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# Super Riemann Surfaces: Uniformization and Teichmüller Theory

Louis Crane[1] and Jeffrey M. Rabin[2]*

[1] The Institute for Advanced Study, Princeton, NJ 08540, USA

[2] Enrico Fermi Institute and Department of Mathematics, The University of Chicago, 5640 S. Ellis Ave., Chicago, IL 60637, USA

**Abstract.** Teichmüller theory for super Riemann surfaces is rigorously developed using the supermanifold theory of Rogers. In the case of trivial topology in the soul directions, relevant for superstring applications, the following results are proven. The super Teichmüller space is a complex super-orbifold whose body is the ordinary Teichmüller space of the associated Riemann surfaces with spin structure. For genus $g > 1$ it has $3g-3$ complex even and $2g-2$ complex odd dimensions. The super modular group which reduces super Teichmüller space to super moduli space is the ordinary modular group; there are no new discrete modular transformations in the odd directions. The boundary of super Teichmüller space contains not only super Riemann surfaces with pinched bodies, but Rogers supermanifolds having nontrivial topology in the odd dimensions as well. We also prove the uniformization theorem for super Riemann surfaces and discuss their representation by discrete supergroups of Fuchsian and Schottky type and by Beltrami differentials. Finally we present partial results for the more difficult problem of classifying super Riemann surfaces of arbitrary topology.

## 1. Introduction

Polyakov's bosonic string theory [1] is a theory of maps from a two-dimensional surface $\Sigma$ into (Euclidean) spacetime, with action

$$S = \int d^2\Sigma \sqrt{g}\, g^{ab} \partial_a X^\mu \partial_b X_\mu,$$
$$X : \Sigma \to R^{26}. \tag{1.1}$$

The world sheet metric $g^{ab}$ is an auxiliary field which permits the action to be

* Enrico Fermi Fellow. Research supported by the NSF (PHY 83-01221) and DOE (DE-AC02-82-ER-40073). Present address: Department of Mathematics, C-012, University of California at San Diego, La Jolla, CA 92093, USA

expressed in local form. Quantization involves functional integration over the fields $g^{ab}$ and $X^{\mu}$. In addition to reparametrization invariance, the action (1.1) has a Weyl invariance under conformal rescalings of the metric

$$g^{ab} \to \Omega g^{ab}, \tag{1.2}$$

with $\Omega$ a positive scalar function on $\Sigma$. To define the functional integral we pick a gauge-fixing slice transverse to the orbits of the Weyl and diffeomorphism groups in the space of metrics. This slice is a realization of the space of conformal equivalence classes of metrics modulo diffeomorphisms on $\Sigma$, which is also the moduli space of Riemann surface structures on $\Sigma$. Hence the amplitudes of the bosonic string theory can be expressed as integrals of various functional determinants over moduli space [2, 3]. Such a representation of the amplitudes allows the use of powerful techniques of algebraic geometry to study their holomorphic structure, investigate their finiteness, and even compute them in terms of theta functions via the Selberg trace formula [2, 4]. Unfortunately, they are divergent.

It is generally believed that the superstring does not suffer from the divergences of the bosonic string and may provide a realistic and predictive theory of all fundamental interactions. Accordingly, there is great interest in generalizing the algebraic geometry of moduli space to the superstring context. Several authors have computed the dimension of the gauge-fixing slice for the superconformal symmetries of $2D$ supergravity using index theorems [5, 6]. This only gives local information about the "super moduli space." The object of this paper is to study the space of super moduli in a global way, as done in the Teichmüller theory of Riemann surfaces. We provide a rigorous foundation for the theory of super Riemann surfaces, proving all the basic results necessary for applications to superstrings.

Teichmüller theory constructs a certain covering space of the moduli space of all complex structures on $\Sigma$. In the course of the construction it is shown that complex structures are in 1-1 correspondence with conformal equivalence classes of metrics. The construction of Teichmüller space proceeds via a passage to the universal covering space of $\Sigma$, which is shown to be the Riemann sphere, complex plane, or upper half plane by the uniformization theorem. The complex structures on $\Sigma$ are then parametrized by representing each generator of $\pi_1(\Sigma)$ by a $PSL(2, C)$ transformation acting on the covering space. The parameters of the $PSL(2, C)$ elements give coordinates on Teichmüller space. The moduli space is obtained as the quotient by the modular group which acts by changing the choice of generators of $\pi_1(\Sigma)$.

Using Friedan's global definition of a super Riemann surface [7] we are able to repeat this entire construction. This is one of the few applications of superspace in physics which seems to require a rigorous mathematical theory of supermanifolds rather than just an intuitive manipulation of anticommuting variables. By employing Rogers' theory of supermanifolds [8–10] we maintain full rigor while staying close to our intuitive notion of superspace. In Sect. 2 we define super Riemann surfaces, specifying in particular their global topology, and describe the supergroup which generalizes $PSL(2, C)$. In Sect. 3 we construct the super Teichmüller space by the procedure outlined above. We show that it is the quotient of a real supermanifold by a $Z_2$ symmetry with fixed points, hence a super-orbifold, and compute its dimension, which agrees with the result from $2D$ supergravity. The description of supertori in terms of superlattices is worked out explicitly with attention to the dependence of the results on spin structure. Some technical aspects of the uniformization theorem are postponed to Sect. 4. In Sect. 5 we show how to describe super Riemann surfaces in terms of Beltrami differentials, deriving the super Beltrami equations and discussing the uniqueness of their solutions. This machinery allows us to embed the double cover of super Teichmüller space in a space of superdifferentials of weight 3/2, thereby exhibiting its complex structure; to represent super Riemann surfaces by Schottky supergroups; and to define a universal super Teichmüller space. It should make possible deeper studies of the geometry of super moduli space as well. Some of our results have been announced by other authors [7, 11, 12], but without the rigorous proofs provided here. In Sect. 6 we briefly consider super Riemann surfaces with nontrivial topology in the anticommuting directions. We sketch arguments for a uniformization conjecture for them and explain why they do not contribute to the superstring path integral. Section 7 contains our conclusions.

