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Recursion Operators and Bi-Hamiltonian Structures in Multidimensions. I

P. M. Santini* and A. S. Fokas

Department of Mathematics and Computer Science and Institute for Nonlinear Studies, Clarkson University, Potsdam, NY 13676, USA

Abstract. The algebraic properties of exactly solvable evolution equations in one spatial and one temporal dimensions have been well studied. In particular, the factorization of certain operators, called recursion operators, establishes the bi-Hamiltonian nature of all these equations. Recently, we have presented the recursion operator and the bi-Hamiltonian formulation of the Kadomtsev-Petviashvili equation, a two spatial dimensional analogue of the Korteweg-deVries equation. Here we present the general theory associated with recursion operators for bi-Hamiltonian equations in two spatial and one temporal dimensions. As an application we show that general classes of equations, which include the Kadomtsev-Petviashvili and the Davey-Stewartson equations, possess infinitely many commuting symmetries and infinitely many constants of motion in involution under two distinct Poisson brackets. Furthermore, we show that the relevant recursion operators naturally follow from the underlying isospectral eigenvalue problems.

1. Introduction

In recent years a deep connection has been discovered [1, 2] between certain nonlinear evolution equations in $1+1$, i.e. in one spatial and one temporal dimensions, and certain linear isospectral eigenvalue (or scattering) equations. These isospectral problems play a central role in developing methods for solving several types of initial value problems of the associated nonlinear evolution equations. The most well known such method, the celebrated inverse scattering transform (IST) method, deals with initial data decaying at infinity. However, the isospectral problem is also crucial for characterizing periodic [3] as well as self similar solutions [4].

It is quite satisfying, from a unified point of view, that the isospectral problems are also central in investigating the "algebraic" properties of the associated

* Permanent address: Dipartimento di Fisica, Università di Roma, La Sapienza, I-00185 Roma, Italy

nonlinear evolution equations: The isospectral problem algorithmically implies a certain linear integrodifferential operator Φ, called the recursion operator. This operator has remarkable properties: Φ maps symmetries into symmetries; Φ has a certain algebraic property [5] which Fuchssteiner [6] calls hereditary and thus generates commuting symmetries; Φ^*, the adjoint of Φ, maps gradients of conserved quantities into gradients of conserved quantities; Φ, admits a symplectic-cosymplectic factorization and thus generates constants of motion in involution [7]; Φ times the first Hamiltonian operator produces the second Hamiltonian [8], hence the associated nonlinear evolution equations are bi-Hamiltonian systems; the eigenfunctions of Φ are also symmetries, which actually characterize the N-soliton solutions [9]; the eigenfunctions of Φ form a complete set [10].

Well-known scattering problems in $|+|$ are the Schrödinger scattering problem, the so-called generalized Zakharov-Shabat (ZS) or Ablowitz-Kaup-Newell-Segur (AKNS) system, and their natural generalization, i.e. the Gel'fand-Dikii operator, and the $N \times N$ AKNS. These isospectral problems are related to several physically important equations, the Korteweg-deVries (KdV), sine-Gordon, nonlinear Schrödinger, modified KdV, Boussinesq, N-wave interaction equations, etc. The above eigenvalue problems have been thoroughly investigated with respect to both the IST method and the associated algebraic properties. The IST of the Schrödinger was investigated in [1, 11], of the AKNS in [12], of the $N \times N$ AKNS in [13–15], and of the Gel'fand-Dikii in [16]. The IST of special important cases of the above systems were investigated in [17] (nonlinear Schrödinger), [18] (modified KdV), [19, 20] (Boussinesq), [21] (3-wave interactions). The recursion operator associated with the Schrödinger equation was obtained by Lenard, of the AKNS in [12], of the Gel'fand-Dikii in [22] and of the $N \times N$ AKNS in [5] and [23]. The general theory of recursion operators and their connection to bi-Hamiltonian formulation has been developed by Magri [8], Gel'fand and Dorfman [24], and Fokas and Fuchssteiner [7]. Other relevant works include [25].

It is also well known that certain two-dimensional generalizations of the above scattering equations are related to physically interesting nonlinear evolution equations in $2+1$ dimensions. In particular, a generalization of the Schrödinger equation is related to the Kadomtsev-Petviashvili (KP) equation (a two-dimensional analogue of the KdV). Similarly, the two-dimensional version of the $N \times N$ AKNS is related to N-wave interactions in $2+1$, the Davey-Stewartson equation (DS) (a two-dimensional analogue of the nonlinear Schrödinger) and the modified KP equation. The IST for the above two scattering problems has been only recently studied [26]. (For other interesting results in this direction see also [27].) In spite of this success, the question of using the scattering equations to obtain recursion operators had remained open. Actually, Zakharov and Konopelchenko [28] have shown that recursion operators of a certain type, naturally motivated from the results in $1+1$, do not in general exist in multidimensions. Recursion operators in $2+1$ dimensions were only known for straightforward examples like the $2+1$ dimension Burgers equation, that can be linearized via a generalized Cole-Hopf transformation [30b]. For a brief review of the literature of the various attempts to obtain recursion operators in $2+1$, we refer the reader to [29]. Here we only note that Konopelchenko and Dubrovsky [30a] were the first to establish the importance of working with $w(x, y_1)w^+(x, y_2)$, as opposed to $w(x, y)w^+(x, y)$, where $w(x, y)$ and $w^+(x, y)$ denote the eigenfunctions of the associated scattering problem and of its adjoint, respectively. They also found a linear equation satisfied by $w(x, y_1)w^+(x, y_2)$. However, they failed to recognize that this equation could actually yield the recursion operator of the entire associated hierarchy of nonlinear equations. Instead, they used the above equation to obtain "local" recursion operators. Thus, the question of studying the remarkably rich structure of the recursion operator, in particular, its connection to symmetries, conservation laws and bi-Hamiltonian operators was not even posed.

Using a suitable generalization, we have recently presented the recursion operator and the two Hamiltonian operators associated with the KP equation [29]. In this paper we present the theory associated with these operators. In particular, the notions of symmetries, gradients of conserved quantities, strong and hereditary symmetries, Hamiltonian operators are generalized to equations in 2+1. Also a simple algorithmic approach is given for obtaining the recursion operator from the scattering problem. As examples of the above theory we study the two-dimensional Schrödinger problem and the 2×2 AKNS problem in two spatial dimensions. The following concrete results are given:

i) The linear eigenvalue problem

$$w_{xx}+q(x,y)w+\alpha w_y=0, \tag{1.1}$$

where α is a constant parameter, gives rise to the *hereditary* recursion operator

$$\Phi_{12}=D^2+q_{12}^+ + Dq_{12}^+D^{-1}+q_{12}^-D^{-1}q_{12}^-D^{-1}, \tag{1.2a}$$

where the operators $q_{12}^{\pm}$ are defined by

$$q_{12}^{\pm} \doteq q_1 \pm q_2 + \alpha(D_1 \mp D_2), \qquad D_i \doteq \frac{d}{dy_i}, \qquad q_i \doteq q(x,y_i), \qquad i=1,2. \tag{1.2b}$$

The operator Φ_{12} admits a factorization in terms of compatible Hamiltonian operators, $\Phi_{12}=\Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$, where $\Theta_{12}^{(1)}=D$ and $\Theta_{12}^{(2)}$ are skew symmetric operators satisfying an appropriate Jacobi identity.

The KP equation

$$q_t=q_{xxx}+6qq_x+3\alpha^2D^{-1}q_{yy} \tag{1.3}$$

is the second member, $n=1$ ($\beta_1=1/2$) of the following hierarchy generated by Φ_{12}

$$q_{1_t}=\beta_n \int_{-\infty}^{\infty} dy_2\delta(y_1-y_2)\Phi_{12}^n\sigma_{12}^{(0)}, \qquad n=0,1,2,\ldots, \tag{1.4}$$

where $\sigma_{12}^{(0)}=(\Phi_{12}D)\cdot 1=q_{1_x}+q_{2_x}+(q_1-q_2)D^{-1}(q_1-q_2)+\alpha D^{-1}(q_{1_{y_1}}-q_{2_{y_2}})$ and $\delta(y_1-y_2)$ is the Dirac delta function. The KP is a bi-Hamiltonian system:

$$q_{1_t}=\int_{-\infty}^{\infty} dy_2\delta(y_1-y_2)\Theta_{12}^{(1)}\gamma_{12}^{(1)}=\int_{-\infty}^{\infty} dy_2\delta(y_1-y_2)\Theta_{12}^{(2)}\gamma_{12}^{(0)}, \tag{1.5}$$

where

$$\gamma_{12}^{(0)}=D^{-1}\sigma_{12}^{(0)}, \qquad \gamma_{12}^{(1)}=D^{-1}\Phi_{12}\sigma_{12}^{(0)}. \tag{1.6}$$

The KP equation possesses two infinite hierarchies of time-independent commuting symmetries and constants of motion. For example, $(\Phi_{12}^{n}\sigma_{12}^{(0)})_{11}$, $n=0,1,2,\ldots$ are symmetries of the KP.

The operator Φ_{12} is the adjoint with respect to an appropriate bilinear form (see Sect. 4) of the "squared eigenfunction" operator. One may verify that

$$\Phi_{12}^{*}w_1w_2^{+}=0, \qquad w_i \doteq w(x,y_i), \tag{1.7}$$

where w^{+} satisfies the adjoint of Eq. (1.1) (see Sect. 4).

ii) The linear eigenvalue problem

$$W_x=JW_y+QW, \tag{1.8}$$

where $J=\alpha\sigma$, $\sigma=\mathrm{diag}(1,-1)$, and Q is a 2×2 off-diagonal matrix containing the potentials $q_1(x,y)$, $q_2(x,y)$, gives rise to the hereditary recursion operator Φ_{12} defined on off-diagonal matrices, where

$$\Phi_{12} \doteq \sigma(P_{12}-Q_{12}^{+}P_{12}^{-1}Q_{12}^{+}), \tag{1.9a}$$

and the operators P_{12}, $Q_{12}^{\pm}$ are defined by

$$P_{12}F_{12} \doteq F_{12_x}-JF_{12_{y_1}}-F_{12_{y_2}}J, \qquad Q_{12}^{\pm}F_{12} \doteq Q_1F_{12} \pm F_{12}Q_2, \tag{1.9b}$$

and $Q_i \doteq Q(x,y_i)$, $i=1,2$. The operator Φ_{12} admits a factorization in terms of Hamiltonian operators, $\Phi_{12}=\Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$, where $\Theta_{12}^{(1)}=\sigma$.

The DS equation

$$iq_t+\tfrac{1}{2}(q_{xx}+\alpha^2q_{yy})=q(\phi-|q|^2); \qquad \phi_{xx}-\alpha^2\phi_{yy}=2|q|_{xx}^2, \tag{1.10}$$

corresponds to $q_2=\bar{q}_1=\bar{q}$, $\beta_2=-\dfrac{i}{4}$, and $n=2$ of the following hierarchy

$$Q_{1t}=\beta_n\int_{\mathbb{R}} dy_2\Phi_{12}^{n}Q_{12}^{-}\sigma. \tag{1.11}$$

The DS equation is also a bi-Hamiltonian system with respect to the two Hamiltonian operators $\Theta_{12}^{(1)}=\sigma$ and $\Theta_{12}^{(2)}=\Phi_{12}\sigma$ defined on off-diagonal matrices. It also possesses two infinite hierarchies of time independent commuting symmetries and constants of motion.

In more detail, this paper is organized as follows: In Sect. 2 we review the algebraic properties of equations in $1+1$. The KdV equation is used as an illustrative example. This is in a sense a summary of [7, 8, 24] although we follow the notation of [7]. In Sect. 3 we derive algorithmically the recursion operators (1.2), (1.9). This derivation is simpler than the one given in [29]; we now use expansions in terms of $d^{\ell}\delta(y_1-y_2)/dy_1^{\ell}$, where δ denotes Dirac's function, as opposed to expansions in terms of λ^{ℓ}. In Sect. 4 we show how Φ_{12} generates extended symmetries σ_{12} and extended gradients of conserved quantities γ_{12}. We then show that σ_{11},γ_{11} are symmetries and gradients of conserved quantities, respectively. Furthermore, the remarkably rich theory associated with the bi-Hamiltonian factorization of Φ_{12} is developed in this section. In developing this theory we use two important notions: a) The role of Frechét derivative is now played by an appropriate directional derivative, which is naturally motivated from the underlying isospectral problem. b) An extended symmetry σ_{12} can be written as $\hat{\sigma}_{12} \cdot 1$, where $\hat{\sigma}_{12}$ is an appropriate operator. The Lie algebra of these operators is closed provided they act on appropriate functions H_{12}. Thus in $2+1$ one is dealing with a Lie algebra of operators as opposed to a Lie algebra of functions. In Sect. 5 we give concrete illustrations of the notions introduced in Sect. 4.

We note that Fuchssteiner and one of the authors (ASF) introduced an alternative way for generating symmetries, the so-called mastersymmetry approach. In particular, it is shown in [31] that for the Benjamin-Ono equation $u_t = K$, the map $[\cdot, \tau]_L$, where the bracket $[\,,]_L$ is defined in (2.16b), $\tau = xK + u^2 + \frac{3}{2}Hu_x$, and H denotes the Hilbert transform, maps symmetries into symmetries. This approach has been applied to KP in [32], and its general theory has been developed in [33] (for other applications see [34]). However, the τ has certain disadvantages: a) The relationship between τ and the eigenvalue problem has not been established. b) τ is not hereditary. c) It is not known if τ can be used to obtain the second Hamiltonian. In [35] we develop further the theory presented here. In particular, we: i) analyze further the Lie algebra of the starting symmetries and use Φ_{12} to generate time-dependent symmetries, ii) use an isomorphism between Lie and Poisson brackets to show that all these symmetries correspond to extended gradients and hence give rise to conserved quantities, iii) show that the τ mentioned above comes from a time dependent symmetry, and since it corresponds to a gradient cannot be used to generate Φ_{12}, iv) find a non-gradient mastersymmetry (for KP it is $\Phi_{12}^{(2)}\delta_{12}$) which can be used to generate Φ_{12}, v) motivate and verify some of the results presented here and in [35] by establishing that equations in $2+1$ are exact reductions of certain nonlocal evolution equations, of which the algebraic properties are straightforward.

Since two central aspects of integrable equations in $2+1$, namely the IST method and the associated algebraic properties, have now successfully been studied, we speculate that essentially all aspects of equations in $1+1$ will be successfully studied for equations $2+1$. (For example, asymptotics and action-angle formulation of KP have been studied in [36].)

2. Review of Algebraic Properties in $1+1$

We consider evolution equations of the form

$$q_t = K(q), \tag{2.1}$$

where q is an element of some space S of functions on the real line vanishing rapidly for $|x| \to \infty$, and K is some differentiable map on this space depending on q, and on derivatives of q with respect to x. We use the KdV equation as an illustrative example:

$$q_t = q_{xxx} + 6qq_x. \tag{2.2}$$

Equation (2.2) admits the following four-parameter Lie-group of transformations

$$x' = e^{\zeta}(x + \alpha + \gamma t), \qquad t' = e^{3\zeta}(t + \beta), \qquad q' = e^{-2\zeta}\left(q + \frac{\gamma}{6}\right).$$

The above transformations (space and time translations, Galilean and scaling transformations) are uniquely characterized by the following infinitesimal generators of *symmetries* [37]:

$$\sigma_1 = q_x, \quad \sigma_2 = q_{xxx} + 6qq_x, \quad \Sigma_1 = 1 + 6tq_x, \quad \Sigma_2 = 2q + xq_x + 3t(q_{xxx} + 6qq_x). \tag{23}$$

Actually, the KdV possesses infinitely many symmetries

$$\sigma_n = \Phi^n \sigma_1, \quad \Sigma_n = \Phi^n \Sigma_1, \quad n = 1, 2, \ldots, \tag{2.4}$$

where Φ, the *recursion operator* (a strong symmetry) of the KdV, is given by

$$\Phi = D^2 + 2q + 2DqD^{-1}, \quad (D^{-1}f)(x) \doteq \int_{-\infty}^{x} f(\xi)d\xi. \tag{2.5}$$

It turns out that Φ has a certain algebraic property, called *hereditary*, which implies that σ_i, σ_j commute. KdV also possess infinitely many constants of motion; the first few are

$$I = \int_{-\infty}^{\infty} \varrho_n dx, \quad \varrho_0 = q, \quad \varrho_1 = \frac{q^2}{2}, \quad \varrho_2 = -\frac{q_x^2}{2} + q^3. \tag{2.6a}$$

It is more convenient to work with the gradients of constants of motion:

$$\langle \operatorname{grad} I, v \rangle = \frac{\partial}{\partial \varepsilon} I(q + \varepsilon v)\bigg|_{\varepsilon = 0}, \quad \text{where} \quad \langle f, v \rangle = \int_{-\infty}^{\infty} fv dx$$

is an appropriate scalar product. The functionals I_1, I_2 imply

$$\gamma_1 = q, \quad \gamma_2 = q_{xx} + 3q^2. \tag{2.6b}$$

Equations (2.3), (2.6b) suggest that $\sigma = D\gamma$, i.e. D is a *Noether* operator for the KdV (it relates symmetries to constants of motion). This follows from the fact that KdV is a Hamiltonian, actually a bi-Hamiltonian, system:

$$q_t = D \operatorname{grad} \int_{-\infty}^{\infty} \left(-\frac{q_x^2}{2} + q^3 \right) dx = (D^3 + 2qD + 2Dq) \operatorname{grad} \int_{-\infty}^{\infty} \frac{q^2}{2} dx. \tag{2.7}$$

The two Poisson brackets associated with the above are

$$\begin{gathered} \{I_i, I_j\} = \langle \operatorname{grad} I_i, \Theta_\ell \operatorname{grad} I_j \rangle, \quad \ell = 1 \text{ or } 2, \\ \Theta_1 = D, \quad \Theta_2 = D^3 + 2qD + 2Dq. \end{gathered} \tag{2.8}$$

It can be verified that $\{,\}$ is skew symmetric and satisfies the Jacobi identity.

The notion of a *conserved covariant* γ is a mathematical generalization of the gradient of a conserved quantity. Namely, if the functional I is conserved with respect to a given evolution, then $\gamma = \operatorname{grad} I$ is a conserved covariant. Conversely, if γ is a conserved covariant and if γ is a gradient function, then its *potential* I is a conserved quantity. For example Σ_1 implies a conserved covariant $\Gamma_1 = x - 6tq$ which is a gradient function, hence it implies a conserved quantity $I = \int_{-\infty}^{\infty} (xq - 3tq^2)dx$. However, Γ_2, corresponding to Σ_2, is not a gradient and hence does not correspond to a usual conservation law.

The above discussion motivates the following definitions:

Definition 2.1. (i) A function σ is a symmetry of (2.1) iff

$$\sigma'[K]-K'(\sigma)=0, \tag{2.9}$$

where prime denotes Frechét derivative, i.e.

$$\sigma'[v] \doteq \frac{\partial}{\partial \varepsilon}\sigma(q+\varepsilon v)\Big|_{\varepsilon=0}. \tag{2.10}$$

(ii) A function γ is a conserved covariant of (2.1) iff

$$\gamma'[K]+K'^{+}[\gamma]=0, \tag{2.11}$$

where K'^{+} is the adjoint of K', namely, $\langle K'^{+}f, g\rangle = \langle f, K'g\rangle$.

(iii) An operator valued function Φ is a recursion operator (strong symmetry) for (2.1) iff

$$\Phi'[K]-[K',\Phi]=0, \tag{2.12}$$

where $[\,,]$ means commutator.

(iv) An operator valued function Θ is called a Noether operator of (2.1) iff

$$\Theta'[K]-\Theta K'^{+}-K'\Theta=0. \tag{2.13}$$

(v) An operator valued function Θ is called a Hamiltonian operator iff it is skew symmetric and it satisfies

$$\langle a, \Theta'[\Theta b]c\rangle + \text{cyclic permutations} = 0. \tag{2.14}$$

vi) An operator valued function Φ is called a hereditary operator iff

$$\Phi'[\Phi v]w - \Phi\Phi'[v]w \quad \text{is symmetric with respect to} \quad v, w. \tag{2.15}$$

(vii) Equation (2.1) is of a Hamiltonian form if it can be written as $q_t = \Theta\gamma$, where Θ is a Hamiltonian operator and γ is a gradient function, i.e. $\gamma' = \gamma'^{+}$.

Proposition 2.1. (i) *If γ is a conserved covariant of* (2.1) *and if γ is a gradient function, then I, the potential of γ, is a conserved quantity for* (2.1).

(ii) *Φ maps σ's to σ's, Φ^{+} maps γ's to γ's, and Θ maps γ's to σ's.*

(iii) *If* (2.1) *is of a Hamiltonian form, then Θ maps γ's to σ's. Furthermore, there is an isomorphism between Lie and Poisson brackets:*

$$[\Theta\gamma_1, \Theta\gamma_2]_L = \Theta \operatorname{grad}\langle \gamma_1, \Theta\gamma_2\rangle, \tag{2.16a}$$

where

$$[a,b]_L \doteq a'[b]-b'[a], \tag{2.16b}$$

and γ_1, γ_2 are gradient functions.

