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Monopoles, Non-Linear σ Models, and Two-Fold Loop Spaces

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Abstract. In this paper we study the topology of $\hat{\mathscr{M}}_k$, the moduli spaces of $SU(2)$ monopoles associated with the Yang–Mills–Higgs and Bogomol'nyi equations, and $\mathscr{H}(m)_k$, non-linear σ models from quantum field theory. Beautiful work of Donaldson [18, 19], Hitchin [24, 25] and Taubes [37, 39, 40] shows that gauge equivalence classes of monopoles correspond to based rational self-maps of the Riemann sphere. Similarly, the non-linear σ models we consider here are based harmonic maps from the Riemann sphere to complex projective m space. In seminal work, Segal [35] studied $\mathscr{R}(m)_k$, the space of based rational maps from the Riemann sphere to complex projective m space of a fixed degree k. Any element of $\mathscr{R}(m)_k$ is clearly an element of $\Omega_k^2 CP(m)$, the space of all based continuous maps from the Riemann sphere to complex projective m space of a fixed degree k, and this assignment gives rise to the natural inclusion of $\mathscr{R}(m)_k$ in $\Omega_k^2 CP(m)$. Segal showed that these natural inclusions are homotopy equivalences through dimension $k(2m-1)$. As the topology of the two-fold loop space $\Omega^2 CP(m)$ is well understood, Segal's result gives a very efficient way to explicitly determine the low dimensional topology of $\mathscr{R}(m)_k$. Thus iterated loop spaces have much to say about the topology of monopoles and non-linear σ models.

In this paper we apply the theory of iterated loop spaces (more precisely, May's C_2 operad spaces [31]) to study $\hat{\mathscr{M}}_k$, $\mathscr{H}(m)_k$ and $\mathscr{R}(m)_k$. Our main technical device is to place a C_2 operad structure on these spaces which is compatible with the usual C_2 operad structure on $\Omega^2 CP(m)$. This will enable us to study the topology of $\mathscr{R}(m)_k$ and thus the topology of $\hat{\mathscr{M}}_k$ and $\mathscr{H}(m)_k$ above the range of the Segal equivalence.

The C_2 operad structure we define here on $\mathscr{R}(m)$ is very similar to the C_4 operad structure defined on the moduli spaces for instantons in [10]. It is worth recalling

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Segal's observation that Morse theory supplies on reason for expecting his theorem to be true (for $m = 1$). That is, Eells and Wood [21] and, independently, Woo [43] showed that the "energy" functional

$$E(f) = \frac{1}{2} \int_{CP(1)} \| df(x) \|^2 \, \mathrm{dvol}$$

on the space of smooth self maps of $CP(1)$ has no critical points (which are precisely the harmonic maps) other than the rational maps which are absolute minima. Thus "infinite dimensional Morse theory", predicts that rational maps should tend to approximate all smooth maps. When $m > 1$ there are harmonic maps (non-linear σ models) which are not rational (holomorphic) [15, 16] [22] and one has the following sequence of *proper* containments $\mathscr{R}(m)_k \subset \mathscr{H}(m)_k \subset \mathrm{Map}_k(CP(1), CP(m))$. In this case, Segal's result implies that $\mathscr{R}(m)_k$ and $\mathrm{Map}_k(CP(1), CP(m))$ are homotopy retracts of $\mathscr{H}(m)_k$ through a range that increases as k grows.

There is a striking similarity between the energy functional with associated rational maps as minima and the Yang–Mills functional with associated instantons (self-dual connections in principal $SU(2)$ bundles over S^4) as minima. This similarity led Atiyah and Jones to their foundational work on the Yang–Mills instanton problem and to make their celebrated, but as yet unsettled, conjecture [7] which generalizes Segal's theorem to the instanton case. One truly remarkable aspect of the theory of Yang–Mills instantons has been the on-going program of Taubes which now includes a proof of a "stable" version of the Atiyah–Jones conjecture [41]. Using a result of Taubes on the existence of tubular neighborhoods for instantons and the linear-algebraic description of instantons due to Atiyah, Drinfeld, Hitchin and Manin [5], we were able [10] to build an operad structure on the moduli spaces of these instantons and to study the associated rich homological structure. At the same time, Fred Cohen explained to us how he had, in an unpublished manuscript, placed an operad structure on $\mathscr{R}(m)$. Furthermore, in a seemingly unrelated direction, Lawson's exciting work on the topology of Chow varieties [28, 29] has, in addition to his many other results, uncovered an operad structure there.

It is apparent that the appearance of operad structures in so many varied moduli problems is no accident and that one should look for other situations where they arise. There are compelling reasons to consider monopoles, that is, the moduli space associated to the functional whose Euler–Lagrange equations are the Yang–Mills–Higgs equations on R^3 and whose minima are the solutions to the Bogomol'nyi equations. First, Taubes' work on monopoles [37, 39, 40] has much the same form as his work on instantons, including a tubular neighborhood theorem which is such an important homotopy tool in these problems. Second, there is also an analog of the ADHM construction for monopoles, namely the Nahm construction [32]. Thus it would be reasonable to proceed in the monopole case precisely as was done for instantons. It is not necessary to carry out that program because the remarkable result of Donaldson [18], which relates the Nahm construction to the space of rational maps, together with deep work of Hitchin [24, 25] and Taubes [37, 39, 40], brings us back full circle to Segal's theorem on rational maps and Cohen's operad structure there!

The monopoles considered here are "Euclidean" monopoles, that is, they are obtained by reducing the self dual Yang–Mills equations on R^4 with respect to a one parameter subgroup R of translations. However, one can also reduce by a circle group to obtain "hyperbolic" monopoles as described by Atiyah [3, 4]. Atiyah showed that hyperbolic monopoles also correspond to based rational maps from $CP(1)$ to itself. Thus, we fully expect that the moduli space of hyperbolic monopoles has the structure of a C_2 operad space.

One amusing consequence of Donaldson's theorem identifying rational maps with monopoles is that the manifold $\mathscr{R}(1)_k$ represents the global minima of two quite differently defined functionals on $\Omega_k^2 CP(1)$ (see 2.1 and 1.4). This is a manifestation of the principle that any two Morse functions on a manifold are equally useful in studying the topology of that manifold.

We now turn to the organization and main results of this paper. In Sect. one we review the fundamental work of Taubes, Hitchin and Donaldson on monopoles culminating in the identification of monopoles with rational maps (see Theorem 1.21). Section two reviews non-linear σ models while Sect. three briefly reviews Segal's theorem. At this point, we write $\mathscr{R}(1)_k$ for both the space of based rational maps of degree k and for the moduli space of monopoles of monopole number k.

In Sect. four we construct a C_2 operad structure on $\mathscr{R}(m)$. This construction, while similar in spirit to our construction on instantons [10], is far simpler for rational maps (and hence monopoles) precisely because rational maps on the Riemann sphere are much simpler than rational maps on $HP(1)$. While the natural inclusion $i(m)_k: \mathscr{R}(m) \to \Omega^2 CP(m)$ is thus a map of C_2 spaces, it is not true that $i(m)$ is a C_2 map. Our main technical result, Theorem 4.16, shows that $i(m)$ is a "homotopy" C_2 map which is sufficient to carry out the computations that occupy the remainder of the paper. The proof is essentially identical to our proof in the instanton case.

Section five reviews basic facts about homology operations in C_2 spaces. We catalog our computational results in Sect. six. Our main result is to generate non-trivial classes in $H_q(\mathscr{R}(m)_k)$ for $q > k(2m-1)$ and describe how the homology of $\mathscr{R}(m)_k$ grows as k increases. We also recover some of the homological information obtained by Segal when $q < k(2m-1)$. We note that Segal's theorem [35] implies that $\mathscr{R}(m)_k$ and $\Omega_k^2 CP(m)$ are homotopy retracts of $\mathscr{H}(m)_k$ up to dimension $k(2m-1)$. Therefore the higher dimensional classes we construct also live in $H_*(\mathscr{H}(m)_k)$.

Finally, in Sect. seven, we give a proof of Proposition 1.22 which seems to have been noted in the literature without proof and which is needed to complete the topological identification of monopoles with rational functions.