## 2. Definitions

We adopt Friedan's definition of a super Riemann surface (SRS) [7], which we make rigorous by combining it with Rogers' general theory of supermanifolds [8–10]. Thus, a SRS will be a complex supermanifold of dimension $(1, 1)$ whose transition functions are superconformal maps. We now explain this definition in detail.

In each coordinate chart of a SRS there will be one complex even coordinate $z$ and one complex odd coordinate $\theta$. These coordinates take their values in a fixed Grassmann algebra $B_L$ having $L$ anticommuting generators $v_1, v_2, \ldots, v_L$. Thus,

$$\begin{aligned} z &= z_0 + z_{ij} v_{ij} + \cdots \equiv z_\Gamma v_\Gamma, \\ \theta &= \theta_i v_i + \theta_{ijk} v_{ijk} + \cdots \equiv \theta_\Gamma v_\Gamma, \end{aligned} \tag{2.1}$$

where $v_\Gamma$ denotes the product of all the $v_i$ whose subscripts appear in the sequence $\Gamma$, and $v_0 \equiv 1$. The coefficients $z_\Gamma, \theta_\Gamma$ are ordinary complex numbers. Sometimes we will use $Z$ to denote either or both of $z$ and $\theta$. The complex number $z_0$ is called the body of $z$, while the remainder $z - z_0$ is its soul; $\theta$ has no body and is pure soul. A SRS can always be viewed as an ordinary complex manifold of dimension $2^L$ by using the $Z_\Gamma$ as complex coordinates. This is a major advantage of Rogers' theory: topological properties of SRS's are as well defined as those of ordinary manifolds.

The transition functions relating coordinates in overlapping charts on the SRS are required to be both complex analytic and superanalytic, meaning that they take the form,

$$\begin{aligned} \tilde{z} &= f(z) + \theta\zeta(z), \\ \tilde{\theta} &= \psi(z) + \theta g(z). \end{aligned} \tag{2.2}$$

Furthermore the component functions $f, \zeta, \psi, g$ have Taylor expansions in powers of the soul of $z$, for example

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \cdots, \tag{2.3}$$

with $f(z_0)$ analytic. This series always terminates because $z - z_0$ is nilpotent, so the component functions are uniquely specified once known for soulless values of $z$.

We are assuming the Grassmann algebra $B_L$ to be finite-dimensional in order to avoid questions of convergence. However, this assumption creates its own technical difficulties stemming from the fact that the derivative $\partial/\partial\theta$ does not obey the Leibniz rule in a finite-dimensional algebra. To see the problem, consider $\partial(\theta v_{12\ldots L})/\partial\theta$. If the Leibniz rule were valid, this derivative would have to be $v_{12\ldots L}$, but the function being differentiated is identically zero! Since we will need the Leibniz rule on several occasions, such as Eq. (2.4) below, we handle this problem by the method suggested recently by Rogers [13]. We restrict the components $f(z_0), \zeta(z_0)$ of all superanalytic functions $F(z, \theta) = f(z) + \theta\zeta(z)$ to take values in the subalgebra $B_{L-1}$ generated by $v_1, v_2, \ldots, v_{L-1}$. Then the components cannot contain any term proportional to $v_{12\ldots L}$ and the problem is resolved. The results to be obtained in this paper will hold for all finite values of $L$ and in the limit $L \to \infty$.

We must now impose the condition that the transition functions (2.2) be superconformal [7, 14, 15]. In each chart there is a derivative operator

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}, \quad D^2 = \frac{\partial}{\partial z}, \tag{2.4}$$

which transforms according to

$$D = (D\tilde{\theta})\tilde{D} + (D\tilde{z} - \tilde{\theta} D\tilde{\theta})\tilde{D}^2. \tag{2.5}$$

By analogy with the behavior of $\partial/\partial z$ on a Riemann surface, we demand that $D$ transform homogeneously, so that $D = (D\tilde{\theta})\tilde{D}$. This imposes the constraint

$$D\tilde{z} = \tilde{\theta} D\tilde{\theta}, \tag{2.6}$$

which becomes explicitly

$$\zeta = g\psi, \quad g^2 = f' + \psi\psi'. \tag{2.7}$$

Thus, a general superconformal map takes the form,

$$\begin{aligned} \tilde{z} &= f + \theta\psi\sqrt{f'}, \\ \tilde{\theta} &= \psi + \theta\sqrt{f' + \psi\psi'}. \end{aligned} \tag{2.8}$$

It is specified by the two $B_{L-1}$-valued analytic functions $f(z_0)$ and $\psi(z_0)$.

There is an equivalent definition of a superconformal map which we will find more useful. Requiring the 1-form $dz + \theta d\theta$ to transform homogeneously leads to the same conditions (2.7) and the transformation law

$$d\tilde{z} + \tilde{\theta} d\tilde{\theta} = (D\tilde{\theta})^2 (dz + \theta d\theta). \tag{2.9}$$

(Our convention for 1-forms is that $d\theta$ commutes with itself but anticommutes with $dz$ and with $\theta$.) This works because $dz + \theta d\theta$ and $d\theta$ constitute the basis of

1-forms dual to the basis $D$, $D^2$ of vector fields. The object $dZ$ defined by Friedan [7] via its transformation law $d\tilde{Z} = (D\tilde{\theta})dZ$ can be viewed as a square root of $dz + \theta d\theta$ in the sense of half-forms.

So far we have defined a *Rogers* SRS whose global topology may be very complicated: in particular, nontrivial topology in the $\theta$ dimensions is possible [9, 10]. For applications to superstrings the more restricted topology of a DeWitt supermanifold [16] is appropriate. To implement this restriction we require that each coordinate chart of the SRS may not be an arbitrary open set but instead must be the Cartesian product of an open set in the $z_0$ plane with the entire complex planes of the other coordinates $Z_\Gamma$. This effectively trivializes the topology in all but the $z_0$ dimension. From now on the unqualified term "SRS" will imply this DeWitt topology. When we discuss Rogers SRS's we will explicitly identify them as such.

