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Scaling Laws in the Vortex Lattice Model of Turbulence*

Alexandre Joel Chorin

Department of Mathematics, University of California, Berkeley, CA 94720, USA

Abstract. The lattice vortex model of the inertial range in turbulence theory is reviewed; the model consists of an array of vortex tubes whose axes coincide with the bonds on a regular lattice, subjected to random stretching and successive scaling, and constrained by conservation laws for energy, specific volume, circulation, helicity, and an energy/vorticity relation. The scaling laws for vorticity are examined in detail, a Hausdorff dimension for the "active" portion of the vortex array is calculated, the origin of intermittency is exhibited, and it is pointed out that the Kolmogorov $-5/3$ power law already accounts for intermittency effects.

Introduction

The inertial range in turbulence is the range of scales far enough from the scale of the driving forces to sustain universal statistics yet not so small that viscous effects are important in their dynamics. The analysis of the inertial range is important for the understanding of turbulence and for the design of practical modeling methods. Numerical calculations designed to elucidate the structure of the inertial range, in particular by vortex methods [2, 5], display surprising patterns of complexity and have not been convincingly reconciled with the qualitative theory of Kolmogorov, Kraichnan, and others [9, 11, 14]. These calculations do provide evidence that vortex tubes stretch, bend and bond into fractal structures.

The lattice model we shall examine stands half way between a straightforward vortex calculation and a qualitative model. It was suggested by the calculations in [5], and affords a way of constraining the simple cascade models of the inertial range to obey the basic conservation properties of the Euler equation. Some aspects of the calculations in [5] have been challenged, in particular by Greengard [10], but their usefulness as a qualitative guide is not impaired, except for one issue

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that will be discussed below. The model explains and reproduces the salient observations made in vortex calculations, in particular intermittency.

The model has been previously presented in [6]. It has been pointed out to the author that the earlier presentation contains a number of gaps that make it hard to understand. The present paper fills these gaps, in particular where the scaling of vortex tubes is concerned, and also repairs an error in the derivation of the Kolmogorov spectrum. It prepares the way for a full-fledged polymeric model of turbulence to be presented later.

Scaling Laws for Vortex Tubes

Consider flow in a periodic domain of period 1 of a fluid with density 1. We shall not write down the equations of motion since they will not be used explicitly. The kinetic energy T of the fluid can be written as [12]

$$T \equiv \int |\mathbf{u}|^2 d\mathbf{x} = \frac{1}{8\pi} \int d\mathbf{x} \int d\mathbf{x}' \frac{\boldsymbol{\xi}(\mathbf{x}) \cdot \boldsymbol{\xi}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \tag{1}$$

where $\mathbf{u}$ is the velocity, $\mathbf{x}$ is the position, $\boldsymbol{\xi} = \operatorname{curl} \mathbf{u}$, is the vorticity, and the integrations are over a periodic box. The enstrophy is defined as

$$Z = \int |\boldsymbol{\xi}|^2 d\mathbf{x}, \tag{2}$$

and the helicity H as

$$H = \int \boldsymbol{\xi} \cdot \mathbf{u} d\mathbf{x}. \tag{3}$$

In the absence of external forces and of viscosity, $dT/dt \leqq 0$, $dH/dt = 0$ and Z is a rapidly increasing function of the time t for "most" data [5].

Before presenting the lattice model, we need some scaling properties of the integrals (1), (2). Consider a vertical cylinder C of height l and a cross section of area $A(r)$, where r is a linear dimension characteristic of that cross section (e.g., if the cross section is circular, $A = \pi r^2$, r = radius). Assume $\boldsymbol{\xi}$ is vertical inside C, $\boldsymbol{\xi} = (0, 0, \xi_0)$, ξ_0 = const. Let $T(r, l)$ be the second integral in (1) evaluated over $C \times C$ and $Z(r, l)$ the integral in (2) evaluated over C. It is understood that $\operatorname{div} \boldsymbol{\xi} \neq 0$ unless $\boldsymbol{\xi}$ is continued beyond C, and thus the first integral in (1) evaluated over C does not necessarily equal $T(r, l)$. We need to know the dependence of T and Z on r, l when the circulation $A(r)\xi_0$ is kept fixed, i.e., $\xi_0 \sim 1/A(r)$.