1. Monopoles

It is well known [27, 39, 40] that the $SU(2)$-Yang–Mills equations on R^4 reduce under "time" translation symmetry to the Yang–Mills–Higgs equations on R^3:

$$D_A * F_A + [\Phi, * D_A \Phi] = 0, \tag{1.1}$$

$$D_A * D_A \Phi = 0, \tag{1.2}$$

where A is a connection on the trivial principal bundle $P = \mathbb{R}^3 \times SU(2)$, and Φ is a section of the Lie algebra adjoint bundle $\wp = P \times_{SU(2)} su(2)$, which satisfies the boundary condition

$$\lim_{\|x\| \to \infty} \| \Phi(x) \| = 1, \tag{1.3}$$

where the norm $\|\cdot\|$ in P is the Euclidean norm on R^3 and the Killing norm on $su(2)$. Equations 1.1 and 1.2 are the Euler Lagrange equations for the Yang–Mills–Higgs functional

$$\mathscr{U}(A, \Phi) = \frac{1}{2} \int_{R^3} [\| F_A \|^2 + \| D_A \Phi \|^2] dx^3. \tag{1.4}$$

We are interested in the space $\mathscr{C}$ of all smooth pairs (A, Φ) such that $\mathscr{U}(A, \Phi) < \infty$ and $1 - \| \Phi \| \in L^6(R^3)$. The topology on $\mathscr{C}$ can be described as follows [39]: Consider the affine space $\mathscr{A}$ of all smooth connections on $R^3 \times SU(2)$ and the space $\Gamma(\wp)$. By fixing the flat connection on $R^3 \times SU(2)$ we can identify $\mathscr{A}$ with the space of smooth sections $\Gamma(\wp \otimes T^*M)$. Now $\Gamma(\wp \otimes T^*M) \times \Gamma(\wp)$ has the product topology where each factor has the weak C^∞-topology. Consider the natural inclusion

$$\iota : \mathscr{C} \to \Gamma(\wp \otimes T^*M) \times \Gamma(\wp) \times R$$

defined by

$$i(A, \Phi) = (A, \Phi, \mathscr{U}(A, \Phi)),$$

and give $\mathscr{C}$ the subspace topology. It is well known [27, 39, 40] that an element $(A, \Phi) \in \mathscr{C}$ defines a homotopy class $[A, \Phi] \in \pi_2(S^2) = Z$ called the *monopole number* k. A curvature computation [23] shows

$$k = \frac{1}{4\pi} \operatorname{tr} \int_{R^3} D_A \Phi \wedge F_A, \tag{1.5}$$

which implies

$$\mathscr{U}(A, \Phi) \geqq 4\pi |k|. \tag{1.6}$$

The monopole number partitions $\mathscr{C}$ into components

$$\mathscr{C} = \coprod_{k \in Z} \mathscr{C}_k, \tag{1.7}$$

and we will be mainly interested in the positive sector

$$\mathscr{C} = \coprod_{k > 0} \mathscr{C}_k. \tag{1.8}$$

Equality holds in (1.6) if and only if (A, Φ) satisfies the Bogomol'nyi equations [9],

$$* F_A = -(\operatorname{sign} k) D_A \Phi. \tag{1.9}$$

Solutions to (1.9) give all minima of the Yang–Mills–Higgs functional (1.4).

The gauge group $\mathscr{G}$ can be identified with the space of smooth maps $g : R^3 \to SU(2)$ again with the weak C^∞-topology. We are more interested in the normal subgroup of based gauge transformations

$$\mathscr{G}^b = \{g \in \mathscr{G} \mid g(0) = 1\},$$

which acts freely on $\mathscr{C}$. Endow $\mathscr{B} = \mathscr{C}/\mathscr{G}^b$ with the quotient topology. Now there is a fibration

$$\begin{array}{ccc} SU(2) & \longrightarrow & \mathscr{G} \\ & & \downarrow \\ & & \mathscr{G}^b \end{array} \tag{1.10}$$

so $SU(2)$ acts on $\mathscr{B}$ and there is an $SU(2)$-equivariant fibration (described in detail in [39])

$$\begin{array}{ccc} \hat{\mathscr{B}} & \longrightarrow & \mathscr{B} \\ & & \downarrow \eta \\ & & S^2, \end{array} \tag{1.11}$$

where $\hat{\mathscr{B}} = \eta^{-1}(\infty)$. Furthermore, since $\mathscr{G}$ leaves the functional (1.4) invariant, it gives a well-defined functional on the spaces $\mathscr{B}$ and $\hat{\mathscr{B}}$.

Now we also have the fibration

$$\begin{array}{ccc} C^\infty(\Omega_k^2 S^2) & \longrightarrow & C_k^\infty(S^2, S^2) \\ & & \downarrow \pi \\ & & S^2, \end{array} \tag{1.12}$$

where π is evaluation at a fixed point $\infty \in S^2$, $C_k^\infty(S^2, S^2)$ is the space of smooth maps of degree k from S^2 to itself and $C^\infty(\Omega_k^2 S^2)$ is the k^{th} component of the subspace of the based maps. Again the topology is the weak C^∞-topology.

The following theorem of Cliff Taubes is fundamental:

Theorem 1.13 [39]. *There is a commutative diagram*

$$\begin{tikzcd} \hat{\mathscr{B}}_k \arrow[r] & \mathscr{B}_k \arrow[dr] & \\ & & S^2 \\ C^\infty(\Omega_k^2 S^2) \arrow[r] \arrow[uu, "\hat{I}_k"] & C_k^\infty(S^2, S^2) \arrow[uu, "I_k"] \arrow[ur] & \end{tikzcd} \tag{1.14}$$

where the vertical maps I_k and $\hat{I}_k$ are homotopy equivalences.

In fact, much more is true [39]. The Taubes map, I_k, is an inclusion, its image $I_k(C_k^\infty(S^2, S^2))$ is a deformation retract of $\mathscr{B}_k$, and $\hat{I}_k$ is the restriction of I_k to $C^\infty(\Omega_k^2 S^2)$. To define I_k, regard $C^\infty(S^2, S^2)$ as a subset of $C^\infty(S^2, su(2))$ by fixing an identification of $su(2)$ with R^3, and let $\beta \in C^\infty(R)$ be a smooth, non-negative bump function which is 1 for $t \leqq \frac{1}{2}$ and 0 for $t \geqq \frac{3}{4}$. Then $I: C^\infty(S^2, S^2) \to \mathscr{B}$ is defined by

$$I(f) = \left(-(1-\beta(|x|)) \left[f\left(\frac{x}{|x|}\right), df\left(\frac{x}{|x|}\right) \right], (1-\beta(|x|)) f\left(\frac{x}{|x|}\right) \right). \tag{1.15}$$

Now $SU(2)$ acts on $\mathscr{B}$ by regarding group elements as constant gauge transformations and on $C^\infty(S^2, S^2)$ by rotations on the image S^2. Note that I is

Ad-equivariant and that the stabilizer of the north pole in $SU(2)$ is S^1. Therefore

$$\hat{\mathscr{B}}/S^1 \simeq C^\infty(\Omega^2 S^2)/S^1 \simeq C^\infty(S^2, S^2)/SU(2) \simeq \mathscr{B}/SU(2) \simeq \mathscr{C}/\mathscr{G}. \tag{1.16}$$

Turning to the first Bogomol'nyi equation (1.9) with $k>0$ (the case $k<0$ can be obtained from the map $\Phi \mapsto -\Phi$), we define the moduli space

$$\mathscr{M}_k = \{(A, \Phi) \in \mathscr{C}_k : (1.9) \text{ is satisfied}\}/\mathscr{G}, \tag{1.17}$$

and the extended moduli space:

$$\hat{\mathscr{M}}_k = \{[A, \Phi] \in \hat{\mathscr{B}}_k : (1.9) \text{ is satisfied}\}, \tag{1.18}$$

where $[A, \Phi]$ denotes the equivalence class of $(A, \Phi) \in \mathscr{C}_k$ under the based gauge group $\mathscr{G}^b$. From (1.16) we see that $\mathscr{M}_k \simeq \hat{\mathscr{M}}_k/S^1$. Actually, it follows that $\hat{\mathscr{M}}_k$ is an S^1 bundle over $\mathscr{M}_k$.

The existence of global solutions to (1.9) satisfying the appropriate boundary conditions was given by Taubes in his Harvard Ph.D. thesis [36] and included in [27]. It was shown that there is a natural inclusion

$$C_k(R^3) \longrightarrow \hat{\mathscr{M}}_k, \tag{1.19}$$

where

$$C_k(R^3) = \{k \text{ distinct points in } R^3\}/\Sigma_k$$

and Σ_k is the symmetric group on k letters. Using the analog of the analysis for instantons of Atiyah and Jones [7], we have the commutative diagram

$$\begin{tikzcd}
C_k(R^2) \arrow[rrr, "S_k"] \arrow[dr] \arrow[dd, "\Sigma"'] & & & \Omega^2_k S^2 \arrow[dd, "\Sigma"] \\
& \hat{\mathscr{M}}_k \arrow[r, "i_k"] & \hat{\mathscr{B}}_k \arrow[ur] \arrow[dr] & \\
C_k(R^3) \arrow[rrr, "S_k"] \arrow[ur] & & & \Omega^3_k S^3
\end{tikzcd} \tag{1.20}$$

where S_k denotes the Segal map [34], which is known to be an equivalence in homology through a range, and Σ is the natural suspension map. Furthermore, Taubes [40] has shown that i_k induces an isomorphism of the homotopy groups $\pi_q(\hat{\mathscr{M}}_k) \cong \pi_q(\hat{\mathscr{B}}_k)$ for $q<|k|$ and an epimorphism for $q=|k|$.