To every (DeWitt) SRS $M$ there is associated a corresponding Riemann surface $M_0$, called the body of $M$, with a particular spin structure. The charts on $M_0$ are the projections on the $z_0$ plane of the charts on $M$, and its transition functions are the bodies $f_0(z_0)$. Then the $f'_0(z_0)$ are the transition functions of the tangent bundle of $M_0$, and a choice of square roots of these functions defines a spin structure [17]. Since the square root of a Grassmann number can be found as an expansion about the square root of its body in powers of its soul, such a choice of square roots is implicit in Eqs. (2.8). $M$ is a fiber bundle over $M_0$ having a vector space as fiber, but it is not strictly speaking a vector bundle because the transition functions need not be linear in the fiber coordinates. Conversely, given a Riemann surface $M_0$ with spin structure, there is a canonical SRS $M$ whose body is $M_0$. The charts on $M$ are the Cartesian products of the charts on $M_0$ with the entire complex planes in the soul coordinates, and its transition functions have $f(z_0)$ equal to the transition functions of $M_0$, and $\psi(z_0) = 0$, with the square roots in Eqs. (2.8) defined via the given spin structure. An important question which will be answered (in the negative) by the Teichmüller theory we will develop is whether every SRS is equivalent to one of these canonical ones.

Because the plane $C$, the Riemann sphere $C^*$, and the upper half-plane $U$ are simply connected, they have unique spin structures. Therefore there are unique canonical SRS's over these Riemann surfaces, which we denote by $SC$, $SC^*$, and $SU$. The group of superconformal automorphisms of $SC^*$ is the natural generalization of the group of fractional linear transformations and will play a central role in our work [7, 14, 15]. We now determine this group, which Friedan calls $S\hat{L}_2$ and which we will denote as $SPL(2, C)$. Each element of the group is specified by functions $f(z_0)$ and $\psi(z_0)$. Certainly the body of $f(z_0)$ must be a Möbius transformation,

$$f_0(z_0) = \frac{a_0 z_0 + b_0}{c_0 z_0 + d_0}. \tag{2.10}$$

At first it seems that there are no constraints on $\psi(z_0)$ or on the soul of $f(z_0)$, because a superanalytic map is invertible whenever its body is: the inverse can be found as an expansion about the inverse of the body in powers of the soul [16]. However the situation is more subtle because of the pole in Eq. (2.10). Certainly an expansion in powers of the soul is not valid if the soul has a pole at a point where the body is finite. The soul may have poles where the body does, but they cannot be poles of arbitrarily high order or again the expansion fails. Indeed, in the neighborhood of a pole we use the transition functions of $SC^*$ to replace $(\tilde{z}, \tilde{\theta})$ by $(-1/\tilde{z}, \tilde{\theta}/\tilde{z})$, which must make the soul as well as the body finite. The most general functions satisfying these conditions are

$$f(z_0) = \frac{az_0 + b}{cz_0 + d}, \quad ad - bc = 1,$$

$$\psi(z_0) = \frac{\gamma z_0 + \delta}{cz_0 + d}, \tag{2.11}$$

depending on three independent even parameters in $B_{L-1}$ and two odd ones. Thus we obtain the general group element

$$\tilde{z} = \frac{az + b}{cz + d} + \theta \frac{\gamma z + \delta}{(cz + d)^2},$$

$$\tilde{\theta} = \frac{\gamma z + \delta}{cz + d} + \frac{\theta}{cz + d}(1 + \tfrac{1}{2}\delta\gamma). \tag{2.12}$$

The group $SPL(2, C)$ so defined is obtained by exponentiating the subalgebra of the Neveu–Schwarz algebra generated by $L_{-1}, L_0, L_1, G_{-1/2}$, and $G_{1/2}$. If any other generators are included, *all* others are produced by commutation, leading to poles of arbitrarily high order, which is unacceptable.

Given three points of $SC^*$, there are exactly two $SPL(2, C)$ transformations which send two of them, as well as the even coordinate of the third, to specified values. For example, the points $(z, \theta) = (0, 0)$, $(1, 0)$ and the even coordinate of $(\infty, 0)$ are fixed by both the identity and the fermionic inversion $I: \tilde{z} = z$, $\tilde{\theta} = -\theta$. The fact that $I$ cannot be distinguished from the identity by its action on these points will lead to a fundamental $Z_2$ ambiguity in the Teichmüller theory in Sects. 3 and 5. An element of $SPL(2, C)$ can be characterized in terms of its fixed points and multiplier, just as is true for ordinary Möbius transformations [12].

Associated to any subgroup $G$ of $SPL(2, C)$ is the group $G_0$ of fractional linear transformations which are the bodies (2.10) of the elements of $G$. If $G$ acts on some SRS $M$, then $G_0$ acts on $M_0$. Specifically, if $x_0$ is a point of $M_0$ and $q_0$ an element of $G_0$, take any point $x$ in the fiber of $M$ lying over $x_0$ and any element $q$ of $G$ with body $q_0$. Then define $q_0 x_0$ to be the body of $qx$, which does not depend on the choice of $x$ and $q$.

## 3. Uniformization

In this section we will prove the uniformization theorem for SRS's, which states that any "metrizable" SRS is $SC^*$ or a quotient of $SC$ or $SU$ by a subgroup of $SPL(2, C)$. As a corollary we learn that the super Teichmüller space is a real super-orbifold and determine its dimension and its body. As an illustration we explicitly work out the groups representing super tori. This genus 1 case is exceptional in that the dimension of the super Teichmüller space depends on spin structure.

Let $M$ be an arbitrary SRS and $M_0$ its body. Since $M$ is a bundle over $M_0$ with contractible fibers, its universal covering space $\hat{M}$ is also such a bundle over $\hat{M}_0$, the universal cover of $M_0$. $\hat{M}$ can be given a SRS structure such that the covering group acts by superconformal transformations and the body is $\hat{M}_0$. By the uniformization theorem for Riemann surfaces we know that $\hat{M}_0$ is $C$, $C^*$, or $U$ [18, 19]. In the next section we will use sheaf cohomology methods to prove that the canonical SRS's $SC$, $SC^*$, and $SU$ are in fact the only SRS's over these Riemann surfaces. This will show that any SRS is a quotient of $SC$, $SC^*$, or $SU$ by a group $G$ of superconformal automorphisms. Furthermore, $G$ is isomorphic to the fundamental group $\pi_1(M)$. Since $M$ is a fiber bundle with a vector space as fiber, this in turn is isomorphic to $\pi_1(M_0)$, a discrete group. Hence $G$ is known to be discrete.