Let $\alpha > 0$ be a real parameter. Define the scaling factors S_1, S_2 by

$$T(r, \alpha l) = S_1(l/r, \alpha) T(r, l),$$

$$T(\alpha r, l) = S_2(l/r, \alpha) T(r, l);$$

where $\boldsymbol{\xi} = (0, 0, \xi_0(\alpha))$, $\xi_0(\alpha) A(\alpha r) = \xi_0(1) A(r)$. S_1, S_2 depend in general on both their arguments, but not on l, r individually as long as the cross sections $A(r)$ are similar. S_1, S_2 are not independent. A brief calculation shows that

$$T(\alpha r, \alpha l) = \alpha T(r, l),$$

while

$$T(\alpha r, \alpha l) = S_1(l/(\alpha r), \alpha) T(\alpha r, l) = S_1(l/(\alpha r), \alpha) S_2(l/r, \alpha) T(r, l),$$

Table 1. Energy scaling factor $S_1(l/r, 2)$ as a function of l/r

l/r	$S_1(l/r, 2)$	$\log S_1(l/r, 2)/\log 2$
0.1	3.83	1.94
0.2	3.72	1.90
0.4	3.44	1.78
0.6	3.27	1.71
0.8	3.15	1.65
1.0	3.06	1.61
1.2	2.99	1.58
1.4	2.93	1.55
1.6	2.88	1.53
1.8	2.85	1.51
2.0	2.81	1.49
2.5	2.76	1.46
3.0	2.70	1.43

and thus

$$\alpha = S_1(l/(\alpha r), \alpha) S_2(l/r, \alpha)\,. \tag{4}$$

Equation (4) has an important consequence: if one pulls on a vortex tube to lengthen it, the energy associated with the vortex increases and thus work has to be done. To the extent that our vortex tube will resemble a polymer, Eq. (4) determines the force law for that polymer. Note that a volume preserving stretching of the tube multiplies the energy T by $S_2(\alpha l/r, 1/\sqrt{\alpha}) \cdot S_1(l/r, \alpha)$. If $l/r \ll 1$ (a "fat vortex"), one can readily see that increasing the length of the tube by a factor α increases the volume over which the integral $T(l, r)$ is evaluated by α^2 while keeping the distribution of values of $|\mathbf{x} - \mathbf{x}'|$ fixed, i.e., $S_1 \sim \alpha^2$, and by (4), $S_2 \sim \alpha^{-1}$. At the other extreme, if $l/r \gg 1$ (a "thin vortex" approximation) a comparison with the expression for the energy of a vortex filament [12] suggests $S_1 \sim \alpha \log \alpha$, $S_2 \sim (\log \alpha)^{-1}$. Thus if one doubles the length of a "fat" vortex one approximately quadruples its contribution to the energy integral, while if one doubles the length of a "thin" vortex one does little more than double its contribution.

In Table 1 we display some numerical values of $S_1(l/r, 2)$ and of

$$q = (\log S_1(l/r, 2)/\log 2)$$

for several intermediate values of l/r, calculated numerically for a cylinder of square cross section. For a given value of l/r, $S_1 \sim \alpha^q$. In [6] it was systematically assumed that the vortex tubes were "fat" without adequate comment. We shall show below that the "fat" vortex assumption is self-consistent.

A simple calculation shows that

$$Z(\alpha r, l) = \frac{1}{\alpha} Z(r, l)\,, \qquad Z(r, \alpha l) = \alpha Z(r, l)\,.$$

We shall refer to r for simplicity as the vortex tube radius even where the cross section is not circular.

Vortex Stretching on a Lattice

The main events that occur in the inertial range are breakdowns of vortical structures into thinner and more convoluted structures on a smaller scale. We shall now produce a lattice model of this breakdown. We shall be assuming that the support of the vorticity can be idealized as a union of vortex tubes. Since presumably every configuration can be approximated by such a union, the loss of generality is not substantial.