(iv) *If Φ is hereditary and Φ is a strong symmetry for σ, then $\Phi^n\sigma_1$, form an abelian algebra.*

(v) *If* (2.1) *is of a bi-Hamiltonian form, then $\Phi = \Theta_2\Theta_1^{-1}$ is a recursion operator of* (2.1).

(vi) *If* (2.1) *is a compatible bi-Hamiltonian system, i.e. if it is bi-Hamiltonian and if* $\Theta_1+\Theta_2$ *is also a Hamiltonian operator, then* $\Phi=\Theta_2\Theta_1^{-1}$ *is hereditary. Furthermore, if* γ_1 *is a conserved gradient of* (2.1), *then* $\Phi^{+n}\gamma_1$ *are also conserved gradients. Thus* (2.1) *possesses infinitely many commuting symmetries and infinitely many conserved quantities in involution.*

Given the isospectral eigenvalue problem associated with (2.1) there is an algorithmic way of obtaining Φ. Furthermore, if Φ has a complete set of eigenfunctions it must be hereditary:

Proposition 2.2. *Let*

$$V_x=U(q,\lambda)V \tag{2.17}$$

be a linear isospectral eigenvalue problem associated with (2.1). *Let* G_λ *denote the gradient of the eigenvalue* λ. *If* G_λ *satisfies*

$$\Psi G_\lambda=\mu(\lambda)G_\lambda\,, \tag{2.18}$$

then $\Phi=\Psi^+$ *is a hereditary operator (provided* G_λ *form a complete set).*

3. Derivation of Recursion Operators

A. The Schrödinger Eigenvalue Problem

Proposition 3.1. *The Schrödinger equation* (1.1) *is associated with the following equation:*

$$\delta_{12}q_{1_t}=D\Psi_{12}T_{12}-2q_{12}^{-}a_{12}\,, \tag{3.1}$$

where $q_{12}^{\pm}$ *are given by* (1.2b), δ *denotes the Dirac delta function,* T, a *are arbitrary functions of the arguments indicated,*

$$\delta_{12}\doteq\delta(y_1-y_2)\,,\qquad T_{12}\doteq T(x,y_1,y_2)\,,\qquad a_{12}\doteq a(y_1,y_2)\,, \tag{3.2}$$

and Ψ_{12} *is given by*

$$\Psi_{12}\doteq D^2+q_{12}^{+}+D^{-1}q_{12}^{+}D+D^{-1}q_{12}^{-}D^{-1}q_{12}^{-}\,. \tag{3.3}$$

To derive the above result first write Eq. (1.1) in matrix form

$$W_x=UW\,,\qquad W\doteq\begin{pmatrix} w \\ w_x \end{pmatrix},\qquad U=\begin{pmatrix} 0 & 1 \\ -q-\alpha D_y & 0 \end{pmatrix}. \tag{3.4}$$

Equation (3.4) is compatible with

$$W_t=VW\,,\qquad V\doteq\begin{pmatrix} A & 2C \\ B & E \end{pmatrix} \tag{3.5}$$

if

$$U_t=V_x-[U,V]\,. \tag{3.6}$$

The operator equation (3.6) implies

$$A_x = B + 2C\hat{q}, \quad E_x = -B - 2\hat{q}C, \quad 2C_x = E - A,$$
$$q_t = -B_x - \hat{q}A + E\hat{q}, \quad \hat{q} \doteq q + \alpha D_y. \tag{3.7}$$

The above equations yield

$$A = -C_x + D^{-1}[C, \hat{q}] + A_0, \quad A_{0x} = 0,$$
$$B = -C_{xx} - [C, q]^+, \tag{3.8}$$

$$E = C_x + D^{-1}[C, \hat{q}] + A_0,$$
$$q_t = C_{xxx} + [\hat{q}, C]_x^+ + [\hat{q}, C_x]^+ + [\hat{q}, D^{-1}[\hat{q}, C]] + A_0\hat{q} - \hat{q}A_0, \tag{3.9}$$

where $[\,,]^+$ is the usual anticommutator of two operators. We represent the operator C by:

$$(Cf)(x, y_1) = \int_{\mathbb{R}} dy_2 T(x, y_1, y_2) f(x, y_2), \tag{3.10}$$

similarly,

$$A_0 f_1 = 2 \int_{\mathbb{R}} dy_2 a_{12} f_2 .$$

Then

$$(\hat{q}_1 C \pm C\hat{q}_1) f_1 = \int_{\mathbb{R}} dy_2 (q_{12}^{\pm} T_{12}) f_2 ,$$
$$[\hat{q}_1, D^{-1}[\hat{q}_1, C]] f_1 = \int_{\mathbb{R}} dy_2 (q_{12}^{-} D^{-1} q_{12}^{-} T_{12}) f_2 , \tag{3.11}$$
$$(A_0\hat{q}_1 - \hat{q}_1 A_0) f_1 = -\int_{\mathbb{R}} dy_2 2 q_{12}^{-} a_{12} f_2 .$$

Hence applying the arbitrary function f to the operator equation (3.9) we obtain

$$\delta_{12} q_{2_t} = T_{12_{xxx}} + (q_{12}^{+} T_{12})_x + q_{12}^{+} T_{12_x} + q_{12}^{-} D^{-1} q_{12}^{-} T_{12} - 2 q_{12}^{-} a_{12} . \tag{3.12}$$

Remark 3.1. It is easily verified that the following important commutator operator relationships are valid:

$$[q_{12}^{-}, h_{12}] = 0, \quad [q_{12}^{+}, h_{12}] = 2\alpha h'_{12}, \quad [\Psi_{12}, h_{12}] = 4\alpha h'_{12}; \tag{3.13}$$

hereafter h_{12} is any arbitrary function $h(y_1 - y_2)$ and h'_{12} denotes its derivative with respect to y_1.

Proposition 3.1 can be used to derive nonlinear evolution equations related to (1.1). One needs only to assume appropriate expansions of T_{12}, a_{12}. We give two examples:

Example 1.

$$T_{12} = \sum_{j=0}^{n} \delta_{12}^{j} T_{12}^{(j)}, \quad T_{12}^{(n)} = C_n, \quad a_{12} = 0, \tag{3.14}$$

where $\delta_{12}^{j} \doteq \partial^j \delta_{12} / \partial y_1^j$, C_n an arbitrary constant. Then

$$q_{1_t} = \beta_n \int_{\mathbb{R}} dy_2 \delta_{12} D \Psi_{12}^{n+1} \cdot 1, \quad n = 1, 2, \ldots . \tag{3.15}$$

To derive (3.15), use Eqs. (3.14) in (3.12) and use (3.13c) with $h_{12}=\delta_{12}$,

$$\delta_{12}q_{2_t}=D\left(\sum_{j=0}^{n}\delta_{12}^{j}\Psi_{12}T_{12}^{(j)}+4\alpha\sum_{j=1}^{n+1}\delta_{12}^{j}T_{12}^{(j-1)}\right).$$

Equating the coefficients of δ_{12}^{n+1} and δ_{12}^{j}, $1\leqq j\leqq n$ to zero, we obtain

$$T_{12_x}^{(n)}=0, \quad T_{12}^{(j-1)}=-\frac{1}{4\alpha}\Psi_{12}T_{12}^{(j)}.$$

Hence

$$T_{12}^{(n-j)}=\left(-\frac{1}{4\alpha}\right)^{j}C_n\Psi_{12}^{j}\cdot 1, \quad \delta_{12}q_{2_t}=\delta_{12}D\Psi_{12}T_{12}^{(0)}=\left(-\frac{1}{4\alpha}\right)^{n}C_n\delta_{12}D\Psi_{12}^{n+1}\cdot 1.$$

Thus (3.15) follows with the normalization $(-1)^n\beta_n=(4\alpha)^{-n}C_n$.

Example 2.

$$T_{12}=\sum_{j=0}^{n}\delta_{12}^{j}T_{12}^{(j)}, \quad T_{12}^{(n)}=0, \quad a_{12}=-\frac{1}{2}C_n\delta_{12}^{n}. \tag{3.16}$$

Then

$$q_{1_t}=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}D\Psi_{12}^{n}D^{-1}q_{\bar{1}2}\cdot 1, \quad n=1,2,\ldots, \tag{3.17}$$

with the normalization $C_n=(-1)^n(4\alpha)^n\beta_n$.

Remark 3.2. 1. The operators Φ_{12}, Ψ_{12} defined by (1.2) and (3.3), respectively, are related via

$$\Phi_{12}D=D\Psi_{12}. \tag{3.18}$$

Hence the hierarchy of Eqs. (3.15) can be written as

$$q_{1_t}=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}D\Psi_{12}^{n+1}\cdot 1=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\Phi_{12}^{n}(\Phi_{12}D)\cdot 1. \tag{3.19}$$

The KP equation corresponds to $n=1$ and $\beta_1=\frac{1}{2}$; the next equation of the class (for $\beta_2=\frac{1}{2}$) is

$$q_t=q_{xxxxx}+10qq_{xxx}+20q_xq_{xx}+30q^2q_x$$
$$+5\alpha^2(2q_{yyx}+D^{-1}(q^2)_{yy}+2q_xD^{-2}q_{yy}+4q_yD^{-1}q_y+4qD^{-1}q_{yy})+5\alpha^4D^{-3}q_{yyyy}.$$

2. Similarly, the hierarchy of Eqs. (3.17) can be written as

$$q_{1_t}=\beta_n\int_{\mathbb{R}}dy\delta_{12}D\Psi_{12}^{n}(D^{-1}q_{\bar{1}2}\cdot 1)=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\Phi_{12}^{n}q_{\bar{1}2}\cdot 1. \tag{3.20}$$

For $n=1$ and $\beta_1=\frac{1}{4}$ the above becomes $q_{1_t}=\alpha q_{1_{y_1}}$, i.e. it corresponds to a y-translation.

B. The 2×2 AKNS in $2+1$

Proposition 3.2. *Equation* (1.8) *is associated with the following equation:*

$$\delta_{12}Q_{2_t}=\sigma\Psi_{12}V_{12_0}, \tag{3.21}$$

where V_{12_0} denotes an arbitrary off-diagonal matrix and the operator Ψ_{12} (acting only on off-diagonal matrices) is given by

$$\Psi_{12} \doteq \sigma(P_{12} - Q_{12}^{-} P_{12}^{-1} Q_{12}^{-}), \quad P_{12}F_{12} \doteq F_{12_x} - JF_{12_{y_1}} - F_{12_{y_2}}J. \tag{3.22}$$

To derive the above note that (1.8) can be written as

$$W_x = \hat{Q}W, \quad \hat{Q} = Q + JD_y. \tag{3.23}$$

Equation (3.23) is compatible with $W_t = \hat{V}W$ if

$$\hat{Q}_t = \hat{V}_x - [\hat{Q}, \hat{V}]. \tag{3.24}$$

We represent the operator $\hat{V}$ by

$$(\hat{V}F)(x, y_1) \doteq \int_{\mathbb{R}} dy_2 V(x, y_1, y_2) F(x, y_2). \tag{3.25}$$

Then $[\hat{Q}, \hat{V}] = \int_{\mathbb{R}} dy_2 (\hat{Q}_{12} V_{12}) F_2$, where $\hat{Q}_{12}F_{12} \doteq Q_1 F_{12} - F_{12}Q_2 + JF_{12_{y_1}} + F_{12_{y_2}}J$. Hence (3.24) implies $\delta_{12} Q_{1_t} = (D - \hat{Q}_{12}) V_{12}$. Splitting this equation into diagonal and off-diagonal parts we obtain

$$\delta_{12} Q_{2_t} = P_{12} V_{12_O} - Q_{12}^{-} V_{12_D}, \quad P_{12} V_{12_D} - Q_{12}^{-} V_{12_O} = 0. \tag{3.26}$$

where V_{12_D} and V_{12_O} are the diagonal and off-diagonal parts of V_{12}. Hence Eq. (3.21) follows.

Remark 3.3. The operator Ψ_{12} satisfies the following important commutator relationship:

$$[\Psi_{12}, h_{12}] F_{12_O} = -2\alpha h'_{12} F_{12_O}, \tag{3.27}$$

where F_{12_O} is the off-diagonal part of the arbitrary matrix function F_{12} and prime denotes differentiation with respect to y_1.

The above relationship follows by considering the diagonal and off-diagonal parts of the following equation

$$[D - \hat{Q}_{12}, h_{12}] F_{12} = -2\alpha h'_{12} \sigma F_{12_O}. \tag{3.28}$$

Remark 3.4. Assuming

$$V_{12_O} = \sum_{j=0}^{n} \delta_{12}^j v_{12}^{(j)}, \quad v_{12}^{(j)} \text{ off-diagonal}, \tag{3.29}$$

Eq. (3.21) implies

$$Q_{1_t} = \sigma \int_{\mathbb{R}} dy_2 \delta_{12} \Psi_{12}^n Q_{12}^{-} v_{12_D}; \quad P_{12} v_{12_D} = 0, \tag{3.30}$$

where v_{12_D} is any diagonal matrix solving (3.30b).

To derive (3.30) note that Eqs. (3.21) and (3.27) imply

$$\delta_{12} Q_{2_t} = \sigma \left(\sum_{j=0}^{n} \delta_{12}^j \Psi_{12} v_{12}^{(j)} - 2\alpha \sum_{j=1}^{n+1} \delta_{12}^j v_{12}^{(j-1)} \right). \tag{3.31}$$

Equating the coefficients of $\delta_{12}^{n+1}, \delta_{12}^j$, $n \geqq j \geqq 1$, to zero we obtain

$$v_{12}^{(n)} = 0, \quad v_{12}^{(0)} = \frac{1}{(2\alpha)^{n-1}} \Psi_{12}^{n-1} v_{12}^{(n-1)}, \quad 2\alpha v_{12}^{(n-1)} = \Psi_{12} v_{12}^{(n)}. \tag{3.32}$$

Equation (3.32c) can be written as

$$2\alpha\sigma v_{12}^{(n-1)}=P_{12}v_{12}^{(n)}-Q_{12}^{-}v_{12_D}, \qquad 0=P_{12}v_{12_D}-Q_{12}^{-}v_{12}^{(n)}, \tag{3.33}$$

where v_{12_D} is an arbitrary diagonal matrix. Hence (3.32c) and (3.32a) imply $v_{12}^{(n-1)}=\left(\frac{-1}{2\alpha}\right)\sigma Q_{12}^{-}v_{12_D}$, where v_{12_D} solves $P_{12}v_{12_D}=0$. Hence $v_{12}^{(0)}=-1/(2\alpha)^n\Psi_{12}^{n-1}\sigma Q_{12}^{-}v_{12_D}$ and the coefficient δ_{12}^{0} imply (3.30).

Remark 3.5. Let Φ_{12} be defined by (1.9a), then one easily verifies that

$$\Phi_{12}\sigma=\sigma\Psi_{12}. \tag{3.34}$$

Equation (3.30), for special choices of v_{12_D} yields hierarchies of integrable equations:

Example 1. Let $v_{12_D}=\sigma$, then (3.30) implies

$$Q_{1_t}=-\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\sigma\Psi_{12}^nQ_{12}^{+}I=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\Phi_{12}^nQ_{12}^{-}\sigma. \tag{3.35}$$

To derive (3.35) note that $Q_{12}^{-}\sigma=-\sigma Q_{12}^{+}$. Also (3.34) implies that $\Phi_{12}^n\sigma=\sigma\Psi_{12}^n$. Hence the integral of Eq. (3.30) implies

$$-\sigma\Psi_{12}^nQ_{12}^{+}I=-\Phi_{12}^n\sigma Q_{12}^{+}I=\Phi_{12}^nQ_{12}^{-}\sigma.$$

Remark 3.6. Equations (3.35) for $n=0,1,2$ become

$$Q_t=\sigma Q, \qquad \beta_0=-\tfrac{1}{2}, \tag{3.36a}$$

$$Q_t=Q_x, \qquad \beta_1=-\tfrac{1}{2}, \tag{3.36b}$$

$$\left.\begin{aligned}Q_t&=-\beta_2[2\sigma(Q_{xx}+\alpha^2Q_{yy})-QA+AQ]\\(D_x-JD_y)A&=-2(D_x+JD_y)\sigma Q^2\end{aligned}\right\}. \tag{3.36c}$$

Equations (3.36c) under the reduction $q_2=\bar{q}_1=q$ yield the DS equation $\left(\beta_2=-\frac{i}{4}\right)$

$$\begin{aligned}iq_t+\tfrac{1}{2}(q_{xx}+\alpha^2q_{yy})&=q(\phi-|q|^2),\\\phi_{xx}-\alpha^2\phi_{yy}&=2|q|_{xx}^2.\end{aligned} \tag{3.37}$$

Example 2. Let $v_{12_D}=I$, then (3.30) implies

$$Q_{1_t}=-\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\sigma\Psi_{12}^nQ_{12}^{+}=\beta_n\int_{\mathbb{R}}dy_2\delta_{12}\Phi_{12}^nQ_{12}^{-}I. \tag{3.38}$$

Equations (3.38) for $n=0,1,2$ become

$$Q_t=0, \tag{3.39a}$$

$$Q_t=\alpha Q_y, \qquad \beta_1=-\tfrac{1}{2}, \tag{3.39b}$$

$$\left.\begin{aligned}Q_t&=\beta_2[-4\alpha\sigma Q_{xy}+BQ-QB]\\(D_x-JD_1)B&=4\alpha\sigma(Q_1^2)_y\end{aligned}\right\}. \tag{3.39c}$$

Equations (3.39c) under the reduction $q_2=\bar{q}_1=\bar{q}$ yield ($\beta_2=-\frac{1}{4}$)

$$\begin{aligned} q_t &= \alpha q_{xy}+uq\,, \\ u_{xx}-\alpha^2 u_{yy} &= 2\alpha|q|^2_{xy}\,. \end{aligned} \tag{3.39d}$$

C. Motivation

A crucial step in deriving the recursion operator associated with the Schrödinger equation was to use an integral representation of the operator C [see Eq. (3.10)]. Also in deriving the theory for recursion operators we will need an appropriate Frechét derivative. Both, the integral representation (3.10) and the above Frechét derivative can be motivated as follows:

Consider

$$w_{xx}+\tilde{q}w+\alpha w_y=0\,; \qquad (\tilde{q}f)(x,y)=\int_{\mathbb{R}} dy_2 q(x,y,y_2)f(x,y_2)\,. \tag{3.40}$$

Equation (1.1) can be thought of as the reduction of (3.40) under $q(x,y_1,y_2)=\delta_{12}q(x,y_1)$. It is clear that $\tilde{q}$ satisfies an equation similar to (3.9) where q is replaced by $\tilde{q}$. Since the operator $\tilde{q}$ has the integral representation (3.40b), one is lead to consider a similar integral representation for the operator C [Eq. (3.10)]. An equation similar to (3.12) is also valid for $\tilde{q}$, where $q^{\pm}_{12}$ are replaced by $\tilde{q}^{\pm}_{12}$,

$$\tilde{q}^{\pm}_{12}f_{12}\doteqdot\int_{\mathbb{R}} dy_3(q_{13}f_{32}\pm f_{13}q_{32})+\alpha(D_1\mp D_2)f_{12}\,. \tag{3.41}$$

The Frechét derivative of $\tilde{q}^{\pm}_{12}f_{12}$ in the direction σ_{12} yields

$$\tilde{q}^{\pm\prime}_{12}[\sigma_{12}]f_{12}\doteqdot\int_{\mathbb{R}} dy_3(\sigma_{13}f_{32}\pm f_{13}\sigma_{32})\,. \tag{3.42}$$

This is precisely the directional derivative we use in Sect. 4. More details on the concept of equations in $2+1$ dimensions as exact reductions of nonlocal evolution equations are presented in [35, Sect. V].