In order for the quotient space $\hat{M}/G$ to be a manifold, $G$ must act properly discontinuously: each point of $\hat{M}$ must have an open neighborhood which does not intersect any of its images under group transformations (other than the identity). If the open neighborhoods in this definition can be arbitrary, then the quotient space in general will be a Rogers SRS. To ensure that the quotient is a DeWitt SRS, the neighborhoods satisfying the above condition must be open in the DeWitt sense: they must be cylinders over open sets in $M_0$. This in turn is possible iff the associated Möbius group $G_0$ acts properly discontinuously on the body $M_0$. A simple example illustrating these points is provided by supersymmetry. The group $G$ generated by

$$\tilde{z} = z + \theta\delta, \quad \tilde{\theta} = \theta + \delta \tag{3.1}$$

for some fixed $\delta$ acts properly discontinuously on $SC$, but $G_0$ consists of the identity map alone. The body is actually fixed by the transformation, and taking the quotient $SC/G$ renders the fibers nonsimply connected while leaving the body unchanged [9].

Since no Möbius transformation acts properly discontinuously on the body $C^*$ of $SC^*$, no new SRS's can be obtained as quotients of $SC^*$. So we know that any SRS is either $SC^*$ or a quotient of $SC$ or $SU$ by a group of superconformal automorphisms which acts properly discontinuously on the body. Unfortunately, it is not true that all superconformal automorphisms of $SC$ and $SU$ belong to $SPL(2, C)$. Since $C$ and $U$ do not contain the point at infinity, we no longer have the constraints on the behavior of superconformal maps at their poles which gave us the group $SPL(2, C)$ for the Riemann sphere. A superconformal automorphism of $SC$ or $SU$ need only have a fractional linear transformation as its body—its soul is unrestricted. For example, the superconformal transformation

$$\tilde{z} = z + 1 + \theta\eta z^n, \quad \tilde{\theta} = \theta + \eta z^n, \tag{3.2}$$

does not belong to $SPL(2, C)$ for $n > 1$, but it is nevertheless an automorphism of $SC$.

For applications to superstrings, however, more than just a SRS structure is required. There must be enough geometric structure to construct an integration measure and invariant Lagrangian for world sheet supergravity. For Riemann surfaces the existence of a metric is automatic, but not every SRS admits a metric generalizing the metric on its body. For physical applications, then, we must restrict ourselves to "metrizable" SRS's. For example, on $SU$ there is a generalization of the Poincaré metric,

$$ds = (\operatorname{Im} z + \tfrac{1}{2}\theta\bar{\theta})^{-1} |dz + \theta d\theta|, \tag{3.3}$$

which is invariant under $SPL(2, R)$ but not under larger superconformal groups [11]. Here $SPL(2, R)$ is the subgroup of $SPL(2, C)$ for which the even parameters are all real ($\bar{a} = a$, etc.) and the odd parameters are restricted by $\bar{\gamma} = i\gamma$, etc. This restriction ensures that the product of two odd parameters will be a real even parameter. Heuristically, one can think of $SPL(2, R)$ as the subgroup of $SPL(2, C)$ which fixes the "superboundary" $SR$ of $SU$, namely the set $\bar{z} = z$, $\bar{\theta} = i\theta$. The superboundary is not the boundary of $SU$ as a manifold because it has *half* the dimension of $SU$ rather than the dimension minus one, but unlike the true boundary $SR$ is a supermanifold. Similarly the metric

$$ds = |dz + \theta d\theta| \tag{3.4}$$

on $SC$ is only invariant under a subgroup of $SPL(2, C)$. Thus the uniformization theorem which is needed for physical applications states that any metrizable SRS is either $SC^*$ or a quotient of $SC$ or $SU$ by a discrete subgroup of $SPL(2, C)$. This subgroup is isomorphic to the fundamental group of the body and is unique up to conjugation.

It is not quite clear that the physical requirement for superstring applications is the existence of the metrics described above, because $2D$ supergravity is not a Riemannian supergeometry based on a metric but instead involves covariant derivatives with torsion. The metrics above also cannot be directly relevant to the heterotic string because they depend on $\bar{\theta}$ as well as $\theta$. However, in $2D$ supergravity one does make use of a frame on the SRS which is constructed from a metric (or the associated zweibein) on its body as well as a gravitino field which can be locally gauged away. Therefore the constant curvature metric on the body must extend to a function of $z$ which is invariant under the bosonic parts of the superconformal automorphisms. The fermionic parts mix this metric with the gravitino. This certainly restricts the bosonic parts of the automorphisms to be those of $SPL(2, C)$ elements, but it restricts the fermionic parts as well since the fermionic parts of two group elements contribute to the bosonic part of their product. In this way one is again led to the metrizability condition. The same is true in "heterotic geometry" [20], although this is a somewhat different construction in which only the bosonic parts of superconformal maps are used as transition functions of the SRS while the fermionic parts act in the tangent space.

In addition to this general argument, a specific one can be given for the $2D$ supergravity describing the nonchiral spinning string. In this case the "metrics" $ds$ above should be reinterpreted as the norms of the even component of the frame field $E^z = dZ^M E^z_M$ on $SU$ and $SC$. Invariance of the "metric" translates into invariance of $E^z$ up to a phase, which is invariance up to a rotation by the tangent space group $U(1)$ in two dimensions. This is certainly necessary for the SRS to inherit a supergravity frame field from its covering space. It is also sufficient, since an odd component $E^\theta$ can be found such that the complete frame is invariant up to a $U(1)$ rotation. The complete frame fields are

$$E^z = dz + \theta d\theta, \quad E^\theta = d\theta, \tag{3.5}$$

for $SC$; and

$$\begin{aligned} E^z &= (\operatorname{Im} z + \tfrac{1}{2}\theta\bar{\theta})^{-1}(dz + \theta d\theta), \\ E^\theta &= (\operatorname{Im} z + \tfrac{1}{2}\theta\bar{\theta})^{-1/2}\, d\theta + \tfrac{1}{2}(i\theta - \bar{\theta})(\operatorname{Im} z + \tfrac{1}{2}\theta\bar{\theta})^{-3/2}(dz + \theta d\theta), \end{aligned} \tag{3.6}$$

for $SU$.

