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Quantization of the Kepler Manifold

Bruno Cordani

Dipartimento di Matematica dell'Università, via Saldini 50, I-20133 Milano, Italy

Abstract. A representation of $SO(2,n+1)$, the maximal finite dimensional dynamical group of the n-dimensional Kepler problem, is obtained by means of (pseudo) differential operators acting on $L_2(S^n)$. This representation is unitary when restricted to $SO(2)\otimes SO(n+1)$, i.e. to the physically relevant subgroup.

1. Introduction

A great number of works have been devoted to the Kepler Problem (KP) but the last word is yet to be said, especially with regard to quantization in the sense of Kostant-Souriau [1–3]. We briefly explain the basic concepts (see e.g. [4–6] and references quoted herein).

The n-dimensional KP, $n\geqq 2$, is the Hamiltonian system on the phase space $T^*(\mathbb{R}^n-\{0\})$ with the Hamiltonian

$$H(q,p)=\frac{1}{2}p^2-\frac{1}{q}, \tag{1.1}$$

q_k and p_k being canonical coordinates. Owing to the collision orbits the flow is not complete. After regularization (that amounts to compactifying each cotangent space to the configuration manifold by adding the point at infinity) and exchange between coordinates and momenta, the phase space becomes symplectomorphic to the so-called "Kepler manifold," i.e. $T^+S^n := T^*S^n$ – null section. This phase space turns out to be a coadjoint orbit (more exactly: the most singular orbit) of the dynamical group $SO(2,n+1)$. For negative energy the maximal compact subgroup $SO(2)\otimes SO(n+1)$ of the dynamical group is physically relevant: its generators are to be identified respectively with the Hamiltonian and the other constant of motion, i.e. angular momentum and Runge-Lenz-Laplace vector.

This analysis at "classical level" allows us to define in an unambiguous way what we mean for "quantization" of the Kepler manifold: an Almost-Unitary Irreducible Representation (AUIR) of $SO(2,n+1)$ through (pseudo) differential operators acting on L_2 functions on the n-dimensional sphere S^n. We do not require the full unitarity, since we are content with the unitarity of the representation of the maximal compact subgroup alone. Unfortunately this program is not easy to realize, because the natural polarization of the coadjoint orbit is not invariant under the action of the dynamical group; or, in other words, because S^n does not carry an effective action of this group.

To escape this difficulty and follow the analysis pursued at classical level in [7], we consider the manifold $S^1 \times S^n$, which is a homogeneous space for $SO(2, n+1)$. The cotangent bundle $T^*(S^1 \times S^n)$ is a union of five homogeneous components under the action of the same group, two of which are the Kepler manifold. But now we can easily construct an AUR of the complementary series that, for a judicious choice of the real parameter, is reducible, while the carrier space of two of the irreducible components is isomorphic to the space of the L_2 functions on S^n.

In Sect. 2 we review and complete the analysis at classical level of [7]. In Sect. 3 we pursue the quantization of the Kepler manifold. In Sect. 4 we add some final comments and remarks, and a comparison with preceding works.

In the sequel, the range of the indices is

$$\begin{aligned} &A, B = -1, 0, \ldots, n+1\,, \\ &\mu, \nu = 0, \ldots, n\,, \\ &\alpha, \beta = 1, \ldots, n+1\,, \\ &h, k = 1, \ldots, n\,. \end{aligned}$$

2. The Classical Case

Let $\eta_{AB} = \mathrm{diag}(- - + \ldots +)$ be the metric tensor of $\mathbb{R}^{2,n+1}$. Define

$$G := \begin{cases} SO_0(2, n+1) & n \text{ even;} \\ SO_0(2, n+1)/\Gamma & n \text{ odd,} \end{cases}$$

where $\Gamma := \{1, -1\}$. Thus G is the identity connected component of the conformal group of the Minkowski space $\mathbb{R}^{1,n}$. Define M as the manifold of *unoriented* generators of the null cone K in $\mathbb{R}^{2,n+1}$. Let $\hat{X}^A$ be the points of $\mathbb{R}^{2,n+1}$ that satisfy the relations

$$\begin{aligned} \hat{X}^\alpha \hat{X}_\alpha &= 1\,, \\ \hat{X}^0 \hat{X}_0 + \hat{X}^{-1} \hat{X}_{-1} &= -1\,. \end{aligned} \tag{2.1}$$

Thus $\hat{X}^A \in K$ parametrize the manifold $\tilde{M}$ of *oriented* generators of K. $\tilde{M}$ is obviously homeomorphic to $S^1 \times S^n$. Consider the mapping $\tilde{M} \mapsto \mathbb{C}^{n+1}$ given by

$$z^\alpha = \hat{X}^\alpha (\hat{X}^0 - i\hat{X}^{-1}): \tag{2.2}$$

$\{z^\alpha\}$ is homeomorphic to M, but also to $S^1 \times S^n$. Since $\tilde{M}$ is connected, it is a double covering of M, i.e. $M = \tilde{M}/\Gamma$, and both have the same topology as $S^1 \times S^n$.