4. Algebraic Properties in $2+1$

The theory of algebraic properties in $2+1$ is based on the following concepts: a) A crucial step in deriving the recursion operator associated with a given two-dimensional eigenvalue problem is the use of an integral representation of operators depending on q and $\partial/\partial y$. In KP for example $\hat{q}\doteqdot q+\alpha\partial/\partial y$ is represented by

$$(q_1+\alpha D_1)f_{12}\doteqdot\int_{\mathbb{R}} dy_3 q_{13}f_{32}\,. \tag{4.1a}$$

The above mapping between an operator and its kernel induces a mapping between derivatives:

$$\hat{q}_{1_d}[\sigma_{12}]f_{12}=\int_{\mathbb{R}} dy_3\sigma_{13}f_{32}\,, \tag{4.1b}$$

where $\hat{q}_{1_d}[\sigma_{12}]$ denotes the directional derivative of the operator valued function $\hat{q}_1$ in the direction σ_{12}. Using an appropriate bilinear form [see (4.7)–(4.8)] Eqs. (4.1) imply

$$\hat{q}^*_1 f_{12}=(q_2-\alpha D_2)f_{12}=\int_{\mathbb{R}} dy_3 f_{13}q_{32}\,, \qquad \hat{q}^*_{1_d}[\sigma_{12}]f_{12}=\int_{\mathbb{R}} dy_3 f_{13}\sigma_{32}\,. \tag{4.2}$$

The recursion operator Φ_{12} depends only on $\hat{q}_1$ and $\hat{q}_1^*$, thus one is able to define $\Phi_{12_d}[\sigma_{12}]$. b) The theory of symmetries for equations in $1+1$ is based on the existence of "starting" symmetries K^0, which via Φ generate infinitely many symmetries. For example, for the KdV $K^0=q_x$. For equations in $2+1$ we find that the starting symmetries K_{12}^0 can be written as $\hat{K}_{12}^0 H_{12}$, where $\hat{K}_{12}^0$ is an operator and H_{12} is a suitable function [for the KP $H_{12}=H_{12}(y_1, y_2)$]. The operators $\hat{K}_{12}^0$ depend only on $\hat{q}_1, \hat{q}_1^*$ and thus $\hat{K}_{12_d}^0$ is well defined. The Lie algebra of the starting operators $\hat{K}_{12}^0$ acting on H_{12} is closed. This fact, which is of fundamental importance for the theory developed both here and in [35], can also be traced back to the integral representation of the fundamental operator $\hat{q}$. For example, Eq. (4.1b) implies:

$$\hat{q}_{1_d}[\sigma_{12}]f_{12}-\hat{q}_{1_d}[f_{12}]\sigma_{12}=\int_{\mathbb{R}} dy_3(\sigma_{13}f_{32}-f_{13}\sigma_{32}).$$

Also using

$$\hat{q}_{1_d}[\hat{q}_1\sigma_{12}]f_{12}=\int_{\mathbb{R}} dy_3(\hat{q}_1\sigma_{12})_{13}f_{32}=\int_{\mathbb{R}^2} dy_3 dy'_3 f_{32}q_{13'}\sigma_{3'3},$$

it follows that

$$\hat{q}_{1_d}[\hat{q}_1\sigma_{12}]f_{12}-\hat{q}_{1_d}[\hat{q}_1 f_{12}]\sigma_{12}=\hat{q}_1\int_{\mathbb{R}} dy_3(\sigma_{13}f_{32}-f_{13}\sigma_{32}).$$

The above equation can be written as

$$[\hat{q}_1 f_{12}, \hat{q}_1\sigma_{12}]_d=\hat{q}_1[\sigma_{12}, f_{12}]_I,$$

where the following brackets have been motivated from the above example:

$$[\hat{K}_{12}^{(1)}H_{12}^{(1)}, \hat{K}_{12}^{(2)}H_{12}^{(2)}]_d \doteq K_{12_d}^{(1)}[\hat{K}_{12}^{(2)}H_{12}^{(2)}]H_{12}^{(1)}-\hat{K}_{12_d}^{(2)}[\hat{K}_{12}^{(1)}H_{12}^{(1)}]H_{12}^{(2)}, \quad (4.3a)$$

$$[H_{12}^{(1)}, H_{12}^{(2)}]_I \doteq \int_{\mathbb{R}} dy_3(H_{13}^{(1)}H_{32}^{(2)}-H_{13}^{(2)}H_{32}^{(1)}). \quad (4.3b)$$

In $1+1$, one considers the Lie algebra of *functions*; in $2+1$ one, instead, considers the Lie algebra of *operators*, thus equations in $2+1$ have richer algebraic structure than equations in $1+1$. c) The recursion operator Φ_{12} and the starting operators $\hat{K}_{12}^0$ have simple commutator relations with δ_{12} or more generally with $h_{12}=h(y_1-y_2)$.

Notation. We will consider exactly solvable evolution equations of the form $q_t=K(q)$, where $q(x, y, t)$ is an element of a suitable space S of functions vanishing rapidly for large x, y. Let K be a differentiable map on this space (we assume for convenience that it does not depend explicitly on x, y, t). The above equation is a member of a hierarchy generated by Φ_{12}, hence more generally, we shall study $q_t=K^{(n)}(q)$. Fundamental in our theory is to write these equations in the form

$$q_{1_t}=\int_{\mathbb{R}} dy_2\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1 \doteq \int_{\mathbb{R}} dy_2\delta_{12}K_{12}^{(n)}=K_{11}^{(n)} \quad (4.4)_n$$

(in the matrix case, 1 is replaced by the identity matrix I), where $K_{12}^{(n)}(q_1, q_2)$ belong to a suitably extended space $\tilde{S}$, and Φ_{12}, $\hat{K}_{12}^0$ are operator valued functions in $\tilde{S}$. For an arbitrary function $K_{12}(q_1, q_2)$ we define the total *Frechét* derivative by

$$K_{12_f}[F] \doteq K_{12_{q_1}}[F_{11}]+K_{12_{q_2}}[F_{22}], \quad (4.5a)$$

where $K_{12_{q_i}}$ denotes the Frechét derivative of K_{12} with respect to q_i, i.e.

$$K_{12_{q_i}}[F_{ii}] \doteq \frac{\partial}{\partial \varepsilon} K_{12}(q_i + F_{ii}, q_j)\Big|_{\varepsilon=0}, \quad i, j = 1, 2, \quad i \neq j. \tag{4.5b}$$

We also define a special *directional* derivative, dictated by the underlying isospectral problem and denoted by K_{12_d}. This derivative is linear, satisfies the Leibnitz rule and is related to the above Frechét derivative by

$$K_{12_d}[\delta_{12} F_{12}] = K_{12_f}[F]. \tag{4.6}$$

For arbitrary functions $f_{12} \in \tilde{S}$ and $g_{12} \in \tilde{S}^*$, where S^* denotes the dual of S, we define the following symmetric *bilinear form*

$$\langle g_{12}, f_{12} \rangle \doteq \int_{\mathbb{R}^3} dx dy_1 dy_2 \, \mathrm{trace}\, g_{21} f_{12}, \quad f_{12}, g_{12} \text{ matrices}, \tag{4.7}$$

where obviously the trace is dropped if f_{12}, g_{12} are scalars. The operator L_{12}^* is called the adjoint of L_{12} with respect to the above bilinear form, iff

$$\langle L_{12}^* g_{12}, f_{12} \rangle = \langle g_{12}, L_{12} f_{12} \rangle. \tag{4.8}$$

For arbitrary functions $f \in S$ and $g \in S^*$, we define the following symmetric *bilinear form*

$$(g, f) \doteq \int_{\mathbb{R}^2} dx dy \, \mathrm{trace}\, g f, \quad f, g \text{ matrices}. \tag{4.9}$$

The operator L^+ is called the adjoint of L with respect to the bilinear form (4.9) iff

$$(L^+ g, f) = (g, Lf). \tag{4.10}$$

Remark 4.1. Definitions (4.7) and (4.9) imply

$$\langle \delta_{12} g_{12}, f_{12} \rangle = \langle g_{12}, \delta_{12} f_{12} \rangle = (g_{11}, f_{11}). \tag{4.11}$$

Let I be a functional given by

$$I = \int_{\mathbb{R}^2} dx dy_1 \, \mathrm{trace}\, \varrho_{11} = \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \, \mathrm{trace}\, \varrho_{12}, \quad \varrho_{12} = \varrho(x, y_1, y_2, t) \in \tilde{S} \tag{4.12}$$

(if ϱ_{12} is a scalar, then omit trace).

The *extended gradient* $\mathrm{grad}_{12} I$ of this functional is defined by

$$\langle \mathrm{grad}_{12} I, \cdot \rangle \doteq I_d[\cdot] = \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \varrho_{12_d}[\cdot]. \tag{4.13}$$

The gradient of I, $\mathrm{grad} I$, is instead defined by

$$(\mathrm{grad} I, \cdot) \doteq I_f[\cdot] = \int_{\mathbb{R}^2} dx dy \varrho_f[\cdot]. \tag{4.14}$$

It is easily seen that a function $\gamma_{12} \in \tilde{S}^*$ is an *extended gradient* function (i.e. it has a *potential* I) iff

$$\gamma_{12_d} = \gamma_{12_d}^*. \tag{4.15a}$$

A function $\gamma \in S$ is a gradient function iff

$$\gamma_f = \gamma_f^+. \tag{4.15b}$$

Some of the above notions make sense only if for certain functions the directional derivative exists. Such functions are called admissible.

Throughout this paper m, n denote non-negative integers.

A. Basic Notions

Definition 4.1. i) An operator valued function L_{12} is called *admissible* if its directional derivative is well defined.

ii) A function K_{12} is called admissible if it can be written as $K_{12}=\hat{K}_{12}H_{12}$, where $\hat{K}_{12}$ is an admissible operator and H_{12} is an appropriate function [for KP, $H_{12}=H_{12}(y_1, y_2)$].

In analogy with Sect. 2 we give the following definitions:

Definition 4.2. Consider the evolution equation

$$q_{1_t}=\int_{\mathbb{R}} dy_2 \delta_{12} K_{12}=K_{11}\,. \tag{4.16}$$

i) The function σ_{12} is called an *extended symmetry* of (4.16) iff

$$\sigma_{12_t}[K]=(\delta_{12}K_{12})_d[\sigma_{12}]\,. \tag{4.17}$$

ii) The function γ_{12} is called an extended *conserved covariant* of (4.16) iff

$$\gamma_{12_t}[K]+(\delta_{12}K_{12})_d^*[\gamma_{12}]=0\,. \tag{4.18}$$

iii) The admissible operator valued function Φ_{12} is called a *strong symmetry* (recursion operator) of (4.16) iff

$$\Phi_{12_t}[K]+[\Phi_{12},(\delta_{12}K_{12})_d]=0\,. \tag{4.19}$$

iv) The admissible operator valued function Θ_{12} is called a *Noether* operator of (4.16) iff

$$\Theta_{12_t}[K]-\Theta_{12}(\delta_{12}K_{12})_d^*-(\delta_{12}K_{12})_d\Theta_{12}=0\,. \tag{4.20}$$

v) The admissible operator valued function Φ_{12} is called a hereditary operator iff

$$\Phi_{12_d}[\Phi_{12}f_{12}]g_{12}-\Phi_{12}\Phi_{12_d}[f_{12}]g_{12} \quad \text{is symmetric with respect to } f_{12}, g_{12} \tag{4.21}$$

Remark 4.2. i) σ_{12} is an extended symmetry of (4.16) iff σ_{12} commutes with $\delta_{12}K_{12}$,

$$[\sigma_{12},\delta_{12}K_{12}]_d=0\,. \tag{4.22}$$

This follows from the fact that $\sigma_{12_d}[\delta_{12}K_{12}]=\sigma_{12_t}[K]$.

ii) If in (4.12), ϱ_{12} is an admissible function, $\varrho_{12}=\hat{\varrho}_{12}H_{12}$; then the functional I depends on H_{12}, $I=I(H_{12})$, and $\gamma_{12}\doteq\text{grad}_{12}I$, defined by (4.13), is also an admissible function $\gamma_{12}=\hat{\gamma}_{12}H_{12}$, enjoying the property (4.15a) for every H_{12}. If, for instance, $I=\int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} q_{12}^{+} D^{-1} q_{12}^{-} H_{12}$ and the directional derivative is defined in (4.13) [see also (4.1b) and (4.2)], then $\gamma_{12}=4D^{-1}q_{12}^{-}H_{12}$ is the corresponding extended gradient.

iii) If γ_{12} in addition to satisfying (4.18) is also an extended gradient function, then its potential I is a conserved quantity of (4.16). This follows from the following:

$$I_t = I_f[K] = I_d[\delta_{12}K_{12}] = \langle \gamma_{12}, \delta_{12}K_{12} \rangle ,$$

where $\gamma_{12} = \mathrm{grad}_{12} I$. The derivative of the above in the arbitrary direction v_{12} is zero if (4.18) holds.

iv) Φ_{12} is a strong symmetry for a_{12} iff

$$\Phi_{12_d}[a_{12}] + [\Phi_{12}, a_{12_d}] = 0 . \tag{4.23a}$$

Hence Eq. (4.21) implies that Φ_{12} is a strong symmetry for $(\delta_{12}K_{12})$ (see Lemma 4.1).

v) Θ_{12} is a Noether operator for a_{12} iff

$$\Theta_{12_d}[a_{12}] - \Theta_{12} a^*_{12_d} - a_{12_d}\Theta_{12} = 0 . \tag{4.23b}$$

Hence Eq. (4.20) implies that Θ_{12} is a Noether operator for $(\delta_{12}K_{12})$ (see Lemma 4.1).

vi) In the above definitions we assume that $\sigma_{12}, \gamma_{12}, \Theta_{12}, \Phi_{12}$ do not explicitly depend on t. Otherwise, $\sigma_{12_f}[K]$ should be replaced by $\partial\sigma_{12}/\partial t + \sigma_{12_f}[K]$; similarly, for $\gamma_{12_f}, \Theta_{12_f}, \Phi_{12_f}$.

Remark 4.3. i) Φ_{12} maps solutions of (4.17) to solutions of (4.17);

ii) Φ^*_{12} maps solutions of (4.18) to solutions of (4.18);

iii) Θ_{12} maps solutions of (4.18) to solutions of (4.17);

iv) if Θ_{12} solves (4.20) and Φ_{12} solves (4.19) then $\Phi^n\Theta_{12}$ also solves (4.20).

Definitions 4.2 make sense only if $(\delta_{12}K_{12})_d$ exists. For equations generated by Φ_{12}, $(\delta_{12}K_{12})_d$ is well defined:

Lemma 4.1. *Assume that the admissible operators Φ_{12} and $\hat{K}^0_{12}$ satisfy the following operator equations*

$$[\Phi_{12}, h_{12}] = -\beta h'_{12} , \tag{4.24a}$$

$$[\hat{K}^0_{12}, h_{12}] = -\tilde{\beta}\hat{S}_{12} h'_{12} , \tag{4.24b}$$

where $\beta, \tilde{\beta}$ are constants, $\hat{S}_{12}$ is some admissible operator, $h_{12} = h(y_1 - y_2)$ and prime denotes derivative with respect to y_1. Then all notions introduced in Definitions 4.2 are well defined for any Eq. $(4.4)_n$. *In particular:*

$$(\delta_{12}\Phi^n_{12}\hat{K}^0_{12} \cdot 1)_d = ((\Phi_{12} + \beta\mathcal{D})^n(\hat{K}^0_{12} + \tilde{\beta}\hat{S}_{12}\mathcal{D})\delta_{12})_d , \tag{4.25}$$

where the operator $\mathcal{D}$ is defined by

$$[\mathcal{D}, \hat{a}_{12}] = 0 , \qquad \mathcal{D} \cdot h_{12} = h'_{12} , \tag{4.26}$$

and $\hat{a}_{12}$ is any admissible operator. Thus

$$(\Phi_{12} + \beta\mathcal{D})^n\delta_{12} = \sum_{\ell=0}^{n} \beta^\ell \binom{n}{\ell} \Phi^{n-\ell}_{12}\delta^\ell_{12} , \qquad \binom{n}{\ell} \doteqdot \frac{n!}{(n-\ell)!\,\ell!} . \tag{4.27}$$

Equations (4.24) *imply that $\delta_{12}\Phi^n_{12}\hat{K}^0_{12} \cdot 1 = (\Phi_{12} + \beta\mathcal{D})^n(\hat{K}^0_{12} + \tilde{\beta}\hat{S}_{12}\mathcal{D})\delta_{12}$ which is an admissible function since Φ_{12}, $\hat{K}^0_{12}$, $\hat{S}_{12}$ are admissible operators.*

Remark 4.4. i) For the two-dimensional AKNS we use two starting operators $\hat{K}^0_{12}$; both of these operators commute with h_{12} (i.e. $\tilde{\beta}=0$). For the two-dimensional Schrödinger we also use two starting operators $\hat{K}^0_{12}$; one of them commutes with h_{12}, the other implies $\tilde{\beta}=\frac{\beta}{2}$, $\hat{S}_{12}=D$.

ii) It is clear that the theory presented here, suitably modified, is also valid for more general commutator relations than the ones given by (4.24). In investigating a new eigenvalue problem one first computes the commutator of Φ_{12} and $\hat{K}^0_{12}$ with h_{12}; one then builds a general theory based on these commutator relations.

iii) We remark that Eq. (4.24a) could be derived directly from the underlying isospectral problem without using the explicit form of Φ_{12}. As an example, in Sect. 4.E we show that the equation $\Phi^*_{12}W_1W_2^+=4\lambda W_1W_2^+$ (which is a direct consequence of the spectral problem $W_{xx}+\hat{q}W=\lambda W$) implies Eq. (4.24a), with $\beta=-4\alpha$.

The usefulness of the extended symmetries and the extended gradients follows from the fact that their reduction yields symmetries and gradients, respectively.

Theorem 4.1. *Assume that the admissible operators* Φ_{12}, $\hat{K}^0_{12}$, *satisfy*

$$[\Phi_{12},\delta_{12}]=-\beta\delta'_{12}, \tag{4.28a}$$

$$[\hat{K}^0_{12},\delta_{12}]=-\tilde{\beta}\hat{S}_{12}\delta'_{12}, \tag{4.28b}$$

where $\beta,\tilde{\beta}$ *are constants,* $\hat{S}_{12}$ *is such that*

$$\hat{S}_{12_d}[\cdot]H_{12}=\hat{S}_{12_f}[\cdot]H_{12}=0$$

and prime denotes derivative with respect to y_1. *Then:*

i) *If* σ_{12} *is an extended symmetry of*

$$q_{1_t}=\int_{\mathbb{R}} dy_2\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1=\int_{\mathbb{R}} dy_2\delta_{12}K^{(n)}_{12}=K^{(n)}_{11}, \tag{4.29}$$

σ_{11} *is a symmetry of* (4.29).

ii) *Similarly, if* γ_{12} *is an extended conserved covariant of* (4.29), γ_{11} *is a conserved covariant of* (4.29).

iii) *If* γ_{12} *is the extended gradient of a conserved quantity of* (4.29), γ_{11} *is the gradient of a conserved quantity of* (4.29).

Proof. We first note that Eqs. (4.28) imply

$$\text{a}_1)\quad \Phi_{12_f}[\;]\delta_{12}g_{12}-\delta_{12}\Phi_{12_f}[\;]g_{12}=0, \tag{4.30a}$$

$$\text{a}_2)\quad \Phi_{12_d}[\delta_{12}\cdot]\delta_{12}g_{12}-\delta_{12}\Phi_{12_d}[\cdot]\delta_{12}g_{12}=0, \tag{4.30b}$$

$$\text{a}_3)\quad (\delta_{12}\hat{K}^0_{12}\cdot 1)_f[\;]=\delta_{12}(\hat{K}^0_{12}\cdot 1)_f[\;], \tag{4.30c}$$

$$\text{a}_4)\quad (\delta_{12}\hat{K}^0_{12}\cdot 1)_d[\delta_{12}\cdot]=\delta_{12}(\delta_{12}\hat{K}^0_{12}\cdot 1)_d[\cdot]. \tag{4.30d}$$

Equations (4.30a), (4.30b) follow from (4.28a) (see Appendix A). Using (4.28b) and the fact that $\hat{S}_{12_f}[\cdot]H_{12}=\hat{S}_{12_d}[\cdot]H_{12}=0$, Eqs. (4.30c), (4.30d) take the form of (4.30a), (4.30b) (with Φ_{12} replaced by $\hat{K}^0_{12}$). However, these equations follow from (4.28b) following a proof similar to the one given in the Appendix A.

a) Equations (4.28a), (4.30a), (4.30c) imply

$$(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_f[\ ]=\delta_{12}(\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_f[\ ]. \qquad (4.31)_n$$

We derive Eq. $(4.31)_n$ by induction: Eq. $(4.31)_0$ is (4.30c). Let subscript L denote any derivative, such that the Leibnitz rule holds. Then

$$(\delta_{12}K^{(n+1)}_{12})_L=(\delta_{12}\Phi_{12}K^{(n)}_{12})_L=(\Phi_{12}\delta_{12}K^{(n)}_{12})_L+\beta(\delta'_{12}K^{(n)}_{12})_L.$$

Hence

$$(\delta_{12}K^{(n+1)}_{12})_L[\ ]=\Phi_{12_L}[\ ]\delta_{12}K^{(n)}_{12}+\Phi_{12}(\delta_{12}K^{(n)}_{12})_L[\ ]+\beta(\delta'_{12}K^{(n)}_{12})_L[\ ]. \qquad (4.32)$$

We assume that $(4.31)_n$ is valid, then applying $\mathscr{D}$ on it, it follows that

$$(\delta'_{12}K^{(n)}_{12})_f[\ ]=\delta'_{12}K^{(n)}_{12_f}[\ ] \qquad (4.33)$$

is also valid: To derive Eq. (4.33) note that Eqs. (4.26) imply

$$\mathscr{D}\delta_{12}\hat{\alpha}_{12}\cdot 1=\delta'_{12}\hat{\alpha}_{12}\cdot 1.$$

Applying the L-derivative on the above we obtain

$$\mathscr{D}(\delta_{12}\hat{\alpha}_{12}\cdot 1)_L[\ ]=(\delta'_{12}\hat{\alpha}_{12}\cdot 1)_L[\ ].$$

The above equation for $L=f$, and (4.26) imply (4.33). Equation $(4.31)_{n+1}$ is valid iff:

$$\Phi_{12_f}[\ ]\delta_{12}G^n+\Phi_{12}(\delta_{12}G^n)_f[\ ]+\beta(\delta'_{12}G^n)_f[\ ]$$
$$=\delta_{12}\Phi_{12_f}[\ ]G^n+(\Phi_{12}\delta_{12}+\beta\delta'_{12})G_{n_f}[\ ].$$

The first terms of the left- and right-hand sides of the above equation are equal because of (4.30a); the second and the third terms are equal because of $(4.31)_n$ and (4.33), respectively.