Since $\Omega^2 S^2$ is a two-fold loop space and

$$C(R^3) = \coprod_{k>0} C_k(R^3)$$

group completes to $\Omega^2 S^2$, it is natural to suspect that, as is the case for instantons [10],

$$\hat{\mathscr{M}} = \coprod_{k>0} \hat{\mathscr{M}}_k$$

has the structure of a C_2 little cubes operad in the sense of May [31]. In Sect. four we show that this is indeed the case. Our main tool in proving this is the following remarkable theorem of Donaldson [18] and Hitchin [24, 25]:

Theorem 1.21. *There is a natural one to one correspondence berween $\hat{\mathcal{M}}_k$ and the space $\mathcal{R}_k$ of based rational maps $f: CP(1) \to CP(1)$ of degree k sending ∞ to 0.*

This correspondence depends on a fixed isomorphism $R^3 \simeq C \times R$. The following stronger result seems to have been noted without proof by Taubes [40, page 476] and Donaldson [19, page 100]. Since we use this result here we shall give a proof in Sect. seven.

Proposition 1.22. *With the natural topologies on $\hat{\mathcal{M}}_k$ and $\mathcal{R}_k$ the correspondence in (1.21) is a homeomorphism.*

2. The Nonlinear σ-Model

Nonlinear σ-models arose in physics as examples of exactly solvable models of a nonlinear quantum field theory. In their simplest form they appear as maps from two-dimensional space-time to an arbitrary manifold that minimize a certain energy functional, or in mathematical terms, harmonic maps. The σ-models considered here are harmonic maps from $CP(1) \simeq S^2$ to $CP(m)$ and were first studied by Din and Zakrzewski [15, 16] and then more rigorously by Eells and Wood [22]. Our presentation follows the latter reference.

On the space $\mathrm{Map}(CP(1), CP(m))$ consider the "energy" functional

$$E(\phi) = \frac{1}{2} \int_{CP(1)} \| d\phi(x) \|^2 \, \mathrm{dvol}, \tag{2.1}$$

where $\|\cdot\|$ denotes the norm with respect to the Fubini-Study metric on $CP(m)$ and dvol is the standard volume form on $CP(1)$. The critical points of E satisfy

$$\operatorname{tr} D d\phi = 0, \tag{2.2}$$

where D denotes the induced connection on $\phi^* TCP(m) \otimes T^* CP(1)$. Solutions to (2.2) are called *harmonic maps*.

The space $\mathrm{Map}(CP(1), CP(m))$ is partitioned by its homotopy classes $\pi_2(CP(m)) \cong Z$ into components which are labeled by the degrees of the maps. The complex structure on $CP(m)$ implies that the energy functional 2.1 can be written as

$$E(\phi) = \frac{1}{2} \int_{CP(1)} (\| \partial\phi(x) \|^2 + \| \bar{\partial}\phi(x) \|^2) \mathrm{dvol} \tag{2.3}$$

and that the difference

$$\frac{1}{2} \int_{CP(1)} (\| \partial\phi(x) \|^2 - \| \bar{\partial}\phi(x) \|^2) \mathrm{dvol} \tag{2.4}$$

depends only on the degree k. Thus the absolute minima are precisely the holomorphic maps, that is, the rational maps from $CP(1)$ to $CP(m)$. In the physics literature (cf. [15, 16]) these are referred to as instanton solutions. More importantly, all minima are rational maps [22] and, for the case $m = 1$, all critical points are minima and hence rational maps [21, 43].

We are more interested in the based maps of degree k:

$$\mathrm{Map}_k^b(CP(1), CP(m)) = \Omega_k^2 CP(m).$$

We have the natural sequence of inclusions

$$\mathscr{R}(m)_k \hookrightarrow \mathscr{H}(m)_k \hookrightarrow \Omega_k^2 CP(m), \tag{2.5}$$

where $\mathscr{R}(m)_k$ and $\mathscr{H}(m)_k$ denote based rational maps and based harmonic maps of degree k from $CP(1)$ to $CP(m)$ respectively. Furthermore, the first inclusion is an equivalence for $m = 1$ and is always *proper* for $m > 1$. Segal's theorem, which is discussed in the next section, shows that the composite $\mathscr{R}(m)_k \hookrightarrow \Omega_k^2 CP(m)$ is a homotopy equivalence through a range that increases as k grows. Thus Segal's result, along with (2.5), implies

Proposition 2.6. *Both $\mathscr{R}(m)_k$ and $\Omega_k^2 CP(m)$ are homotopy retracts of $\mathscr{H}(m)_k$ through a range that increases as k grows.*

In Sect. six we will use this proposition to extract information on the homology of $\mathscr{H}(m)_k$.

3. Rational Maps and Segal's Theorem

Segal, in his fundamental paper [35], considered the space of based rational self-maps of the Riemann sphere of degree k and the natural inclusion of this space in the space of all continuous based self-maps of the Riemann sphere of degree k. We have seen ((1.21) and (1.22)) that this space and its associated natural inclusion are of fundamental importance in the study of monopoles. Segal's main result, which we state below, represents a guiding philosophy in the study of the topology of moduli spaces of global solutions to partial differential equations. Briefly put, as k increases the finite dimensional space of analytic objects (here rational maps or monopoles) becomes an increasingly better homotopy approximation to the infinite dimensional space of continuous objects (here all continuous maps or smooth pairs (A, Φ)). In practice, much is usually known about the range space and this classical knowledge may then be used to obtain topological information about the analytic moduli spaces.

Let $f(z) = (p(z)/q(z))$ be the quotient of two monic polynomials of degree k, where $p(z)$ and $q(z)$ have no roots in common. $f(z)$ is then a based self-map of $C^1 \cup \infty = S^2$ which sends ∞ to 1 and thus, by forgetting the analytic structure, $f(z)$ determines an element in $\Omega_k^2 S^2$. If we denote the space of all such $f(z)$ by $\mathscr{R}_k$ and the natural inclusion induced by the forgetful map by i_k, we may then state Segal's first theorem in [35].

Theorem 3.1 [35].

$$i_k : \mathscr{R}_k \to \Omega_k^2 S^2$$

is a homotopy equivalence up to dimension k.

The work of Donaldson [18], Hitchin [24, 25] and Taubes [39, 40] described in Sect. one shows that Segal's theorem is also a theorem about the moduli spaces $\hat{\mathscr{M}}_k$. In fact, while Segal's original proof of this theorem was purely topological, the work of Donaldson, Hitchin and Taubes, taken together, gives a purely "monopole theoretic" proof of (3.1). This allows us to adopt the notational convention that $\hat{\mathcal{M}}_k$ and $\mathcal{R}_k$ are identical. We do so for the remainder of this paper.

As $S^2 = CP(1)$, we can equivalently write $f(z)$ in terms of homogeneous coordinates as $f(z) = [p(z), q(z)]$, which immediately generalizes to define rational maps of S^2 into $CP(m)$ as follows: Let $\mathcal{R}(m)_k$ denote the maps $f(z) = [p_0, p_1, \ldots, p_m]$ where each p_i is a monic polynomial of degree k and none of the p_i and p_j have a common root for $i \neq j$. Again, each such $f(z)$ is an element of $\Omega_k^2 CP(m)$, and if we denote the natural inclusion by $i(m)_k$, we may state Segal's second theorem.

Theorem 3.2 [35].

$$i(m)_k\colon \quad \mathcal{R}(m)_k \to \Omega_k^2 CP(m)$$

is a homotopy equivalence up to dimension $k(2m-1)$.

We conclude this section by pointing out that the homology of $\Omega_k^2 CP(m)$ is well known, and therefore it should be possible to use all the $i(m)$'s following a program similar to that in [10] to obtain homological information about $\mathcal{R}(m)_k$ and $\mathcal{H}(m)_k$ as well as $i(m)_k\colon \mathcal{R}(m)_k \to \mathcal{H}(m)_k \to \Omega_k^2 CP(m)$. This program is carried out in the remaining sections of this paper.

4. Operads and Rational Maps

In this section we define a C_2 little cubes operad action on

$$\mathcal{R}(m) = \coprod \mathcal{R}(m)_k,$$

the disjoint union of the spaces of degree k based rational functions studied by Segal [35]. We are indebted to Fred Cohen for pointing out this structure to us. We then observe that this operad structure is homotopy compatible with the natural inclusion of the space of based rational maps into the space of all based maps. This observation will then permit us to make new computations on the homology of $\mathcal{R}(m)$ which we catolog in Sect. six.

We begin by recalling some basic facts about the theory of iterated loop spaces and the machinery which we will need to tie $\mathcal{R}$ more precisely into this theory. Actually, we expand our attention from iterated loop spaces to May's C_n operad spaces [31] which are based on Boardman and Vogt's little n-cube spaces [8]. We then show that $\mathcal{R}$ is a C_2 space and that, up to homotopy, this C_2 structure is compatible with the natural C_2 structure on $\Omega^2 CP(m)$ under the natural inclusion $i(m)$.