Let H_0 be the analytic subgroup of G with Lie algebra $\mathscr{H} := \mathscr{M} \oplus \mathscr{A} \oplus \mathscr{N}$, where

$\mathscr{M}$ = Lie algebra of the Lorentz group $SO(1, n)$,
$\mathscr{A}$ = Lie algebra of the dilation group,
$\mathscr{N}$ = Lie algebra of the conformal translation group.

Let H be the subgroup of $SO_0(2,n+1)$ that leaves invariant a point of M. It is well known, and easy to check, that $H_0 \subset H$. Moreover, since the reflection in the origin $\hat{X}^A \mapsto -\hat{X}^A$ leaves M invariant and is connected with the identity of $SO_0(2,n+1)$ for n odd, we have

$$H = \begin{cases} H_0 & n \text{ even} \\ H_0 \otimes \Gamma & n \text{ odd}. \end{cases}$$

Therefore $M = SO_0(2,n+1)/H = G/H_0$. M is also referred to as "the conformal compactification of $\mathbb{R}^{1,n}$," since it is identified with the Minkowski space with a null cone at infinity adjoined [8]. If we define $\tilde{G}$ as the double covering of G, i.e. if

$$\tilde{G} := \begin{cases} \mathrm{Spin}_0(2,n+1) & n \text{ even} \\ SO_0(2,n+1) & n \text{ odd}, \end{cases}$$

we have $\tilde{M} = \tilde{G}/H_0$.

H is a non-minimal parabolic subgroup of G and $\mathscr{H}$ a parabolic subalgebra of $\mathscr{G}$ (Lie algebra of G). Parabolic subalgebras are important since they give a polarization of coadjoint orbits, and, as shown by Kostant and Kirillov [2, 9–11], polarizations are a fundamental tool to construct UIRs of Lie groups. Let for a moment G be an arbitrary Lie group, $\mathscr{G}$ and $\mathscr{G}^*$ the corresponding Lie algebra and dual algebra respectively.

Definition [10]. A real invariant polarization $\mathscr{H}$ of $f \in \mathscr{G}^*$ is a subalgebra of $\mathscr{G}$ such that

1) $\dim \mathscr{H} = \frac{1}{2}(\dim \mathscr{G} + \dim \mathscr{G}_f)$,
2) $\langle f, [\mathscr{H}, \mathscr{H}] \rangle = 0$,
3) $\mathscr{H}$ is $\mathrm{Ad}(G_f)$-invariant, where G_f is the isotropy subgroup of G with respect to f.

In our case $\mathscr{H}$ satisfies the even more stringent conditions of the following theorem, due to Wolf [12].

Theorem 1. *Let H_0 be a closed subgroup of G with Lie algebra $\mathscr{H}$ such that*

1) *as above in the Definition,*
2') $\langle f, \mathscr{H} \rangle = 0$,
3') $G_f \subset H_0$.

Then the coadjoint orbit $\mathcal{O}_f$ of G through f is equivariantly diffeomorphic to an open G-orbit in $T^*(G/H_0)$.

If, as in the present case, $\mathscr{G}/\mathscr{H}$ is an abelian, and not only a nilpotent subalgebra, we have an equivariant *symplectomorphism* between $\mathcal{O}_f$, equipped with the Kirillov form

$$\omega_f(u,v) := \langle f, [u,v] \rangle, \qquad u, v \in \mathscr{G},$$

and $T^*(G/H_0)$, equipped with the canonical symplectic form.

Identify $\mathscr{G}$ and $\mathscr{G}^*$ by means of the Cartan-Killing form. Let f_- and f_+ be a conformal translation of timelike and spacelike type respectively. By Theorem 1 we have ($\simeq$ means "is symplectomorphic to")

i) $\mathcal{O}_{f_-} \simeq T^*M$ with timelike covectors future pointing,
ii) $\mathcal{O}_{(-f_-)} \simeq T^*M$ with timelike covectors past pointing,
iii) $\mathcal{O}_{f_+} \simeq T^*M$ with spacelike covectors.

These orbits are $(2n+2)$-dimensional. We now come to the (two) $2n$-dimensional orbit(s) considered by Onofri [13] and called the "Kepler manifold."