Consider a three-dimensional cubic lattice with vertices $(i\delta, j\delta, k\delta)$, i,j,k integers, $1 \leqq i,j,k \leqq m$, $m\delta = 1$. Consider a vortex tube made up of oriented segments of finite thickness whose center lines coincide with the bonds of the lattice; its configuration is continued periodically in all directions. Assume the tube is connected and does not intersect itself, i.e., each vertex of the lattice connects either 0 or 2 segments. Write $I=(i,j,k)$ in short instead of $(i\delta, j\delta, k\delta)$; denote by $u_I = u_{i,j,k}$ a horizontal active segment whose leftmost end coincides with I ("active" means "belonging to the tube", "horizontal" means "parallel to the $x = i\delta$ axis", "leftmost" means "corresponding to the smallest value of x"). Similarly, denote by w_I a vertical segment and by v_I a segment that is transverse. Denote by $\tilde{u}_I, \tilde{v}_I, \tilde{w}_I$ a variable attached to these segments and taking the value $+1$ if the segments are oriented in the direction of increasing x, y, z and the value -1 otherwise.

Endow this configuration with an energy T, constructed so as to mimic Eq. (1):

$$T = T_E + T_S,$$

$$T_E = \sum_I \sum_{I' \neq I} \tilde{u}_I \tilde{u}_I \frac{1}{|I-I'|} + \sum_I \sum_{I' \neq I} \tilde{v}_I \tilde{v}_{I'} \frac{1}{|I-I'|} + \sum_I \sum_{I' \neq I} \tilde{w}_I \tilde{w}_{I'} \frac{1}{|I-I'|}, \tag{5}$$

$$T_S = \sum_I S(u_I) + \sum_I S(v_I) + \sum_I S(w_I),$$

where $|I-I'|$ is the distance between I and I'. The sums in T_E mimic Eq. (1), the immaterial factor $(8\pi)^{-1}$ has been omitted, and the subscript E stands for "exchange." The system being periodic, each segment should interact with every other segment and with all periodic images of every other segment; to save effort we keep only the largest of these interactions (for details of implementation, see [6]). The domain of integration in formula (1) includes values of $\mathbf{x}$ and $\mathbf{x}'$ belonging to the same segment, giving rise to a self energy S; denote this self energy by $S(u_I)$, $S(v_I)$, $S(w_I)$, depending on whether the segment is horizontal, transverse or vertical. $S(u_I)$, etc. depend on the cross section of the segment as well as on its length; assume that at the initial time these self-energies have been computed. Note that self energy is different from "self induction." A straight vortex tube has no self-induced motion but there is energy in the velocity field that it creates.

We shall now stretch the vortex tube by a sequence of elementary stretchings; the end result of the sequence will be compared with the breakdown of an "eddy." Pick an active segment at random, with all segments having an equal chance of being picked. Consider the possible stretching of this segment into a U-shaped configuration of three segments (Fig. 1). There are four such configurations for every segment. Pick one of them at random. If the proposed configuration leads to self-intersection of the vortex tube, reject it and pick another segment. If it does not lead to self-intersection, accept it if a certain energy constraint is satisfied. This energy constraint will be described shortly.

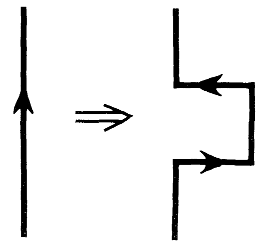

Fig. 1. An elementary stretching

Note that self-intersection is rejected because it would lead to infinite energy and also violate the conservation of helicity. Helicity is conserved as long as the "degree of knottedness" of the tubes is fixed, which will certainly be the case if they are not allowed to self-intersect [15]. Euler's equations, as long as their solutions are smooth, forbid self-intersection. If a small viscosity is allowed, self-intersection can happen but the mechanism is at present obscure (see e.g. [1]). Some recent theories of quantum turbulence gave a fundamental role to self-intersections of vortex lines but the agreement of these theories with experiment has now been shown to be a numerical artifact [4].

