq n\}, \tag{3.2}$$

3)
$$\lim_{n\to\infty} \sup_{\mathbf{p}\in R^3} \int d^3 p_1 k_i(\mathbf{p}, \mathbf{p}_1) X_n = 0, \tag{3.3}$$

where $i=1,2$ *and* X_n *is the characteristic function of the set* $R^6 - \Omega_n$.
Then the K operator is compact in $L^2(\mathbf{p})$.

To use Lemma 3.1 it suffices to check whether the conditions 1)–3) are satisfied by the integral kernels (2.12) and (2.13).

We begin with point 1) and we prove:

Lemma 3.2. *Under the assumption* (2.7) *the following estimate holds:*

$$\sup_{\mathbf{p}\in R^3} \int d^3 p_1 k_i(\mathbf{p}, \mathbf{p}_1) < \infty, \qquad i=1,2. \tag{3.4}$$

In the proof of the Lemma 3.2 and Theorem 3.1 we will frequently make use of simple algebraic estimates, which are listed and proved in Appendix A.

Proof of Lemma 3.2. For $i=1$ and $i=2$ estimations are made separately; in both cases we first find a bound on $k_i(\mathbf{p}, \mathbf{p}_1)$ as a function of $\mathbf{p}$ and $\mathbf{v} := \mathbf{p}_1 - \mathbf{p}$; next we integrate it over $\mathbf{v}$ to show (3.4).

1) $i=1$. From (2.12) we have:

$$k_1(\mathbf{p}, \mathbf{p}_1) = \frac{g s^{1/2}}{2M^2 K_2(M) p_0 p_{10}} \exp(-d) \int_0^\pi d\theta \sin\theta\, \sigma(g, \theta), \tag{3.5}$$

where $d := (\tau + \tau_1)/2$. From (2.7), making use of inequalities A.2 and A.8 from Appendix A, we obtain:

$$k_1(\mathbf{p}, \mathbf{p}_1) \leqq \frac{(v^2+M^2)^{1/2}}{M^4 K_2(M)} \exp[-(p_0+v)/12] \int_0^\pi d\theta (\sin\theta)^{\gamma+1} (B g^{\beta+1} + B' g^{1-\alpha}), \tag{3.6}$$

where $v = |\mathbf{v}|$.

Using inequalities (A.2) and (A.13) it is easy to show that:

$$k_1(\mathbf{p}, \mathbf{p}_1) \leqq [h_1'(v) + h_1''(v)(1 + p_0^{\alpha-1})] \exp(-p_0/12), \tag{3.7}$$

where

$$h_1'(v) := \frac{2B}{M^4 K_2(M)} \left[\frac{\gamma+4}{\gamma+2}\right] v^{\beta+1} (v^2+M^2)^{1/2} \exp(-v/12), \tag{3.8}$$

$$h_1''(v) := \frac{2B'}{M^4 K_2(M)} \left[\frac{\gamma+4}{\gamma+2}\right] (v^2+M^2)^{1/2} \exp(-v/12) h_1(v), \tag{3.9}$$

and

$$h_1(v):=\begin{bmatrix} v^{1-\alpha} & \text{for} \quad \alpha\leqq 1 \\ [(v+2M)/M^2]^{\alpha-1}v^{1-\alpha} & \text{for} \quad \alpha\geqq 1 \end{bmatrix}. \tag{3.10}$$

Since $\alpha<4$ [see (2.7)], then h_1' and h_1'' are integrable over $\mathbf{v}$ and from (3.10) the final estimate is obtained:

$$\begin{aligned}\sup_{\mathbf{p}\in R^3}\int d^3p_1k_1(\mathbf{p},\mathbf{p}_1)&\leqq\int d^3vh_1'(v)\sup_{\mathbf{p}\in R^3}\exp(-p_0/12)\\ &\quad+\int d^3vh_1''(v)\sup_{\mathbf{p}\in R^3}[\exp(-p_0/12)(1+p_0^{\alpha-1})]\\ &\leqq\int d^3vh_1'(v)+6^6\int d^3vh_1''(v)<\infty\,.\end{aligned} \tag{3.11}$$

Thus for $i=1$ Lemma 3.2 is proved.

2) $i=2$. From (2.13) we have:

$$\begin{aligned}k_2(\mathbf{p},\mathbf{p}_1)=&\frac{1}{8M^2K_2(M)}\frac{s^{3/2}}{gp_0p_{10}}\int_0^\infty dx\exp[-d(1+x^2)^{1/2}]I_0(ax)\\ &\times\sigma\left[\frac{g}{\sin(\psi/2)},\psi\right]\left[x\frac{1+(1+x^2)^{1/2}}{(1+x^2)^{1/2}}\right],\end{aligned} \tag{3.12}$$

where $a:=|\mathbf{p}\wedge\mathbf{p}_1|/(2g)$. From inequality (A.12) we obtain:

$$\begin{aligned}k_2(\mathbf{p},\mathbf{p}_1)\leqq&\frac{2^{\gamma+\zeta+1}}{M^2K_2(M)}\left[\frac{g^2+M^2}{g^2}\right]^{(\zeta+1)/2}[B(g^2+M^2)^{\beta/2}+B'M^{-\alpha}]\\ &\times\frac{g^2+M^2}{p_0p_{10}}\int_0^\infty dx\exp[-d(1+x^2)^{1/2}]I_0(ax)\left[\frac{(1+x^2)^{1/2}}{x}\right]^\zeta x(1+x^2)^{\chi/2},\end{aligned} \tag{3.13}$$

where

$$\zeta:=\max[-\gamma,(\alpha-\gamma)/2,\alpha/2],\qquad \chi:=\max[0,-(\alpha+\gamma)/2,(\beta-\gamma)/2].$$

To estimate the integral in (3.13) we use the following proposition:

Proposition 3.1. *Denote by* J:

$$J:=\int_0^\infty dx\exp[-d(1+x^2)^{1/2}]I_0(ax)x\left[\frac{(1+x^2)^{1/2}}{x}\right]^\xi(1+x^2)^{\eta/2}.$$

Then

$$\begin{aligned}J\leqq 3^{\xi+\eta}&\left[(2-\xi)^{-1}\exp(-d/4)+2(8/M^2)^\eta(1+2/M)\frac{(6g/v)^2}{(M^2+g^2)}d^{\eta+1}\right]\\ &\times\exp(-v/24),\end{aligned} \tag{3.14}$$

where $0\leqq\eta<1$ *and* $0\leqq\xi<2$.

Proof is given in Appendix B.

Since $0\leqq\chi<1$ and $0<\zeta<2$, then from Proposition 3.1 we get the inequality:

$$\begin{aligned}k_2(\mathbf{p},\mathbf{p}_1)\leqq&\frac{2^{\gamma+\zeta+1}}{M^2K_2(M)}\left[\frac{g^2+M^2}{g^2}\right]^{(\zeta+1)/2}[B(g^2+M^2)^{\beta/2}+B'M^{-\alpha}]\\ &\times 3^{\zeta+\chi}\left[\frac{g^2+M^2}{M^2(2-\zeta)}\exp(-d/4)+2(8/M^2)^\chi(1+2/M)(6g/v)^2\frac{d^{\chi+1}}{p_0p_{10}}\right]\\ &\times\exp(-v/24).\end{aligned} \tag{3.15}$$