Theorem 2. *Let $f_0 := f_- + f_+$ be a null conformed translation; then $\mathcal{O}_{(\mp f_0)}$ are symplectomorphic to T^+S^n, endowed with the canonical symplectic form.*

For the proof see [7]. Let $N_\mp$ be the submanifold of null non-vanishing covectors in T^*M pointing into the past and into the future respectively. Since the null cone in $\mathbb{R}^{1,n}$, the cotangent space to a point of M, is diffeomorphic to $\mathbb{R}^n - \{0\}$, we have

iv) $\mathcal{O}_{(-f_0)} \simeq N_-/(S^1/\Gamma)$,
v) $\mathcal{O}_{(+f_0)} \simeq N_+/(S^1/\Gamma)$.

The reduction $T^*\tilde{M} \mapsto T^+S^n$ may be interpreted as the reduction of the *extended* phase space of a mechanical system to the phase space by means of the pseudo-energy integral. In fact, let us consider the geodesic motion in $T^*\tilde{M}$, where the metric of $S^1 \times S^n$ is the usual pseudo-Riemannian: restriction to $\tilde{N}$ is equivalent to fixing the pseudo-energy and dividing by S^1 to dividing out by the flow.

Choose in $T^*\tilde{M}$ the canonical coordinates $\{x^\mu, y_\nu\}$, where x^0 is an angle that parametrizes S^1, and x^k local coordinates on S^n obtained through stereographic projection. Restriction to $\tilde{N}_\mp$ reads as

$$y_0 = \mp \tfrac{1}{2} y(x^2+1). \tag{2.3}$$

The right-hand member of (2.3) is the Hamiltonian of the geodesic flow on S^n. Let us consider the canonical transformation $\mathscr{C}$

$$q_k = y_0 y_k, \tag{2.4a}$$

$$p_k = -\frac{x_k}{y_0}, \tag{2.4b}$$

$$q_0 = -y_0^3\left[x_0 - \frac{\langle x, y\rangle}{y_0}\right], \tag{2.4c}$$

$$p_0 = \frac{1}{2y_0^2}. \tag{2.4d}$$

$\mathscr{C}$ may be viewed as the composition of three canonical transformations: a) that given by exchanging coordinates and momenta: b) that given by (2.4a, b), equivalent to an "energy rescaling;" c) that given by (2.4d). Note that (2.4c) is forced by requiring canonicity. Now the restriction to $\tilde{N}_\mp$ reads as

$$p_0 + H(q,p) = 0, \tag{2.5}$$

where

$$H(q,p) = \frac{1}{2}p^2 \mp \frac{1}{q}. \tag{2.6}$$

This approach permits us to handle the KP for every sign of the energy and to introduce the regularization parameter x^0 without postulating it (see [7] for more details).

3. The Quantization of the Kepler Manifold

Our starting point is the induced representation $R_\lambda := \mathrm{Ind}_{H_0}^{\tilde{G}}(1 \otimes e^\lambda \otimes 1)$, where the representation is unitary for λ pure imaginary. Notice that this is *not* the usual *parabolic-induced*, since the parabolic subgroup corresponding to $\mathscr{H}$ is $H_0 \otimes \Gamma$. So we expect that R_λ is fully reducible and the direct sum of two components indexed by Γ. But, as we shall see, for $\lambda = -1$ this is not the whole story.

Let us explicitly construct the representation R_λ. Let $J_{AB} = -J_{BA}$ be a basis for the Lie algebra $\mathscr{G}$ and $v^\nu_{AB}\partial_\nu$ $\left(\text{with } \partial_\nu = \dfrac{\partial}{\partial x^\nu}\right)$ the corresponding conformal vector field on $\mathbb{R}^{1,n}$. If

$$
\begin{aligned}
\hat{X}^{-1} &= \cos x^0 , \\
\hat{X}^0 &= \sin x^0 , \\
\hat{X}^k &= \frac{2x^k}{x^2+1} , \\
\hat{X}^{n+1} &= \frac{x^2-1}{x^2+1} ,
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
\hat{Y}_{-1} &= -\sin x^0 \partial_0 , \\
\hat{Y}_0 &= \cos x^0 \partial_0 , \\
\hat{Y}_k &= \tfrac{1}{2}(x^2+1)\partial_k - x_k x^h \partial_h , \\
\hat{Y}_{n+1} &= x^h \partial_h ,
\end{aligned}
\tag{3.2}
$$

then

$$v^\nu_{AB}\partial_\nu = \hat{X}_B \hat{Y}_A - \hat{X}_A \hat{Y}_B . \tag{3.3}$$