b) Equations (4.28a), (4.30b), (4.30d) imply

$$(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_d[\delta_{12}\cdot]=\delta_{12}(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_d[\cdot]. \qquad (4.34)_n$$

To derive Eq. $(4.34)_n$ we use again induction. Equation $(4.34)_0$ is (4.30c). Assume that $(4.34)_n$ is valid, then applying the operator $\mathscr{D}$ on it, it follows that

$$(\delta'_{12}K^{(n)}_{12})_d[\delta_{12}\cdot]=\delta_{12}(\delta'_{12}K^{(n)}_{12})_d[\cdot]+\delta'_{12}(\delta_{12}K^{(n)}_{12})_d[\cdot]. \qquad (4.35)$$

Using (4.35) it follows that Eq. $(4.34)_{n+1}$ is valid if

$$\Phi_{12_d}[\delta_{12}\cdot]\delta_{12}K^{(n)}_{12}+\Phi_{12}(\delta_{12}K^{(n)}_{12})_d[\delta_{12}\cdot]+\beta(\delta'_{12}K^{(n)}_{12})_d[\delta_{12}\cdot]$$
$$=\delta_{12}\Phi_{12_d}[\cdot]\delta_{12}K^{(n)}_{12}+(\Phi_{12}\delta_{12}+\beta\delta'_{12})(\delta_{12}K^{(n)}_{12})_d[\cdot]+\delta_{12}\beta(\delta'_{12}K^{(n)}_{12})_d[\cdot].$$

The first term of the left- and right-hand sides of the above equation are valid because of (4.30b); the second and the remainder terms because of $(4.34)_n$ and (4.35), respectively.

c) Equations (4.28), (4.30), $(4.34)_n$, $(4.31)_n$, and (4.6) imply:

$$\delta_{12}(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_d[\cdot]=(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_d[\delta_{12}\cdot]=(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_f[\cdot]$$
$$=\delta_{12}(\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_f[\cdot]. \qquad (4.36)$$

Using the definitions of symmetries and extended symmetries and Eq. (4.30c–d), the first part of Theorem 4.1 follows:

$$\sigma_{11_t}=\int_{\mathbb{R}} dy_2\delta_{12}\sigma_{12_t}=\int_{\mathbb{R}} dy_2\delta_{12}(\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_d[\sigma_{12}]$$

$$=\int_{\mathbb{R}} dy_2\delta_{12}(\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_f[\sigma]=K_{11_f}^{(n)}[\sigma_{11}].$$

The derivation of ii) is similar to the derivation of i): It follows from the equations

$$(\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_f^*[\ ]=\delta_{12}(\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_f^*[\ ],\tag{4.37a}$$

$$(\delta_{12}\Phi_{12}^n\hat{K}_{12}^{(0)}\cdot 1)_d^*[\delta_{12}\cdot]=\delta_{12}(\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_d^*[\cdot],\tag{4.37b}$$

which are direct consequences of Eqs. $(4.31)_n$, $(4.34)_n$, (4.6), (4.7), and (4.8). Then

$$\gamma_{11_t}=\int_{\mathbb{R}} dy_2\delta_{12}\gamma_{12_t}=-\int_{\mathbb{R}} dy_2\delta_{12}(\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_d^*[\gamma_{12}]$$

$$=-\int_{\mathbb{R}} dy_2(\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_d^*[\delta_{12}\gamma_{12}]=-\int_{\mathbb{R}} dy_2(\delta_{12}\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_f^*[\gamma]$$

$$=-\int_{\mathbb{R}} dy_2\delta_{12}(\Phi_{12}^n\hat{K}_{12}^0\cdot 1)_f^*[\gamma]=-K_{11_f}^{(n)*}[\gamma].$$

The derivation of iii) follows from ii) and the fact that if γ_{12} is an extended gradient function γ_{11} is a gradient function: Recall that γ_{12} is an extended gradient iff $\gamma_{12_d}[\ ]=\gamma_{12_d}^*[\ ]$, namely iff $\langle\gamma_{12_d}[g_{12}],f_{12}\rangle=\langle g_{12},\gamma_{12_d}[f_{12}]\rangle$. Letting $f_{12}\to\delta_{12}f_{12}$ and $g_{12}\to\delta_{12}g_{12}$, we obtain $(\gamma_{11_f}[g_{11}],f_{11})=(g_{11},\gamma_{11_f}[f_{11}])$ which implies that $\gamma_{11_f}=\gamma_{11_f}^{+}$ (γ_{11} is a gradient). Moreover, one could easily show that if $\gamma_{12}=\text{grad}_{12}I$, then $\gamma_{11}=\text{grad}\,I$.

Another important property of the extended symmetries is given by the following theorem:

Theorem 4.2. *If σ_{12} is an extended symmetry of Eq.* (4.29), *then $\sigma_{12}=0$ is an auto-Bäcklund Transformation for Eq.* (4.29). *In equation $\sigma_{12}=0$, q_1 and q_2 are viewed as two different solutions of* (4.29).

Proof. If σ_{12} is an extended symmetry of Eq. (4.29) and $\sigma_{12}=0$, then $D_t\sigma_{12}=\frac{\partial\sigma_{12}}{\partial t}+\sigma_{12_f}[K]=0$, which implies the result.

Remark 4.5. Theorems 4.1 and 4.2 show that the symmetries and the auto-Bäcklund Transformations of an equation originate from the same entity: the extended symmetry. This remarkable connection between symmetries and auto-Bäcklund Transformations exists also in $1+1$ dimensions. If we consider as an example the classes of evolution equations in $2+1$ dimensions (3.19), (3.17), (3.35), and (3.38), then extended symmetries and gradients for the corresponding $1+1$ dimensional systems are still defined by Eqs. (4.17) and (4.18), in which the operators $(\delta_{12}K_{12})_d$ and $(\delta_{12}K_{12})_d^*$ are evaluated at $\alpha=0$. For $\alpha=0$ Φ_{12} is indeed the operator that generates Bäcklund Transformations in $1+1$ dimensions [38].

The above theorems imply that it is useful to have an effective way of generating extended symmetries and extended gradients of conserved quantities.

For equations in $1+1$ one makes fundamental use of the following two notions: a) if Φ is hereditary it generates infinitely many commuting symmetries. b) If Φ admits a factorization in terms of compatible Hamiltonian operators it generates infinitely many constants of motion in involution. Both the above notions are extended to equations in $2+1$.

B. Characterization of the Starting Symmetry $\hat{K}^0_{12}\cdot H_{12}$ through the Recursion Operator Φ_{12}

Fundamental role in the theory presented in this paper is played by a hereditary operator Φ_{12} and a starting symmetry $\hat{K}^0_{12}H_{12}$. It is interesting that the recursion operator Φ_{12} algorithmically implies $\hat{K}^0_{12}H_{12}$. Furthermore, if Φ_{12} is hereditary, it is also a strong symmetry for $\hat{K}^0_{12}H_{12}$.

Definition 4.3. A starting symmetry associated with the recursion operator Φ_{12} is $\hat{K}^0_{12}H_{12}$, where the admissible operator $\hat{K}^0_{12}$ and the function H_{12} satisfy

$$\Phi_{12}\hat{S}_{12}\cdot H_{12}=\hat{K}^0_{12}H_{12}\,, \qquad \hat{S}_{12}\cdot H_{12}=0\,, \tag{4.38}$$

and $\hat{S}_{12}$ is an invertible operator, of course, on a space of functions excluding $\mathrm{Ker}\,\hat{S}_{12}\ni H_{12}$.

Examples. 1. For the KP hierarchies, $\hat{S}_{12}=D$ and/or $\hat{S}_{12}=D(q^-_{12})^{-1}D$. This implies

$$\hat{K}^0_{12}=Dq^+_{12}+q^-_{12}D^{-1}q^-_{12}\,, \qquad \hat{S}_{12}=D\,, \tag{4.39a}$$

$$\hat{K}^0_{12}=q^-_{12}\,, \qquad \hat{S}_{12}=D(q^-_{12})^{-1}D\,, \tag{4.39b}$$

with H_{12} any solution of $DH_{12}=0$.

2. For the DS hierarchies $\hat{S}_{12}=(Q^+_{12})^{-1}P_{12}$. This implies

$$\hat{K}^0_{12}=Q^-_{12}\sigma \quad \text{and/or} \quad \hat{K}^0_{12}=Q^-_{12}\,, \tag{4.40}$$

with H_{12} any diagonal matrix solving $P_{12}H_{12}=0$.

For the results presented in this paper we only use a subclass of solutions of $DH_{12}=0$ and $P_{12}H_{12}=0$, given by $H_{12}=h_{12}\doteqdot h(y_1-y_2)$ and $H_{12}=h_{12}(aI+b\sigma)$, a, b constants, respectively. More general solutions of the above equations are used in [35] and give rise to time-dependent symmetries.

Lemma 4.2. *If $\hat{K}^0_{12}H_{12}$ is a starting symmetry associated with the hereditary operator Φ_{12}, then Φ_{12} is a strong symmetry of $\hat{K}^0_{12}H_{12}$.*

Proof. Since Φ_{12} is hereditary,

$$\Phi_{12_d}[\Phi_{12}f_{12}]g_{12}-\Phi_{12}\Phi_{12_d}[f_{12}]g_{12} \quad \text{is symmetric in } f_{12}, g_{12}\,. \tag{4.41}$$

Letting $g_{12}=\hat{S}_{12}\cdot H_{12}$ we obtain

$$\Phi_{12_d}[\Phi_{12}\hat{S}_{12}H_{12}]f_{12}-\Phi_{12}\Phi_{12_d}[\hat{S}_{12}H_{12}]f_{12}-\Phi_{12_d}[\Phi_{12}f_{12}]\hat{S}_{12}H_{12}$$
$$+\Phi_{12}\Phi_{12_d}[f_{12}]\hat{S}_{12}H_{12}=0\,.$$

Using $\Phi_{12}\hat{S}_{12}H_{12}=\hat{K}^0_{12}H_{12}$, $\hat{S}_{12}H_{12}=0$ and its consequence $\hat{S}_{12_d}[f_{12}]H_{12}=0$, for every f_{12}, we obtain

$$\Phi_{12_d}[\hat{K}^0_{12}H_{12}]f_{12}-(\hat{K}^0_{12}H_{12})_d[\Phi_{12}f_{12}]+\Phi_{12}(\hat{K}^0_{12}H_{12})_d[f_{12}]=0, \quad \forall f_{12}, \tag{4.42}$$

thus Φ_{12} is a strong symmetry of $\hat{K}^0_{12}H_{12}$.

C. Hereditary Symmetries

Theorem 4.3. *Assume that the admissible hereditary operator Φ_{12} and its associated starting symmetry $\hat{K}^0_{12}H_{12}$, defined via*

$$\Phi_{12}\hat{S}_{12}H_{12}=\hat{K}^0_{12}H_{12}, \qquad \hat{S}_{12}H_{12}=0 \tag{4.43}$$

satisfy

$$[\Phi_{12},h_{12}]=-\beta h'_{12}, \tag{4.44a}$$

$$[\hat{K}^0_{12},h_{12}]=-\tilde{\beta}\hat{S}_{12}h'_{12}, \tag{4.44b}$$

where $\beta,\tilde{\beta}$ are constants, $\hat{S}_{12}$ is an admissible operator, $h_{12}=h(y_1-y_2)$ and prime denotes derivative with respect to y_1. Further assume that

$$[\hat{K}^0_{12}H^{(1)}_{12},\hat{K}^0_{12}H^{(2)}_{12}]_d=0, \quad \textit{for} \quad [H^{(1)}_{12},H^{(2)}_{12}]_I=0, \tag{4.44c}$$

where $[\ ]_d$, $[\ ]_I$ are defined by (4.3) *and h_{12} belongs to H_{12}. Then*

$$[\Phi^m_{12}\hat{K}^0_{12}H^{(1)}_{12},\Phi^n_{12}\hat{K}^0_{12}H^{(2)}_{12}]_d=0, \quad \textit{for} \quad [H^{(1)}_{12},H^{(2)}_{12}]_I=0. \tag{4.45a}$$

Furthermore,

$$\Phi^m_{12}\hat{K}^0_{12}\cdot 1 \quad \textit{are extended symmetries of } (4.4)_n, \tag{4.45b}$$

for all nonnegative integers m,n.

Proof. In analogy with the results of $1+1$ one easily verifies that if $K^{(1)}_{12}$, $K^{(2)}_{12}$ commute, Φ_{12} is hereditary and Φ_{12} is a strong symmetry for both $K^{(1)}_{12}$ and $K^{(2)}_{12}$, then $\Phi^n_{12}K^{(1)}_{12}$, $\Phi^m_{12}K^{(2)}_{12}$ also commute, for all m,n. Using these results with $K^{(1)}_{12}=\hat{K}^0_{12}H^{(1)}_{12}$, $K^{(2)}_{12}=\hat{K}^0_{12}H^{(2)}_{12}$ one immediately proves (4.45a) above. To prove (4.45b) we note that (4.44) imply

$$\delta_{12}K^{(n)}_{12}=\sum_{\ell=0}^{n} b_{n,\ell}\Phi^{n-\ell}_{12}\hat{K}^0_{12}\delta^{\ell}_{12}, \tag{4.46}$$

where $b_{n,\ell}$ depend on $\beta,\tilde{\beta}$ (see Appendix B). Hence

$$[\Phi^m_{12}\hat{K}^0_{12}\cdot 1,(\Phi_{12}+\beta\mathscr{D})^n\delta_{12}\hat{K}^0_{12}\cdot 1]_d=\left[\Phi^m_{12}\hat{K}^0_{12}\cdot 1,\ \sum_{\ell=0}^{n} b_{n,\ell}\Phi^{n-\ell}_{12}\hat{K}^0_{12}\cdot\delta^{\ell}_{12}\right]_d=0. \tag{4.47}$$

Equation (4.47) follows from (4.45a) since $[1,\delta^{\ell}_{12}]_I=0$ for all nonnegative integers ℓ. The left-hand side of Eq. (4.47) equals

$$(\Phi^m_{12}\hat{K}^0_{12}\cdot 1)_d[\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1]-(\delta_{12}\Phi^n_{12}\hat{K}^0_{12}\cdot 1)_d[\Phi^m_{12}\hat{K}^0_{12}\cdot 1];$$

but the first term of the above equals $(\Phi^m_{12}\hat{K}^0_{12}\cdot 1)_f[K^{(n)}]$, hence (4.45b) follows.

It turns out that the recursion operators associated with both the two-dimensional Schrödinger and the two-dimensional 2×2 AKNS are hereditary. Actually, isospectral eigenvalue equations always yield hereditary operators (see Sect. 4E).

Remark 4.6. If Φ_{12} generates two classes of evolution equations $(4.4)_n$, corresponding to two different starting points $\hat{M}_{12}$ and $\hat{N}_{12}$, and if, in addition to (4.44), we have

$$[\hat{M}_{12}H_{12}^{(1)}, \hat{N}_{12}H_{12}^{(2)}]_d=0, \quad \text{for} \quad [H_{12}^{(1)}, H_{12}^{(2)}]_I=0, \tag{4.48}$$

then $\Phi_{12}^m\hat{M}_{12}\cdot 1$ and $\Phi_{12}^m\hat{N}_{12}\cdot 1$ are extended symmetries for both classes of evolution equations.

D. Bi-Hamiltonian Systems

Definition 4.4. i) An admissible operator Θ_{12} is called a Hamiltonian (inverse symplectic) operator iff

a) $$\Theta_{12}^*=-\Theta_{12}, \tag{4.49a}$$

b) it satisfies the Jacobi identity with respect to the bracket

$$\{a_{12}, b_{12}, c_{12}\} \doteq \langle a_{12}, \Theta_{12_d}[\Theta_{12}b_{12}]c_{12}\rangle, \tag{4.49b}$$

for arbitrary a_{12}, b_{12}, c_{12}.

ii) An Eq. (4.16) is of a Hamiltonian form (or is a Hamiltonian system) if it can be written as

$$q_{1_t}=\int_{\mathbb{R}} dy_2\delta_{12}\Theta_{12}\gamma_{12}, \tag{4.50}$$

where Θ_{12} is a Hamiltonian operator and γ_{12} is an extended gradient function of the form $\gamma_{12}=\hat{\gamma}_{12}\cdot 1$ [with, of course, $(\hat{\gamma}_{12}H_{12})_d=(\hat{\gamma}_{12}H_{12})_d^*$].

The associated Poisson bracket is given by:

$$\{I^{(1)}, I^{(2)}\}_H \doteq \langle \mathrm{grad}_{12}I^{(1)}, \Theta_{12}\,\mathrm{grad}_{12}I^{(2)}\rangle, \tag{4.51}$$

where the functional $I^{(i)}$ is given by $I^{(i)}=\int_{\mathbb{R}^3} dxdy_1dy_2\delta_{12}\hat{\varrho}_{12}^{(i)}H_{12}^{(i)}$.

Remark 4.7. If Θ_{12} satisfies a), b) above then the Poisson bracket (4.51) is skew symmetric and satisfies the Jacobi identity.

Proposition 4.1. *Let*

$$G_{12}=\Theta_{12}f_{12}, \quad \Theta_{12} \text{ skew symmetric}. \tag{4.52}$$

Then for arbitrary a_{12}, b_{12} the following identities are valid.

$$\begin{aligned} \mathrm{a}_1)\ &\langle b_{12}, (\Theta_{12_d}[G_{12}]-\Theta_{12}(G_{12})_d^*-(G_{12})_d^*\Theta_{12})a_{12}\rangle \\ &=\{b_{12}, f_{12}, a_{12}\}+\{f_{12}, a_{12}, b_{12}\}+\{a_{12}, b_{12}, f_{12}\} \\ &\quad+\langle b_{12}, \Theta_{12}(f_{12_d}-f_{12_d}^*)\Theta_{12}a_{12}\rangle. \end{aligned} \tag{4.53}$$

Let Θ_{12} *be Hamiltonian and let* a_{12}, b_{12} *be extended gradient functions. Then*

a$_2$) $$[\Theta_{12}a_{12}, \Theta_{12}b_{12}]_d = \Theta_{12}\,\mathrm{grad}_{12}\langle a_{12}, \Theta_{12}b_{12}\rangle. \tag{4.54}$$

These identities imply:

a$_3$) *If* Θ_{12} *is a Hamiltonian operator and* f_{12} *is an extended gradient, then* Θ_{12} *is a Noether operator for* G_{12}.

a$_4$) *If* Θ_{12} *is a Hamiltonian operator and it is a Noether operator for* G_{12} *then* f_{12} *is an extended gradient function.*

The above results are exactly analogous to those in $1+1$ and thus their derivation is omitted.

The above results can be used for any Hamiltonian system as soon as the commutator $[\Theta_{12}, H_{12}]$ is specified. However, for a completely integrable Hamiltonian system additional results are valid.