Applying inequalities (A.2), (A.4), (A.8), (A.9), we obtain the final estimate:

$$k_2(\mathbf{p}_1, \mathbf{p}_1) \leqq \frac{3^{\gamma+6}}{M^2 K_2(M)} (v^2+M^2)^{(\zeta+1)/2} [B(v^2+M^2)^{\beta/2} + B'M^{-\alpha}] \exp(-v/24)$$
$$\times \left[\frac{v^2+M^2}{M^2(2-\zeta)} \left[\frac{2M+v}{M^2}\right]^{1+\zeta} v^{-(1+\zeta)} p_0^{1+\zeta} \exp(-d/4) \right.$$
$$\left. + 72(8/M^2)^{\chi}(1+2/M) v^{-2} g^{1-\zeta} (p_0 p_{10})^{(\chi-1)/2} \left[\frac{d^2}{p_0 p_{10}}\right]^{(\chi+1)/2} \right]$$
$$\leqq h_2'(v) p_0^{1+\zeta} \exp(-p_0/24) + h_2''(v,y) p_0^{(\chi-1)/2}, \tag{3.16}$$

where $y := \mathbf{v} \cdot \mathbf{p}/(vp)$ and

$$h_2'(v) := \frac{3^{\gamma+6}}{M^2 K_2(M)} (v^2+M^2)^{(\zeta+1)/2} [B(v^2+M^2)^{\beta/2} + B'M^{-\alpha}]$$
$$\times \exp(-v/24) \frac{v^2+M^2}{M^2(2-\zeta)} \left[\frac{2M+v}{M^2}\right]^{1+\zeta} v^{-(1+\zeta)}, \tag{3.17}$$

$$h_2''(v,y) := \frac{72 \cdot 3^{\gamma+6}}{M^2 K_2(M)} U_0^{1+\chi} (8/M^2)^{\chi} (1+2/M) M^{(\chi-1)/2}$$
$$\times \exp(-v/24)(4+v/M)^{(1+\chi)/2} [B(v^2+M^2)^{\beta/2} + B'M^{-\alpha}]$$
$$\times (v^2+M^2)^{(\zeta+1)/2} h_2(v,y), \tag{3.18}$$

and

$$h_2(v,y) := \begin{cases} v^{-(1+\zeta)}(1-|y|)^{-(1+\chi)/4} & \text{for} \quad \zeta \leqq 1 \\ \left[\dfrac{2M+v}{M}\right]^{\zeta-1} v^{-(\zeta+1)} (1-|y|)^{-(2\zeta-1+\chi)/4} & \text{for} \quad \zeta \geqq 1 . \end{cases} \tag{3.19}$$

Notice that $(1+\chi)/4 < 1/2$, and that for $\zeta \geqq 1 : (2\zeta - 1 + \chi)/4 < 1$, so h_2 is integrable over y. It is easy now to show that functions h_2' and h_2'' are integrable over $\mathbf{v}$, and we have:

$$\sup_{\mathbf{p} \in R^3} \int d^3p_1 k_2(\mathbf{p}, \mathbf{p}_1) \leqq \sup_{\mathbf{p} \in R^3} [\exp(-p_0/24) p_0^{1+\zeta}] \int d^3v h_2'(v)$$
$$+ \sup_{\mathbf{p} \in R^3} [p_0^{(\chi-1)/2})] \int d^3v h_2''(v,y)$$
$$\leqq 9^6 \int d^3v h_2'(v) + M^{(\chi-1)/2} \int d^3v h_2''(v,y) < \infty . \tag{3.20}$$

Equation (3.20) completes the proof of our Lemma 3.2.

As a result of Lemma 3.2 and the Young lemma [19], we find:

Corollary 3.1. *The K operator is bounded in $L^2(\mathbf{r}, \mathbf{p})$.*

Lemma 3.3. *Let us assume that $\Omega_n = A_n \cap B_n$, where*

$$A_n = \{(\mathbf{p}, \mathbf{p}_1) : |\mathbf{p} - \mathbf{p}_1| \geqq 1/n\}, \tag{3.1}$$

$$B_n = \{(\mathbf{p}, \mathbf{p}_1) : |\mathbf{p}| \leqq n\}. \tag{3.2}$$

Then $k_i \in L^2(\Omega_n)$ $\forall n = 1, 2, 3, \ldots$.

Proof. It is sufficient to show that:

$$\int_{\Omega_n} d^3p_1 d^3p k_i^2(\mathbf{p},\mathbf{p}_1) \leqq \infty\,. \tag{3.21}$$

From (3.1), (3.2), (3.11), and (3.20), we obtain:

$$\begin{aligned}\int_{\Omega_n} d^3p_1 d^3p k_i^2(\mathbf{p},\mathbf{p}_1) &= \int_{v \geqq 1/n} d^3v \int_{p \leqq n} d^3p k_i^2(\mathbf{p},\mathbf{p}_1)\\ &\leqq 16\pi^2 9^{12} n^3 \int_0^1 dy \int_{1/n}^{\infty} dv\, v^2 \Bigg[h_i' + h_i'' \Bigg\{ \delta_{i1} + \delta_{i2} M^{(\chi-1)/2}\\ &\times \left[(1-|y|) \frac{n^2+M^2}{M^2} \right]^{(\zeta-1)/2} \Bigg\} \Bigg]^2 .\end{aligned} \tag{3.22}$$

Using (3.10), (3.16) and (3.17), (3.18), (3.19) it is easy to show that both integrals in (3.22) converge, which ends the proof of Lemma 3.3.

Lemma 3.4.

$$\lim_{n\to\infty} \sup_{\mathbf{p}\in R^3} \int d^3p_1 k_i(\mathbf{p},\mathbf{p}_1) X_n = 0\,, \tag{3.3}$$

where $i = 1, 2$ *and* X_n *is the characteristic function of the set* $R^6 - \Omega_n$.

Proof. We will use the relation:

$$\sup_{\mathbf{p}\in R^3} \int d^3p_1 k_i(\mathbf{p},\mathbf{p}_1) X_n \leqq \sup_{\mathbf{p}\in R^3} \int_{v \leqq 1/n} d^3v k_i(\mathbf{p},\mathbf{p}_1) + \sup_{p \geqq n} \int d^3v k_i(\mathbf{p},\mathbf{p}_1)\,. \tag{3.23}$$

From (3.7) and (3.16) it follows that:

$$\begin{aligned}k_i(\mathbf{p},\mathbf{p}_1) \leqq{}& h_i'(v)\,(1+p_0^{\zeta+1})\exp(-p_0/24) + h_i''(v,y)\\ &\times [(1+p_0^{\alpha-1})\exp(-p_0/24) + p_0^{(\chi-1)/2}]\,.\end{aligned} \tag{3.24}$$

For $p \geqq n$, (3.24) gives:

$$k_i(\mathbf{p},\mathbf{p}_1) < 4^{12}[h_i'(v) + h_i''(v,y)]\exp(-n/48) + h_i''(v,y)(1/n)^{(1-\chi)/2}\,. \tag{3.25}$$

Since h_i' and h_i'' are integrable over $\mathbf{v}$, then we obtain:

$$\lim_{n\to\infty} \sup_{p \geqq n} \int d^3v k_i(\mathbf{p},\mathbf{p}_1) = 0\,. \tag{3.26}$$

To evaluate

$$\lim_{n\to\infty} \sup_{\mathbf{p}\in R^3} \int_{v \leqq 1/n} d^3v k_i(\mathbf{p},\mathbf{p}_1)\,,$$

we first estimate $k_i(\mathbf{p},\mathbf{p}_1)$ using (3.24):

$$k_i(\mathbf{p},\mathbf{p}_1) \leqq (4^{12} + M^{(\chi-1)/2})[h_i'(v) + h_i''(v,y)]\,, \tag{3.27}$$

and next examine the behaviour of h_i' and h_i'' for $v \leqq 1/n$. From (3.7), (3.16) and (3.17), (3.18), (3.19) it is easy to show that:

$$h_1'(v) + h_1''(v,y) \leqq C_1 + C_1^* v^{1-\alpha}\,, \tag{3.28}$$

$$\begin{aligned}h_2'(v) + h_2''(v,y) \leqq{}& C_2 v^{-(1+\zeta)} + C_2^* v^{-(1+\zeta)}\\ &\times [(1-|y|)^{-(\chi+1)/4} + (1-|y|)^{-(2\zeta-1+\chi)/4}]\,,\end{aligned} \tag{3.29}$$

where

$$C_1 := 2[M^4 K_2(M)]^{-1} [(\gamma+4)/(\gamma+2)] (1+M^2)^{1/2} (B+B'), \tag{3.30}$$

$$C_1^* := 2[M^4 K_2(M)]^{-1} \frac{\gamma+4}{\gamma+2} (1+M^2)^{1/2} B' \left[\frac{1+2M}{M^2}\right]^{\alpha-1}, \tag{3.31}$$

$$C_2 := \frac{3^{\gamma+6}}{M^2 K_2(M)} (2-\zeta)^{-1} (1+M^2)^{(\zeta+1)/2} [B(1+M^2)^{\beta/2} + B' M^{-\alpha}] \times \frac{v^2+M^2}{M^2} \left[\frac{2M+v}{M^2}\right]^{1+\zeta}, \tag{3.32}$$

$$C_2^* := \frac{3^{\gamma+10}}{M^2 K_2(M)} U_0^{1+\chi} (1+M^2)^{(\zeta+1)/2} [B(1+M^2)^{\beta/2} + B' M^{-\alpha}] \times (8/M^2)^{\chi} (1+2/M) M^{(\alpha-1)/2} (4+1/M)^{(3+\chi)/2}. \tag{3.33}$$

Now, using (3.28) and (3.29), we estimate the integrals:

$$J_i := \int_{v \leqq 1/n} d^3 v [h_i'(v) + h_i''(v, y)], \tag{3.34}$$

with results:

$$J_1 \leqq 4\pi[(C_1/3)(1/n)^3 + C_1^*(4-\alpha)^{-1}(1/n)^{4-\alpha}], \tag{3.35}$$

$$J_2 \leqq 4\pi[[C_2/(2-\zeta)] (1/n)^{2-\zeta} + 2[C_2^*/(2-\zeta)] (1/n)^{2-\zeta} \times (1+2/(5-2\zeta-\chi))]. \tag{3.36}$$

Thus

$$\lim_{n\to\infty} J_i = 0, \tag{3.37}$$

and from (3.37) we obtain

$$\lim_{n\to\infty} \sup_{\mathbf{p}\in R^3} \int_{v \leqq 1/n} d^3 v k_i(\mathbf{p}, \mathbf{p}_1) \leqq \lim_{n\to\infty} [(4^{12} + M^{(\alpha-1)/2}) J_i] = 0. \tag{3.38}$$

Equations (3.26) and (3.38) end the proof of Lemma 3.4.
Thus the Theorem 3.1 is proved.