We now examine the change in energy T due to a proposed stretching. T_E changes because all interactions involving the old segment are deleted and interactions involving the new segment are added. T_S changes because the self-energy changes as the vortex tube becomes thinner, and because there are now three segments with self energy instead of one. To calculate the increase in self-energy we need to know the radii of the tubes. We shall see below that it is self consistent, under appropriate constraints on m, to assume that the vortex segments are close to being "fat". After stretching, each segment has a cross-section 1/3 smaller than the original cross-section and its radius is $1/\sqrt{3}$ smaller than previously, and thus its self-energy is $\sqrt{3}\times$ its original self energy. In moderately fat vortices, the new energy is $C\times$ old energy, $1<C<\sqrt{3}$. The self-energy of the segments can only increase.

Consider a vortex tube that is initially vertical, with the $S(w_I)$ initially small. Perform a stretching. T_E will decrease, since the vertical contribution to T_E will decrease and the only horizontal contribution will involve segments pointing in opposite directions (Fig. 1) and will be negative. The initial increase in T_S will be small. If one keeps on performing stretchings (discarding possible self-intersections) $S(u_I)$, $S(v_I)$, $S(w_I)$ will be increasing and eventually it is plausible that the energy $T=T_E+T_S$ will return to its original value. If it does and when it does, we have a new vortex tube; larger, thinner and more complicated than the initial tube, but with the same energy, volume, helicity and connectivity. For detail of computer implementation, see again [6].

The criterion for accepting stretchings is then as follows: as long as there are no self-intersections and the energy T is below the initial energy, accept the stretchings. If the energy after a proposed stretching exceeds the initial energy, reject it and stop until further notice.

Note that the criterion we have just presented encourages the formation of configurations for which T_E is negative, to make up for the increase in T_S. An

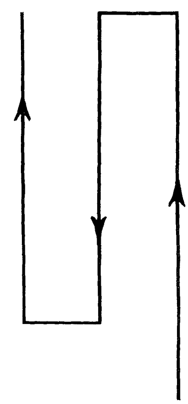

Fig. 2. A configuration that provides a negative contribution to T_E

example of such a configuration is found in Fig. 2. This creates folds ("hairpins" in the usual terminology) whose presence is well known from numerical and physical experiment. The need for this folding to occur as Z increases while the energy is kept fixed has been established in [5, 6] on the basis of potential-theoretical considerations. Here we have a discrete explanation of the reason why vortex stretching coupled with energy conservation leads to the formation of hairpins. A one dimensional "cartoon" of this fact has been presented in [7]. The intuitive reason for the folds is that if the vorticity stretches while energy is fixed cancellation must occur between the velocities induced by the stretched elements of vorticity.

Scaling of the Lattice

Once the calculation of the preceding section has come to a halt, no further stretching with conservation of energy is possible on the given lattice. One could hope that if the mesh were refined then stretching would become possible again into the newly created smaller scale bonds. Presumably, one would have then to keep on refining the mesh until one's computer memory were saturated – the standard difficulty in turbulence calculations. Instead, one could attempt a scaling – simultaneously halve δ, the bond length, and focus attention on an eight of the computational domain (see [5, 6]) by picking it out of the domain, making sure it contains active segments, and then throwing out the rest of the segments and replacing them by a periodic continuation of what has been kept. To simplify the book-keeping, one can scale up the piece that has been kept to the original lattice.

The periodic continuation is made to provide boundary conditions for the calculation. It clearly changes the calculation in ways that are not well controlled. Some of the more obvious difficulties can be suppressed by changing units at the beginning of a scaled calculation so that the energy remains fixed, but this is not a full solution of the problem. Such problems with boundary conditions are common in scaling transformation (see e.g. [3]). This is the biggest gap in the justification of the model. Some confidence in the validity of the process can be obtained by noting that the calculation reproduces certain bounds on the allowed support of the vorticity that are derived through potential theory (see [5, 6] for details).