Definition 4.1. ([8], [31]). Let I^n be the unit n-cube and let $J^n = \dot{I}^n$ be the interior of I^n. An open little n cube is an affine embedding f of J^n into J^n with parallel axes. Then $C_n(j)$ is the space of j-tuples $(f_1, \ldots, f_j)$ of open little n cubes with disjoint images in $J^n \subset I^n$ (with the subspace topology inherited from $\mathrm{Map}(\coprod^j J^n, J^n)$).

Notice that the symmetric group Σ_j acts on $C_n(j)$ by permuting the disjoint images of the little n cubes.

Let ΩX be the space of based loops on X; that is, the space of based maps $f:(S^1, 1)\to(X, *)$ with the compact open topology. Recall ΩX is a topological "group" where the group operation is classically called the loop sum. We may iterate this construction to obtain $\Omega^n(X)=\Omega(\Omega\ldots(\Omega X))$, the space of n-fold iterated loops on X. Equivalently, we identify $\Omega^n X$ with the space of maps $(I^n, \partial I^n)\to(X, *)$. Now $C_n(j)$ acts on $(\Omega^n X)^j$, the j-fold Cartesian product of $\Omega^n X$, in the following way:

$$\vartheta_{n,j}:\quad C_n(j)\times_{\Sigma_j}(\Omega^n X)^j\to\Omega^n X \tag{4.2}$$

is defined by mapping the image of the i^{th} little n-cube in $J^n\subset I^n$ via the i^{th} coordinate function of $(\Omega^n X)^j$ into X and mapping the complement of the images of all j little n-cubes in I^n to the base point $*\in X$. If Σ_j acts on $(\Omega^n X)^j$ by permuting the j coordinates, then it is clear that $\vartheta_{n,j}$ is well defined.

These structures satisfy many other compatibility conditions; for example, the following diagram (see May [31]) is known to commute.

$$\begin{tikzcd}
C_n(k)\times C_n(J)\times(\Omega^n X)^j \arrow[r, "\delta\times\mathrm{id}"] \arrow[dd, "\mathrm{id}\times\mu"'] & C_n(j)\times(\Omega^n X)^j \arrow[d, "\vartheta_{n,j}"] \\
& \Omega^n X \\
C_n(k)\times C_n(J, \Omega^n X) \arrow[r, "\mathrm{id}\times\vartheta_{n,J}"] & C_n(k)\times(\Omega^n X)^k \arrow[u, "\vartheta_{n,k}"']
\end{tikzcd} \tag{4.3}$$

Here $J=(j_1,\ldots,j_k)$, $\sum_{l=1}^{k} j_l=j$, $C_n(J)=C_n(j_1)\times\cdots\times C_n(j_k)$, $C_n(J, \Omega^n X)=C_n(j_1)\times(\Omega^n X)^{j_1}\times\cdots\times C_n(j_k)\times(\Omega^n X)^{j_k}$, μ is the shuffle homeomorphism, and $\delta: C_n(k)\times C_n(j_1)\times\cdots\times C_n(j_k)\to C_n(j)$ is defined by $\delta(g; f_1,\ldots,f_k)=g(f_1+\cdots+f_k)$, where $+$ denotes disjoint union. Thus δ places the j_i disjoint little n-cubes of $f_i\in C_n(j_i)$ homeomorphically into the interior of the i^{th} little n-cube of $g\in C_n(k)$. Also, note that (4.3) is equivariant with respect to the obvious actions of the various symmetric groups $\Sigma_k, \Sigma_j, \Sigma_{ji}$.

Restricting $\vartheta_{n,2}$ to a fixed point $c\in C_n(2)$, we recover the standard "loop sum" map

$$*:\quad \Omega^n X\times\Omega^n X\to\Omega^n X,$$

which makes $\Omega^n X$ a topological "group". The main thrust of iterated loop space theory is that the presence of the additional loop factors on $\Omega^n X$ further enriches its topological structure as exhibited in (4.3).

These structure maps are formalized in both Boardman and Vogt's [8] and May's [31] theories of iterated loop spaces. More precisely, C_n, the union over all $j\geqq 0$ of the $C_n(j)$'s together with the structure maps $\delta: C_n(k)\times C_n(j_1)\times\cdots\times C_n(j_k)\to C_n(j)$, is an operad (see [31], Definition 1.1 and Theorem 4.1). Furthermore, the structure maps $\{\vartheta_{n,j}\}$, together with higher compatibilities such as (4.3), make $\Omega^n X$ a C_n space (see [31], Definition 1.2, Lemma 1.4 and Theorem 5.1). We now need the following key definition of May.

Definition 4.4 [31] Y is a C_n space if it is equipped with structure maps

$\vartheta_j: C_n(j) \times Y^j \to Y$ for all $j \geqq 0$ such that

(a) The following (the analogue of diagram 4.3) commutes:

$$\begin{tikzcd} C_n(k) \times C_n(J) \times Y^j \arrow[r, "\delta \times \mathrm{id}"] \arrow[dd, "\mathrm{id} \times \mu"'] & C_n(j) \times Y^j \arrow[d, "\vartheta_{n,j}"] \\ & Y \\ C_n(k) \times C_n(J, Y) \arrow[r, "\mathrm{id} \times \vartheta_{n,J}"] & C_n(k) \times Y^k \arrow[u, "\vartheta_{n,k}"'] \end{tikzcd} \tag{4.5}$$

(b) $\vartheta_1(1, y) = y$ for all $y \in Y$.
(c) $\vartheta_j(c\sigma, z) = \vartheta_j(c, \sigma z)$ for $c \in C_n(j), z \in Y^j$ and $\sigma \in \Sigma^j$.

Again Y^j denotes the j-fold Cartesian product of Y. In addition, May defines a C_n map between C_n spaces to be map $f:(Y, \vartheta) \to (Y', \vartheta')$ such that the following diagram commutes:

$$\begin{tikzcd} C_n(j) \times Y^j \arrow[r, "\vartheta_j"] \arrow[d, "\mathrm{id} \times f^j"'] & Y \arrow[d, "f"] \\ C_n(j) \times (Y')^j \arrow[r, "\vartheta'_j"] & Y' \end{tikzcd} \tag{4.6}$$

The following fundamental recognition theorem of May [31] relates C_n operad spaces to more familiar objects, namely n-fold loop spaces.

Theorem 4.7. *Every connected C_n operad space has the weak homotopy type of an n-fold loop space.*

In order to place a C_2 operad structure on $\mathscr{R}(m)$ we begin by defining a loop sum map

$$*: \quad \mathscr{R}(m)_{k_1} \times \mathscr{R}(m)_{k_2} \to \mathscr{R}(m)_{k_1+k_2} \tag{4.8}$$

as follows: Let α be a fixed homeomorphism of $C \simeq R^2$ with the open unit square centered at $(0,2) \in C \simeq R^2$ and let β be a fixed homeomorphism of $C \simeq R^2$ with the open unit square centered at $(0,-2) \in C \simeq R^2$.

Definition 4.9. Let $f = [p_0, p_1, \ldots, p_m] \in \mathscr{R}(m)_{k_1}$ and $g = [q_0, q_1, \ldots, q_m] \in \mathscr{R}(m)_{k_2}$, then

$$f * g \in \mathscr{R}(m)_{k_1+k_2}$$

is the unique rational function

$$[\alpha p_0 \beta q_0, \alpha p_1 \beta q_1, \ldots, \alpha p_m \beta q_m],$$

where the roots of αp_j are precisely the image under α of the roots of p_j and the roots of βq_j are precisely the image under β of the roots of q_j.

It is now easy to extend this loop sum map to obtain maps

$$\vartheta: \quad C_2(p) \times_{\Sigma_p} (\mathscr{R}(m)_k)^p \to \mathscr{R}(m)_{pk}, \tag{4.10}$$

which will permit us to define the C_2 operad structure (with associated homology operations) on $\mathscr{R}$.

We think of little cubes in I^2 as big cubes in $C^1 \simeq R^2$ in the obvious way. Thus a point in $C_2(p)$ is equivalent with p disjoint open cubes in C^1 (with sides parallel to the axes).

Definition 4.11. Let $f_i \in \mathscr{R}(m)_k$ for $1 \leqq i \leqq p$. Then

$$\vartheta(c_{1\ldots p}, f_1, \ldots, f_p) \in \mathscr{R}(m)_{pk}$$

is the unique rational function whose $p(m+1)k$ roots (in homogeneous coordinates) are uniquely determined by where the map $c_{1\ldots p} \in C_2(p)$ sends the $(m+1)k$ roots (in homogeneous coordinates) of the f_i's.

Remarks.

1. When $p = 2$ and $c_{1,2}$ is fixed, we recover the loop sum map $*$.
2. It is clear that $\vartheta(c_{1\ldots p}, f_1, \ldots, f_p)$ is a well-defined element of $\mathscr{R}(m)_{pk}$ as the roots of the distinct coordinate polynomials of the f_i's are placed in disjoint cubes by the $c_{1\ldots p}$.