4. Relativistic Hard and Soft Interactions

The general results obtained in the previous section are valid for the whole class of cross sections defined by (2.7). However, as is well known from the nonrelativistic kinetic theory, more refined mathematical analysis of the Boltzmann equation seems to be sensitive to further assumptions on a form of function σ [8, 21].

Thus, following the nonrelativistic Grad's procedure [6], we shall introduce in this section two classes of functions $\sigma(g, \theta)$, corresponding to the so-called hard and soft interactions. The distinction between both of them is due to a different collision frequency behaviour: for hard interactions the collision frequency (2.11) satisfies the relation:

$$\nu(\mathbf{p}) \geqq \nu_0 > 0, \tag{4.1}$$

while for soft interactions we have:

$$\nu(\mathbf{p}) \leqq \nu_0 \quad \text{and} \quad \nu(\mathbf{p}) \to 0 \quad \text{for} \quad |\mathbf{p}| \to \infty. \tag{4.2}$$

Hard interactions are especially interesting not only because of some physical motivations, but also from a mathematical point of view. Namely, for this class of functions $\sigma(g,\theta)$ a more refined spectral analysis of the operator $L+(\mathbf{p}/p_0)\nabla$ can be performed by means of a perturbation expansion. In contrast to the nonrelativistic operator $\mathbf{p}\nabla$, the relativistic streaming operator $(\mathbf{p}/p_0)\nabla$ is bounded in $L^2(\mathbf{p})$, since $|\mathbf{p}/p_0| \leqq 1$. Thus for hard interactions analytical perturbation theory can be applied to calculate explicitly an asymptotic form of the relativistic distribution function for long time and small space gradients. Details of this calculation will be given in a forthcoming paper.

In the non-relativistic physics Grad [6] has defined as the *hard interactions* those for which the cross section obeys:

$$\sigma(g,\theta) > B\frac{g^{\varepsilon}}{1+g}, \tag{4.3}$$

and as the *soft interactions* these with:

$$\sigma(g,\theta) < B' g^{\varepsilon-2}, \tag{4.4}$$

where $0<\varepsilon<1$ (see Fig. 2).

Grad has shown [6] that the dependence of the collision frequency ν on momentum is qualitatively different for both cases. For hard interactions ν is bounded from below:

$$\nu(\mathbf{p}) > \nu_0, \tag{4.5}$$

while for soft interactions ν is bounded from above:

$$\nu(\mathbf{p}) < \nu'(\mathbf{p}) \leqq \nu_0, \tag{4.6}$$

where ν_0 is a positive constant and $\nu'(\mathbf{p})\to 0$ for $|\mathbf{p}|\to\infty$.

The aim of this section is to examine the behaviour of the collision frequency $\nu(\mathbf{p})$ in the relativistic case and to establish the meaning of relativistic hard and soft interactions.

The main result of this section is contained in Theorem 4.1:

Theorem 4.1A. *Let us assume that* $\exists \gamma > -2$, $0 \leqq \beta < \gamma+2$,

$$\sigma(g,\theta) > B\frac{g^{\beta+1}}{c_0+g}\sin^{\gamma}\theta. \tag{4.7}$$

Then the collisional frequency obeys:

$$\nu(\mathbf{p}) > \nu_0\left[\frac{p_0}{M}\right]^{\beta/2} \geqq \nu_0, \tag{4.8}$$

where c_0 and ν_0 are constants.

Theorem 4.1B. *Let us assume that* $\exists 0<\alpha<4$, $\gamma>-2$*:*

$$\sigma(g,\theta) < B' g^{-\alpha}\sin^{\gamma}\theta. \tag{4.9}$$

Then:

$$\nu(\mathbf{p}) < \nu_0\left[\frac{p_0}{M}\right]^{-\varepsilon/2} \leqq \nu_0, \tag{4.10}$$

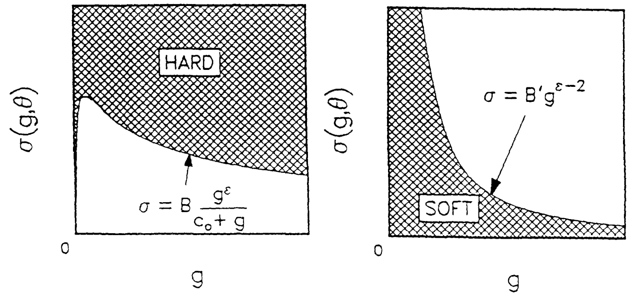

Fig. 2. The non-relativistic momentum dependence of the scattering cross section σ. All functions σ lying in the shaded regions correspond to the hard and soft interactions respectively. In general $0<\varepsilon<1$; on the figure $\varepsilon=1/2$

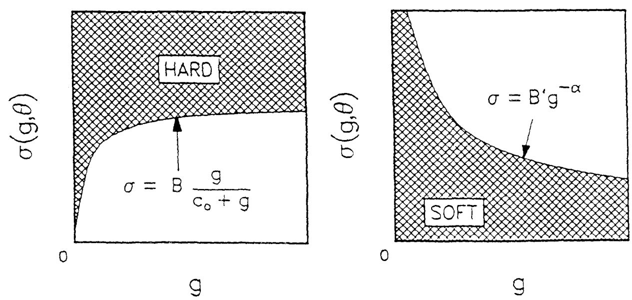

Fig. 3. The relativistic momentum dependence of the scattering cross section σ. All functions σ lying in the shaded regions correspond to the hard and soft interactions respectively. In general $0<\alpha<4$; on the figure $\alpha=1/2$

where

$$\varepsilon=\begin{cases}\alpha & \text{for} \quad 0<\alpha<3, \\ \alpha-2 & \text{for} \quad 3<\alpha<4, \\ \delta+1, & \text{where} \quad 0<\delta<1, \quad \text{for} \quad \alpha=3.\end{cases} \tag{4.11}$$

Comparing (4.7) and (4.9) with (4.3) and (4.4) we notice that the relativistic collision frequency depends on a form of the cross section in a different way than it is in the non-relativistic case (see Fig. 3).

Since Grad's distinction between hard and soft interactions is based on different properties of the collision frequency for corresponding types of interactions, in the relativistic theory the meaning of hard and soft interactions should be redefined by using the relations (4.7) and (4.9)[3]. Roughly speaking, *in the relativistic theory interactions are softer than in the non-relativistic one.*

---

[1] Note that conditions (4.7) and (4.9) can be formulated in a more general way, i.e.:

$\exists B_0,\ \varepsilon_0>0,\ \gamma>-2$ such that:

$$\sigma(g,\theta)>\inf_{x\in[0,\varepsilon_0]} B_0\frac{g^x}{c_0+g^x}\sin^\gamma\theta\,, \tag{4.7a}$$

$\exists B_0>0,\ 0<\varepsilon_0<\varepsilon_1<4,\ \gamma>-2$ such that:

$$\sigma(g,\theta)<\sup_{x\in[\varepsilon_0,\varepsilon_1]} B_0 g^{-x}\sin^\gamma\theta\,. \tag{4.9a}$$

Although by adopting conditions (4.7a) and (4.9a) instead of (4.7) and (4.9) the class of soft and hard interactions is enlarged, but still the distinction is not exclusive. Thus in this work we shall use less general, but simpler definitions (4.7) and (4.9)