It is easy to see that the rescaled calculation may allow some additional stretching of the vortex line without an increase of the energy simply by creating new allowed configurations, but that after a few scalings vortex stretching will come to a halt. Indeed, in each scaling each segment doubles its length; every term in T_E is replaced by four terms, each making approximately half of the original contribution to T_E (because the distance $|I-I'|$ has doubled). Thus T_E doubles, and to make the scaling consistent T_S should double. Each old segment is now replaced by two segments, and thus after rescaling each new segment (whose ancestor was a half segment) retains the value [$S(u_I)$, $S(v_I)$, or $S(w_I)$] of its parent segment. It is clear that T_E is bounded from below by a value obtained from a perfect antiferromagnetic arrangement of tubes, and since the S's are monotonically increasing there will soon be no way to stretch the tube further without violating energy conservation. In order for the stretching to continue indefinitely, the segments must be allowed to bunch up more tightly than is possible on a regular lattice that fills out all of space. This is the origin of intermittency – vortex stretching and energy conservation lead not only to folding of the vortex tubes, but to a bunching of the stretched tubes in an ever decreasing fraction of the available volume. A one dimensional model of this situation also was presented in [7]. This remark may also explain why some turbulence models based on global Fourier expansions (that assume a regular lattice in physical space) fail to exhibit intermittency.

Note that in each scaling, like the one we have just described, the spatial scale is reduced by a factor $1/2$; in a single stretching the radius of a vortex tube is reduced by a factor $1/\sqrt{3}=0.577$. The "fat vortex" approximation is self-consistent only if the scale of the calculation shrinks faster than the vortex radius, i.e., if between two scalings the numbers of active segments increases by a factor less than $3\sqrt{3/2}$ on the average. If the stretching comes to a halt, as we have just described, the "fat vortex" approximation is automatically self consistent. When intermittency is introduced this conclusion will have to be reconsidered. If the scale of the calculation shrinks faster than the vortex radius and stretching does take place, some of the tubes must explode into subtubes that stretch independently, a real effect that we shall ignore.

Scaling with Intermittency

We shall bunch up the vortex tubes by scaling up the eighth of the calculation that is retained at each scaling into volume $\beta=2^D/2^3$, $0<D<3$. D is a similarity dimension, analogous to the one used in [9, 14]. Such bunching is observed in the calculations in [2, 5]. We shall make the arbitrary assumption that the smaller volume has a cubic shape, with side $d=\sqrt[3]{\beta}=2^{(D-3)/3}$. If one compares the terms in T_E, T_S in the smaller volume with their values in a unit cube, one sees that the terms in T_E are multiplied by d (d^2 because each segment is d times shorter, d^{-1} because the distance between segments has decreased). In the "fat vortex" approximation each term in T_S is multiplied by d^2 because its energy is a quadratic function of its length, then by $\sqrt{d}$ because its radius is multiplied by $\sqrt{d}$, giving a total factor $d^{5/2}$. Note that the scaling here is different from the one that results from considering $T(\alpha r, \alpha l)$, $\alpha=d$, in Eq. (4). To obtain $T(\alpha r, \alpha l)$ one multiplies each length by α without regard to conservation of volume, while here volume is conserved. Thus, if $T_n = (T_S)_n + (T_E)_n$ is the energy after the n-th scaling, when intermittency is taken into account we find

$$T_{n+1} = (T_S)_{n+1} + (T_E)_{n+1} = d^{5/2}(T_S)_n + d(T_E)_n .$$

When $d < 1$, T_S becomes smaller compared to T_E and the stretching can proceed. The "active" part of the volume, i.e., the part in which vorticity keeps on stretching, occupies a shrinking part of the total volume, characterized by the scaling dimension D. Potential theory [5, 6] shows that $D > 1$. Note that in general $T_{n+1} < T_n$, as indeed must be the case since the vorticity in those parts of the total volume where stretching no longer takes place still contributes to the total energy.

The smaller D (and thus d) the smaller T_S, and thus more stretching can occur. Z_n, the enstrophy after n scalings, is thus a function of D. The relation $(Z_{n+1}/Z_n) = 4(T_{n+1}/T_n)$ holds for homogeneous turbulence and provides an implicit equation for D. This relation is a consequence of the relation $\xi = \mathrm{curl}\,\mathbf{u}$. The resulting equation for D was solved numerically in [6], under the "fat vortex" scaling, and provided the estimate $D \sim 2.3$. Finite values of r/l lead to larger values of D, and thus $D \sim 2.3$ is an underestimate, consistent with the usual guess $D \sim 2.5$ [9, 13, 14]. The arguments above show that $D < 3$.