We point out that this construction depends on having based maps and is not well-defined if we consider the related space of all rational functions mapping to all unbased maps of S^2 to $CP(m)$. This is analogous to the situation for instantons in the Yang–Mills theory as explained in [10].

It is now routine to verify

Theorem 4.12. *Let $\vartheta: C_2(p) \times_{\Sigma_p} (\mathscr{R}(m)_k)^p \to \mathscr{R}(m)_{pk}$ be given by (4.11). Then $(\mathscr{R}(m), \vartheta)$ is a C_2 operad space.*

Since $\mathscr{R}(m)$ is not connected we cannot immediately apply May's recognition theorem 4.7. However, May's theory still implies that $\mathscr{R}(m)$ has a classifying space. Hence it follows from Segal's theorem that

Corollary 4.13. $\Omega B\mathscr{R}(m) \simeq \Omega^2 CP(m)$.

This corollary is just a special case of the classical "group completion" theorem contained in the works of Barratt, May, Milgram, Priddy, Quillen, Segal and others in the late 1960's and early 1970's. The reader who is unfamiliar with this body of work would do well to consult Adam's book [1, chapter 3] on this point. Corollary 4.13 is equivalent to the statement that the direct limit of the $\mathscr{R}(m)_k$'s is homotopy equivalent to $\Omega_0^2 CP(m)$, which is also an immediate consequence of Segal's theorem.

It is interesting to note that the existence of the operad structure for rational functions is trivial to prove while its anolog for instantons was quite involved and delicate [10, Sect. five]. The reason for the difference is that C is abelian and thus rational functions on the Riemann sphere are completely specified by their poles and zeros, while the ADHM construction [5] for instantons is given in terms of quaternions. The failure of H to be abelian endows the ADHM maps in

$\mathrm{Map}(HP(1), HP(m))$ with a more subtle structure which is not obviously compatible with an operad structure. Thus the difficulty in placing an operad structure on monopoles has been hidden in Theorem 1.21.

Hence we find the natural inclusion $i(m): \mathscr{R}(m) \to \Omega^2 CP(m)$ is a map of C_2 spaces. However, while both $\mathscr{R}(m)$ and $\Omega^2 CP(m)$ are C_2 spaces $i(m)$ is not a C_2 map! This is evident even at the loop sum level in trying to construct a map $\mathscr{R}(m)_{k_1} \times \mathscr{R}(m)_{k_2} \xrightarrow{*} \mathscr{R}(m)_{k_1+k_2}$ that commutes with the map $\Omega^2_{k_1} CP(m) \times \Omega^2_{k_2} CP(m) \xrightarrow{*} \Omega^2_{k_1+k_2} CP(m)$ given by $\vartheta_2(c, f, g) = f * g$ for any fixed $c \in C_2(2)$. The problem arises because $\vartheta_2(c, f, g)$ is constant on a rather large set while, of course, any element of $\mathscr{R}_{k_1+k_2}$ is analytic. Fortunately, this defect can be remedied up to homotopy, which is sufficient for all computational purposes.

Theorem 4.14. *The following diagram homotopy commutes*

$$\begin{array}{ccc} \Omega^2_{k_1} CP(m) \times \Omega^2_{k_2} CP(m) & \xrightarrow{*} & \Omega^2_{k_1+k_2} CP(m) \\ \uparrow & & \uparrow \\ \mathscr{R}(m)_{k_1} \times \mathscr{R}(m)_{k_1} & \xrightarrow{*} & \mathscr{R}(m)_{k_1+k_2} \end{array} \tag{4.15}$$

where the vertical arrows are given by the natural inclusions $i(m)_j$ and $*$ is the standard loop sum map on $\Omega^2 CP(m)$.

Theorem 4.16. *The following diagram homotopy commutes*:

$$\begin{array}{ccc} C_2(p) \times_{\Sigma_p} (\Omega^2_n CP(m))^p & \xrightarrow{\vartheta} & \Omega^2_{pn} CP(m) \\ \uparrow & & \uparrow \\ C_2(p) \times_{\Sigma_p} (\mathscr{R}_n)^p & \xrightarrow{\vartheta} & \mathscr{R}_{pn} \end{array} \tag{4.17}$$

Proofs of (4.14) and (4.16). The proofs are completely analogous to that of Theorems 6.3 and 6.10 of [10]. There is a continuous way, which depends only on the $c_{1\dots p}$'s and not on the q_i's, of deforming $\vartheta(c_{1\dots p}, q_1, \dots, q_p)$, first to a map of $c_{1\dots p}$ into $CP(m)$ which is constant off the little cubes, and then to a map which is defined by the q_i's on those little cubes. Details are left to the reader.

5. Homology Operations on C_2-Spaces

Theorem 4.16 shows that $\mathscr{R}(m)$ is a C_2 space which is homotopy compatible with the standard C_2 operad structure on $\Omega^2 CP(m)$. The homology theory of C_n spaces is very rich. In particular, it is well-known that C_n spaces admit homology operations above and beyond the duals of the Steenrod operations. Araki and Kudo [2] were the first to discover and study such operations which generalize the Pontrjagin product induced by loop sum when $p = 2$. Browder [12] obtained more complete information when $p = 2$ and also studied operations when p is an odd prime. Dyer and Lashof [20] then studied the algebra of stable operations (on infinite loop spaces) for all primes p. These stable operations (which naturally give rise to unstable operations on iterated loop spaces) and their rich algebraic structure have made the Dyer–Lashof algebra a fundamental tool in algebraic topology. May's theory of iterated loop spaces [31, 13] shows that all the homology operations mentioned above live naturally on C_n operad spaces. Finally, Cohen [13] has given a complete treatment of all the mod p homology operations on C_2 spaces that we consider in this paper. We now recall the definitions of these homology operations and various fundamental facts which we will need in the next two sections. We refer the reader to [13] for a complete treatment of this theory.

Definition 5.1. Let X be a C_{n+1} space with $x \in H_q(X, Z/p)$ and $y \in H_r(X, Z/p)$. Then define

a. For $i < n$

$$Q_{i(p-1)}(x) = \vartheta_{p*}(e_{i(p-1)} \otimes x^p) \in H_{pq+i(p-1)}(X, Z/p)$$

and, for p odd,

$$Q_{i(p-1)-1}(x) = \vartheta_{p*}(e_{i(p-1)-1} \otimes x^p) \in H_{pq+i(p-1)-1}(X, Z/p).$$

b. For $p = 2$ and $s < q$

$$Q^s(x) = 0,$$

while if $s \geqq q$, then

$$Q^s(x) = Q_{s-q}(x).$$

c. For $p > 2$ and $2s < q$

$$Q^s(x) = 0,$$

while if $2s \geqq q$ then

$$Q^s(x) = (-1)^s v(q) Q_{(2s-q)(p-1)}(x).$$

d. For $2s \leqq q$

$$\beta Q^s(x) = 0,$$

while if $2s > q$

$$\beta Q^s(x) = (-1)^s v(q) Q_{(2s-q)(p-1)-1}(x),$$

where $v(q) = (-1)^{q(q-1)(p-1)/4}(((p-1)/2)!)^q$.

e. For $p = 2$

$$\xi_n(x) = \vartheta_2(e_n \otimes x \otimes x) \in H^{2q+n}(X, Z/p).$$

f. For p odd

$$\xi_n = (-1)^{(n+q)/2} v(q) \vartheta_{p*}(e_{n(p-1)} \otimes x^p) \in H_{pq+n(p-1)}(X, Z/p).$$

g. For p odd and $n + q$ even

$$\zeta_n = (-1)^{(n+q)/2} v(q) \vartheta_{p*}(e_{n(p-1)-1} \otimes x^p) \in H_{pq+n(p-1)-1}(X, Z/p).$$

h.

$$\lambda_n(x, y) = (-1)^{nq+1} \psi_*(\iota \otimes x \otimes y) \in H_{n+q+r}(X, Z/p).$$

Here $\psi: C_{n+1}(2) \times X \times X \to X$ is the Σ_2 equivariant map without the $Z/2$ quotient action on the domain and $\iota \in H_n(C_{n+1}(2), Z/p) \cong H_n(S^n, Z/p)$ is the fundamental class.

Remarks:

1. Part a. defines the operations which come from the stable Dyer–Lashof operations [20].
2. Parts b. and c. provide a dictionary for passing between lower notation $(Q_a(x))$ and upper notation $(Q^b(x))$ which was invented by May to simplify many computational formulae involving iterated operations, especially at odd primes. We shall not strictly adhere to one convention but rather pass freely to whichever notation can be used most easily to state our results.
3. Part h. defines the Browder operation [12].
4. The cells $e_i \in H_i(C_n(p)/\Sigma_p, Z/p(q))$ are dual to the i-dimensional generator in the image of $H^i(B\Sigma_p, Z/p(q)) \to H^i(C_n(p)/\Sigma_p, Z/p(q))$, see [13].
5. $Q_0(x) = x^p$, the p-fold Pontrjagin product of x with itself.
6. In general, the top operation ξ_n behaves very much like a Dyer–Lashof operation (ξ_n is precisely $Q_{n(p-1)}$ if X is a C_{n+2} operad space). Theorem 1.3 of [13, pages 217–218] catalogs the precise differences. As $\mathscr{R}(m)$ is a C_2 space, $n = 1$ in our computations. Thus ξ_1 and ζ_1, which we write as Q_{p-1} and βQ_{p-1} respectively, are the relevant operations to consider.