Having proven the uniformization theorem, we can now describe the super Teichmüller space for genus $g$, $ST_g$. It is defined as the space of marked metrizable SRS's $M$ having compact bodies of genus $g$, where a marking is a specific choice of generators for $\pi_1(M)$. Dropping the marking defines the super moduli space. For genus $g > 1$, representing the SRS as a quotient of $SU$ by a (Fuchsian) subgroup $G$ of $SPL(2, R)$ having $2g$ generators $q_1, q_2, \ldots, q_{2g}$ provides $6g$ even and $4g$ odd parameters. These parameters are not all independent, however, since there is the freedom of an overall conjugation as well as a single relation among the generators of the fundamental group. Indeed, by an $SPL(2, R)$ conjugation we can move the fixed points of $q_1$ to $(z, \theta) = (0, 0)$, $(\infty, 0)$ and specify the even coordinate of the attractive fixed point of $q_2$. There are in fact two conjugations that do this, differing by the inversion $I$. [In principle we are free to conjugate by superconformal automorphisms of $SU$ lying outside $SPL(2, R)$ as well, but such a conjugation would take the $q_i$ out of $SPL(2, R)$.] The group relation

$$q_1 q_2 q_1^{-1} q_2^{-1} \ldots q_{2g-1} q_{2g} q_{2g-1}^{-1} q_{2g}^{-1} = 1 \tag{3.7}$$

then imposes three more even and two odd conditions. Each point of $ST_g$ can therefore be described by $6g$–6 independent even parameters and $4g$–4 odd ones, and this can be done in two different ways depending on which of the two possible conjugations is chosen. Since conjugation by $I$ changes the signs of the odd parameters of an $SPL(2, C)$ element while leaving the even ones unchanged, the two descriptions are related by changing the signs of the odd parameters. This means that $ST_g$ has a double cover which is a real supermanifold of dimension ($6g$–6, $4g$–4). It is also a complex supermanifold, but we will not be able to prove this until Sect. 5. The only restrictions on the values of the parameters come from the requirement that the group act properly discontinuously on the body, which affects only the bodies of the parameters. Thus the soul coordinates are unrestricted, so the supermanifold is of the DeWitt type. $ST_g$ itself is a super-orbifold, the singular points being those whose odd parameters vanish. The body of $ST_g$ is identified with the equivalence classes of canonical SRS's, whose transition functions are soulless. But these are in 1-1 correspondence with Riemann surfaces with spin structure. Hence the body of $ST_g$ is precisely the ordinary Teichmüller space of Riemann surfaces with spin structure, which is a $2^{2g}$-sheeted covering of the Teichmüller space $T_g$. This incidentally implies that any metrizable SRS is a deformation of a canonical one. This is the analogue of Rothstein's theorem that any complex (Berezin-Leites-Kostant) supermanifold is a deformation of a vector bundle [21].

In a previous version of this paper we indicated that by omitting metrizability from the definition of super Teichmüller space, one would obtain an object which is infinite-dimensional and not a supermanifold. Recently, however, Hodgkin has shown that metrizability is a superfluous assumption for SRS's having compact bodies; precisely the same space $ST_g$ is defined without including any metrizability requirement [22]. This is surprising because infinitely many parameters should be required to describe $2g$ superconformal automorphisms of $SU$ representing the generators of $\pi_1$ of an arbitrary SRS. The implication of Hodgkin's result is that any set of $2g$ automorphisms obeying the relation (3.7) can in fact be brought into $SPL(2, R)$ by a conjugation.

Next we can discuss the reduction of $ST_g$ to the super moduli space. In order to obtain coordinates on $ST_g$ we had to choose a specific set of generators for the fundamental group of a SRS, then represent these generators by $SPL(2, R)$ elements. To pass to super moduli space we must eliminate the dependence on the choice of generators. There will be infinitely many points of $ST_g$ which describe the same group in terms of different sets of generators. The transformations of $ST_g$ which take such points into one another form the super modular group, and super moduli space is the quotient of $ST_g$ by this group. Because the fundamental group of a SRS is isomorphic to that of its body, changing the choice of generators of $\pi_1(M)$ is equivalent to changing the choice of generators of $\pi_1(M_0)$. Hence the super modular group is isomorphic to the ordinary modular group for genus $g$. (This does *not* mean that it acts trivially on the odd coordinates of $ST_g$. The groups are isomorphic, but they act on different spaces.) No "super extension" of the modular group appears for SRS's with the DeWitt topology, although it certainly would for general Rogers SRS's. The body of super moduli space is again a $2^{2g}$-sheeted covering of the moduli space of the body, but is branched at points where the Riemann surface has automorphisms taking one spin structure into another.

We have not yet discussed the super Teichmüller space in the genus 1 case. We will now work it out explicitly as a concrete illustration of the above ideas. The existence of a conformal Killing spinor for the trivial spin structure in this case means that the corresponding sheet of $ST_1$ will have a different dimension than the other sheets.