Proof of Theorem 4.1A. From (2.11) and (2.12) we have:

$$\nu(\mathbf{p}) = [2M^2K_2(M)]^{-1} \int d^3p_1 \frac{gs^{1/2}}{p_0 p_{10}} \exp(-\tau_1) \int_0^{\pi} d\theta \sin\theta\, \sigma(g,\theta). \tag{4.12}$$

Using assumption (4.7) and inequalities (A.13), (A.14) we obtain:

$$\begin{aligned} \nu(\mathbf{p}) &> \frac{4B}{|\gamma|+2} \frac{1}{2M^2K_2(M)} \int d^3p_1 \frac{p_1}{p_0^{1/2}p_{10}^{3/2}} \exp(-\tau_1) \frac{g^{\beta+2}}{c_0+g} \\ &> \frac{8\pi B \exp(-2MU_0)}{2(|\gamma|+2)M^2K_2(M)} \int_0^{\infty} dp_1 \frac{\exp(-2U_0p_1)}{p_0^{1/2}p_{10}^{3/2}} p_1^3 \int_{-1}^{1} dz \frac{g^{\beta+2}}{c_0+g}, \end{aligned} \tag{4.13}$$

where $z := \mathbf{p} \cdot \mathbf{p}_1/(pp_1)$. The integral over z may be easily estimated:

$$\begin{aligned} \int_{-1}^{1} dz \frac{g^{\beta+2}}{c_0+g} &\geqq 2^{-(\beta+2)/2} \int_{-1}^{0} dz \frac{(p_0p_{10} - pp_1z - M^2)^{(\beta+2)/2}}{c_0 + p_0^{1/2}p_{10}^{1/2}} \\ &\geqq 2^{-(\beta+2)/2} \frac{(p_{10}-M)^{(\beta+2)/2}}{p_{10}^{1/2} + c_0M^{-1/2}} p_0^{(\beta+1)/2}. \end{aligned} \tag{4.14}$$

Thus from (4.13) and (4.14) we obtain:

$$\nu(\mathbf{p}) > \nu_0' \left[\frac{p_0}{M}\right]^{\beta/2}, \tag{4.15}$$

where

$$\begin{aligned} \nu_0' := &\frac{2^{-(\beta-4)/2}\pi B}{|\gamma|+2} M^{\beta/2}[2M^2K_2(M)]^{-1} \exp(-2MU_0) \\ &\times \int_0^{\infty} dp_1 \frac{(p_{10}-M)^{(\beta+2)/2} \exp(-2U_0p_1)p_1^3}{(p_{10}^{1/2} + c_0M^{-1/2})p_{10}^{3/2}}. \end{aligned} \tag{4.16}$$

Thus Theorem 4.1A is proved.

Proof of Theorem 4.1B. Using assumption (4.9) and inequalities (A.13), (A.14) we obtain from (4.12) the estimation:

$$\begin{aligned} \nu(\mathbf{p}) &< \frac{\gamma+4}{\gamma+2} [M^2K_2(M)]^{-1} B' \int d^3p_1 \frac{g^{1-\alpha}s^{1/2}}{p_0p_{10}} \exp(-p_{10}/3) \\ &\leqq 2\pi 2^{(\alpha+1)/2} \frac{\gamma+4}{\gamma+2} [M^2K_2(M)]^{-1} B' p_0^{-1/2} \\ &\times \int_0^{\infty} dp_1 \left[\frac{p_1^2 \exp(-p_{10}/3)}{p_{10}^{1/2}} \int_{-1}^{1} dz (p_0p_{10} - pp_1z - M^2)^{(1-\alpha)/2}\right]. \end{aligned} \tag{4.17}$$

The last integral may be estimated using the following proposition:

Proposition 4.1. *Let us denote:*

$$J_\eta := \int_{-1}^{1} dz (a_1 - a_2 z)^\eta, \tag{4.18}$$

where $\eta \in R$, $a_1 = p_0 p_{10} - M^2$, $a_2 = pp_1$. *Then*

i) *for* $-1 < \eta < 1/2$:

$$J_\eta \leqq 4(\eta+1)^{-1}(a_1+a_2)^\eta \leqq 16(\eta+1)^{-1}[p_{10}^\eta + (p_1^2/p_{10})^\eta]p_0^\eta, \tag{4.19}$$

and

ii) *for* $-3/2 < \eta < -1$:

$$\begin{aligned} J_\eta &\leqq 2(-\eta-1)^{-1}(a_1+a_2)^{-1}(a_1-a_2)^{1+\eta} \\ &\leqq 8(-\eta-1)^{-1} M^{2(1+\eta)} p_{10}^{-\eta} p_1^{-2} |p-p_1|^{2(1+\eta)} p_0^{-2-\eta}. \end{aligned} \tag{4.20}$$

Proof is given in Appendix C.

Using now Proposition 4.1 with $\eta = (1-\alpha)/2$, we obtain from (4.17) the following estimates:

i) For $0 < \alpha < 3$:

$$\nu(\mathbf{p}) < \nu_1 p_0^{-\alpha/2}, \tag{4.21}$$

where

$$\begin{aligned} \nu_1 = &\frac{2^{(\alpha+13)/2}\pi \cdot B'}{M^2 K_2(M)(3-\alpha)} \frac{\gamma+4}{\gamma+2} \int_0^\infty dp_1 \frac{\exp(-p_{10}/3)}{p_{10}^{1/2}} \\ &\times p_1^2[p_{10}^{(1-\alpha)/2} + (p_1^2/p_{10})^{(1-\alpha)/2}]. \end{aligned} \tag{4.22}$$

ii) For $3 < \alpha < 4$:

$$\begin{aligned} \nu(\mathbf{p}) < &\frac{2^{(\alpha+11)/2}\pi \cdot B'}{M^2 K_2(M)(\alpha-3)} \frac{\gamma+4}{\gamma+2} M^{3-\alpha} \int_0^\infty dp_1 \frac{\exp(-p_{10}/3)}{p_{10}^{1/2}} \\ &\times p_{10}^{(\alpha-1)/2} |p-p_1|^{3-\alpha} p_0^{(\alpha-6)/2}. \end{aligned} \tag{4.23}$$

The last integral is estimated using the relation:

$$\exp(-p_{10}/6)p_{10}^{(\alpha-2)/2} < 6. \tag{4.24}$$

Thus we have:

$$\begin{aligned} &\int_0^\infty dp_1 \frac{\exp(-p_{10}/3)}{p_{10}^{1/2}} p_{10}^{(\alpha-1)/2} |p-p_1|^{3-\alpha} < 6 \int_0^\infty dp_1 |p-p_1|^{3-\alpha} \exp(-p_{10}/6) \\ &< 6 \int_0^p dp_1 |p-p_1|^{3-\alpha} + 6 \int_p^{p+M} dp_1 |p-p_1|^{3-\alpha} + 6M^{3-\alpha} \int_{p+M}^\infty dp_1 \exp(-p_{10}/6) \\ &= 6\left[2M^{3-\alpha}\exp[-(p+M)/6] + \frac{M^{4-\alpha}+p^{4-\alpha}}{(4-\alpha)}\right] \\ &< 12[(4-\alpha)^{-1} + \exp(-M/6)/M] p_0^{4-\alpha}. \end{aligned} \tag{4.25}$$

Then from (4.23) and (4.25) we obtain:

$$\nu(\mathbf{p}) < \nu_2 p_0^{(\alpha-6)/2}, \tag{4.26}$$

where

$$\nu_2 = 12 \frac{2^{(\alpha+11)/2}\pi \cdot B'}{M^2 K_2(M)(\alpha-3)} \frac{\gamma+4}{\gamma+2} M^{3-\alpha}[(4-\alpha)^{-1} + \exp(-M/6)/M]. \tag{4.27}$$

iii) For $\alpha=3$ we use the relation:

$$g^3 \leqq \max[g^{-(3+\varepsilon)}, g^{-(1+\varepsilon)}] \tag{4.28}$$

with $0<\varepsilon<1$, to apply the results of i) and ii). We obtain:

$$\nu(\mathbf{p}) < \nu_3 p_0^{-(1+\varepsilon)/2}, \tag{4.29}$$

where

$$\nu_3 = \max(\nu_1, \nu_2). \tag{4.30}$$

Equation (4.30) ends the proof of Theorem 4.1.

From Theorems 3.1 and 4.1 it follows that the collisional operator L is self-adjoint and non-positive in $L^2(\mathbf{p})$. Moreover, for soft interactions L is a bounded operator with essential spectrum $\sigma_{ess}(L)=[-\nu_{max}, 0]$; for hard interactions L is unbounded and $\sigma_{ess}(L)=[-\infty, -\nu_{min}]$. Location of $\sigma_{ess}(L)$ is a direct consequence of the Schechter theorem on the behaviour of the essential spectrum under compact perturbation [22].