At this point a remark is in order. The calculus of C_n homology operations is quite intricate. We have tried to present the results that follow with no further computational prerequisites. We refer the reader who wishes to make further calculations based on the results of this paper to Cohen's concise yet encyclopedic treatment [13, pages 213–219] of the calculus.

We conclude this section with the following classical facts that we will use in the next two sections. To describe $H_*(\Omega^2 S^2, Z/p)$, recall that the identity map $S^2 \to S^2$ represents the base point in the 1 component $\Omega_1^2 S^2$ and thus a distinguished homology class $[1] \in H_0(\Omega_1^2 S^2, Z/p)$. Furthermore, if x and y are homology classes carried by the k and l components of $\Omega^2 S^2$, then $x * y$ and $Q_i(x)$ are carried by the $k + l$ and pk components respectively. The following is classical.

Theorem 5.2. *$H_*(\Omega^2 S^2, Z/2) \cong Z/2([1], Q_I(1))$, a polynomial algebra over $Z/2$, under the loop sum Pontrjagin product, on generators $[1]$ and $Q_I(1) = Q_1 Q_1 \dots Q_1(1)$.*

Furthermore, the natural fibration

$$S^1 \to S^{2m+1} \to CP(m)$$

splits after being looped once and we have the following splitting

$$\Omega^2 CP(m) \cong \Omega^2 S^{2m+1} \times Z,$$

where $\Omega^2 S^{2m+1}$ is connected and Z indexes the components of $\Omega^2 CP(m)$. We also have the classical

Theorem 5.3. $H_*(\Omega^2 CP(m), Z/2) \cong H_*(\Omega^2 S^{2m+1} \times Z, Z/2)$

$$\cong Z/2([1], \iota_{2m-1}, Q_{I_1}(\iota_{2m-1}))$$

a polynomial algebra over $Z/2$, under the loop sum Pontrjagin product, on generators

[1], ι_{2m-1}, and $Q_{I_l}(\iota_{2m-1}) = Q_1 Q_1 \dots Q_1(\iota_{2m-1})$, where I_l has length l.
As is customary, we have written Q_1 for ξ_1. When $m=1$, (5.2) is a special case of 5.3 and $\iota_1 = Q_1(1) * [-1]$.

To state the analog of (5.3) for odd primes we need a bit more notation. Let $\beta^{\varepsilon_1} Q^{s_1} \dots \beta^{\varepsilon_l} Q^{s_l}(x) = Q^I(x)$ be an iterated mod p operation on x and let $\Lambda(\cdot)$ denote the tensor product over Z/p of polynomial algebras on even dimensional generators and of exterior algebras on odd dimensional generators.

Theorem 5.4. *Let p be an odd prime. As algebras, under the loop sum Pontrjagin product:*

$$H_*(\Omega^2 CP(m), Z/p) \cong \Lambda([1], Q^{I_l}(\iota_{2m-1}), \beta Q^{I_l}(\iota_{2m-1}))$$

for $I_l = (0, p^{l-1}, 0, p^{l-2}, \dots, 0, 1)$; that is, $\varepsilon_j = 0$ and $s_j = p^{l-j}$ for all $1 \leqq j \leqq l$.

6. Homology Calculations

We are now ready to use the results of the previous sections to construct many new non-trivial classes in $H_*(\mathscr{R}(m)_k, Z/p)$ for all primes p. As mentioned in the introduction we recapture the classes discovered by Segal for $* < k(2m-1)$ as well as generate new classes above the range of the equivalence induced by the natural inclusion. Theorems 4.14 and 4.16 imply that the following diagrams commute for all $m \geqq 1$:

$$\begin{tikzcd} H_s(\Omega^2_k CP(m), Z/p) \otimes H_t(\Omega^2_l CP(m), Z/p) \arrow[r, "*"] & H_{s+t}(\Omega^2_{k+l} CP(m), Z/p) \\ H_s(\mathscr{R}(m)_k, Z/p) \otimes H_t(\mathscr{R}(m)_l, Z/p) \arrow[u, "i_{k*} \otimes \iota_{l*}"] \arrow[r, "*"] & H_{s+t}(\mathscr{R}(m)_{k+l}, Z/p) \arrow[u, "\iota_{k+l*}"'] \end{tikzcd} \tag{6.1}$$

and

$$\begin{tikzcd} H_s(\Omega^2_k CP(m), Z/p) \arrow[r, "Q_{p-1}"] & H_{ps+p-1}(\Omega^2_{pk} CP(m), Z/p) \\ H_s(\mathscr{R}(m)_k, Z/p) \arrow[u, "i_{k*}"] \arrow[r, "Q_{p-1}"] & H_{ps+p-1}(\mathscr{R}(m)_{pk}, Z/p) \arrow[u, "\iota_{pk*}"'] \end{tikzcd} \tag{6.2}$$

Lemma 6.3. *There exists an element $\iota_{2m-1} \in H_{2m-1}(\mathscr{R}(m)_1, Z/p)$ such that ι_{2m-1} is sent to $\iota_{2m-1} \in H_{2m-1}(\Omega^2_1 CP(m), Z/p)$ under the natural inclusion i_1.*

Proof. This follows immediately from Segal's Theorem.

If we start with the generator $\iota_{2m-1} \in H_{2m-1}(\mathscr{R}(m)_1, Z/p)$ and compute iterated operations on ι_{2m-1} and loop sums of such elements by using the commutivity of diagrams (6.1) and (6.2) and the known C_2 structure of $H_*(\Omega^2 CP(m), Z/p)$, we may construct many non-zero homology classes in $H_q(\mathscr{R}(m)_k, Z/2)$, including new classes where $q > k(2m-1)$. The following two theorems summarize the homology generated by the operations.

Theorem 6.4. *$H_*(\mathscr{R}(m)_k, Z/2)$ contains elements of the form*

$$Q_{I_1}(\iota_{2m-1})*\cdots*Q_{I_n}(\iota_{2m-1})*[k-l] \tag{6.5}$$

for all sequences $(I_1,\ldots,I_n)$ such that $l=\sum_{m=1}^{n} 2^{l(Im)} \leqq k$. Here each I_m is one of the sequences given in (5.3). Furthermore, the image of this class in $H_*(\Omega^2 CP(m)_k, Z/2)$ is given by the element of the same name.
and

Theorem 6.6. *$H_*(\mathscr{R}(m)_k, Z/p)$ contains elements of the form*

$$\beta^{\varepsilon_1} Q^{I_1}(\iota_{2m-1})*\cdots*\beta^{\varepsilon_n} Q^{I_n}(\iota_{2m-1})*[k-l] \tag{6.7}$$

for all sequences $(I_1,\ldots,I_n)$ such that $l=\sum_{m=1}^{n} p^{l(I_m)} \leqq k$. Here each I_m is one of the sequences given in (5.4) and each ε_j may be either 0 or 1. Furthermore, the image of this class in $H_*(\Omega^2 CP(m)_k, Z/p)$ is given by the element of the same name.

Remark. We may use diagrams (6.1) and (6.2) to generate many classes of the form $Q_{p-1}(x*y)$ however, the Cartan formula [13] decomposes these classes into a sum of classes of the form (6.5) or (6.7).

In particular, we have recovered both a weak form of Segal's theorem in homology, and a version of [35, Proposition 5.3].

Corollary 6.8. *$(i(m)_k)_*$ is surjective for $* \leqq k(2m-1)$.*

Corollary 6.9. *The inclusions*

$$j(k_1,k_2)\colon H_*(\mathscr{R}(m)_{k_1}, Z/p)\to H_*(\mathscr{R}(m)_{k_2}, Z/p)$$

induced by loop summing with $[k_2-k_1]$ are injective on the image of

$$i(m)_{k_1}\colon H_*(\mathscr{R}(m)_{k_1}, Z/p)\to H_*(\Omega^2_{k_1} CP(m), Z/p).$$

It is easy to list the classes generated by Theorems 6.4 and 6.6 of dimension greater than $k(2m-1)$ and to obtain a better bound on how fast the homology of the space of rational maps, equivalently the moduli space of monopoles, is growing. We close our analysis by highlighting the top dimensional non-trivial classes generated by this method.

Corollary 6.10. *Let $k=2^j$. Then*

$$Q_1 Q_1 \cdots Q_1(\iota_{2m-1})\in H_q(\mathscr{R}(m)_k, Z/2)$$

has non-zero image in $H_q(\Omega^2 CP(m)_k, Z/2)$. Here $q=2^{j+1}m-1$.