Proposition 4.2. *Let*

$$\hat{\gamma}_{12}^{(m)} \doteq (\Phi_{12}^*)^m \Theta_{12}^{-1}\hat{K}_{12}^0, \quad \gamma_{12}^{(m)} \doteq \hat{\gamma}_{12}^{(m)}\cdot 1, \quad \hat{K}_{12}^{(n)} \doteq \Phi_{12}^n \hat{K}_{12}^0. \tag{4.55}$$

Assume that Θ_{12} *is Hamiltonian, its inverse exists and that* $\hat{\gamma}_{12}^{(m)}H_{12}$ *are extended gradients. Further assume that Eqs.* (4.4) *are valid. Then*

i) $$\langle \hat{\gamma}_{12}^{(m)}H_{12}^{(1)}, \hat{K}_{12}^{(n)}H_{12}^{(2)}\rangle = \langle \hat{\gamma}_{12}^{(m)}H_{12}^{(1)}, \Theta_{12}\hat{\gamma}_{12}^{(n)}H_{12}^{(2)}\rangle = 0, \tag{4.56}$$

ii) $$(\gamma_{11}^{(m)}, K_{11}^{(n)}) = 0, \quad \text{if} \quad [H_{12}^{(1)}, H_{12}^{(2)}]_I = 0. \tag{4.57}$$

Proof. Since the hereditary operator Φ_{12} is a strong symmetry for the starting symmetry $\hat{K}_{12}^0 H_{12}$ that satisfies (4.4c), then $[\hat{K}_{12}^{(m)}H_{12}^{(1)}, \hat{K}_{12}^{(n)}H_{12}^{(2)}]_d = 0$ if $[H_{12}^{(1)}, H_{12}^{(2)}]_I = 0$. Then (4.56) follows from Proposition 4.1a$_2$). Equation (4.57) follows from (4.56) choosing $H_{12}^{(1)} = 1$ and $H_{12}^{(2)} = \delta_{12}^{\ell}$:

$$(\gamma_{11}^{(m)}, K_{11}^{(n)}) = \langle \gamma_{12}^{(m)}, \delta_{12}K_{12}^{(n)}\rangle = \left\langle \hat{\gamma}_{12}^{(m)}\cdot 1, \sum_{s=0}^{n} b_{n,s}\Phi_{12}^{n-s}\hat{K}_{12}^0\delta_{12}^s\right\rangle = 0.$$

Theorem 4.4. *Let* $\Theta_{12}^{(1)}$, $\Theta_{12}^{(2)}$, $\Theta_{12}^{(1)} + \Theta_{12}^{(2)}$ *be Hamiltonian operators and assume that* $\Theta_{12}^{(1)}$ *is invertible. Then*

i) $\Phi_{12} = \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$ *is a hereditary operator.*

ii) $\Phi_{12}^n\Theta_{12}^{(1)}$, *are Hamiltonian operators.*

iii) *If* $\hat{\gamma}_{12}^0 H_{12} \doteq (\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^0 H_{12}$ *is an extended gradient function and if Eqs.* (4.44) *hold, then Eq.* $(4.4)_n$ *is a bi-Hamiltonian system having* $\Theta_{12}^{(1)}, \Theta_{12}^{(2)}$ *as Noether operators.*

Furthermore, all functions $\gamma_{12}^{(m)}$

$$\gamma_{12}^{(m)} \doteq \hat{\gamma}_{12}^{(m)}\cdot 1, \quad \hat{\gamma}_{12}^{(m)} \doteq (\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^{(m)}, \quad \hat{K}_{12}^{(m)} \doteq \Phi_{12}^m\hat{K}_{12}^0 \tag{4.58}$$

are extended gradients of conserved quantities in involution under the two Poisson brackets defined by

$$\{I^{(m)}, I^{(n)}\} \doteq \langle \delta_{12}\gamma_{12}^{(m)}, \Theta_{12}\gamma_{12}^{(n)}\rangle, \quad \Theta_{12} = \Theta_{12}^{(1)} \text{ or } \Theta_{12}^{(2)}. \tag{4.59}$$

Proof. The derivation of the above results is analogous to similar results for equations in $1+1$ (see for example [7]). With respect to iii) above we note that $\hat{K}_{12}^{(n)}H_{12} = \Phi_{12}^n\Theta_{12}^{(1)}\hat{\gamma}_{12}^0 H_{12}$, hence $\Phi_{12}^n\Theta_{12}^{(1)}$ is a Noether operator for $\Phi_{12}^n\hat{K}_{12}^0 H_{12}$;

the arbitrariness of H_{12} and (4.46) imply that $\Phi^n_{12}\Theta^{(1)}_{12}$ is a Noether operator for $(4.4)_n$; hence $(4.4)_n$ is a Hamiltonian system with $\Phi^n_{12}\Theta^{(1)}_{12}$ as a Noether operator. However, Φ_{12} is a strong symmetry for $\hat{K}^0_{12}H_{12}$, hence Φ^n_{12} is a strong symmetry for $\hat{K}^0_{12}H_{12}$. Since $\Phi^n_{12}\Theta^{(1)}_{12}$ is Noether and Φ^n_{12} is a strong symmetry $\Theta^{(1)}_{12}$ is also Noether. Thus $\Theta^{(2)}_{12}=\Phi_{12}\Theta^{(1)}_{12}$ is also a Noether operator. Furthermore, $K^{(n)}_{12}=\Phi^{n-m}_{12}\Theta^{(1)}_{12}\gamma^{(m)}_{12}$, and the operator $\Phi^{n-m}_{12}\Theta^{(1)}_{12}$ is both Noether and Hamiltonian, thus $\hat{\gamma}^{(m)}_{12}H_{12}$ are extended gradient functions (using Proposition 4.1).

It now trivially follows [since Theorem 4.3 implies that $K^{(m)}_{12}$ are extended symmetries of $(4.4)_n$] that $\gamma^{(m)}_{12}$ are conserved covariants of $(4.4)_n$. Moreover, Proposition 4.2 implies:

$$\begin{aligned}\{I^{(m)},I^{(n)}\}_H&=\langle\hat{\gamma}^{(m)}_{12}H^{(1)}_{12},\Theta^{(1)}_{12}\hat{\gamma}^{(n)}_{12}H^{(2)}_{12}\rangle\\&=\langle\hat{\gamma}^{(m)}_{12}H^{(1)}_{12},\Theta^{(2)}_{12}\gamma^{(n-1)}_{12}H^{(2)}_{12}\rangle=0\,,\quad\text{if}\quad[H^{(1)}_{12},H^{(2)}_{12}]_I=0\,,\end{aligned}$$

and the choice $H^{(1)}_{12}=\delta^{(\ell)}_{12}$, $H^{(2)}_{12}=1$ yields

$$\{I^{(m)},I^{(n)}\}\doteq\langle\delta_{12}\gamma^{(m)}_{12},\Theta_{12}\gamma^{(n)}_{12}\rangle=0\,,\qquad\Theta_{12}=\Theta^{(1)}_{12}\text{ or }\Theta^{(2)}_{12}\,. \tag{4.60a}$$

Namely $\gamma^{(n)}_{12}$, are extended gradients of conserved quantities in involution. If $[\Theta_{12},\delta_{12}]=0$, then

$$(\gamma^{(m)}_{11},\Theta_{11}\gamma^{(n)}_{11})=0\,. \tag{4.60b}$$

Combining Theorems 4.1–4.4, we obtain the following important theorem.

Theorem 4.5. *Let $\Theta^{(1)}_{12}+v\Theta^{(2)}_{12}$ be a Hamiltonian operator for all constant values of v. Assume that $\Theta^{(1)}_{12}$ is invertible. Define*

$$\Phi_{12}\doteq\Theta^{(2)}_{12}(\Theta^{(1)}_{12})^{-1}\,,\qquad K^{(n)}_{12}\doteq\Phi^n_{12}\hat{K}^0_{12}\cdot 1\,,\qquad\gamma^0_{12}\doteq(\Theta^{(1)}_{12})^{-1}K^0_{12}\,. \tag{4.61}$$

Assume that the operator Φ_{12} and its associated starting symmetry $\hat{K}^0_{12}H_{12}$ satisfy (4.44). *Further assume that $\gamma^{(0)}_{12}$ is an extended gradient function. Then*

i) *Equations $(4.4)_n$ are bi-Hamiltonian systems.*

ii) $K^{(m)}_{12}\doteq\Phi^m_{12}\hat{K}^0_{12}\cdot 1$, $\gamma^{(m)}_{12}=(\Phi^*_{12})^m\gamma^0_{12}$ *are extended symmetries and extended gradients of conserved quantities, respectively, for Eq.* $(4.4)_n$.

iii) $K^{(m)}_{11}$ *and* $\gamma^{(m)}_{11}$ *are symmetries and gradients of conserved quantities in involution for* $q_{1_t}=K^{(n)}_{11}$.

iv) $K^{(m)}_{12}=0$ *are auto-Bäcklund Transformations for Eq.* $(4.4)_n$.

v)
$$[K^{(m)}_{11},K^{(n)}_{11}]_f=0\,, \tag{4.62a}$$

$$\{I^{(m)},I^{(n)}\}\doteq\langle\delta_{12}\gamma^{(m)}_{12},\Theta_{12}\gamma^{(n)}_{12}\rangle=0\,,\qquad\Theta_{12}=\Theta^{(1)}_{12}\ \textit{or}\ \Theta^{(2)}_{12}\,, \tag{4.62b}$$

where

$$[a,b]_f=a_f[b]-b_f[a]\,. \tag{4.62c}$$

E. Isospectral Problems Yield Hereditary Operators

Section 4.C illustrates the importance of hereditary operators. For equations in $1+1$, isospectral problems yield hereditary operators. A similar construction is possible for equations in $2+1$. Furthermore, this construction also provides us with a simple commutation relation of the type (4.24a) between Φ_{12} and h_{12}.

Proposition 4.3. *Let*

$$\frac{dV}{dx} = U(\hat{q}, \lambda)V \tag{4.63}$$

be an isospectral two-dimensional problem; $\hat{q}$ is an operator depending on $q(x, y)$ and $\partial/\partial y$; λ is an eigenvalue. Assume that $(G_\lambda)_{12}$, the extended gradient of λ satisfies

$$\Psi_{12}(G_\lambda)_{12} = \mu(\lambda)(G_\lambda)_{12}\,. \tag{4.64}$$

Then if $\Phi_{12} \doteqdot \Psi^*_{12}$ has a complete set of eigenfunctions, it is hereditary operator.

Instead of deriving this result we illustrate it by two examples. The interested reader is referred to [5]. A proof of completeness should follow a two-dimensional version of the method developed by [10].

The derivation of Eq. (4.24a) from Eqs. (4.63) and (4.64) is also illustrated in an example.

Example 1. Consider the isospectral problem

$$v_{1xx} + (q_1 + \alpha D_{y_1})v_1 = \lambda v_1\,. \tag{4.65}$$

Let $\hat{q}_1 \doteqdot q_1 + \alpha D_{y_1}$ and consider the directional derivative of (4.65):

$$v_{1_{xx_d}}[\;] + \hat{q}_{1_d}[\;]v_1 + \hat{q}_1 v_{1_d}[\;] = \lambda v_{1_d}[\;] + \lambda_d[\;]v_1\,.$$

Multiplying the above by v_1^+, where v_1^+ satisfies the adjoint of (4.65), with respect to the bilinear form (4.9), integrating with respect to $dy_1 dx$, and assuming $\int_{\mathbb{R}^2} dx dy_1 v_1 v_1^+ = 1$ it follows that

$$\lambda_d[f_{12}] = \int_{\mathbb{R}^2} dx dy_1 v_1^+ \hat{q}_{1_d}[f_{12}]v_1\,. \tag{4.66}$$

Using (4.1b) to evaluate $\hat{q}_{1_d}[f_{12}]v_1$ it follows that

$$\lambda_d[f_{12}] = \int_{\mathbb{R}^3} dx dy_1 dy_2 v_2 v_1^+ f_{12}\,.$$

Hence, using $\lambda_d[f_{12}] = \int_{\mathbb{R}^3} dx dy_1 dy_2 (\operatorname{grad}\lambda)_{21} f_{12}$, it follows that

$$(\operatorname{grad}\lambda)_{12} = v_1 v_2^+\,. \tag{4.67}$$

Since Φ_{12} defined by (1.2a) satisfies [29]

$$\Phi^*_{12} v_1 v_2^+ = 4\lambda v_1 v_2^+\,, \tag{4.68}$$

it follows that Φ_{12} is hereditary.

Example 2. Consider the isospectral problem

$$V_{1_x} - JV_{1_y} - Q_1 V_1 = \lambda J V_1\,, \tag{4.69}$$

where J, Q are defined in (1.8). In analogy with (4.66) and assuming $\operatorname{tr}\int_{\mathbb{R}^2} dx dy_1 V_1^+ J V_1 = 1$, we find

$$\lambda_d[F_{12}] = \operatorname{tr}\int_{\mathbb{R}^2} dx dy_1 V_1^+ \hat{Q}_{1_d}[F_{12}]V_1\,.$$

Hence, using $\hat{Q}_{1_d}[F_{12}]G_{12}=\int_{\mathbb{R}} dy_3 F_{13}G_{32}$, it follows that

$$\lambda_d[F_{12}]=\operatorname{tr}\int_{\mathbb{R}^3} dx dy_1 dy_2 V_1^+ F_{12} V_2 .$$

Thus

$$(\operatorname{grad}\lambda)_{12}=V_1 V_2^+ .$$

Since $R_{12}\doteq D-\hat{Q}_{12}$ satisfies

$$R_{12}V_1V_2^+=\lambda\hat{J}V_1V_2^+, \qquad \hat{J}F_{12}\doteq JF_{12}-F_{12}J, \tag{4.70}$$

it follows that $(R_{12}^{-1}\hat{J})^*=\hat{J}^*(R_{12}^{-1})^*=\hat{J}R_{12}^{-1}$ is hereditary (see [39] for the analogous result in $1+1$ dimensions).

Now we show that Eqs. (4.65) and (4.68) imply

$$[\Phi_{12}, h_{12}]=4\alpha h'_{12}, \qquad h_{12}=h(y_1-y_2). \tag{4.71}$$

First, we recall that Eq. (4.68) follows from Eq. (4.65): Eq. (4.68) and its adjoint $V^+_{2_{xx}}+(q_2-\alpha D_2)V_2^+=\lambda V_2^+$ imply

$$V_{1_{xx}}V_2^+ +(q_1+\alpha D_1)V_1V_2^+=\lambda V_1V_2^+, \tag{4.72a}$$

$$V_1V^+_{2_{xx}}+(q_2-\alpha D_2)V_1V_2^+=\lambda V_1V_2^+, \tag{4.72b}$$

$$V_{1_{xx}}V^+_{2_x}+(q_1+\alpha D_1)V_1V^+_{2_x}=\lambda V_1V^+_{2_x}, \tag{4.73a}$$

$$V_{1_x}V^+_{2_{xx}}+(q_2-\alpha D_2)V_{1_x}V_2^+=\lambda V_{1_x}V_2^+ . \tag{4.73b}$$

Adding Eqs. (4.72a) and (4.72b), Eqs. (4.73a) and (4.73b), and subtracting Eq. (4.72b) from Eq. (4.72a) we obtain, respectively,

$$(D^2+q^+_{12})V_1V_2^+=2V_{1_x}V^+_{2_x}+2\lambda V_1V_2^+, \tag{4.74a}$$

$$V_{1_x}V^+_{2_x}=-\frac{D^{-1}}{2}q^+_{12}DV_1V_2^+-\frac{D^{-1}}{2}q^-_{12}(V_1V^+_{2_x}-V_{1_x}V_2^+)+\lambda V_1V_2^+, \tag{4.74b}$$

$$V_1V^+_{2_x}-V_{1_x}V_2^+=D^{-1}q^-_{12}V_1V_2^+ . \tag{4.74c}$$

Using Eqs. (4.74b–c) into Eq. (4.74a) we finally obtain the eigenvalue equation (4.68).

Now, by virtue of the commutation relations $[q_1+\alpha D_1, h_{12}]=[q_2-\alpha D_2, h_{12}]=\alpha h'_{12}$, Eqs. (4.72) and (4.73) are still valid replacing $V_1\to V_{12}\doteq h_{12}V_1$, $V_2^+\to V^+_{12}\doteq h_{12}V_2^+$ and $\lambda\to\lambda_{12}\doteq\lambda+2\alpha h'_{12}/h_{12}$; then $\Phi^*_{12}V_{12}V^+_{12}=4\lambda_{12}V_{12}V^+_{12}$, namely

$$\begin{aligned}\Phi^*_{12}V_{12}V^+_{12}&=\Phi^*_{12}h^2_{12}V_1V_2^+=(h^2_{12}\Phi^*_{12}+[\Phi^*_{12},h^2_{12}])V_1V_2^+\\&=(4\lambda h^2_{12}+8\alpha h'_{12}/h_{12})V_1V_2^+ .\end{aligned}$$

Using Eq. (4.68) and the completeness of the eigenfunctions of Φ^*_{12}, Eq. (4.71) follows.

5. Applications

In this section we apply the theory developed in the previous sections to the classes of evolutions associated with the Schrödinger eigenvalue problem (1.1) and with the 2×2 AKNS problem (1.8).

Some interesting details of the explicit calculations concerning these two examples are separately presented in Appendix C.

An isospectral problem [e.g. (1.1)] yields a recursion operator Φ_{12} [e.g. (1.2a)]. This operator must be hereditary (see Sect. 4.E). The isospectral problem also yields a basic operator $\hat{q}_{12}$; the integral representation of this operator implies a directional derivative $\hat{q}_{1_d}$. Using the bilinear form (4.7), $\hat{q}_1^*$, $\hat{q}_{1_d}^*$ are also obtained.

i) In investigating the time-independent symmetries of the hierarchies associated with Φ_{12} one then needs to: a) Find the starting symmetries $\hat{K}_{12}^0 H_{12}$ associated with Φ_{12} (see Sect. 4.B). b) Calculate the commutator relations of Φ_{12}, $\hat{K}_{12}^0$ with h_{12}. c) Compute the Lie algebra of the starting symmetries. Then Theorems 4.1, 4.3 yield hierarchies of infinitely many commuting symmetries.

ii) In investigating the Hamiltonian nature of the hierarchies associated with Φ_{12} one, in addition to the above, also needs to: a) Prove that $\Theta_{12}^{(1)}$, $\Theta_{12}^{(2)}$, where $\Phi_{12}=\Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}$, are compatible Hamiltonian operators. b) Verify that the starting covariants are extended gradients. Then Theorem 4.4 yields hierarchies of infinitely many involutionary conserved quantities.

A. The Schrödinger Eigenvalue Problem

The spectral problem (1.1) yields the hereditary operator

$$\Phi_{12}=D^2+q_{12}^+ + Dq_{12}^+D^{-1}+q_{12}^-D^{-1}q_{12}^-D^{-1}, \tag{5.1a}$$

where

$$q_{12}^{\pm} \doteq q_1 \pm q_2 + \alpha(D_1 \mp D_2). \tag{5.1b}$$

The integral representation of the basic operator $\hat{q}_1$ implies an appropriate directional derivative:

$$\hat{q}_1 f_{12} \doteq (q_1+\alpha D_1)f_{12} = \int_{\mathbb{R}} dy_3 q_{13} f_{32}, \qquad \hat{q}_{1_d}[\sigma_{12}]f_{12} = \int_{\mathbb{R}} dy_3 \sigma_{13} f_{32}. \tag{5.2}$$

The adjoint of Eq. (5.2) implies

$$\hat{q}_1^* f_{12} = (q_1-\alpha D_2)f_{12} = \int_{\mathbb{R}} dy_3 f_{13} q_{32}, \qquad \hat{q}_{1_d}^*[\sigma_{12}]f_{12} = \int_{\mathbb{R}} dy_3 f_{13}\sigma_{32}. \tag{5.3}$$

Combining the above we obtain the following derivative:

$$\begin{aligned} a_{12}(\hat{q})_d[f_{12}] &= \frac{\partial}{\partial\varepsilon} a_{12}(q_{12}^{\pm}+\varepsilon f_{12}^{\pm})\Big|_{\varepsilon=0}, \\ f_{12}^{\pm} g_{12} &= \int_{\mathbb{R}} dy_3 (f_{13}g_{32} \pm g_{13}f_{32}), \end{aligned} \tag{5.4}$$

which satisfies the projective property (4.6).

i) Let us first investigate the time-independent symmetries of the equations generated by Φ_{12}.

a) Equation (4.33) yields

$$\hat{S}_{12}=D\,,\qquad H_{12}=H_{12}(y_1,y_2)\,, \tag{5.5a}$$

and starting operators $\hat{K}^0_{12}$ given by

$$\hat{N}_{12}\doteq q^-_{12}\,,\qquad \hat{M}_{12}\doteq Dq^+_{12}+q^-_{12}D^{-1}q^-_{12}\,. \tag{5.5b}$$

b) The commutators of Φ_{12} with h_{12} imply the following operator equations:

$$[\Phi_{12},h_{12}]=4\alpha h'_{12}\,,\qquad [\hat{N}_{12},h_{12}]=0\,,\qquad [\hat{M}_{12},h_{12}]=2\alpha Dh'_{12}\,. \tag{5.6}$$

Hence, if

$$N^{(n)}_{12}\doteq \Phi^n_{12}\hat{N}_{12}\cdot 1\,,\qquad M^{(n)}_{12}\doteq \Phi^n_{12}\hat{M}_{12}\cdot 1\,, \tag{5.7}$$

then Eq. (4.46) yields

$$\delta_{12}N^{(n)}_{12}=\sum_{\ell=1}^{n}(-4\alpha)^{\ell}\binom{n}{\ell}\Phi^{n-\ell}_{12}\hat{N}_{12}\delta^{\ell}_{12}\,, \tag{5.8a}$$

$$\delta_{12}M^{(n)}_{12}=\sum_{\ell=1}^{n} b_{n,\ell}\Phi^{n-\ell}_{12}\hat{M}_{12}\delta^{\ell}_{12}\,,\qquad b_{n,\ell}\doteq(-4\alpha)^{\ell}\sum_{j=0}^{\ell}2^{-j}\binom{n-j}{\ell-j} \tag{5.8b}$$

(see Appendix B).

c) The Lie algebra of the starting symmetries is given by

$$[\hat{N}_{12}H^{(1)}_{12},\hat{N}_{12}H^{(2)}_{12}]_d=-\hat{N}_{12}H^{(3)}_{12}\,,\qquad [\hat{N}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\hat{M}_{12}H^{(3)}_{12}\,,$$

$$[\hat{M}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\Phi_{12}\hat{N}_{12}H^{(3)}_{12}\,,\qquad H^{(3)}_{12}\doteq[H^{(1)}_{12},H^{(2)}_{12}]_I\,, \tag{5.9}$$

where $[\,,\,]_d$, $[\,,\,]_I$ are defined by (4.3).

ii) We now investigate the Hamiltonian structure of the equations generated by Φ_{12}:

a) $\Phi_{12}\Theta^{(1)}_{12}=\Theta^{(1)}_{12}\Phi^*_{12}$, where

$$\Theta^{(1)}_{12}=D\,,\qquad \Phi^*_{12}=D^2+q^+_{12}+D^{-1}q^+_{12}D+D^{-1}q^-_{12}D^{-1}q^-_{12}=D^{-1}\Phi_{12}D=\Psi_{12}\,.$$

We first note that both $\Theta^{(1)}_{12}=D$ and $\Theta^{(2)}_{12}=\Phi_{12}D$ are skew symmetric:

$$\Theta^{(1)*}_{12}=-D=-\Theta^{(1)}_{12}\,,\qquad \Theta^{(2)*}_{12}=(\Phi_{12}D)^*=-D\Phi^*_{12}=-\Phi_{12}D=-\Theta^{(2)}_{12}\,.$$