A super torus is obtained as the quotient of $SC$ by a subgroup of $SPL(2, C)$ having two generators of the form (2.12). This subgroup must act properly discontinuously on the body $C$, which requires $c_0 = 0$, $a_0^2 = 1$, and $b_0 \neq 0$. If the super torus is to be metrizable, then invariance of the metric (3.4) strengthens these conditions to $c = 0$, $a^2 = 1$, $\gamma = 0$, and $b_0 \neq 0$. $SPL(2, C)$ elements with $c = \gamma = 0$ take the form,

$$\tilde{z} = a^2 z + ab + a^2 \theta\delta,$$
$$\tilde{\theta} = a(\theta + \delta). \tag{3.8}$$

We represent the two generators by the ordered triples $(a, b, \delta)$ and $(a', b', \delta')$. The choice of signs for $a$ and $a'$ determines the spin structure. The composition law for the transformations (3.8) is

$$(a', b', \delta')(a, b, \delta) = \left( a'a, a'b + \frac{b'}{a} + a'\delta\delta', \delta + \frac{\delta'}{a} \right). \tag{3.9}$$

Next we need to determine how much the generators can be simplified by conjugation, and when such generators commute ($\pi_1$ of the torus is Abelian). We find

$$(A,B,\Delta)^{-1}(a,b,\delta)(A,B,\Delta)=[a,bA^{-2}+(a+1)\Delta\delta A^{-1},(1-a)\Delta+\delta A^{-1}], \quad (3.10)$$

and

$$\begin{aligned}&(a',b',\delta')^{-1}(a,b,\delta)^{-1}(a',b',\delta')(a,b,\delta)\\&\quad=[1,(aa'+a+a'-1)\delta\delta',(1-a')\delta-(1-a)\delta'],\end{aligned} \quad (3.11)$$

where we have used $a^2=a'^2=1$. Note that the commutator (3.11) is a pure soul transformation. This means that if we dropped the requirement that the generators commute the quotient space would be a Rogers SRS.

Consider first the case of a nontrivial spin structure, so that $a$ and $a'$ are not both 1; for definiteness assume $a=-1$. Then by a conjugation (3.10) we can set $b$ to unity and $\delta$ to zero, whereupon commutativity requires $\delta'=0$ as well. Hence the generators can be chosen to be $(-1,1,0)$ and $(\pm 1,b',0)$, with one complex even parameter and no odd ones. The three sheets of $ST_1$ describing nontrivial spin structures are supermanifolds of complex dimension $(1,0)$.

For the trivial spin structure $a=a'=1$, the conjugation (3.10) can set $b$ to unity but cannot set $\delta$ to zero. This reflects the existence of the conformal Killing spinor. Nevertheless, $\delta$ *can* be set to zero by conjugation with the $SPL(2,C)$ element

$$\tilde{z}=z+\theta\delta z, \quad \tilde{\theta}=\theta+\delta z, \quad (3.12)$$

without changing the form of the other generator. Commutativity imposes no restriction on $\delta'$ in this case, so the generators can be chosen to be $(1,1,0)$ and $(1,b',\delta')$, with one even and one odd complex parameter. However, this sheet of $ST_1$ is not quite a supermanifold of complex dimension $(1,1)$. Conjugation by the inversion $I$ shows that changing the sign of $\delta'$ does not change the super torus. The trivial sheet of $ST_1$ is obtained by identifying $(b',\delta')$ with $(b',-\delta')$ in the parameter space. Since the points with $\delta'=0$ are fixed by this $Z_2$ symmetry, the result is a super-orbifold.

It is clear from the super torus example that small changes in the group parameters of a DeWitt SRS can produce a Rogers SRS. The super Teichmüller space sits inside a larger space of Rogers SRS's and its boundary contains Rogers SRS's with nontrivial topology in soul directions. In general one can approach the boundary by sending the group parameters toward values for which the action on the body is not properly discontinuous. During this process one can always adjust the souls of the parameters in such a way that the action on the entire supermanifold remains properly discontinuous. The boundary point then represents a Rogers supermanifold. In effect, the souls of the group parameters can be used to regulate the pinch singularity on the body, replacing it with a compactification in the soul dimensions. This type of nonsingular representation of pinched surfaces may be convenient in some applications.

## 4. Deformation of SRS Structure

In this section we will complete the proof of the uniformization theorem by showing that $SC^*$, $SC$, and $SU$ are the unique SRS's having simply connected Riemann surfaces as bodies. Our method is essentially to treat an arbitrary SRS over $C^*$, $C$, or $U$ as a deformation of the canonical one. We then use the standard sheaf cohomology methods of deformation theory to show that the deformation is trivial. This means that the transition functions of the SRS can be brought to the canonical soulless form by redefining the coordinates in the charts. An excellent if lengthy account of deformation theory for complex manifolds can be found in [23].

Let $M$ be a SRS with simply connected body $M_0$. If $\{U^\alpha\}$ is an atlas of charts for $M_0$, then the cylinders over the $U^\alpha$ are charts on $M$. The structure of $M$ is completely characterized by the functions $f^{\alpha\beta}(z_0^\beta)$ and $\psi^{\alpha\beta}(z_0^\beta)$ which specify the superconformal transformation from $Z^\beta$ to $Z^\alpha$ in the region above $U^\alpha \cap U^\beta$. Of course, many different sets of functions describe the same SRS, because a superconformal redefinition of coordinates in any chart will change the transition functions but not the SRS structure. Hence a SRS corresponds to a collection of transition functions modulo coordinate redefinitions.

The consistency condition satisfied by the transition functions on triple overlaps $U^\alpha \cap U^\beta \cap U^\gamma$ is of crucial importance. Changing coordinates from $Z^\gamma$ to $Z^\beta$ and then to $Z^\alpha$ must give the same result as changing directly from $Z^\gamma$ to $Z^\alpha$. This imposes the "cocycle conditions,"

$$f^{\alpha\gamma} = f^{\alpha\beta}(f^{\beta\gamma}) + \psi^{\beta\gamma}\psi^{\alpha\beta}(f^{\beta\gamma})\sqrt{f^{\alpha\beta\prime}(f^{\beta\gamma})}, \tag{4.1a}$$

$$\psi^{\alpha\gamma} = \psi^{\alpha\beta}(f^{\beta\gamma}) + \psi^{\beta\gamma}\sqrt{f^{\alpha\beta\prime}(f^{\beta\gamma}) + \psi^{\alpha\beta}(f^{\beta\gamma})\psi^{\alpha\beta\prime}(f^{\beta\gamma})}. \tag{4.1b}$$

The summation convention will not apply to the Greek indices in this and subsequent equations. We expand the transition functions in the basis of $B_L$ and consider separately their soul components $f_\Gamma^{\alpha\beta}, \psi_\Gamma^{\alpha\beta}, \Gamma \neq 0$, showing by induction on the length of the sequence $\Gamma$ that they can all be set to zero by superconformal coordinate redefinitions. This will show that $M$ is equivalent to a canonical SRS. (We need not consider the body component $f_0^{\alpha\beta}$, since it is well known that $M_0$ admits only one complex structure.)