Corollary 6.11. *Let $k=p^j$. Then*

$$Q^{p^{j-1}m} Q^{p^{j-2}m}\cdots Q^{pm} Q^m(\iota_{2m-1})\in H_q(\mathscr{R}(m)_k, Z/p)$$

has non-zero image in $H_q(\Omega^2 CP(m)_k, Z/p)$. Here $q=(2p^j)m-1$.

Thus in the special case for monopoles we see ((6.10) and (6.11)) that, while the Segal homotopy equivalence holds for $q<\frac{1}{4}d$, where d is the dimension of $\hat{\mathscr{M}}_k$, there are non-trivial classes in $H_q(\hat{\mathscr{M}}_k)$ for $q<\frac{1}{2}d$ for infinitely many k.

Recall that when $m = 1$ the results stated above hold equally well both for spaces of rational maps and for moduli spaces of monopoles. We conclude this section with the observation that (3.2) and (2.5) imply that when $m > 1$ the results stated above hold equally well for spaces both of rational and of harmonic maps.

7. Proof of Proposition 1.22

The proof of 1.22 amounts to keeping track of continuity while tracing through the Donaldson–Hitchin–Nahm correspondence between monopoles and rational maps. We do this in three steps: Based rational maps $\overset{1}{\Leftrightarrow}$ equivalence classes of Nahm complexes $\overset{2}{\Leftrightarrow}$ certain algebraic curves on the space of lines in $R^3 \overset{3}{\Leftrightarrow}$ Gauge equivalence classes of $SU(2)$-monopoles.

We begin by giving Donaldson's definition of a Nahm complex [18]: The triple (α, β, v) is called a *Nahm complex* if $v \in C^k$ and α and β are smooth maps from the open interval $(0, 2)$ to the $k \times k$ complex matrices $M^k(C)$ which satisfy:

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0, \tag{7.1}$$

$$\frac{d(\alpha + \alpha^*)}{ds} + 2([\alpha, \alpha^*] + [\beta, \beta^*]) = 0, \tag{7.2}$$

where $\alpha(2 - s) = \alpha^T(s), \beta(2 - s) = \beta^T(s), \alpha$ and β are meromorphic in some neighborhood of $s = 0$ and $s = 2$ with simple poles at 0 and 2, and with residues a and b respectively at $s = 0$. In addition, $\mathrm{Tr}(a) = 0$ and v is an eigenvector of a with eigenvalue $-(k - 1)/4$. Furthermore, $\{v, bv, \ldots, b^{k-1}v\}$ span C^k.

The topology on the set N_k of Nahm complexes is that induced as a subspace of $\mathrm{Map}((0, 2), M^k(C)) \times \mathrm{Map}((0, 2), M^k(C)) \times C^k$. Consider the space of continuous maps $\mathrm{Map}([0, 2], GL(k, C))$ and let $\mathscr{H}$ denote the subset of maps that are smooth on $(0, 2)$ and satisfy $g^T(s)g(2 - s) = I$. $\mathscr{H}$ acts on N_k by:

$$\alpha' = g\alpha g^{-1} - \frac{1}{2}\frac{dg}{ds}g^{-1}, \tag{7.3}$$

$$\beta' = g\beta g^{-1}, \tag{7.4}$$

$$v' = g(0)v \tag{7.5}$$

for $g \in \mathscr{H}$. We are interested in the quotient space $N_k/\mathscr{H}$.

Donaldson [18] shows that there is a one correspondence between based rational maps of degree $k, \mathscr{R}_k$, and elements of $N_k/\mathscr{H}$ given as follows: Consider the open subset of pairs $(B, w) \in M^k(C) \times C^k$, where B is symmetric and which satisfies the condition that w generates C^k as a $C[B]$ module. The group $O(k, C)$ acts on this subspace by conjugation on $M^k(C)$ and by the standard action on C^k. Let $\mathscr{Q}_k$ denote the quotient space. Donaldson shows that the map $\mathscr{Q}_k \to \mathscr{R}_k$ defined by

$$(B, w) \mapsto f(z) = w^T(zI - B)^{-1}w$$

is a bijection. To obtain the correspondence between $\mathcal{Q}_k$ and $N_k/\mathcal{H}$, consider the unique "covariantly constant" map $u:(0,2)\to C^k$.

$$D_\alpha u \equiv \frac{1}{2}\frac{du}{ds} + \alpha u = 0, \tag{7.6}$$

which satisfies $u(s) \sim vs^{(k-1)/2}$ as $s\to 0$. If

$$B = \beta(1), \tag{7.7}$$

and

$$w = u(1), \tag{7.8}$$

then the elements of $N_k/\mathcal{H}$ are in one to one correspondence with the elements of $\mathcal{Q}_k$.

Lemma 7.9. *The Donaldson bijection described above is a homeomorphism between $N_k/\mathcal{H}$ and $\mathcal{R}_k$.*

Proof. That $\mathcal{R}_k$ and $\mathcal{Q}_k$ are homeomorphic follows easily from the explicit form of $f(z)$. To describe the homeomorphism between $\mathcal{Q}_k$ and $N_k/\mathcal{H}$, we first remove the regular singular point at $s=0$ (a similar argument works at $s=2$) in (7.6) by the substitution $u(s)=s^{(k-1)/2}y(s)$, for then y satisfies a regular equation in $[0,1]$, viz.,

$$\frac{dy}{ds} + ry = 0, \tag{7.10}$$

where r is analytic at $s=0$ and $y(0)=v$.

Near $s=0$ we can represent (α,β,v) as

$$\alpha(s) = \frac{a}{s} + \tfrac{1}{2}g(s)^{-1}\frac{dg}{ds}, \tag{7.11}$$

$$\beta(s) = \frac{b}{s} + g(s)^{-1}(B-b)g(s), \tag{7.12}$$

$$v = \lim_{s\to\infty} s^{(-k-1)/2}u(s) = y(0), \tag{7.13}$$

where $g(1)=I$, the identity element. Thus the continuity of the map $[B,w]\mapsto[\alpha,\beta,v]$ follows from (7.11), (7.12), (7.13) and the smoothness of y as a function of its initial condition $y(1)=w$.

To show that the inverse map $[\alpha,\beta,v]\mapsto[B,w]$ is continuous amounts, by (7.7) and (7.8), to showing that y is a continuous function of r. Suppose for $i=1$ and 2 we have

$$\frac{dy_i}{ds} + r_i y_i(s) = 0,$$

with $y_i(0)=v_i$. Then we can write

$$y_1'(s) - y_2'(s) = -r_1(s)(y_1(s)-y_2(s)) + (r_2(s)-r_1(s))y_2(s).$$

From Gronwall's lemma, [30, page 87], we obtain an estimate

$$|y_1(s)-y_2(s)| \leqq (|v_1-v_2| + \|y_2\|_{L^\infty(0,1)}\|r_2-r_1\|_{L^1(0,s)})\operatorname{Exp}\|r_1\|_{L^1(0,s)}.$$

This proves continuity since, for regular functions, the compact-open topology coincides with the uniform topology on [0, 1].

The correspondence 2 given by Hitchin [24, 25] (see also [26]) depends crucially on the existence of a certain holomorphic curve S, the spectral curve, in the twistor space T of lines in R^3, which is known to be just the holomorphic tangent bundle of the Riemann sphere, $TCP(1)$. Hitchin [24, 25] shows how Nahm's equations, (7.1) and (7.2), can be solved by certain algebraic curves S in $TCP(1)$. The gauge group $\mathscr{H}$ can be used to choose $\alpha = \alpha^*$ and the stabilizer of this condition in $\mathscr{H}$ is $O(k, R)$. Consequently, we have the homeomorphism

$$N_k/\mathscr{H} \simeq N_k^0/O(k, R) \equiv \hat{\mathscr{N}}_k,$$

where

$$N_k^0 = \{(\alpha, \beta, v) \in N_k \mid \alpha = \alpha^* \quad \text{and} \quad \bar{v} = e^{i\theta} v\}.$$

Since the "real" eigendirection is determined completely by α, we have an S^1 bundle

$$\begin{array}{c} S^1 \to \hat{\mathscr{N}}_k \\ \downarrow \\ \mathscr{N}_k \end{array} \tag{7.14}$$

where $\mathscr{N}_k$ is the space of $O(k, R)$ conjugacy classes of pairs (α, β) which satisfy

$$\frac{d\beta}{ds} + 2[\alpha, \beta] = 0, \tag{7.15}$$

$$\frac{d\alpha}{ds} + [\beta, \beta^*] = 0, \tag{7.16}$$

$$\alpha(s) = \alpha(s)^*, \quad \alpha(2-s) = \alpha^T(s) \quad \text{and} \quad \beta(2-s) = \beta^T(s), \tag{7.17}$$

where α and β are meromorphic functions on a neighborhood of [0, 2] with simple poles at $s = 0$ and 2 whose residues a and b define an irreducible representation of dimension k of $su(2)$.