5. Existence of the Solution to the Linearized Relativistic Boltzmann Equation

We show in this section that for σ fulfilling our main assumption (2.7) the initial value problem for the Eq. (2.5) has a weak solution in $L^2(\mathbf{r}, \mathbf{p})$, which is unique and global in time. Boundedness of the K operator in $L^2(\mathbf{r}, \mathbf{p})$ and non-negativity of the function $\nu(\mathbf{p})$ allow us to construct for the relativistic equation the semigroup representation of the solution, analogous to that given by Ukai [23], Ellis and Pinsky [8], Nishida and Imai [24], and Nishida [9] for the non-relativistic Boltzmann equation. We will follow the notation of Nishida [9] to simplify comparison of our results and non-relativistic ones.

Let $H^1(\mathbf{r})$, $l \geqq 0$, denote the Sobolev space of these $L^2(\mathbf{r})$ functions, derivatives of which, up to and including order l, belong to $L^2(\mathbf{r})$. We denote as $\hat{H}^l(\mathbf{k})$ the Fourier transform of $H^l(\mathbf{r})$ with a norm:

$$\|f(\mathbf{k})\|_{\hat{H}^l(\mathbf{k})} = \|(1+k^2)^{1/2}\hat{f}(\mathbf{k})\|_{L^2(\mathbf{k})} \equiv \|f(\mathbf{r})\|_{H^l(\mathbf{r})}. \tag{5.1}$$

We denote by H the $L^2(\mathbf{r}, \mathbf{p})$ space. Let us introduce the partial Fourier transform in $\mathbf{r}$ of $f \in H$ as follows:

$$\hat{\mathrm{f}}(\mathbf{k}, \mathbf{p}) = (2\pi)^{-3/2} \int d^3r \exp[-i\mathbf{k}\mathbf{r}] f(\mathbf{r}, \mathbf{p}) \tag{5.2}$$

and denote $\hat{H} = \{\hat{f} : f \in H\}$. For $\hat{f} \in \hat{H}$ we define:

$$\|\hat{f}\|^2 = \int d^3k\, d^3p |\hat{f}(\mathbf{k}, \mathbf{p})|^2 = \|f\|^2. \tag{5.3}$$

Finally we put $H_0 = H$ and for $l>0$ we define H_l as the Hilbert subspace of H consisting of all $H^l(\mathbf{r})$-valued L^2-functions of $\mathbf{p}$ with the norm:

$$\|f\|_l = [\int d^3p \|f(\cdot, p)\|^2_{H^l(\mathbf{r})}]^{1/2} = [\iint d^3k\, d^3p (1+k^2)^{1/2} |f(\mathbf{k}, \mathbf{p})|^2]^{1/2}. \tag{5.4}$$

The LRBE (2.5) can be rewritten as follows:

$$\partial_t f(\mathbf{r}, \mathbf{p}, t) = Bf(\mathbf{r}, \mathbf{p}, t), \tag{5.5}$$

where
$$B = L - \frac{\mathbf{p}}{p_0}\nabla . \tag{5.6}$$

For $f \in H_l$, $l \geqq 0$, we introduce a Fourier transformed equation:
$$\partial_t \hat{f}(\mathbf{k}, \mathbf{p}, t) = B_{\mathbf{k}} \hat{f}(\mathbf{k}, \mathbf{p}, t), \tag{5.7}$$
where the operator $B_{\mathbf{k}}$ is defined as follows:
$$B_{\mathbf{k}} = -\frac{i\mathbf{kp}}{p_0} + L . \tag{5.8}$$

For each $\mathbf{k} \in R^3$ $B_{\mathbf{k}}$ is in general an unbounded operator on $L^2(\mathbf{p})$ with a domain:
$$D(B_{\mathbf{k}}) = \{ f \in L^2(\mathbf{p}) : \nu(\mathbf{p}) f(\mathbf{p}) \in L^2(\mathbf{p}) \} . \tag{5.9}$$

An important property of $B_{\mathbf{k}}$ is given by:

Theorem 5.1. *For each* $\mathbf{k} \in R^3$ *the operator* $B_{\mathbf{k}}$ *generates a strongly continuous contraction semigroup on* $L^2(\mathbf{p})$ *such that for any* $t \geqq 0$ *and* $f \in L^2(\mathbf{p})$ *the following relation holds:*
$$\| \exp(tB_{\mathbf{k}}) f(\mathbf{p}) \|_{L^2(\mathbf{p})} \leqq \| f(\mathbf{p}) \|_{L^2(\mathbf{p})} . \tag{5.10}$$

The operators $B_{\mathbf{k}}$ can then be used to construct an explicit representation of the semigroup $\exp(tB)f$, for $f \in H_l(\mathbf{r}, \mathbf{p})$. It is provided by:

Theorem 5.2. *The operator B generates a strongly continuous contraction semigroup on* $H_l(\mathbf{r}, \mathbf{p})$, $l \geqq 0$, *given explicitly as:*
$$\exp(tB) f(\mathbf{r}, \mathbf{p}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \exp(i\mathbf{kr}) \exp(tB_{\mathbf{k}}) \hat{f}(\mathbf{k}, \mathbf{p}), \tag{5.11}$$

where for $t \geqq 0$ *and for every* $f(\mathbf{r}, \mathbf{p}) \in H_l$, $l \geqq 0$, *the following inequality holds:*
$$\| \exp(tB) f(\mathbf{r}, \mathbf{p}) \|_l \leqq \| f(\mathbf{r}, \mathbf{p}) \|_l . \tag{5.12}$$

Proofs of the Theorems 5.1 and 5.2 can be easily deduced from their non-relativistic counterparts due to Nishida [9] and Ellis and Pinsky [8].

It is now easy to see that $f(\mathbf{r}, \mathbf{p}, t) = \exp(tB) f(\mathbf{r}, \mathbf{p})$ is a solution to the LRBE with the initial value $f(\mathbf{r}, \mathbf{p})$. Thus we have:

Corollary 5.1. *For every initial data* $f(\mathbf{r}, \mathbf{p}) \in H_l$ *with* $l \geqq 0$ *there exists globally in time a unique, nonincreasing in* H_l *norm, solution to* LRBE *represented by Eq.* (5.11).

This is only a weak solution to the Boltzmann equation because we cannot guarantee differentiability with respect to the $\mathbf{r}$ variable. For initial data from H_l with $l > 5/2$, by virtue of the Sobolev embedding theorem, this solution is also a classical one [25].

In our proof of the causality of the LRBE [26] we have used a different representation of the solution (5.11). Global validity of that representation for cross-sections fulfilling condition (2.7) is provided by the following proposition:

Proposition 5.1. *For cross-sections fulfilling condition* (2.7) *the solution to the* LRBE *with arbitrary initial data* $f(\mathbf{r}, \mathbf{p}) \in L^2(\mathbf{r}, \mathbf{p})$ *can be represented by a norm converging* *series of the form:*

$$f(\mathbf{r},\mathbf{p},t)=\sum_0^\infty f^{(n)}(\mathbf{r},\mathbf{p},t), \tag{5.13}$$

where

$$f^{(0)}(\mathbf{r},\mathbf{p},t)=\exp[-\nu(\mathbf{p})t]f(\mathbf{r}-t\mathbf{p}/p_0,\mathbf{p}), \tag{5.14}$$

$$f^{(n)}(\mathbf{r},\mathbf{p},t)=\int_0^t ds\exp[(t-s)\nu(\mathbf{p})]q^{(n-1)}(\mathbf{r}-(t-s)\mathbf{p}/p_0,\mathbf{p},s), \tag{5.15}$$

$$q^{(n)}(\mathbf{r},\mathbf{p},t)=K[f^{(n)}(\mathbf{r},\mathbf{p},t)]. \tag{5.16}$$

Proof. In order to obtain Eq. (5.13) we first transform the linearized Boltzmann equation into the following integral equation:

$$f(\mathbf{r},\mathbf{p},t)=\exp(tA)f(\mathbf{r},\mathbf{p})+\int_0^t ds\exp[(t-s)A]K[f(\mathbf{r},\mathbf{p},s)]. \tag{5.17}$$