Consider first Eq. (4.1b) for the leading terms $\psi_i^{\alpha\beta}$. It simplifies to

$$\psi_i^{\alpha\gamma}(z_0^\gamma) = \psi_i^{\alpha\beta}(z_0^\beta) + \psi_i^{\beta\gamma}(z_0^\gamma)\sqrt{f_0^{\alpha\beta\prime}(z_0^\beta)}, \tag{4.2}$$

where we have used the fact that the $f_0^{\alpha\beta}$ are the transition functions for the body. To see the significance of this equation, multiply both sides by $e^\alpha$, a local section over $U^\alpha$ for the line bundle of spinors whose square is the tangent bundle to $M_0$. The transformation law of $e^\alpha$ absorbs the square root in (4.2), and we obtain

$$e^\alpha\psi_i^{\alpha\gamma}(z_0^\gamma) = e^\alpha\psi_i^{\alpha\beta}(z_0^\beta) + e^\beta\psi_i^{\beta\gamma}(z_0^\gamma). \tag{4.3}$$

This shows that the $e^\alpha\psi_i^{\alpha\beta}$ define a cocycle in the first Cech cohomology group of $M_0$ with coefficients in the sheaf of sections of the spin bundle. This cohomology is known to be trivial when $M_0$ is simply connected. For the sphere this follows from the Riemann–Roch theorem, while for $C$ and $U$ it follows from the triviality of holomorphic line bundles including the spin bundle [19]. Since the cohomology is trivial, the cocycle is exact,

$$e^\alpha\psi_i^{\alpha\beta} = e^\beta\eta_i^\beta - e^\alpha\eta_i^\alpha, \tag{4.4}$$

or

$$\psi_i^{\alpha\beta} = \sqrt{f_0^{\alpha\beta\prime}}\, \eta_i^\beta - \eta_i^\alpha. \tag{4.5}$$

This relation implies that the leading terms $\psi_i^{\alpha\beta}$ can be set to zero by the superconformal redefinition of coordinates

$$\begin{aligned} \tilde{z}^\alpha &= z^\alpha + \theta^\alpha \eta_i^\alpha v_i, \\ \tilde{\theta}^\alpha &= \theta^\alpha \sqrt{1 + \eta_i^\alpha \eta_j^{\alpha\prime} v_{ij}} + \eta_i^\alpha v_i. \end{aligned} \tag{4.6}$$

Of course, this redefinition may also alter the higher-order terms in the transition functions, but these are dealt with at a later stage of our inductive argument.

The same type of reasoning applies to the leading soul terms of the even functions $f^{\alpha\beta}$. From Eq. (4.1a) we have, using the fact that $\psi_i^{\alpha\beta}$ has already been set to zero,

$$f_{ij}^{\alpha\gamma}(z_0^\gamma) = f_{ij}^{\alpha\beta}(z_0^\beta) + f_{ij}^{\beta\gamma}(z_0^\gamma) f_0^{\alpha\beta\prime}(z_0^\beta). \tag{4.7}$$

This shows that $f_{ij}^{\alpha\beta}\partial/\partial z_0^\alpha$ defines a cocycle in the first cohomology group of $M_0$ with coefficients in the sheaf of holomorphic vector fields. Again the triviality of this cohomology group means that the $f_{ij}^{\alpha\beta}$ can all be set to zero by superconformal coordinate redefinitions. The induction proceeds in this manner until the souls of the transition functions have all been set to zero. Equations (4.2) and (4.7) continue to hold at higher orders, so that no new cohomology groups appear. This completes the proof of uniqueness of the canonical SRS's $SC^*$, $SC$, and $SU$.

It is possible to generalize this analysis to SRS's whose bodies are nonsimply connected. After all, the dimensions of the cohomology groups which appeared are $6g$–$6$ and $4g$–$4$ for genus $g$, which indicates that an alternative derivation of the dimension of super Teichmüller space should be possible along these lines. It was by carrying out such a derivation that Hodgkin determined the dimension and structure of $ST_g$ without making the metrizability assumption [22].

## 5. Super Beltrami Differentials

The representation of Riemann surfaces in terms of Beltrami differentials is an extremely powerful technique which permits straightforward proofs of deep results which are difficult to obtain by other methods [24]. It is therefore important to develop the analogous representation for SRS's. Just as Beltrami differentials characterize the various possible Riemann surface structures on a given smooth 2-manifold, super Beltrami differentials will describe different metrizable SRS structures on a given smooth ($G^\infty$) supermanifold. The $G^\infty$ structure of a SRS is obtained by relaxing the requirements that the transition functions be complex analytic and superconformal. They may instead be arbitrary functions of $z$, $\bar{z}$, $\theta$, and $\bar{\theta}$ which are polynomials in $\theta$ and $\bar{\theta}$ with coefficients admitting Taylor expansions in the souls of $z$ and $\bar{z}$. In this section we will derive the super Beltrami equations and discuss their solution. We will then be able to exhibit the complex structure of $ST_g$ via a Bers embedding theorem, show that metrizable SRS's can be represented by Schottky supergroups, and define the notion of universal super Teichmüller space.