Furthermore, the bracket

$$\begin{aligned}\{a_{12},b_{12},c_{12}\}&=\langle a_{12},\Theta^{(2)}_{12\,d}[\Theta^{(2)}_{12}b_{12}]c_{12}\rangle\\ &=\langle a_{12},(\Theta^{(2)}_{12}b_{12})^+D+D(\Theta^{(2)}_{12}b_{12})^+ +(\Theta^{(2)}_{12}b_{12})^-D^{-1}q^-_{12}+q^-_{12}D^{-1}(\Theta^{(2)}_{12}b_{12})^-)c_{12}\rangle\end{aligned}$$

satisfies the Jacobi identity. Also $\Theta^{(1)}_{12}$, $\Theta^{(2)}_{12}$ are compatible.

b) $\hat{\gamma}^0_{12}H_{12}=D^{-1}q^-_{12}H_{12}$ and $\hat{\gamma}^0_{12}=D^{-1}\hat{M}_{12}H_{12}$ are extended gradient functions. Thus the Theorems 4.1–4.4 imply:

Proposition 5.1. *Consider the two compatible Hamiltonian operators* $\Theta^{(1)}_{12}=D$ *and*

$$\Theta^{(2)}_{12}=D^3+q^+_{12}D+Dq^+_{12}+q^-_{12}D^{-1}q^-_{12}\,,$$

and define

$$\Phi_{12} \doteq \Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1} = D^2 + q_{12}^+ + Dq_{12}^+D^{-1} + q_{12}^-D^{-1}q_{12}^-D^{-1},$$

$$\hat{N}_{12}^{(n)} \doteq \Phi_{12}^n \hat{N}_{12}, \quad \hat{M}_{12}^{(m)} \doteq \Phi_{12}^n \hat{M}_{12}, \quad \hat{\gamma}_{12}^{(n)} \doteq (\Theta_{12}^{(1)})^{-1}\hat{N}_{12} \quad \textit{and/or} \quad (\Theta_{12}^{(1)})^{-1}\hat{M}_{12}^{(n)},$$

where the starting operator $\hat{N}_{12}$ *and* $\hat{M}_{12}$ *are defined by* $\hat{N}_{12} \doteq q_{12}^-$ *and* $\hat{M}_{12} = Dq_{12}^+ + q_{12}^-D^{-1}q_{12}^-$. *Then*

i) $M_{12}^{(m)} \doteq \hat{M}_{12}^{(m)} \cdot 1$ *and* $N_{12}^{(m)} \doteq \hat{N}_{12}^{(m)} \cdot 1$ *are extended symmetries for both classes of evolution equations*

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} N_{12}^{(n)} = N_{11}^{(n)}, \tag{5.10a}$$

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} M_{12}^{(n)} = M_{11}^{(n)}; \tag{5.10b}$$

namely

$$[M_{12}^{(m)}, \delta_{12}K_{12}^{(n)}]_d = [N_{12}^{(m)}, \delta_{12}K_{12}^{(n)}]_d = 0, \tag{5.11}$$

where $K_{12}^{(n)} = N_{12}^{(n)}$ *and/or* $M_{12}^{(n)}$.

ii) $\gamma_{12}^{(m)} \doteq \hat{\gamma}_{12}^{(m)} \cdot 1$ *are extended gradients of conserved quantities of both classes of evolution equations* (5.10), *namely*

$$\gamma_{12_d}^{(m)}[\delta_{12}K_{12}^{(n)}] + (\delta_{12}K_{12}^{(n)})_d^*[\gamma_{12}^{(m)}] = 0, \tag{5.12a}$$

$$(\hat{\gamma}_{12}^{(m)}H_{12})_d = (\hat{\gamma}_{12}^{(m)}H_{12})_d^*, \quad H_{12_x} = 0, \tag{5.12b}$$

*where * indicates the adjoint operation with respect to the bilinear form*

$$\langle f_{12}, g_{12} \rangle \doteq \int_{\mathbb{R}^3} dx dy_1 dy_2 f_{21} g_{12}. \tag{5.13}$$

iii) *The two classes of evolution equations* (5.10) *are bi-Hamiltonian, namely they can be written in the form*

$$q_{1_t} = \int_{\mathbb{R}} dy_2 \delta_{12} \Theta_{12}^{(1)} \gamma_{12}^{(n)} = \int_{\mathbb{R}} dy_2 \delta_{12} \Theta_{12}^{(2)} \gamma_{12}^{(n-1)}. \tag{5.14}$$

iv) $M_{11}^{(m)}$ *and* $N_{11}^{(m)}$ *are infinitely many commuting symmetries of the classes of evolution equations* (5.10), *namely*

$$[M_{11}^{(m)}, M_{11}^{(n)}]_f = [M_{11}^{(m)}, N_{11}^{(n)}]_f = [N_{11}^{(m)}, N_{11}^{(n)}]_f = 0. \tag{5.15}$$

v) $\gamma_{11}^{(m)}$ *are infinitely many gradients of conserved quantities of the equations* (5.10), *namely*

$$\gamma_{11_f}^{(m)}[K_{11}^{(n)}] + K_{11_f}^{(n)+}[\gamma_{11}^{(m)}] = 0, \tag{5.16a}$$

$$\gamma_{11_f}^{(m)} = \gamma_{11_f}^{(m)+}, \tag{5.16b}$$

where $^+$ *indicates the operation of adjoint with respect to the bilinear form*

$$(f, g) \doteq \int_{\mathbb{R}^2} dx dy f g. \tag{5.17}$$

The corresponding conserved quantities are in involution with respect to the Poisson brackets

$$\{I^{(n)}, I^{(m)}\} \doteq \langle \delta_{12}\gamma_{12}^{(n)}, \Theta_{12}\gamma_{12}^{(m)} \rangle, \quad \Theta_{12} = \Theta_{12}^{(1)} \text{ or } \Theta_{12}^{(2)}; \tag{5.18a}$$

if

$$\Theta_{12}=\Theta_{12}^{(1)}, \quad \langle\delta_{12}\gamma_{12}^{(n)}, D\gamma_{12}^{(m)}\rangle=(\gamma_{11}^{(n)}, D\gamma_{12}^{(m)}). \tag{5.18b}$$

vi) *The equations* $M_{12}^{(m)}=0$ *and* $N_{12}^{(m)}=0$ *are Bäcklund Transformations for both classes of evolution equations* (5.10).

B. The 2×2 *AKNS Problem*

The spectral problem (1.8) yields the hereditary operator

$$\Phi_{12}=\sigma(P_{12}-Q_{12}^{+}P_{12}^{-}Q_{12}^{+}) \tag{5.19}$$

acting on off-diagonal matrices, where

$$Q_{12}^{\pm}F_{12}\doteq Q_1F_{12}\pm F_{12}Q_{12}, \tag{5.20a}$$

$$P_{12}F_{12}\doteq F_{12_x}-JF_{12_{y_1}}-F_{12_{y_2}}J. \tag{5.20b}$$

The integral representation of the basic operator $\hat{Q}_1\doteq Q_1+JD_1$, implies an appropriate directional derivative:

$$\hat{Q}_1F_{12}\doteq(Q_1+JD_1)F_{12}=\int_{\mathbb{R}} dy_3Q_{13}F_{32}, \quad \hat{Q}_{1_d}[\sigma_{12}]F_{12}=\int_{\mathbb{R}} dy_3\sigma_{13}F_{32}, \tag{5.21}$$

and the adjoint of Eqs. (5.21) imply

$$\hat{Q}_1^*F_{12}=F_{12}Q_2-F_{12_{y_2}}J=\int_{\mathbb{R}} dy_3F_{13}Q_{32}, \quad \hat{Q}_{1_d}^*[\sigma_{12}]F_{12}=\int_{\mathbb{R}} dy_3F_{13}\sigma_{32}. \tag{5.22}$$

Then the reduction to the space of off-diagonal matrices performed in Sect. 3 induces the following derivative of the operator Φ_{12}:

$$\Phi_{12_d}[G_{12}]=-\sigma(G_{12}^{+}P_{12}^{-1}Q_{12}^{+}+Q_{12}^{+}P_{12}^{-1}G_{12}^{+}), \tag{5.23a}$$

$$G_{12}^{\pm}F_{12}\doteq\int_{\mathbb{R}} dy_3(G_{13}G_{32}\pm F_{13}G_{32}). \tag{5.23b}$$

Again the Leibnitz rule and property (4.6) are satisfied.

i) The investigation of the time-independent symmetries of the evolution equations generated by Φ_{12} gives the following results.

a) Equations (4.38) yield $\hat{S}_{12}=(Q_{12}^{+})^{-1}P_{12}$, the starting operators $\hat{K}_{12}^0$ are given by

$$\hat{N}_{12}\doteq Q_{12}^{-}, \quad \hat{M}_{12}\doteq Q_{12}^{-}\sigma, \tag{5.24}$$

and H_{12} is diagonal and such that $P_{12}H_{12}=0$.

b) The commutators of Φ_{12} with h_{12} imply the following operator equations:

$$[\Phi_{12},h_{12}]=-2\alpha h'_{12}, \quad [\hat{N}_{12},h_{12}]=[\hat{M}_{12},h_{12}]=0, \tag{5.25}$$

valid on arbitrary off-diagonal matrices. Hence, if

$$N_{12}^{(n)}\doteq\Phi_{12}^n\hat{N}_{12}\cdot I, \quad M_{12}^{(n)}\doteq\Phi_{12}^n\hat{M}_{12}\cdot I, \tag{5.26}$$

then Eq. (4.46) yields

$$\delta_{12}N_{12}^{(n)}=\sum_{\ell=1}^{n}(2\alpha)^{\ell}\binom{n}{\ell}\Phi_{12}^{n-\ell}\hat{N}_{12}\delta_{12}^{\ell}\,, \tag{5.27a}$$

$$\delta_{12}M_{12}^{(n)}=\sum_{\ell=1}^{n}(2\alpha)^{\ell}\binom{n}{\ell}\Phi_{12}^{n-\ell}\hat{M}_{12}\delta_{12}^{\ell}\,. \tag{5.27b}$$

c) The Lie algebra of the starting symmetries is given by

$$[\hat{N}_{12}H_{12}^{(1)},\hat{N}_{12}H_{12}^{(2)}]_d=-\hat{N}_{12}H_{12}^{(3)}\,,\qquad [\hat{N}_{12}H_{12}^{(1)},\hat{M}_{12}H_{12}^{(2)}]_d=-\hat{M}_{12}H_{12}^{(3)}\,,$$
$$[\hat{M}_{12}H_{12}^{(1)},\hat{M}_{12}H_{12}^{(2)}]_d=-\hat{N}_{12}H_{12}^{(3)}\,,\qquad H_{12}^{(3)}\doteq[H_{12}^{(1)},H_{12}^{(2)}]_I\,. \tag{5.28}$$

ii) We now investigate the Hamiltonian structure of the equations generated by Φ_{12}:

a) $\Phi_{12}\Theta_{12}^{(1)}=\Theta_{12}^{(1)}\Phi_{12}^{*}$, where

$$\Theta_{12}^{(1)}=\sigma\,,\qquad \Phi_{12}^{*}=\sigma(P_{12}-Q_{12}^{-}P_{12}^{-1}Q_{12}^{-})=\sigma^{-1}\Phi_{12}\sigma=\Psi_{12}\,; \tag{5.29}$$

notice that on the space of off-diagonal matrices $\sigma F_{12}=\frac{1}{2}[\sigma,F_{12}]$, $\Theta_{12}^{(1)}=\sigma$ and $\Theta_{12}^{(2)}=\Phi_{12}\Theta_{12}^{(1)}$ are skew-symmetric in the space of off-diagonal matrices:

$$\langle F_{12},\sigma G_{12}\rangle=-\langle\sigma F_{12},G_{12}\rangle\,,$$

and

$$\Theta_{12}^{(2)*}=(\Phi_{12}\sigma)^{*}=-\sigma\Phi_{12}^{*}=-\Phi_{12}\sigma=-\Theta_{12}^{(2)}\,.$$

Furthermore, the bracket $\{A_{12},B_{12},C_{12}\}\doteq\langle A_{12},\Theta_{12d}^{(2)}[\Theta_{12}^{(2)}B_{12}]C_{12}\rangle$ satisfies the Jacobi identity and $\Theta_{12}^{(1)}$, $\Theta_{12}^{(2)}$ are compatible.

b) $\hat{\gamma}_{12}^{0}H_{12}=(\Theta_{12}^{(1)})^{-1}\hat{K}_{12}^{0}H$ ($\hat{K}_{12}^{0}=\hat{N}_{12}$ or $\hat{M}_{12}$) are extended gradients, thus Theorems 4.1–4.4 imply:

Proposition 5.2. *Consider the two compatible Hamiltonian operators $\Theta_{12}^{(1)}=\sigma$ and $\Theta_{12}^{(2)}=P_{12}-Q_{12}^{-}P_{12}^{-1}Q_{12}^{-}$ acting on off-diagonal matrices, and define*

$$\Phi_{12}\doteq\Theta_{12}^{(2)}(\Theta_{12}^{(1)})^{-1}=\sigma(P_{12}-Q_{12}^{+}P_{12}^{-1}Q_{12}^{+})\,,\qquad \hat{N}_{12}^{(n)}\doteq\Phi_{12}^{n}\hat{N}_{12}\,,$$
$$\hat{M}_{12}^{(n)}\doteq\Phi_{12}^{n}\hat{M}_{12}\,,\qquad \hat{\gamma}_{12}^{(n)}\doteq(\Theta_{12}^{(1)})^{-1}\hat{N}_{12}^{(n)}\quad\text{and/or}\quad(\Theta_{12}^{(1)})^{-1}\hat{M}_{12}^{(n)}\,,$$

where the starting operators $\hat{N}_{12}$ and $\hat{M}_{12}$ are defined by $\hat{N}_{12}\doteq Q_{12}^{-}$ and $\hat{M}_{12}\doteq Q_{12}^{-}\sigma$. Then the results i)–vi) of Proposition 5.1 are all valid for the two classes of evolution equations

$$Q_{1_t}=\int_{\mathbb{R}}dy_2\delta_{12}N_{12}^{(n)}=N_{11}^{(n)}\,, \tag{5.30a}$$

$$Q_{1_t}=\int_{\mathbb{R}}dy_2\delta_{12}M_{12}^{(n)}=M_{11}^{(n)}\,, \tag{5.30b}$$

introducing trace in the right-hand side Eqs. (5.13) and (5.17) and replacing (5.18b) by

$$\Theta_{12}=\Theta_{12}^{(1)}=\sigma\,,\qquad \langle\delta_{12}\gamma_{12}^{(n)},\sigma\gamma_{12}^{(m)}\rangle=(\gamma_{11}^{(n)},\sigma\gamma_{11}^{(m)})\,.$$

Appendix A

Now we show that the assumptions (4.30a), (4.30b) follow from (4.28a), without using the explicit form of the operator. We show this for the recursion operator associated with the Schrödinger eigenvalue problem.

Admissibility requires Φ_{12} to depend on $q_{12}^{\pm}$, moreover, (4.28a) and (3.13) imply that Φ_{12} depends linearly on q_{12}^{+}. Then, without loss of generality we have

$$\Phi_{12_d}[f_{12}]g_{12}=\sum_j c_j f_{12}^{+} d_j g_{12}+\sum_s p_s(q_{12}^{-}) f_{12}^{-} r_s(q_{12}^{-}) g_{12}\,, \tag{A.1a}$$

$$\Phi_{12_f}[f]g_{12}=\sum_j c_j(f_{11}+f_{22}) d_j g_{12}+\sum_s p_s(q_{12}^{-})(f_{11}-f_{22}) r_s(q_{12}^{-}) g_{12}\,, \tag{A.1b}$$

where c_i, d_i are arbitrary functions of D, D^{-1}; p_s, r_s are arbitrary functions of q_{12}^{-} and $f_{12}^{\pm}$ are defined in (5.4b).

Then the commutation property $[q_{12}^{-}, h_{12}]=0$ implies

$$\Phi_{12_d}[h_{12} f_{12}]\delta_{12} g_{12}=h_{12}\Phi_{12_d}[f_{12}]\delta_{12} g_{12}\,, \tag{A.2a}$$

$$\Phi_{12_f}[f] h_{12} g_{12}=h_{12}\Phi_{12_f}[f] g_{12}\,. \tag{A.2b}$$

Appendix B

In this appendix we show that equations

$$[\Phi_{12}, h_{12}]=-\beta h'_{12}\,, \qquad h_{12}=h(y_1-y_2)\,, \tag{B.1a}$$

$$[\hat{K}_{12}^{0}, h_{12}]=-\tilde{\beta}\hat{S}_{12} h'_{12}\,, \tag{B.1b}$$

and some additional notions concerning the associated spectral problem, imply

$$\delta_{12} K_{12}^{(n)}=\sum_{\ell=0}^{n} b_{n,\ell}\Phi_{12}^{n-\ell}\hat{K}_{12}^{0}\delta_{12}^{\ell} \tag{B.2}$$

for suitable constants $b_{n,\ell}$.

We first observe that the case $\tilde{\beta}=0$ is particularly simple; indeed, in this case

$$\delta_{12} K_{12}^{(n)}=\delta_{12}\Phi_{12}^{n}\hat{K}_{12}^{0}\cdot 1=(\Phi_{12}+\beta\mathscr{D})^{n}\hat{K}_{12}^{0}\delta_{12}=\sum_{\ell=0}^{n} b_{n,\ell}\Phi_{12}^{n-\ell}\hat{K}_{12}^{0}\delta_{12}^{\ell}\,, \tag{B.3a}$$

$$b_{n,\ell}\doteq\beta^{\ell}\binom{n}{\ell}. \tag{B.3b}$$

This is the case for the two classes of evolution equations associated with the two-dimensional AKNS problem and for Eqs. (3.20). For the KP class (3.19), $\hat{K}_{12}^{0}=\hat{M}_{12}\doteq Dq_{12}^{+}+q_{12}^{-}D^{-1}q_{12}^{-}$, $\tilde{\beta}=\beta/2=-2\alpha$, $\hat{S}_{12}=D$ and the result (B.2) is less straightforward.

In order to obtain it, we first show that

$$\Phi_{12}^{n}\Gamma_{12}\cdot 1=0\,, \qquad \forall n\geqq 0\,; \qquad \Gamma_{12}\doteq\Phi_{12}D-\hat{M}_{12}\,. \tag{B.4}$$

This result could be easily derived using the explicit form of Φ_{12} and $\hat{M}_{12}$. Here we give a different derivation using the underlying spectral problem (and the consequent eigenvalue equation satisfied by Φ^*_{12}). This derivation is similar in spirit to the one of (B.1a) presented in Sect. 4.E.

From Eq. (4.38), it follows that Γ_{12} can be written as

$$\Gamma_{12}=\Delta_{12}D, \qquad \Delta_{12}H_{12}\neq 0. \tag{B.5}$$

The operator Δ_{12}, which is part of Φ_{12}, is admissible depending on D, D^{-1}, $q^{\pm}_{12}$. If for any admissible operator L_{12}, we define $L^{(0)}_{12}$ as $L^{(0)}_{12}\doteq L_{12}|_{q=0}$, then

$$\Phi^n_{12}\Gamma_{12}\cdot 1=\Phi^n_{12}\Delta_{12}D\cdot 1=\Phi^n_{12}\Delta^{(0)}_{12}D\cdot 1=D\Psi^n_{12}\Delta^{(0)}_{12}\cdot 1, \tag{B.6}$$

since $D^{-1}qD\cdot 1=0$ and $[L^{(0)}_{12},D]=0$. On the other hand, if $q=0$, $w=1$ solves Eq. (1.1) and its adjoint, then Eq. (1.7) implies that

$$\Psi^{(0)}_{12}\cdot 1=0 \quad (\text{and } \Delta^{(0)}_{12}\cdot 1=0). \tag{B.7}$$

Equations (B.7) imply $D\Psi^n_{12}\Delta^{(0)}_{12}\cdot 1=0$ which is equivalent to (B.4).