In Eq. (5.17) we have introduced an auxiliary operator A of the following form:

$$A=\frac{\mathbf{p}}{p_0}\cdot\nabla-\nu(\mathbf{p}), \tag{5.18}$$

and for $g(\mathbf{r},\mathbf{p})\in L^2(\mathbf{r},\mathbf{p})$ we have:

$$\exp(tA)g(\mathbf{r},\mathbf{p})=\exp[-\nu(\mathbf{p})t]g(\mathbf{r}-t\mathbf{p}/p_0,\mathbf{p}). \tag{5.19}$$

Iterating Eq. (5.17) we obtain a formal solution in the form given by Eq. (5.13). Estimating this series term by term in the $L^2(\mathbf{r},\mathbf{p})$ norm one easily sees its convergence with the following upper bound:

$$\|f(\mathbf{r},\mathbf{p},t)\|_{L^2(\mathbf{r},\mathbf{p})}\leqq\exp(\|K\|t)\,\|f(\mathbf{r},\mathbf{p})\|_{L^2(\mathbf{r},\mathbf{p})}, \tag{5.20}$$

with $\|K\|$ being the norm of the operator K. Applying now Theorem 5.2 we see that in fact estimate (5.20) can be improved and the following inequality holds:

$$\|f(\mathbf{r},\mathbf{p},t)\|_{L^2(\mathbf{r},\mathbf{p})}\leqq\|f(\mathbf{r},\mathbf{p})\|_{L^2(\mathbf{r},\mathbf{p})}. \tag{5.21}$$

We can conclude then:

Corollary 5.2. *For cross-sections fulfilling condition* (2.7) *the solution to the* LRBE *with arbitrary initial data* $f(\mathbf{r},\mathbf{p})\in L^2(\mathbf{r},\mathbf{p})$ *is causal.*

For the proof and precise definition of causality see [26].

Appendix A

In this appendix inequalities used in the proofs of Theorems 3.1 and 4.1 are listed and proved.

Inequality A.1.

$$p_0(1-|y|)^{1/2}\leqq p_{10}\leqq p_0+v, \tag{A.1.1}$$

where

$$v=|\mathbf{v}|,\quad \mathbf{v}:=\mathbf{p}_1-\mathbf{p},\quad y:=\mathbf{v}\cdot\mathbf{p}/(vp). \tag{A.1.2}$$

Proof. From (A.1.2) we have

$$p_0^2(1-|y|) \leqq p_0^2(1-y^2) \leqq (py+v)^2 + p_0^2 - p^2y^2 = p_{10}^2$$
$$= p_0^2 + 2yvp + v^2 \leqq (p_0+v)^2 .$$

Inequality A.2.

$$\frac{M^2}{(v+2M)p_0} v \leqq g \leqq \frac{v}{2} . \tag{A.2.1}$$

Proof. It is easy to check that v^μ is a space-like vector and $v^\mu v_\mu = -4g^2$, which immediately gives the upper bound on g in (A.2.1). The lower bound in (A.2.1) is easily obtained from the following estimates:

$$g^2 = v^2(p_0+p_{10})^{-2}[p_0^2 - p^2y^2 + (p_0 p_{10} - p_0^2 - ypv)/2]$$
$$\geqq v^2(p_0+p_{10})^{-2}[p_0^2 - p^2y^2] \geqq v^2(v+2M)^{-2}(M/p_0)^2[p_0^2 - p^2y^2] \tag{A.2.2}$$

and:

$$v^2(v+2M)^{-2}(M/p_0)^2[p_0^2 - p^2y^2] \geqq v^2(v+2M)^{-2}M^2(M/p_0)^2 , \tag{A.2.3}$$

where to obtain (A.2.2) Inequality A.1 was used.

Inequality A.3.

$$\frac{M(1-|y|)^{1/2}}{(v+2M)} v \leqq g . \tag{A.3.1}$$

Proof. Since

$$v^2(v+2M)^{-2}(M/p_0)^2[p_0^2 - p^2y^2] \geqq v^2(v+2M)^{-2}M^2(1-|y|) ,$$

then from (A.2.2) we get (A.3.1).

Inequality A.4.

$$g^\lambda \leqq v^\lambda + \left[\frac{v+2M}{Mv(1-|y|)^{1/2}}\right]^{-\lambda} \quad \textit{for} \quad \lambda \in R . \tag{A.4.1}$$

Proof is a trivial consequence of (A.2.1) and (A.3.1).

Inequality A.5. *Let us denote:*

$$d := (\tau + \tau_1)/2 \quad \textit{and} \quad a := |\mathbf{p} \wedge \mathbf{p}_1|/(2g) . \tag{A.5.1}$$

Then we have:

$$d^2 - a^2 \geqq M^2/4 + v^2(\tfrac{3}{4}M^2 + g^2)/(36g^2) . \tag{A.5.2}$$

Proof. Equation (2.15) gives for $d^2 - a^2$ the expression:

$$d^2 - a^2 = M^2 + g^2 + (\tau - \tau_1)^2(4g^2)^{-1} . \tag{A.5.3}$$

In the following we use the relation:

$$(\tau - \tau_1)^2 + 4g^2 \geqq v^2(U_0 - U)^2 \geqq v^2/9 \tag{A.5.4}$$

to estimate (A.5.3):

$$d^2 - a^2 \geqq M^2/4 + [(\tau - \tau_1)^2 + 4g^2]\,(\tfrac{3}{4}M^2 + g^2)/(4g^2)$$
$$\geqq M^2/4 + v^2(\tfrac{3}{4}M^2 + g^2)/(36g^2) . \tag{A.5.5}$$

Thus (A.5.2) is proved.

Inequality A.6.

$$d^2 - a^2 \geqq M^2/4 + v^2/36. \tag{A.6.1}$$

Proof follows immediately from Inequality A.5.

Inequality A.7.

$$d^2 - a^2 \geqq M^2. \tag{A.7.1}$$

Proof is a trivial consequence of (A.5.3).

Inequality A.8.

$$d \geqq v/6; \qquad d \geqq p_0/6. \tag{A.8.1}$$

Proof follows from A.6 and the relation:

$$d \geqq \tau/2 \geqq (U^0 - U)p_0/2 \geqq p_0/6. \tag{A.8.2}$$

Inequality A.9.

$$d \leqq U_0(1-|y|)^{-1/4}(4+v/M)^{1/2}(p_0 p_{10})^{1/2}. \tag{A.9.1}$$

Proof. From Inequality A.1 it is easy to show that:

$$\begin{aligned} d^2 &\leqq U_0^2(p_0+p_{10})^2 \leqq U_0^2 p_0 p_{10}[(1-|y|)^{-1/2} + v/p_0 + 3] \\ &\leqq U_0^2(1-|y|)^{-1/2}(4+v/M)p_0 p_{10}. \end{aligned} \tag{A.9.2}$$

Inequality A.10.

$$\frac{g}{(g^2+M^2)^{1/2}}(1+x^2)^{-1/4} \leqq \sin\frac{\psi}{2} \leqq \left[\frac{g}{M}\right]^{1/2}\left[\frac{1+x^2}{x^2}\right]^{1/4}. \tag{A.10.1}$$

Proof. Using the inequality:

$$1 + x^2 - [(g^2-M^2)/(g^2+M^2)]^2 \geqq x \cdot 4gM/(g^2+M^2), \tag{A.10.2}$$

it is easy to show that

$$\begin{aligned} &[(g^2+M^2)(1+x^2)^{1/2}]^{-1} \leqq 2[((g^2-M^2)+(g^2+M^2)(1+x^2)^{1/2}]^{-1} \\ &= \left(\sin\frac{\psi}{2}\Big/ g\right)^2 = 2(g^2+M^2)^{-1}[(1+x^2)^{1/2} + (g^2-M^2)/(g^2+M^2)] \\ &\times [1+x^2-(g^2-M^2)^2/(g^2+M^2)^2]^{-1} \leqq (gM)^{-1}[(1+x^2)/x^2]^{1/2}, \end{aligned}$$

which immediately gives (A.10.1).