It should be noted that the object whose similarity dimension has just been computed and the object whose similarity dimension was estimated in [5] are not the same. In [5] we estimated the dimension of the support of a fraction $(1 - \varepsilon)$ of Z while here we are estimating the dimension of the collection of cubes whose interior contains the support of all but a negligible fraction of Z. It is plausible that the two objects are equal, but the calculation in [5] was marred by the uncertainty as to what happens when $\varepsilon \to 0$, as was pointed out to the author by Greengard [10]; the present estimate is the better estimate.

The self-consistency of the "fat vortex" approximation depends on the number of stretching per scaling not being large, and thus on m, the number of lattice nodes, not being large. This restricts the model with "fat" vortices to small m, where m^3 is the number of lattice nodes, i.e., to the case where the cascade is local in scale. Recently, the validity of this assumption has been challenged [16]. The Yakhot/Orszag model could presumably be accommodated by a "thin vortex" approximation. The calculations in [6] were made mostly with $m = 4$ and $m = 6$, for which values the potential theoretical result $D > 1$ for finite energy is recovered. However, it should be pointed out that for D near 1 the restriction posed by the use of the "fat" vortex model is more severe than near $D \sim 2.5$, as can indeed be seen from the results in [6].

Some Comments on the Kolmogorov Spectrum

The spectrum derived by Kolmogorov for the inertial range is $E(k) \sim \varepsilon^{2/3} k^{-5/3}$, where k is a wave number and ε is the rate of energy dissipation. A dynamical derivation of this law was given by Kraichnan [11]: consider a sequence of eddies of decreasing sizes l_n, $n = 1, 2, \ldots, l_{n+1} < l_n$, with characteristic velocities u_n and characteristic energies $E_n \sim u_n^2$. Suppose that in a time $t_n \sim l_n/u_n$, the "turnover time" of the n-th eddy, that eddy yields its energy to the eddy of the next size; then

$$E_n/t_n \sim u_n^3/l_n \sim \varepsilon\,, \tag{6}$$

$$u_n^2 \sim \varepsilon^{2/3} l_n^{2/3}\,, \tag{7}$$

and the Kolmogorov law follows by a Fourier transform. Equation (6) is very attractive because it is a plausible caricature of the Euler equations: if one considers an eddy of size l, then $\mathbf{u}_x \sim u/l$, $(\mathbf{u} \cdot \nabla)\mathbf{u} \sim u^2/l$, $-\operatorname{grad} p \sim (\mathbf{u} \cdot \nabla)\mathbf{u}$, $\mathbf{u}\partial_t \mathbf{u} \sim \varepsilon \sim u^3/l$. Note however that Eqs. (6) and (7) contain a paradox. If indeed all of the energy in eddies of size l_n move to eddies of size l_{n+1} [as is assumed in deriving Eq. (6)], then the amount of energy in eddies of size l_n is on the average equal to $E_0 t_n/T$, where T is a characteristic decay time for the vortical structures, E_0 is the constant available energy, and $t_n \sim l_n/u_n$, $u_n \sim \sqrt{E} = \text{const}$. Thus $E_n \sim l_n$, and $E(k) \sim k^{-2}$, contradicting Eq. (7). The paradox can be resolved if some energy is left behind at each step of the cascade, as a result of the integral constraints discussed above and the resulting intermittency, and furthermore if the characteristic times and lengths are also modified by the intermittency. Thus the difference between a k^{-2} and $k^{-5/3}$ spectrum can be ascribed to intermittency and there is no need for further intermittency corrections as in [9, 14]. This argument was made in [6] in a more precise way, but the more detailed analysis is implausible and must be abandoned. Furthermore, assumptions about the contribution of the energy left behind as a result of intermittency to the structure of the inertial range may well lead to a reconciliation of the cascade picture with the experimental data on the higher statistics of the flow in the inertial range.

Conclusion

We have provided a lattice vortex model of the inertial range that explains many of the results of direct numerical calculations. A number of omissions and an error in earlier presentations have been corrected, in particular, the scaling properties of vortex tubes and the relation between the model and the Kolmogorov law have been made explicit. The use of the lattice model in turbulence modeling will be explained in a subsequent paper.

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