We are mainly interested in the non-compact generators $J_{\alpha -1} \pm iJ_{\alpha 0}$. Equation (3.3) gives the corresponding vector field as

$$v^\pm_\alpha = -e^{\pm ix^0}[\hat{Y}_\alpha \pm i\hat{X}_\alpha \partial_0] . \tag{3.4}$$

Let $\{r, x^\nu\}$ be coordinates on the null cone K. The linear action of $a \in SO_0(2, n+1)$ on $\mathbb{R}^{2,n+1}$ sends $\{r, x^\nu\}$ into $\{r', x'^\nu\}$, with $r' = \mu(a,x)r$, where $\mu(a,x)$ is a so-called *multiplier* [14]. Defining the *infinitesimal multiplier* [14] as

$$\tau_{AB} := \frac{\partial}{\partial a^{AB}} \mu(a,x)|_{a=e} , \tag{3.5}$$

we easily obtain

$$\tau^\pm_\alpha = e^{\pm ix^0} \hat{X}_\alpha , \tag{3.6}$$

whereas, for a belonging to the compact subgroup,

$$\tau_{-10}=0=\tau_{\alpha\beta}, \tag{3.7}$$

since $r'=r$. Thus the infinitesimal representation corresponding to R_λ is given by

$$U_{AB}:=v^{\mu}_{AB}\partial_\mu+\left(\frac{n+1}{2}+\lambda\right)\tau_{AB}. \tag{3.8}$$

These operators, acting on functions $\phi:\tilde{M}\mapsto C$, are skew-hermitian with respect to the natural Riemannian measure of M, when λ is purely imaginary. Notice that R_λ, when restricted to the maximal compact subgroup, is skew-hermitian for each λ.

As a basis for the representation space, we may assume functions of the type

$$\phi_{lm}:=h_l(\hat{X})e^{imx^0}, \tag{3.9}$$

with $l\in Z_+$ and $m\in\begin{cases}\frac{1}{2}Z & n \text{ even}\\ Z & n \text{ odd}\end{cases}$. Here h_l is a harmonic homogeneous polynomial (h.h.p.) of degree l, i.e.

$$\frac{\partial}{\partial X^\alpha}\frac{\partial}{\partial X_\alpha}h_l(X)=0 \quad \text{(harmonicity)},$$

$$X^\alpha\frac{\partial}{\partial X^\alpha}h_l(X)=lh_l(X) \quad \text{(homogeneity)}.$$

It is well known that a h.h.p. of degree l restricted to S^n gives a spherical harmonic of the same degree. Equation (3.8) gives

$$U_{-10}=-\partial_0, \tag{3.10a}$$

$$U_{\alpha\beta}=X_\beta\frac{\partial}{\partial X^\alpha}-X_\alpha\frac{\partial}{\partial X^\beta}, \tag{3.10b}$$

$$U^{\pm}_\alpha=-e^{\pm ix^0}\left[\frac{\partial}{\partial X^\alpha}\mp mX_\alpha-\left(l+\frac{n+1}{2}+\lambda\right)X_\alpha\right]. \tag{3.10c}$$

Define $\mathscr{F}_{lm}$ as the space of the functions of type (3.9): for every fixed couple (l,m) it is the representation space of the maximal compact subgroup. Define $\mathscr{F}:=\oplus_{lm}\mathscr{F}_{lm}$. The main result of the present work is the following

Theorem 3. *For $\lambda=-1$ the two subspaces of $\mathscr{F}$ determined by the pairs $(l,m^\spadesuit)$ and $(l,m^\clubsuit)$, with*

$$m^\spadesuit:=l+\frac{n-1}{2}, \qquad m^\clubsuit:=-m^\spadesuit, \tag{3.11}$$

are invariant under the action of the representation (3.10) *and are manifestly isomorphic to the space of the smooth functions on S^n.*

Proof. Let us consider the case $(l,m^\spadesuit)$: the proof for the other case is similar. We have

$$U^-_\alpha=-e^{-ix^0}\frac{\partial}{\partial X^\alpha}. \tag{3.12}$$

Since $\partial/\partial X^\alpha$ commutes with the Laplacian $\frac{\partial}{\partial X^\beta}\frac{\partial}{\partial X_\beta}$, we have that $\partial h_l/\partial X^\alpha$ is a h.h.p. of degree $(l-1)$. Therefore

$$U_\alpha^- \mathscr{F}_{lm^\bullet} \subset \mathscr{F}_{l-1\,m^\bullet-1}, \qquad \forall l. \tag{3.13}$$

Since $\hat{X}^\beta \hat{X}_\beta = 1$, we may write

$$U_\alpha^+ = -e^{ix^0}\left[(\hat{X}^\beta \hat{X}_\beta)\frac{\partial}{\partial X^\alpha} - 2m^\bullet X_\alpha\right]. \tag{3.14}$$

It easy to check that if we apply the operator in the square brackets to a h.h.p. of degree l, we obtain one of degree $(l+1)$. Thus

$$U_\alpha^+ \mathscr{F}_{lm^\bullet} \subset \mathscr{F}_{l+1\,m^\bullet+1}, \qquad \forall l \tag{3.15}$$

and the theorem is proved.