Inequality A.11.

$$\frac{x}{2(1+x^2)^{1/2}} \leqq \cos\frac{\psi}{2}. \tag{A.11.1}$$

Proof is obtained by the estimate:

$$\frac{x^2}{4(1+x^2)} \leqq \frac{x^2}{[1+(1+x^2)^{1/2}]^2} \leqq \frac{(1+x^2)^{1/2}-1}{(1+x^2)^{1/2}+(g^2-M^2)^2/(g^2+M^2)^2} = \left[\cos\frac{\psi}{2}\right]^2. \tag{A.11.2}$$

Inequality A.12.

$$\sigma\left[\frac{g}{\sin\psi/2},\psi\right] \leqq 2^{\gamma}\sigma_0\cdot[B(g^2+M^2)^{\beta/2}+B'M^{-\alpha}], \tag{A.12.1}$$

where

$$\zeta := \max[-\gamma, (\alpha-\gamma)/2, \alpha/2], \qquad \chi := \max[0, -(\alpha+\gamma)/2, (\beta-\gamma)/2]$$

and

$$\sigma_0 := 2^{\zeta}\left[\frac{(1+x^2)^{1/2}}{x}\right]^{\zeta}(1+x^2)^{\chi/2}\left[\frac{(g^2+M^2)}{g^2}\right]^{\zeta/2}. \tag{A.12.2}$$

Proof. From assumption (2.7) it follows that

$$\sigma\left[\frac{g}{\sin\psi/2},\psi\right] \leqq 2^{\gamma}[b(g,\psi)+b'(g,\psi)], \tag{A.12.3}$$

where

$$b(g,\psi) := Bg^{\gamma}[\cos\psi/2)]^{\gamma}\left[\frac{g}{\sin\psi/2}\right]^{\beta-\gamma}, \tag{A.12.4}$$

$$b'(g,\psi) := B'g^{\gamma}[\cos\psi/2)]^{\gamma}\left[\frac{\sin\psi/2}{g}\right]^{\alpha+\gamma}. \tag{A.12.5}$$

Bounds on b and b' depend on sign of γ, $\beta-\gamma$, and $\alpha+\gamma$; so the six different estimates on b and b' are obtained from Inequalities A.10 and A.11:

1. $0 \leqq \gamma \leqq \beta$.

$$\begin{aligned} b(g,\psi) &\leqq Bg^{\gamma}(1+x^2)^{(\beta-\gamma)/4}(g^2+M^2)^{(\beta-\gamma)/2} \\ &\leqq B(1+x^2)^{(\beta-\gamma)/4}(g^2+M^2)^{\beta/2} \leqq B(g^2+M^2)^{\beta/2}\sigma_0. \end{aligned} \tag{A.12.6}$$

2. $\gamma \geqq \beta$.

$$b(g,\psi) \leqq Bg^{\beta} \leqq B(g^2+M^2)^{\beta/2} \leqq B(g^2+M^2)^{\beta/2}\sigma_0. \tag{A.12.7}$$

3. $\gamma \leqq 0$.

$$\begin{aligned} b(g,\psi) \leqq\, & B\cdot 2^{-\gamma}g^{\gamma}(1+x^2)^{(\beta-\gamma)/4}(g^2+M^2)^{(\beta-\gamma)/2} \\ & \times[(1+x^2)^{1/2}/x]^{-\gamma} \leqq B\cdot 2^{\zeta}[(1+x^2)^{1/2}/x]^{-\gamma} \\ & \times(1+x^2)^{(\beta-\gamma)/4}[(g^2+M^2)/g^2]^{-\gamma/2}(g^2+M^2)^{\beta/2} \\ \leqq\, & B(g^2+M^2)^{\beta/2}\sigma_0. \end{aligned} \tag{A.12.8}$$

4. $\gamma \geqq 0$.

$$\begin{aligned} b'(g,\psi) &\leqq B'[(\sin\psi/2)/g]^{\alpha} \leqq B'[(1+x^2)^{1/2}/x]^{\alpha/2}(Mg)^{-\alpha/2} \\ &\leqq B'[(1+x^2)^{1/2}/x]^{\alpha/2}[(g^2+M^2)/g^2]^{\alpha/4}M^{-\alpha} \\ &\leqq B'M^{-\alpha}\sigma_0. \end{aligned} \tag{A.12.9}$$

5. $-\alpha \leqq \gamma \leqq 0$.

$$b'(g,\psi) \leqq B'[2(1+x^2)^{1/2}/x]^{(\alpha-\gamma)/2}(M/g)^{(\alpha-\gamma)/2}M^{-\alpha} \leqq B'M^{-\alpha}\sigma_0. \tag{A.12.10}$$

6. $\gamma \leqq -\alpha$.

$$b'(g,\psi) \leqq B'[2(1+x^2)^{1/2}/x]^{-\gamma} g^{\gamma}(1+x^2)^{-(\alpha+\gamma)/4}(g^2+M^2)^{-(\alpha+\gamma)/2}$$
$$\leqq B'[2(1+x^2)^{1/2}/x]^{-\gamma}(1+x^2)^{-(\alpha+\gamma)/4}[(g^2+M^2)/g^2]^{-\gamma/2}M^{-\alpha} \leqq B'M^{-\alpha}\sigma_0 . \quad \text{(A.12.11)}$$

The estimates (A.12.6)–(A.12.11) immediately give (A.12.1).

Inequality A.13. *For* $\gamma > -2$*:*

$$4/(|\gamma|+2) \leqq \int_0^{\pi} d\theta (\sin\theta)^{\gamma+1} \leqq 2(\gamma+4)/(\gamma+2) . \quad \text{(A.13.1)}$$

Proof is easily obtained by changing variables: $z=\cos\theta$ and using estimate: $1-|z| \leqq 1-z^2 \leqq 1$.

Inequality A.14.

$$p_0 p_1^2/p_{10} \leqq s \leqq 4p_0 p_{10} . \quad \text{(A.14.1)}$$

Proof.

$$p_0 p_1^2/p_{10} \leqq 2p_0(p_{10}-M) \leqq 2(p_0 p_{10} - \mathbf{p}\mathbf{p}_1 + M^2) = s \leqq 4p_0 p_{10} .$$

Appendix B

In this appendix the proof of Proposition 3.1 is given.

Proposition 3.1. *Denote by J:*

$$J := \int_0^{\infty} dx \exp[-d(1+x^2)^{1/2}] I_0(ax) x \left[\frac{(1+x^2)^{1/2}}{x}\right]^{\xi} (1+x^2)^{\eta/2} .$$

Then:

$$J \leqq 3^{\xi+\eta}\left[(2-\xi)^{-1}\exp(-d/4) + 2(8/M^2)^{\eta}(1+2/M)\frac{(6g/v)^2}{(M^2+g^2)} d^{\eta+1}\right]$$
$$\times \exp(-v/24) , \quad \text{(B.1)}$$

where $0 \leqq \eta < 1$ *and* $0 \leqq \xi < 2$.

Proof. The idea is to estimate the integrated function in such a way that its estimator can be integrated for large arguments. To this aim we split the integral: $\int_0^{\infty} = \int_0^{1/2} + \int_{1/2}^{\infty}$ and investigate its two parts separately.

For $x \leqq 1/2$, making use of Inequalities A.5–A.7, we have:

$$\exp[-d(1+x^2)^{1/2}] I_0(ax) \leqq \exp(-d/4)\exp(-v/24) . \quad \text{(B.2)}$$

For $x \geqq 1/2$ we estimate: $[(1+x^2)^{1/2}/x]^\xi < 3^\xi$ and next: $\int_{1/2}^{\infty} \leqq \int_{0}^{\infty}$. Thus the following relation holds:

$$J \leqq 3^{\xi+\eta}\left[(2-\xi)^{-1}\exp(-d/4)\exp(-v/24)+(8/M^2)^\eta d^\eta \times \int_0^\infty dx x I_0(ax)\exp[-\varrho(1+x^2)^{1/2}]\right], \tag{B.3}$$

where $\varrho:=(d^2-M^2/4)^{1/2}$ and the upper bound on function $G(z):=\exp[-(d-\varrho)z]z^\eta$ was inserted:

$$G(z) \leqq (8/M^2)^\eta d^\eta . \tag{B.4}$$

The integral in (B.3) may be calculated exactly [20]:

$$J' := \int_0^\infty dx x I_0(ax)\exp[-\varrho(1+x^2)^{1/2}] = \varrho[1+(\varrho^2-a^2)^{-1/2}] \times (\varrho^2-a^2)^{-1}\exp[-(\varrho^2-a^2)^{1/2}], \tag{B.5}$$

and next estimated making use of Inequality A.8:

$$J' \leqq 2(1+2/M)(6g/v)^2(M^2+g^2)^{-1} d\exp(-v/24). \tag{B.6}$$

Equation (B.6) ends the proof of Proposition 3.1.

Appendix C

In this appendix the proof of Proposition 4.1 is given.