We need to show that $\hat{\mathscr{N}}_k$ is homeomorphic to $\hat{\mathscr{M}}_k$. First we show that $\mathscr{N}_k$ and $\mathscr{M}_k$ are homeomorphic. Hitchin associates to each $[\alpha, \beta] \in \mathscr{N}_k$ a real algebraic curve S in $TCP(1)$ as follows: Let $\zeta \in C$ be the coordinate on $CP(1)$ centered at 0 and (η, ζ) be the standard coordinates in $TCP(1)$. To each $[\alpha, \beta]$ we associate

$$R(\zeta) = \beta - 2\alpha\zeta + \beta^* \zeta^2 \tag{7.18}$$

and

$$R_+(\zeta) = \alpha + \beta^* \zeta, \tag{7.19}$$

which satisfy the isospectral deformation equation

$$\frac{dR}{ds} = 2[R_+, R] \tag{7.20}$$

and the reality condition

$$R(\zeta)^* = \zeta^2 R(\bar{\zeta}^{-1}). \tag{7.21}$$

Then to each $\{\alpha,\beta\}\in\mathcal{N}_k$ we associate a divisor of a section

$$\phi([\alpha,\beta])=\det(\eta I+R(\zeta))$$

of the linear system $|\pi^*\mathcal{O}(2k)|$ on $TCP(1)$. This defines the genus $(k-1)^2$ real curve S by the equation

$$\det(\eta I+R(\zeta))=0. \tag{7.22}$$

The isospectral deformation equation guarantees that S is an invariant of Nahm's equations and thus so is its spectrum. Hitchin shows that S must satisfy certain conditions (B1–B4 of [25]):

1. S has no multiple components.
2. S is real with respect to the standard real structure on $TCP(1)$.
3. If L^s is the line bundle on $TCP(1)$ with transition functions $e^{s\eta/\zeta}$ with respect to the standard coordinates (η,ζ), then L^2 is trivial on S and

$$L^s(k-1)=L^s\otimes\pi^*\mathcal{O}(k)$$

is real.
4. $H^0(S,L^s(k-2))=0$ for all $s\in(0,2)$.

It follows from (7.18) and (7.22) that S is a continuous function of the equivalence class $\{\alpha,\beta\}\in\mathcal{N}_k$. Conversely, given a compact algebraic curve S in the linear system $|\pi^*\mathcal{O}(2k)|$ which satisfies conditions B1–B4, it must have the form (7.22) and hence determines (α,β) up to an $O(k,R)$ transformation. In making a choice of representative, α and β appear as the coefficients of $R(\zeta)$ and thus depend continuously on S.

It remains to show that the space of compact algebraic curves S in the linear system $|\pi^*\mathcal{O}(2k)|$ satisfying conditions B1–B4 is homeomorphic to $\mathcal{M}_k$. Following Hitchin [25], we pick a representative $\{\nabla,\Phi\}\in\mathcal{M}_k$ and define a rank two holomorphic vector bundle $\tilde{E}$ on $TCP(1)$ as follows: Pick a point $z\in TCP(1)$, let γ_z be the oriented line in R^3 represented by z, and u be the unit tangent vector along γ^z. Define

$$\tilde{E}_z=\{s\in\Gamma(\gamma_z,E)|(\nabla_u-i\Phi)s=0\}.$$

Then $\tilde{E}$ satisfies the following conditions:

1. $\tilde{E}$ is trivial on every real section.
2. $\tilde{E}$ has a symplectic structure.
3. $\tilde{E}$ has an anti-holomorphic linear map $\sigma:\tilde{E}_z\to\tilde{E}_{\tau z}$ such that $\sigma^2=-1$, where $\tau(\eta,\zeta)=(-(\bar{\eta}/\zeta^2),-(1/\zeta))$.
4. If t parameterizes the line γ_z and $L_z^{\pm}$ denotes the subspace of $\tilde{E}_z$ such that $s(t)\to 0$ as $t\to\pm\infty$, then L^+ is a holomorphic subbundle of $\tilde{E}$ isomorphic to $L^1(-k)$. Thus $\tilde{E}$ can be represented by an extension

$$0\to L^1(-k)\to\tilde{E}\to L^1(-k)\to 0.$$

The spectral curve is then defined by

$$S=\{z\in TCP(1)|L_z^*=L_z^-\}. \tag{7.23}$$

Remarks.

1. Hitchin [24] showed that *every* holomorphic rank two bundle on $TCP(1)$ satisfying the above conditions comes from a solution to the Bogomol'nyi equations, (1.9), satisfying certain boundary conditions.
2. Taubes (private communication) has shown that the boundary conditions used by Hitchin (see [24, page 589] or [25, page 146]) are equivalent to those used here; that is, those of [39, 40].
3. The spectral curve S depends only on the gauge class $\{A, \Phi\}$.

Now, arguing precisely as in Lemma 7.9, the sections s satisfying the ordinary differential equation

$$(\nabla_u - i\Phi)s = 0 \tag{7.24}$$

depend continuously on (A, Φ), where A is a connection 1-form associated to ∇ with respect to a local trivialization. Hence the line bundles $L^{\pm}$ and thus the spectral curve S vary continuously with (A, Φ). It then follows that the map $\mathcal{M}_k \to \mathcal{N}_k$ described by Hitchin is continuous.

To finish the proof that $\mathcal{N}_k$ and $\mathcal{M}_k$ are homeomorphic it suffices to prove

Lemma 7.25. *The map given by Nahm's* ADHM *construction is continuous.*

Proof. The connection coefficients and Higgs field are given by Nahm [32] as

$$A = \int_0^2 y^* D y \, ds \tag{7.26}$$

and

$$\Phi = \int_0^2 y^* s y \, ds, \tag{7.27}$$

where

1. D denotes the derivative with respect to $x \in R^3$.
2. $y \in \ker \Delta^* \cap L^2(0,2) \otimes C^k \otimes H$.
3. $\Delta^* = i(d/ds) - iT + \bar{x}$.
4. $T = (\beta - \beta^*)i - (\beta + \beta^*)k + 2\alpha j$.

Hitchin shows ([25, pages 150–151]) that the residue of the pole at zero of T decomposes into the direct sum of two irreducible representations of $su(2)$, namely labeling by dimension, $[k+1] \oplus [k-1]$. The regular solution of Δ^* at 0 is the representation $[k+1]$ and $y \sim s^{(k+1)/2}$ near 0. We can remove the pole at zero by the substitution $y(s,x) = s^{(k+1)/2} f(s,x)$ and, in the representation $[k+1]$, the initial value $f(0,x)$ is the lowest weight vector of $\mathrm{Res}_{s=0}(-iT)$. We restrict ourselves here to the interval $[0,1]$ but a similar analysis works for the interval $[1,2]$.

We need to show that (A_i, Φ_i) for $i = 1,2$ are close in $\mathcal{M}_k$ whenever the Nahm complexes (α_i, β_i) are close in N_k^0. We note that here (A, Φ) depend only on the equivalence class $\{\alpha, \beta\}$. From (7.26), (7.27) and the discussion above, it suffices to show that $D^j f_1$ and $D^i f_2$ are always close for any $j \in Z$ and for all x in a compact set $K \subset R^3$. Writing $\Delta^* y = 0$ as a real equation, we have

$$\frac{df_i}{ds}(s,x) = (r_i(s) + l(x)) f_i(s,x),$$

where $r_i(s)$ is analytic near $s=0$ and $l(x)$ is a linear function of x. From a standard theorem in ODE's $f_i(s,x)$ is C^∞ in the parameter x, and thus we seen that $D^j f_i$ satisfies a similar equation, namely

$$\frac{dD^j f_i}{ds} = (r_i(s) + l(x)) D^j f_i + j c_0 D^{j-1} f_i. \tag{7.28}$$

Again, from Gronwall's lemma we obtain an estimate

$$|D^j f_1(s,x) - D^j f_2(s,x)| \leqq (A + BC) D,$$

where

1. $A = c_1 \| D^{j-1} f_1 - D^{j-1} f_2 \|_{L^1(0,s)}$,
2. $B = \| r_1 - r_2 \|_{L^1(0,s)}$,
3. $C = \| D^j f_2 \|_{L^\infty(0,s)}$,
4. $D = \operatorname{Exp} \| r_1 + l(x) \|_{L^1(0,s)}$.

This estimate is valid for all $s \in [0,1]$ and for all $x \in K$. The result then follows by induction (and also noting that $r_i(s)$ are linear functions of α_i and β_i). This concludes the proof of the lemma.

Thus $\hat{\mathcal{M}}_k$ and $\hat{\mathcal{N}}_k \cong \mathcal{R}_k$ are S^1 bundles over homeomorphic spaces ($\mathcal{M}_k$ and $\mathcal{N}_k$). Since both bundles have sections, the proof of Proposition 1.22 is complete.

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