Equation (B.4) and Eqs. (B.1) imply (B.2). In fact, multiplying Eq. (B.4) by h_{12} and using Eqs. (B.1) we obtain

$$(\Phi_{12}+\beta\mathscr{D})^{n+1}D\cdot h_{12}=(\Phi_{12}+\beta\mathscr{D})^n(\hat{M}_{12}+\tilde{\beta}\mathscr{D}D)\cdot h_{12}. \tag{B.8}$$

The above can be written in the following recursive way:

$$A_{n+1}(h_{12})=B_n(h_{12})+A_n(\tilde{\beta}h'_{12}), \tag{B.9}$$

where

$$A_n(h_{12})\doteq\sum_{\ell=0}^{n}\beta^\ell\binom{n}{\ell}\Phi^{n-\ell}_{12}D\cdot h^{(\ell)}_{12}, \qquad A_0(h_{12})=0, \tag{B.10a}$$

$$B_n(h_{12})\doteq\sum_{\ell=0}^{n}\beta^\ell\binom{n}{\ell}\Phi^{n-\ell}_{12}\hat{M}_{12}h^{(\ell)}_{12}, \qquad B_0(h_{12})=\hat{M}_{12}h_{12}, \tag{B.10b}$$

$$h^{(\ell)}_{12}\doteq\frac{\partial^\ell h_{12}}{\partial y^\ell_1}. \tag{B.10c}$$

The solution $A_{n+1}(h_{12})=\sum_{s=0}^{n}B_{n-s}(\tilde{\beta}^s h^{(s)}_{12})$ of Eqs. (B.9) and (B.10) implies Eq. (B.2). Indeed,

$$\begin{aligned}\delta_{12}K^{(n)}_{12}&=\delta_{12}\Phi^n_{12}\hat{M}_{12}\cdot 1=\delta_{12}\Phi^{n+1}_{12}D\cdot 1=A_{n+1}(\delta_{12})\\&=\sum_{s=0}^{n}B_{n-s}(\tilde{\beta}^s\delta^s_{12})=\sum_{\ell=0}^{n}b_{n,\ell}\Phi^{n-\ell}_{12}\hat{M}_{12}\delta^\ell_{12},\end{aligned} \tag{B.11}$$

where

$$b_{n,\ell}\doteq\sum_{s=0}^{\ell}\beta^{\ell-s}\tilde{\beta}^s\binom{n-s}{\ell-s}. \tag{B.12}$$

For example, for the KP equation ($\hat{M}=Dq^+_{12}+q^-_{12}D^{-1}q^-_{12}$):

$$\delta_{12}M^{(1)}_{12}=\delta_{12}\Phi_{12}\hat{M}_{12}\cdot 1=\Phi_{12}\hat{M}_{12}\delta_{12}-6\alpha\hat{M}_{12}\delta'_{12}, \tag{B.13a}$$

and for the DS equation ($\hat{M}_{12}=Q_{12}^{-}\sigma$):

$$\delta_{12}M_{12}^{(2)}=\delta_{12}\Phi_{12}^{2}\hat{M}_{12}\cdot I=\Phi_{12}^{2}\hat{M}_{12}\delta_{12}+4\alpha\Phi_{12}\hat{M}_{12}\delta_{12}'+4\alpha^{2}\hat{M}_{12}\delta_{12}^{2}. \tag{B.13b}$$

Finally, we use again Eq. (B.4) to derive the following interesting equation:

$$\Phi_{12}^{n+1}D\cdot h_{12}=\sum_{s=0}^{n}(\tilde{\beta}-\beta)^{s}\Phi_{12}^{n-s}\hat{M}_{12}h_{12}^{(s)}. \tag{B.14}$$

Multiplying Eq. (B.4) by h_{12} and using (B.1a), we obtain

$$\Phi_{12}^{j}h_{12}(\Phi_{12}^{n-j+1}D\cdot 1-\Phi_{12}^{n-j}\hat{M}_{12}\cdot 1)=0, \qquad j\leqq n. \tag{B.15}$$

Equation (B.15) for $j=n$ and Eqs. (B.1) imply

$$\Phi_{12}^{n+1}D\cdot h_{12}=\Phi_{12}^{n}\hat{M}_{12}\cdot h_{12}+(\tilde{\beta}-\beta)\Phi_{12}^{n}D\cdot h_{12}^{(1)}, \tag{B.16}$$

and hence Eq. (B.14).

Remark B.1. i) Equation (B.14) contains (B.4) if $h_2=1$.

ii) Equation (B.14) can be used to obtain (B.2), (B.12) in an alternative way. In fact,

$$\delta_{12}M_{12}^{(n)}=\delta_{12}\Phi_{12}^{n+1}D\cdot 1=\sum_{r=0}^{n}\beta^{r}\binom{n+1}{r}\Phi_{12}^{n+1-r}D\cdot h_{12}^{(r)}$$
$$=\sum_{\ell=0}^{n}\left[\sum_{s=0}^{\ell}\beta^{\ell-s}\binom{n+1}{\ell-s}(\tilde{\beta}-\beta)^{s}\right]\Phi_{12}^{n-\ell}\hat{M}_{12}h_{12}^{(\ell)}=\sum_{\ell=0}^{n}b_{n,\ell}\Phi_{12}^{n-\ell}\hat{M}_{12}\cdot h_{12}^{(\ell)},$$

since the identity

$$\binom{n-s}{\ell-s}=\sum_{\nu=s}^{\ell}(-1)^{\nu-s}\binom{\nu}{s}\binom{n+1}{\ell-\nu}, \qquad s\leqq\ell\leqq n, \tag{B.17}$$

implies that

$$\sum_{s=0}^{\ell}\beta^{\ell-s}(\tilde{\beta}-\beta)^{s}\binom{n+1}{\ell-s}=\sum_{s=0}^{\ell}\beta^{\ell-s}\tilde{\beta}^{s}\binom{n-s}{\ell-s}, \qquad \ell\leqq n.$$

Appendix C

In this appendix we define explicitly the directional derivative introduced in Sect. 4 for the KP and DS classes. Then we use it to verify some of the results contained in this paper.

C1. Evolution Equations Associated with the KP Equation

The directional derivative of the basic operators $q_{12}^{\pm}\div q_1\pm q_2+\alpha(D_1\mp D_2)$ associated with the non-stationary Schrödinger problem (1.1) is the usual Frechét derivative with respect to the kernel q_{12} of their integral representation:

$$q_{12}^{\pm}g_{12} = \int_{\mathbb{R}} dy_3(q_{13}g_{32} \pm g_{13}q_{32}), \qquad q_{12} = \delta_{12}q_1 + \alpha\delta'_{12}, \tag{C.1a}$$

$$q_{12_a}^{\pm}[f_{12}]g_{12} = f_{12}^{\pm}g_{12}, \tag{C.1b}$$

$$f_{12}^{\pm}g_{12} \doteq \int_{\mathbb{R}} dy_3(f_{13}g_{32} \pm g_{13}f_{32}). \tag{C.1c}$$

In order to make explicit calculations, it is convenient to use the following basic identities of this algebra of integral operators

$$a_{12}^{\pm}b_{12} = \pm b_{12}^{\pm}a_{12}, \tag{C.2a$_\pm$}$$

$$(a_{12}^{\pm}b_{12}^{\pm} - b_{12}^{\pm}a_{12}^{\pm})c_{12} = (a_{12}^{-}b_{12})^{-}c_{12} = -c_{12}^{-}a_{12}^{-}b_{12}, \tag{C.2b$_\pm$}$$

$$(a_{12}^{+}b_{12}^{-} \mp b_{12}^{\mp}a_{12}^{\pm})c_{12} = (a_{12} \mp b_{12})^{\pm}c_{12} = \pm c_{12}^{\pm}a_{12}^{\mp}b_{12}; \tag{C.2c$_\pm$}$$

where a_{12}, b_{12}, c_{12} are arbitrary functions of x, y_1, y_2 decaying at ∞ and $a_{12}^{\pm}, b_{12}^{\pm}, c_{12}^{\pm}$ are the corresponding integral operators defined in (C.1c).

The integral representations (C.1a) imply that the basic operators $q_{12}^{\pm}$ can replace $a_{12}^{\pm}$ (and/or $b_{12}^{\pm}, c_{12}^{\pm}$) in Eqs. (C.2). For instance, if $a_{12}^{\pm} = f_{12}^{\pm}$, $b_{12}^{\pm} = q_{12}^{\pm}$, and $c_{12}^{\pm} = H_{12}^{\pm}$, the identity (C.2c)$_-$ becomes

$$f_{12}^{\pm}q_{12}^{-}H_{12} + q_{12}^{+}f_{12}^{-}H_{12} + H_{12}^{-}q_{12}^{+}f_{12} = 0, \tag{C.3}$$

where we have also used Eq. (C.2a)$_+$ to replace $f_{12}^{+}q_{12}$ by the expression $q_{12}^{+}f_{12}$ in which the kernel q_{12} does not appear explicitly.

It is worthwhile to remark that formulas (C.2) can also be interpreted as matrix identities in which a, b, c are matrices and the $\pm$ operations denote anticommutator and commutator:

$$a^{\pm}b \doteq ab \pm ba. \tag{C.4}$$

Interpreting the operation $a_{12}^{\pm}b_{12}$ as in (C.4), the recursion operator (1.2) of the KP class becomes the recursion operator

$$\Phi = D^2 + q^{+} + Dq^{+}D^{-1} + q^{-}D^{-1}q^{-}D^{-1} \tag{C.5}$$

associated with the $N \times N$ matrix Schrödinger problem in 1 dimension and introduced by Calogero and Degasperis [38]. Then important properties of the recursion operator of the KP, like its hereditariness (4.21), are equivalent to the corresponding properties of the matrix operator (C.5)! This important connection is explained from the fact that the $2+1$ dimensional systems considered here can be viewed as reductions of certain evolution equations nonlocal in y. These equations are directly connected to matrix evolution equations (see Sect. 5 of [35]).

Now we use Eqs. (C.2) to verify some results concerning the symmetries and the bi-Hamiltonian structure of Eqs. (3.19) and (3.20).

a) Φ_{12} is a strong symmetry of $\hat{N}_{12}H_{12}$, where $\hat{N}_{12} = q_{12}^{-}$ and $H_{12_x} = 0$ (this result is a consequence of Lemma 4.2; but here it is verified directly).

$$\Phi_{12_d}[q_{12}^- H_{12}]f_{12} - (q_{12}^- H_{12})_d[\Phi_{12} f_{12}] + \Phi_{12}(q_{12}^- H_{12})_d[f_{12}]$$
$$= (q_{12}^- H_{12})^+ f_{12} + D(q_{12}^- H_{12})^+ D^{-1} f_{12}$$
$$+ (q_{12}^- H_{12})^- D^{-1} q_{12}^- D^{-1} f_{12} + q_{12}^- D^{-1}(q_{12}^- H_{12})^- D^{-1} f_{12}$$
$$-(D^2 f_{12} + q_{12}^+ f_{12} + Dq_{12}^+ D^{-1} f_{12} + q_{12}^- D^{-1} q_{12}^- D^{-1} f_{12})^- H_{12}$$
$$+(D^2 + q_{12}^+ + Dq_{12}^+ D^{-1} + q_{12}^- D^{-1} q_{12}^- D^{-1}) f_{12}^- H_{12} = 0, \quad \text{since:}$$

the terms without $q_{12}^\pm$ give

$$-f_{12_{xx}}^- H_{12} + D^2 f_{12}^- H_{12} = 0;$$

the terms linear in $q_{12}^\pm$ give

$$(q_{12}^- H_{12})^+ f_{12} + D(q_{12}^- H_{12})^+ D^{-1} f_{12} - (q_{12}^+ f_{12})^- H_{12} - D(q_{12}^+ D^{-1} f_{12})^- H_{12}$$
$$+ q_{12}^+ f_{12}^- H_{12} + Dq_{12}^+ D^{-1} f_{12}^- H_{12} = f_{12}^+ q_{12}^- H_{12} + q_{12}^+ f_{12}^- H_{12} + H_{12}^- q_{12}^+ f_{12}$$
$$D((D^{-1} f_{12})^+ q_{12}^- H_{12} + q_{12}^+ (D^{-1} f_{12})^- H_{12} + H_{12}^- q_{12}^+ D^{-1} f_{12}) = 0,$$

using Eq. (C.3);

the terms quadratic in $q_{12}^\pm$ give

$$(q_{12}^- H_{12})^- D^{-1} q_{12}^- D^{-1} f_{12} + H_{12}^- q_{12}^- D^{-1} q_{12}^- D^{-1} f_{12}$$
$$+ q_{12}^- D^{-1}(-(D^{-1} f_{12})^- q_{12}^- H_{12} + q_{12}^- D^{-1} f_{12}^- H_{12})$$
$$= (-q_{12}^- H_{12}^- + H_{12}^- q_{12}^- + (q_{12}^- H_{12})^-) D^{-1} q_{12}^- D^{-1} f_{12} = 0.$$

b) The Lie algebra of the starting symmetries is given by the following equations:

$$[\hat{N}_{12} H_{12}^{(1)}, \hat{N}_{12} H_{12}^{(2)}]_d = -\hat{N}_{12} H_{12}^{(3)}, \qquad [\hat{N}_{12} H_{12}^{(1)}, \hat{M}_{12} H_{12}^{(2)}]_d = -\hat{M}_{12} H_{12}^{(3)},$$
$$[\hat{M}_{12} H_{12}^{(1)}, \hat{M}_{12} H_{12}^{(2)}]_d = -\Phi_{12} \hat{N}_{12} H_{12}^{(3)}, \qquad H_{12}^{(3)} \doteq [H_{12}^{(1)}, H_{12}^{(2)}]_l = (H_{12}^{(1)})^- H_{12}^{(2)}, \tag{C.6}$$

where

$$\hat{N}_{12} \doteq q_{12}^-, \qquad \hat{M}_{12} \doteq Dq_{12}^+ + q_{12}^- D^{-1} q_{12}^-, \qquad H_{12_x} = 0.$$

Equation (C.6a) holds, since,

$$[q_{12}^- H_{12}^{(1)}, q_{12}^- H_{12}^{(2)}]_d = (q_{12}^- H_{12}^{(2)})^- H_{12}^{(1)} - (q_{12}^- H_{12}^{(1)})^- H_{12}^{(2)}$$
$$= -(H_{12}^{(1)})^- q_{12}^- H_{12}^{(2)} + (H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)} = -q_{12}^- (H_{12}^{(1)})^- H_{12}^{(2)},$$

using (C.2b)$_-$. Equation (C.6b) holds since:

$$[q_{12}^- H_{12}^{(1)}, (Dq_{12}^+ + q_{12}^- D^{-1} q_{12}^-) H_{12}^{(2)}]_d$$
$$= ((Dq_{12}^+ + q_{12}^- D^{-1} q_{12}^-) H_{12}^{(2)})^- H_{12}^{(1)} - D(q_{12}^- H_{12}^{(1)})^+ H_{12}^{(2)}$$
$$-(q_{12}^- H_{12}^{(1)})^- D^{-1} q_{12}^- H_{12}^{(2)} - q_{12}^- D^{-1}(q_{12}^- H_{12}^{(1)})^- H_{12}^{(2)}$$
$$= -D((H_{12}^{(1)})^- q_{12}^+ H_{12}^{(2)} + (H_{12}^{(2)})^+ q_{12}^- H_{12}^{(1)}) - (H_{12}^{(1)})^- q_{12}^- D^{-1} q_{12}^- H_{12}^{(2)}$$
$$+(D^{-1} q_{12}^- H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)} + q_{12}^- D^{-1}(H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)}$$
$$= -Dq_{12}^+ (H_{12}^{(1)})^- H_{12}^{(2)} + q_{12}^- D^{-1}(-(H_{12}^{(1)})^- q_{12}^- H_{12}^{(2)} + (H_{12}^{(2)})^- q_{12}^- H_{12}^{(1)})$$
$$= -\hat{M}_{12} (H_{12}^{(1)})^- H_{12}^{(2)}.$$

The verification of Eq. (C.6c) is left to the reader.

The notion of an extended symmetry σ_{12} of the evolution equation $q_{1_t}=\int_{\mathbb{R}} dy_2\delta_{12}K_{12}^{(n)}=K_{11}^{(n)}$ plays an important role in 2+1 dimensions. σ_{12} is a solution of the equation

$$\sigma_{12_f}[K^{(n)}]=(\delta_{12}K_{12}^{(n)})_d[\sigma_{12}], \tag{C.7a}$$

where

$$(\delta_{12}K_{12}^{(n)})_d \doteq \sum_{\ell=0}^{n} b_{n,\ell}(\Phi_{12}^{n-\ell}\hat{K}_{12}^{0}\delta_{12}^{\ell})_d . \tag{C.7b}$$

Again the use of Eqs. (C.2) and the property

$$(\delta_{12}^{n})^{\pm}f_{12}=(D_1^n\pm(-1)^n D_2^n)f_{12} \tag{C.8}$$

simplify the calculations of the operator (C.7b).

c) σ_{12} is an extended symmetry of

i) the wave equation $q_{1_t}=M_{11}^{(0)}=2q_{1_x}$ iff

$$\sigma_{12_f}[2q_x]=2D\sigma_{12}; \tag{C.9a}$$

ii) the KP equation $q_{1_t}=M_{11}^{(1)}=2(q_{1_{xxx}}+6q_1q_{1_x}+3\alpha^2D^{-1}q_{1_{y_1y_1}})$ iff

$$\begin{aligned}\sigma_{12_f}[2(q_{xxx}+6qq_x+3\alpha^2D^{-1}q_{yy})]&=2[D^3+6D(q_1+q_2)-3\alpha(D^{-1}(q_{1_{y_1}}-q_{2_{y_2}}))\\&+6\alpha(q_1-q_2)D^{-1}(D_1+D_2)+6\alpha D^{-1}(D_1+D_2)^2]\sigma_{12}.\end{aligned} \tag{C.9b}$$

$$\begin{aligned}(\delta_{12}K_{12}^{(0)})_d[f_{12}]&=(\hat{M}_{12}\delta_{12})_d[f_{12}]=Df_{12}^{+}\delta_{12}+f_{12}^{-}D^{-1}q_{12}^{-}\delta_{12}\\&+q_{12}^{-}D^{-1}f_{12}^{-}\delta_{12}=2Df_{12}.\end{aligned}$$

$$\begin{aligned}(\delta_{12}K_{12}^{(1)})_d[f_{12}]&=(\Phi_{12}\hat{M}_{12}\delta_{12}-6\alpha\hat{M}_{12}\delta_{12}')_d[f_{12}]\\&=\Phi_{12_d}[f_{12}]\hat{M}_{12}\delta_{12}+\Phi_{12}(\hat{M}_{12}\delta_{12})_d[f_{12}]-6\alpha(\hat{M}_{12}\delta_{12}')_d[f_{12}]\\&=(f_{12}^{+}+Df_{12}^{+}D^{-1}+f_{12}^{-}D^{-1}q_{12}^{-}D^{-1}+q_{12}^{-}D^{-1}f_{12}^{-}D^{-1})(Dq_{12}^{+}+q_{12}^{-}D^{-1}q_{12}^{-})\delta_{12}\\&\quad+D^2+q_{12}^{+}+Dq_{12}^{+}D^{-1}+q_{12}^{-}D^{-1}q_{12}^{-}D^{-1})(Df_{12}^{+}+f_{12}^{-}D^{-1}q_{12}^{-}+q_{12}^{-}D^{-1}f_{12}^{-})\delta_{12}\\&\quad-6\alpha(Df_{12}^{+}+f_{12}^{-}D^{-1}q_{12}^{-}+q_{12}^{-}D^{-1}f_{12}^{-})\delta_{12}'\\&=2[D^3+6D(q_1+q_2)-3\alpha(D^{-1}(q_{1_{y_1}}-q_{2_{y_2}}))\\&\quad+6\alpha(q_1-q_2)D^{-1}(D_1+D_2)+6\alpha^2D^{-1}(D_1+D_2)^2],\end{aligned}$$

since, for instance:

$$\begin{aligned}f_{12}^{+}Dq_{12}^{+}\delta_{12}&=(Dq_{12})^{+}f_{12}^{+}\delta_{12}-\delta_{12}^{-}f_{12}^{-}q_{12}=2(q_1+q_2)_xf_{12},\\Df_{12}^{+}q_{12}^{+}\delta_{12}&=2Df_{12}^{+}q_{12}=2Dq_{12}^{+}f_{12},\\Df_{12}^{+}\delta_{12}'&=D(\delta_{12}')^{+}f_{12}=D(D_1-D_2)f_{12},\\f_{12}^{-}D^{-1}q_{12}^{-}\delta_{12}'&=-(D^{-1}q_{12}^{-}\delta_{12}')^{-}f_{12}=(D^{-1}(\delta_{12}')^{-}q_{12})^{-}f_{12}\\&=(D^{-1}(D_1+D_2)q_{12})^{-}f_{12}=(D^{-1}(q_{1_{y_1}}-q_{2_{y_2}}))f_{12},\\q_{12}^{-}D^{-1}f_{12}^{-}\delta_{12}'&=-q_{12}^{-}D^{-1}(\delta_{12}')^{-}f_{12}=-q_{12}^{-}D^{-1}(D_1+D_2)f_{12},\end{aligned}$$

and we have used, for the first and only time in this appendix, the explicit representation (C.1a) of q_{12}.