Proposition 4.1. *Let us denote:*

$$J_\eta := \int_{-1}^{1} dz(a_1-a_2z)^\eta, \tag{C.1}$$

where $\eta \in R$, $a_1 = p_0 p_{10} - M^2$, $a_2 = pp_1$. *Then*

i) *for* $-1<\eta<1/2$:

$$J_\eta \leqq 4(\eta+1)^{-1}(a_1+a_2)^\eta \leqq 16(\eta+1)^{-1}[p_{10}^\eta + (p_1^2/p_{10})^\eta]p_0^\eta, \tag{C.2}$$

and

ii) *for* $-3/2<\eta<-1$:

$$J_\eta \leqq 2(-\eta-1)^{-1}(a_1+a_2)^{-1}(a_1-a_2)^{1+\eta} \leqq 8(-\eta-1)^{-1} \times M^2(1+\eta)p_{10}^{-\eta}p_1^{-2}|p-p_1|^{2(1+\eta)}p_0^{-2-\eta}. \tag{C.3}$$

Proof. $J_\eta = [(a_1+a_2)^{\eta+1}-(a_1-a_2)^{\eta+1}]/[a_2(\eta+1)]$, so:

i) Using Inequality A.14 we have:

A) $\eta<0$.

$$\begin{aligned} J_\eta &= (a_1+a_2)^{\eta+1}\{1-[1-2a_2/(a_1+a_2)]^{\eta+1}\}/[a_2(\eta+1)] \\ &\leqq (a_1+a_2)^{\eta+1} 2a_2/(a_1+a_2)/[a_2(\eta+1)] = 2(\eta+1)^{-1}(a_1+a_2)^\eta \\ &\leqq 16(\eta+1)^{-1}p_{10}^\eta p_0^\eta . \end{aligned} \tag{C.4}$$

B) $0 \leqq \eta < 1/2$.

$$\begin{aligned} J_\eta &= \{2a_2(a_1-a_2)^\eta + (a_1+a_2)[(a_1+a_2)^\eta - (a_1-a_2)^\eta]\}/[a_2(\eta+1)] \\ &= 2(a_1-a_2)^\eta/(\eta+1) + J_{\eta-1}(a_1+a_2)\eta/(\eta+1) \\ &\leqq 4(\eta+1)^{-1}(a_1+a_2)^\eta \leqq 16(\eta+1)^{-1}(p_1^2/p_{10})^\eta p_0^\eta . \end{aligned} \tag{C.5}$$

ii) It is easy to show that:

$$(a_1-a_2)^{-1} \leqq 2p_0p_{10}M^{-2}(p-p_1)^2 . \tag{C.6}$$

Using (C.6) and Inequality A.14 we obtain:

$$\begin{aligned} J_\eta &= \frac{1}{a_2(-\eta-1)} \frac{1}{(a_1-a_2)^{-\eta-1}} \left[1 - \left[1 - \frac{2a_2}{a_1+a_2}\right]^{-\eta-1}\right] \\ &\leqq \frac{1}{a_2(-\eta-1)} \frac{1}{(a_1-a_2)^{-\eta-1}} \frac{2a_2}{a_1+a_2} \\ &= 2(-\eta-1)^{-1}(a_1+a_2)^{-1}(a_1-a_2)^{1+\eta} \\ &\leqq 8(-\eta-1)^{-1}M^{2(1+\eta)}p_{10}^{-\eta}p_1^{-2}|p-p_1|^{2(1+\eta)}p_0^{-2-\eta} . \end{aligned}$$

Thus Proposition 4.1 is proved.

Acknowledgements. This work was supported in part by the Polish Ministry of Science & High Education, Grant CPBP01.02.

References

1. de Groot, S.R., van Leeuwen, W.A., van Weert, Ch.G.: Relativistic kinetic theory-principles and applications. Amsterdam: North-Holland 1980
2. Stewart, J.M.: Non-equilibrium relativistic kinetic theory. In: Lecture Notes in Physics, Vol. 10. Berlin, Heidelberg, New York: Springer 1971
3. Bichteler, K.: On the Cauchy problem of the relativistic Boltzmann equation. Commun. Math. Phys. **4**, 352–364 (1967)
4. Bancel, D.: Problème de Cauchy pour l'équation de Boltzmann en relativité générale. Ann. Inst. Henri Poincaré **18**, 263–284 (1973)
5. Bancel, D., Choquet-Bruhat, Y.: Existence, uniqueness and local stability for the Einstein-Maxwell-Boltzmann system. Commun. Math. Phys. **33**, 83–96 (1973)
6. Grad, H.: In: Third International Rarefield Gas Symposium. Vol. 1, p. 26. Lauremann, J.A. (ed.). New York: Academic Press 1963
7. Grad, H.: Asymptotic theory of the Boltzmann equation. Phys. Fluids **6**, 147–181 (1963); Grad, H.: Asymptotic equivalence of the Navier-Stokes and non-linear Boltzmann equations. Proc. Symp. Appl. Math., Am. Math. Soc. **17**, 154–183 (1965)
8. Ellis, R.S., Pinsky, M.A.: The first and the second fluid approximations to the linearized Boltzmann equation. J. Math. Pures. Appl. **54**, 125–156 (1975);
   Ellis, R.S., Pinsky, M.A.: The projection of the Navier-Stokes equations upon the Euler equations. J. Math. Pures. Appl. **54**, 157–182 (1975)
9. Nishida, T.: Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation. Commun. Math. Phys. **61**, 119–148 (1978)
10. Kawashima, S., Matsumara, A., Nishida, T.: On the fluid-dynamical approximation to the Boltzmann equation at the level of the Navier-Stokes equations. Commun. Math. Phys. **70**, 97–124 (1979)
11. Jutner, F.: Ann. Phys. (Leipzig) **34**, 856 (1911); **35**, 145 (1911)

12. Abramovitz, M., Stegun, I.A.: Handbook of mathematical functions with formulas, graphs and mathematical tables. New York: Dover 1965
13. Ekiel-Jeżewska, M.L.: In: Recent developments in non-equilibrium thermodynamics. In: Lecture Notes in Physics, Vol. 199. Berlin, Heidelberg, New York: Springer 1984
14. van Weert, Ch.G.: Generalized hydrodynamics from relativistic kinetic theory. Physica. **111**A, 537–552 (1982)
15. Drange, H.B.: The linearized Boltzmann collision operator for cut-off potentials. SIAM J. Appl. Math. **29**, 665–676 (1975)
16. Cercignani, C.: Mathematical methods in kinetic theory. New York: Plenum/London, McMillan 1969
Cercignani, C.: Theory and application of the Boltzmann equation. Edinburgh: Scottish Academic Press; New York: Elsevier 1975
17. Grad, H.: Rarefied gas dynamics. London, Oxford, New York: Pergamon Press 1960
18. Dijkstra, J.J., van Leeuwen, W.A.: Mathematical aspects of relativistic kinetic theory. Physica **90**A, 450–486 (1978)
19. Hutson, V.C.L., Pym, J.S.: Applications of functional analysis and operator theory. London, New York: Academic Press 1980
20. Prudnikov, A.P., Britchkov, Yu.A., Maritchev, O.I.: Intiegraly i Riady, Specjalnyje Funkcji. Moskva: Nauka 1983 (in Russian)
21. Caflisch, R.E.: The Boltzmann equation with a soft potential. Commun. Math. Phys. **74**, 71–95 (1980) and **74**, 97–109 (1980)
22. Schechter, M.: On the essential spectrum of an arbitrary operator. J. Math. Anal. Appl. **13**, 205–215 (1966)
23. Ukai, S.: On the existence of global solutions of mixed problem for nonlinear Boltzmann equation. Proc. Jpn. Acad. **50**, 179–184 (1974)
24. Nishida, T., Imai, L.: Global solutions to the initial value problem for the nonlinear Boltzmann equation. Publ. RIMS Kyoto, **12**, 229–239 (1976)
25. Adams, R.A.: Sobolev spaces. New York: Academic Press 1975
26. Dudyński, M., Ekiel-Jeżewska, M.L.: Causality of the linearized relativistic Boltzmann equation. Phys. Rev. Lett. **55**, 2831–2834 (1985)
Dudyński, M., Ekiel-Jeżewska, M.L.: Causality problem in the relativistic kinetic theory. In: Recent developments in nonequilibrium thermodynamics. Lecture Notes in Physics, Vol. 253. Berlin, Heidelberg, New York: Springer 1986
27. Dudyński, M., Ekiel-Jeżewska, M.L.: Relativistic Boltzmann equation – mathematical and physical aspects. In: Relativistic fluid dynamics. Lecture Notes in Physics. Berlin, Heidelberg, New York: Springer (to appear)

Communicated by J. L. Lebowitz

Received June 16, 1987