Let us start with an arbitrary couple (l, m), with $m \neq \mp m^\bullet$. Since $\hat{X}_\alpha h_l(\hat{X})$ is decomposable into spherical harmonics of degree $l+1, l-1, l-3, \ldots$, we see that the total space $\mathscr{F}$ of the representation splits into two disjoint components. We illustrate all that in the following figure.

l $\Downarrow$				…	$-\frac{n-1}{2}$	$\Leftarrow$ m $\Rightarrow$ …	$\frac{n-1}{2}$	…			
0	♢	♡	♢	♡	♣	♡	♠	♡	♢	♡	♢
1	♡	♢	♡	♣	♡	♢	♡	♠	♡	♢	♡
2	♢	♡	♣	♡	♢	♡	♢	♡	♠	♡	♢
3	♡	♣	♡	♢	♡	♢	♡	♢	♡	♠	♡
⋮	♣	♡	♢	♡	♢	♡	♢	♡	♢	♡	♠

Every suit represents one of the spaces $\mathscr{F}_{lm}$ and like suits represent invariant subspaces of $\mathscr{F}$: notice however that R_{-1} is *fully* reducible in the *red* suits, but only reducible in the *black* suits (we might display this graphically by adjoining to every black the corresponding red suit).

4. Some Remarks

a) We claim to have obtained an AUIR of $\tilde{G}$ by means of pseudo differential operators acting on $L_2(S^n)$. In fact, consider, for example, the energy operator iU_{-10}: the eigenvalues of iU_{-10} acting on $\mathscr{F}_{lm^\bullet}$ are $\left(l + \frac{n-1}{2}\right)$ with multiplicity $\frac{(2l+n-1)(l+n-2)!}{l!(n-1)!}$ and equal to those of $\sqrt{\Delta_{S^n} + \left(\frac{n-1}{2}\right)^2}$ acting on $L_2(S^n)$. (This result has been obtained also by Akyildiz [15].) The use of pseudo differential operators is obviously due to the fact that our dynamical group does not act effectively on S^n. Theorem 3 is the representation-theoretic analogue of the reduction $T^*\tilde{M} \mapsto T^+S^n$: in fact the two subspaces of the Theorem are annihilated by the operators

$$i\frac{\partial}{\partial x^0} \pm \sqrt{\Delta_{S^n} + \left(\frac{n-1}{2}\right)^2},$$

and this is the "quantum" analogue of the "classical" equations (2.3).

b) There are papers concerned with the computation of the spectrum and multiplicity of the energy operator. We refer to [16] for the 3-dimensional case and [17] for the n-dimensional one. We stress however that in these works only the so-called "pre-quantization" is carried out.

c) The representation (3.10) can be realized by means of differential operators acting on $\mathscr{F}_{lm'}$. Taking into account the homogeneity property, we easily find the representation of [18].

d) The philosophy of our approach is reminiscent of that of Onofri [13], but differs in that while we start from a reducible representation of the supplementary series, Onofri starts from the discrete series whose carrier space is the space of the sections of a holomorphic line bundle over the Kaehler manifold $SO(2,n+1)/SO(2)\otimes SO(n+1)$. To obtain, through a limit process, the supplementary from the discrete series is not an easy task which seems difficult to realize in a rigorous way (in fact the choice $l_0=-1$, basically equivalent to our $\lambda=-1$, is made by Onofri in an heuristic way).

e) Since the representation (3.10) is unitary with respect to the natural pseudo-Riemannian measure of $S^1\times S^n$ only when restricted to the maximal compact subgroup, it is reasonable to ask if we can obtain an UIR by changing the scalar product. The present author has not been able to give a definitive answer to this problem. Lastly, note the following interesting fact: writing the Schrödinger equation for the KP in the momentum representation (as in the classical work of Fock [19]), the potential term gives rise to a convolution integral of the type

$$\int\frac{\phi(p')d^np'}{|p-p'|^{n-1}}.$$

But this looks exactly like the interwining operator entering in the study of the representations of the supplementary series. We hope to clarify this link (if any) elsewhere.

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