In order to investigate the Hamiltonian structure of the equations generated by Φ_{12}, in addition to Eqs. (C.2) we use the following properties:

$$a_{12}^{\pm *} = \pm a_{12}^{\pm}, \quad q_{12}^{\pm *} = \pm q_{12}^{\pm}. \tag{C.10}$$

These properties follow from the definitions (C.1c), (C.1a), and (4.8):

$$\begin{aligned}\langle f_{12}, a_{12}^{\pm} g_{12}\rangle &= \int_{\mathbb{R}^4} dx dy_1 dy_2 dy_3 f_{21}(a_{13}g_{32} \pm g_{13}a_{32})\\ &= \int_{\mathbb{R}^4} dx dy_1 dy_2 dy_3 (f_{23}a_{31} \pm f_{31}a_{23}) g_{12}\\ &= \pm \langle a_{12}^{\pm} f_{12}, g_{12}\rangle .\end{aligned}$$

d) $\hat{\gamma}_{12}^0 H_{12} = D^{-1}\hat{K}_{12}^0 H_{12}$ ($\hat{K}_{12}^0 = \hat{N}_{12}$ and $\hat{M}_{12}$) are extended gradients, namely $(\hat{\gamma}_{12}^0 H_{12})_d^* = (\hat{\gamma}_{12}^0 H_{12})_d$.

i) If $\hat{K}_{12}^0 = \hat{N}_{12}$, then $(\hat{\gamma}_{12}^0 H_{12})_d[g_{12}] = D^{-1} g_{12}^- H_{12}$ and

$$\begin{aligned}\langle f_{12}, (\hat{\gamma}_{12}^0 H_{12})_d[g_{12}]\rangle &= \langle f_{12}, D^{-1} g_{12}^- H_{12}\rangle = \langle D^{-1} f_{12}, H_{12}^- g_{12}\rangle\\ &= -\langle H_{12}^- D^{-1} f_{12}, g_{12}\rangle = \langle D^{-1} f_{12}^- H_{12}, g_{12}\rangle\\ &= \langle (\hat{\gamma}_{12}^0 H_{12})_d[f_{12}], g_{12}\rangle .\end{aligned}$$

ii) If $\hat{K}_{12}^0 = \hat{M}_{12}$, then

$$(\hat{\gamma}_{12}^0 H_{12})_d[g_{12}] = (g_{12}^+ + D^{-1} g_{12}^- D^{-1} g_{12}^- + D^{-1} q_{12}^- D^{-1} g_{12}^-) H_{12}$$

and

$$\begin{aligned}\langle f_{12}, (\hat{\gamma}_{12}^0 H_{12})_d[g_{12}]\rangle &= \langle f_{12}, g_{12}^+ H_{12} + D^{-1} g_{12}^- D^{-1} q_{12}^- H_{12} + D^{-1} q_{12}^- D^{-1} g_{12}^- H_{12}\rangle\\ &= \langle f_{12}, (H_{12}^+ - D^{-1}((D^{-1} q_{12}^- H_{12})^- + q_{12}^- D^{-1} H_{12}^-)) g_{12}\rangle\\ &= \langle (H_{12}^+ - [(D^{-1} q_{12}^- H_{12})^- + H_{12}^- D^{-1} q_{12}^-] D^{-1}) f_{12}, g_{12}\rangle\\ &= \langle (H_{12}^+ - D^{-1}((D^{-1} q_{12}^- H_{12})^- + q_{12}^- H_{12}^- D^{-1})) f_{12}, g_{12}\rangle\\ &= \langle (\hat{\gamma}_{12}^0 H_{12})_d[f_{12}], g_{12}\rangle .\end{aligned}$$

e) In [35] we show that

$$\gamma_{12}^{(n)} = \mathrm{grad}_{12} I_n , \tag{C.11a}$$

$$\begin{aligned} I_n &\doteq \frac{1}{2(2n+3)}\langle \gamma_{12}^{(n+1)}, \delta_{12}\rangle = \frac{1}{2(2n+3)} \int_{\mathbb{R}^3} dx dy_1 dy_2 \delta_{12} \gamma_{12}^{(n+1)}\\ &= \frac{1}{2(2n+3)} \int_{\mathbb{R}^2} dx dy_1 \gamma_{11}^{(n+1)}, \end{aligned} \tag{C.11b}$$

where $\gamma_{12}^{(n)} \doteq D^{-1} K_{12}^{(n)}$ and $\hat{K}_{12}^0 = \hat{M}_{12}$. Here we directly verify this result for $n=0$,

$$\begin{aligned} I_{0_d}[f_{12}] &= \tfrac{1}{6}\langle \delta_{12}, \gamma_{12_d}^{(1)}[f_{12}]\rangle\\ &= \tfrac{1}{6}\langle \gamma_{12_d}^{(1)*}[\delta_{12}], f_{12}\rangle = \tfrac{1}{6}\langle \gamma_{12_d}^{(1)}[\delta_{12}], f_{12}\rangle\\ &= \tfrac{1}{6}\langle \Phi_{12_d}^*[\delta_{12}]\gamma_{12}^{(0)} + \Phi_{12}^* \hat{\gamma}_{12_d}^{(0)}[\delta_{12}]\cdot 1, f_{12}\rangle\\ &= \tfrac{1}{6}\langle 4\gamma_{12}^{(0)} + 2\Phi_{12}^* \cdot 1, f_{12}\rangle = \langle \gamma_{12}^{(0)}, f_{12}\rangle , \end{aligned} \tag{C.12}$$

which implies that $\gamma_{12}^{(0)}=\text{grad}_{12} I_0$. (In this derivation we have used the property $\gamma_{12_d}^{(1)*}=\gamma_{12_d}^{(1)}$.)

f) The bracket $\{a_{12}, b_{12}, c_{12}\} \doteq \langle a_{12}, \Theta_{12_d}^{(2)}[\Theta_{12}^{(2)} b_{12}] c_{12}\rangle$, $\Theta_{12}^{(2)} \doteq \Phi_{12} D$ satisfies the Jacobi identity for every a_{12}, b_{12}, c_{12}. Here we only display some of the calculations for the linear terms in $q_{12}^{\pm}$.

$$\begin{aligned}&\langle a_{12}, [(q_{12}^{+} D b_{12} + D q_{12}^{+} b_{12})^{+} D + D(q_{12}^{+} D b_{12} + D q_{12}^{+} b_{12})^{+} \\ &\quad + (D^3 b_{12})^{-} D^{-1} q_{12}^{-} + q_{12}^{-} D^{-1} (D^3 b_{12})^{-}] c_{12} \\ &\quad + \text{cyclic permutations of } a_{12}, b_{12}, c_{12} \\ &= \{a_{12}, b_{12}, c_{12}\} + \langle [D(q_{12}^{+} D c_{12} + D q_{12}^{+} c_{12})^{+} + (q_{12}^{+} D c_{12} + D q_{12}^{+} c_{12})^{+} D \\ &\quad - q_{12}^{-} D^{-1} (D^3 c_{12})^{-} - (D^3 c_{12})^{-} D^{-1} q_{12}^{-}] b_{12}, a_{12}\rangle \\ &\quad + \langle c_{12}, (D b_{12})^{+} (q_{12}^{+} D a_{12} + D q_{12}^{+} a_{12}) + D b_{12}^{+} (q_{12}^{+} D a_{12} + D q_{12}^{+} a_{12}) \\ &\quad - (D^{-1} q_{12}^{-} b_{12})^{-} D^3 a_{12} - q_{12}^{-} D^{-1} b_{12}^{-} D^3 a_{12}\rangle = \langle a_{12}, L_{12}(b_{12}, c_{12})\rangle,\end{aligned}$$

where

$$\begin{aligned}L_{12}(b_{12}, c_{12}) \doteq\ & (q_{12}^{+} D b_{12} + D q_{12}^{+} b_{12}) D c_{12} + D(q_{12}^{+} D b_{12} + D q_{12}^{+} b_{12})^{+} c_{12} \\ & + (D^3 b_{12})^{-} D^{-1} q_{12}^{-} c_{12} + q_{12}^{-} D^{-1} (D^3 b_{12})^{-} c_{12} \\ & + D(q_{12}^{+} D c_{12} + D q_{12}^{+} c_{12})^{+} b_{12} + (q_{12}^{+} D c_{12} + D q_{12}^{+} c_{12})^{+} D b_{12} \\ & - q_{12}^{-} D^{-1} (D^3 c_{12})^{-} b_{12} - (D^3 c_{12})^{-} D^{-1} q_{12}^{-} b_{12} - D q_{12}^{+} (D b_{12})^{+} c_{12} \\ & - q_{12}^{+} D (D b_{12})^{+} c_{12} + D q_{12}^{+} b_{12}^{+} D c_{12} + q_{12}^{+} D b_{12}^{+} D c_{12} \\ & - D^3 (D^{-1} q_{12}^{-} b_{12})^{-} c_{12} - D^3 b_{12}^{-} D^{-1} q_{12}^{-} c_{12}.\end{aligned}$$

Using Eqs. (C.2), it is possible to show that $L_{12}(b_{12}, c_{12})=0$, $\forall b_{12}, c_{12}$.

C2. Evolution Equations Associated with the DS Equation

As in the previous case, it is easy to check from their definitions

$$Q_{12}^{\pm} G_{12} \doteq Q_1 G_{12} \pm G_{12} Q_2 = \int_{\mathbb{R}} dy_3 (Q_{13} G_{32} \pm G_{13} Q_{32}), \qquad Q_{12} = \delta_{12} Q_1, \tag{C.13a}$$

$$Q_{12_d}^{\pm}[F_{12}] G_{12} = F_{12}^{\pm} G_{12}, \tag{C.13b}$$

$$F_{12}^{\pm} G_{12} \doteq \int_{\mathbb{R}} dy_3 (F_{13} G_{32} \pm G_{13} F_{32}), \tag{C.13c}$$

that the operators $Q_{12}^{\pm}$ and $F_{12}^{\pm}$ satisfy Eqs. (C.2) and (C.10). Moreover, it is possible to show that the operator P_{12}, defined by

$$P_{12} F_{12} \doteq F_{12_x} - J F_{12_{y_1}} - F_{12_{y_2}} J, \tag{C.14}$$

satisfies the following equations

$$P_{12} F_{12}^{\pm} G_{12} = (P_{12} F_{12})^{\pm} G_{12} + F_{12}^{\pm} P_{12} G_{12}, \tag{C.15a}$$

$$\begin{aligned}P_{12}^{-1} F_{12}^{\pm} G_{12} &= (P_{12}^{-1} F_{12})^{\pm} G_{12} - P_{12}^{-1} (P_{12}^{-1} F_{12})^{\pm} P_{12} G_{12} \\ &= F_{12}^{\pm} P_{12}^{-1} G_{12} - P_{12}^{-1} (P_{12} F_{12})^{\pm} P_{12}^{-1} G_{12}.\end{aligned} \tag{C.15b}$$

Now we use Eqs. (C.13), (C.2), and (C.15) to verify some result concerning symmetries and bi-Hamiltonian structure of Eqs. (3.35) and (3.38).

a) Φ_{12} is a strong symmetry for $\hat{K}^0_{12}H_{12}$, where $\hat{K}^0_{12}=\hat{N}_{12}\doteqdot Q^-_{12}$ and $P_{12}H_{12}=0$, H_{12} diagonal.

$$\begin{aligned}&\Phi_{12_d}[Q^-_{12}H_{12}]F_{12}-(Q^-_{12}H_{12})_d[\Phi_{12}F_{12}]+\Phi_{12}(Q^-_{12}H_{12})_d[F_{12}]\\&\quad=-\sigma[(Q^-_{12}H_{12})^+P^{-1}_{12}Q^+_{12}+Q^+_{12}P^{-1}_{12}(Q^-_{12}H_{12})^+]F_{12}\\&\qquad-(\sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12})F_{12})^-H_{12}+\sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12})F^-_{12}H_{12}=0\,,\quad\text{since:}\end{aligned}$$

the terms without $Q^{\pm}_{12}$ give

$$-\sigma(P_{12}F_{12})^-H_{12}+\sigma P_{12}F^-_{12}H_{12}=0\,;$$

the terms with $Q^{\pm}_{12}$ give

$$\begin{aligned}&-\sigma[((Q^-_{12}H_{12})^++H^-_{12}Q^+_{12})P^{-1}_{12}Q^+_{12}F_{12}+Q^+_{12}P^{-1}_{12}(F^+_{12}Q^-_{12}H_{12}-Q^+_{12}F^-_{12}H_{12})]\\&\quad=-\sigma Q^+_{12}P^{-1}_{12}(H^-_{12}Q^+_{12}F_{12}+F^+_{12}Q^-_{12}H_{12}+Q^+_{12}F^-_{12}H_{12})=0\end{aligned}$$

(in order to show that Φ_{12} is a strong symmetry for $\hat{K}^0_{12}H_{12}$, where $\hat{K}^0_{12}=\hat{M}_{12}\doteqdot Q^-_{12}\sigma$, it is enough to replace H_{12} by σH_{12} in the previous calculation).

b) The Lie algebra of the starting operators (on H_{12}) is given by the following equations:

$$\begin{aligned}&[\hat{N}_{12}H^{(1)}_{12},\hat{N}_{12}H^{(2)}_{12}]_d=-\hat{N}_{12}H^{(3)}_{12}\,,\qquad[\hat{N}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\hat{M}_{12}H^{(3)}_{12}\,,\\&[\hat{M}_{12}H^{(1)}_{12},\hat{M}_{12}H^{(2)}_{12}]_d=-\hat{N}_{12}H^{(3)}_{12}\,,\qquad H^{(3)}_{12}\doteqdot[H^{(1)}_{12},H^{(2)}_{12}]_I=(H^{(1)}_{12})^-H^{(2)}_{12}\,,\end{aligned}\tag{C.16}$$

where

$$\hat{N}_{12}\doteqdot Q^-_{12}\,,\qquad\hat{M}_{12}\doteqdot Q^-_{12}\sigma\,,\qquad P_{12}H^{(i)}_{12}=0\,,\quad H^{(i)}_{12}\ \text{diagonal}\,,\quad i=1,2,3\,,$$

$$\begin{aligned}[Q^-_{12}H^{(1)}_{12},Q^-_{12}H^{(2)}_{12}]_d&=(Q^-_{12}H^{(2)}_{12})^-H^{(1)}_{12}-(Q^-_{12}H^{(1)}_{12})^-H^{(2)}_{12}\\&=-H^{(1)-}_{12}Q^-_{12}H^{(2)}_{12}+H^{(2)-}_{12}Q^-_{12}H^{(1)}_{12}\\&=-Q^-_{12}(H^{(1)}_{12})^-H^{(2)}_{12}\,.\end{aligned}$$

Equations (C.16b) and (C.16c) are obtained replacing $H^{(2)}_{12}$ by $\sigma H^{(2)}_{12}$ and $H^{(i)}_{12}$ by $\sigma H^{(i)}_{12}$, $i=1,2$, respectively, in the derivation of (C.16a).

c) The operator

$$\Phi_{12}\doteqdot\sigma(P_{12}-Q^+_{12}P^{-1}_{12}Q^+_{12})\,,\tag{C.17}$$

defined on off-diagonal matrices, is hereditary, namely

$$\Phi_{12_d}[\Phi_{12}F_{12}]G_{12}-\Phi_{12}\Phi_{12_d}[F_{12}]G_{12}\quad\text{is symmetric in }F_{12},G_{12}\,.\tag{C.18}$$

In order to show it, we make use of Eqs. (C.2), (C.15) and of

$$(\sigma F_{12})^{\pm}G_{12}=\begin{cases}\sigma F^{\pm}_{12}G_{12}\,, & G_{12}\ \text{diagonal}\,,\\ \sigma F^{\mp}_{12}G_{12}\,, & G_{12}\ \text{off-diagonal}\,.\end{cases}\tag{C.19}$$

Here we display the calculations for the terms linear in $Q^{\pm}_{12}$:

$$-(\sigma P_{12}F_{12})^{+}P^{-1}_{12}Q^{+}_{12}G_{12}-Q^{+}_{12}P^{-1}_{12}(\sigma P_{12}F_{12})^{+}G_{12}$$
$$+\sigma P_{12}(F^{+}_{12}P^{-1}_{12}Q^{+}_{12}G_{12}+Q^{+}_{12}P^{-1}_{12}F^{+}_{12}G_{12})$$
$$=\sigma(Q^{-}_{12}P^{-1}_{12}(P_{12}F_{12})^{-}G_{12}+F^{+}_{12}Q^{+}_{12}G_{12}+P_{12}Q^{+}_{12}P^{-1}_{12}F^{+}_{12}G_{12}),$$

which is symmetric in F_{12}, G_{12}, since

$$F^{+}_{12}G_{12}=G^{+}_{12}F_{12},$$

$$Q^{-}_{12}P^{-1}_{12}(P_{12}F_{12})^{-}G_{12}+F^{+}_{12}Q^{+}_{12}G_{12}$$
$$=Q^{-}_{12}F^{-}_{12}G_{12}+Q^{-}_{12}P^{-1}_{12}(P_{12}G_{12})^{-}F_{12}+F^{+}_{12}Q^{+}_{12}G_{12}$$
$$=G^{+}_{12}Q^{+}_{12}F_{12}+Q^{-}_{12}P^{-1}_{12}(P_{12}G_{12})^{-}F_{12}.$$

d) σ_{12} is an extended symmetry of
i) $Q_{1_t}=M^{(0)}_{11}=-2\sigma Q_1$, iff

$$\sigma_{12_f}[-2\sigma Q]=-2\sigma\delta_{12}, \tag{C.20a}$$

ii) $Q_{1_t}=M^{(1)}_{11}=-2Q_{1_x}$, iff

$$\sigma_{12_f}[-2Q_x]=-2D\sigma_{12}. \tag{C.20b}$$

$$(\delta_{12}\hat{M}_{12}\cdot 1)_d[F_{12}]=(Q^{-}_{12}\sigma\delta_{12})_d[F_{12}]=F^{-}_{12}\sigma\delta_{12}$$
$$=-\sigma F^{+}_{12}\delta_{12}=-2\sigma F_{12}.$$

$$(\delta_{12}\hat{M}^{(1)}_{12})_d[F_{12}]=(\Phi_{12}Q^{-}_{12}\sigma\delta+2\alpha Q^{-}_{12}\sigma\delta'_{12})_d[F_{12}]$$
$$=\Phi_{12_d}[F_{12}]Q^{-}_{12}\sigma\delta_{12}+\Phi_{12}Q^{-}_{12_d}[F_{12}]\sigma\delta_{12}+2\alpha Q^{-}_{12_d}[F_{12}]\sigma\delta'_{12}$$
$$=-\sigma[(F^{+}_{12}P^{-1}_{12}Q^{+}_{12}+Q^{+}_{12}P^{-1}_{12}F^{+}_{12})Q^{-}_{12}\sigma\delta_{12}$$
$$-(P_{12}-Q^{+}_{12}P^{-1}_{12}Q^{+}_{12})F^{-}_{12}\sigma\delta_{12}+F^{+}_{12}\delta'_{12}I]$$
$$=(-2P_{12}-2\alpha\sigma(D_1-D_2))F_{12}=-2DF_{12},$$

since, for instance,

$$\sigma P_{12}F^{-}_{12}\sigma\delta_{12}=-P_{12}F^{+}_{12}\delta_{12}I=-2P_{12}F_{12},$$
$$-\sigma Q^{+}_{12}P^{-1}_{12}Q^{+}_{12}F^{-}_{12}\sigma\delta_{12}=Q^{-}_{12}P^{-1}_{12}Q^{-}_{12}F^{+}_{12}\delta_{12}I, \quad 2Q^{-}_{12}P^{-1}_{12}Q^{-}_{12}F_{12},$$
$$F^{+}_{12}\delta'_{12}I=(D_1-D_2)F_{12},$$
$$-\sigma Q^{+}_{12}P^{-1}_{12}F^{+}_{12}Q^{-}_{12}\sigma\delta_{12}=Q^{-}_{12}P^{-1}_{12}F^{-}_{12}Q^{+}_{12}\delta_{12}I=2Q^{-}_{12}P^{-1}_{12}F^{-}_{12}Q_{12}$$
$$=-2Q^{-}_{12}P^{-1}_{12}Q^{-}_{12}F_{12},$$

having used the properties

$$G^{\pm}_{12}\sigma=-\sigma G^{\mp}_{12}, \quad G_{12} \text{ off-diagonal},$$
$$Q^{\pm}_{12}\sigma=-\sigma Q^{\mp}_{12},$$
$$(I\delta^{n}_{12})^{\pm}F_{12}=(D^{n}_{1}\pm(-1)^{n}D^{n}_{2})F_{12}.$$

e) $\hat{\gamma}^0_{12}H_{12} \div \sigma\hat{K}^0_{12}H_{12}$ ($\hat{K}^0_{12}=\hat{N}_{12}$ and/or $\hat{M}_{12}$) are extended gradients, namely $(\hat{\gamma}^0_{12}H_{12})^*_d=(\hat{\gamma}^0_{12}H_{12})_d$.

i) If $\hat{\gamma}^0_{12}=\sigma\hat{N}_{12}=\sigma Q^-_{12}$, then $(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]=\sigma G^-_{12}H_{12}=-\sigma H^-_{12}G_{12}$, and

$$\begin{aligned}\langle F_{12},(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]\rangle &= -\langle F_{12},\sigma H^-_{12}G_{12}\rangle=\langle -\sigma H^-_{12}F_{12},G_{12}\rangle\\ &=\langle(\hat{\gamma}^0_{12}H_{12})_d[F_{12}],G_{12}\rangle;\end{aligned}$$

ii) If $\hat{\gamma}^0_{12}=\sigma\hat{M}_{12}=\sigma Q^-_{12}\delta=-Q^+_{12}$, then

$$(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]=-G^+_{12}H_{12}=-H^+_{12}G_{12},$$

and

$$\begin{aligned}\langle F_{12},(\hat{\gamma}^0_{12}H_{12})_d[G_{12}]\rangle &=\langle F_{12},-H^+_{12}G_{12}\rangle=\langle -H^+_{12}F_{12},G_{12}\rangle\\ &=\langle(\hat{\gamma}^0_{12}H_{12})_d[F_{12}],G_{12}\rangle.\end{aligned}$$

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