Let $(z, \theta)$ and $(w, \phi)$ be two sets of coordinates on $SU$, related by a $G^\infty$ but not necessarily superconformal transformation

$$w = w(z, \bar{z}, \theta), \quad \phi = \phi(z, \bar{z}, \theta). \tag{5.1}$$

Then the invariant metric which appears in one set of coordinates as

$$(\operatorname{Im} w + \tfrac{1}{2}\phi\bar{\phi})^{-1}|dw + \phi d\phi| \tag{5.2}$$

will appear in the other set as

$$\lambda(z, \bar{z}, \theta, \bar{\theta})|dz + \mu(z, \bar{z}, \theta)d\bar{z} + \nu(z, \bar{z}, \theta)d\theta|. \tag{5.3}$$

Conversely, if the two Beltrami coefficients $\mu, \nu$ are given then the map which reduces (5.3) to the standard form (5.2) satisfies the super Beltrami equations,

$$w_{\bar{z}} + \phi\phi_{\bar{z}} = \mu(w_z + \phi\phi_z), \quad -w_\theta + \phi\phi_\theta = \nu(w_z + \phi\phi_z), \tag{5.4}$$

where the subscripts indicate differentiation.

We write the super Beltrami equations in terms of components by expanding

$$\begin{aligned} w &= w^0 + \theta w^1, \quad \phi = \phi^1 + \theta\phi^0, \\ \mu &= \mu^0 + \theta\mu^1, \quad \nu = \nu^1 + \theta\nu^0. \end{aligned} \tag{5.5}$$

We then obtain four component equations

$$w^0_{\bar{z}} + \phi^1\phi^1_{\bar{z}} = \mu^0(w^0_z + \phi^1\phi^1_z), \tag{5.6a}$$

$$w^1_{\bar{z}} + \phi^0\phi^1_{\bar{z}} - \phi^1\phi^0_{\bar{z}} = \mu^0(w^1_z + \phi^0\phi^1_z - \phi^1\phi^0_z) + \mu^1(w^0_z + \phi^1\phi^1_z), \tag{5.6b}$$

$$-w^1 + \phi^1\phi^0 = \nu^1(w^0_z + \phi^1\phi^1_z), \tag{5.6c}$$

$$(\phi^0)^2 = \nu^1(-w^1_z + \phi^1\phi^0_z - \phi^0\phi^1_z) + \nu^0(w^0_z + \phi^1\phi^1_z). \tag{5.6d}$$

Since all the functions entering these equations are $G^\infty$, one need only solve for their dependence on $z_0, \bar{z}_0$ to determine them completely. Therefore one can read $z_0$ for $z$ everywhere in the equations. The unknowns are two even and two odd $B_{L-1}$-valued functions of $z_0, \bar{z}_0$.

We will discuss the solution of the super Beltrami equations when the Beltrami coefficients are specified in all of $SC$. If they are only known in $SU$, as above, some extension into the rest of $SC$ must be prescribed. In terms of the components defined in Eqs. (5.5) the appropriate extension is

$$\begin{aligned} \mu^0(\bar{z}) &= \bar{\mu}^0(z), \quad \mu^1(\bar{z}) = -i\bar{\mu}^1(z), \\ \nu^0(\bar{z}) &= \bar{\nu}^0(z), \quad \nu^1(\bar{z}) = -i\bar{\nu}^1(z). \end{aligned} \tag{5.7}$$

The symmetry of this extension implies that the solution in $SC$ will fix the superboundary $SR$ and will map $SU$ to itself.

One easily checks that all solutions of the trivial super Beltrami equations with $\mu = 0$, $\nu = \theta$ are superconformal maps. It follows that the solution of the general equations is unique up to composition with a superconformal automorphism of $SC$. Here again we encounter the problem that $SPL(2, C)$ is not the full group of superconformal automorphisms of $SC$. We will deal with this by imposing strong enough boundary conditions on the solutions at infinity to guarantee that solutions actually extend to $SC^*$. Such solutions will be unique up to $SPL(2, C)$ transformations.

How does one actually solve the super Beltrami equations given the functions $\mu(z, \bar{z}, \theta)$ and $v(z, \bar{z}, \theta)$? The general technique for solving algebraic or differential equations involving Grassmann-valued functions is to solve the body of the equation first and do perturbation theory in the souls of all functions. Since the souls are nilpotent, the resulting perturbation series always terminate and give the exact solution. The generators $v_i$ of the Grassmann algebra play the role of the "small" parameters of the perturbation. To begin, the body of Eq. (5.6a) is

$$(w_0^0)_{\bar{z}} = \mu_0^0 (w_0^0)_z. \tag{5.8}$$

This is an ordinary Beltrami equation which can be solved for $w_0^0$ provided that $|\mu_0^0(z)| < 1$ and given boundary conditions such as $w_0^0(0)$, $w_0^0(1)$, $w_0^0(\infty)$. Then the body of Eq. (5.6d),

$$(\phi_0^0)^2 = v_0^0 (w_0^0)_z, \tag{5.9}$$

is a purely algebraic equation determining $\phi_0^0$. Provided that $v_0^0(z)$ does not vanish there are two solutions differing in sign. The equations have now been solved to zeroth order in the $v_i$. To proceed to first order, solve the algebraic Eq. (5.6c) for $w^1$ and substitute into Eq. (5.6b), obtaining to first order

$$2\phi_0^0 \phi_{i\bar{z}}^1 = (v_i^1 w_{0z}^0)_{\bar{z}} + \mu_0^0 [2\phi_0^0 \phi_{iz}^1 - (v_i^1 w_{0z}^0)_z] + \mu_i^1 w_{0z}^0. \tag{5.10}$$

This is an inhomogeneous Beltrami equation which has a unique solution for $\phi_i^1$ given the boundary conditions $\phi_i^1(0) = \phi_i^1(\infty) = 0$ [25]. Then $w_i^1$ can be computed algebraically from Eq. (5.6c). This procedure can be continued until the map $W(Z)$ is completely determined. At each order it requires the solution of one inhomogeneous Beltrami equation. There are precisely two solutions given the values of $w_0(0)$, $w_0(1)$, and $w_0(\infty)$, and the boundary conditions that the souls of $w^0$ and $\phi^1$ vanish at $z = z_0 = 0$ and $\infty$. If $W = (w, \phi)$ is one solution, the other is $IW = (w, -\phi)$.

If we consider a family of Beltrami coefficients $\mu$, $v$ which are $G^\infty$ functions of some parameters, then the solutions of the super Beltrami equations also have $G^\infty$ dependence on the parameters. This can be seen by treating the parameters as additional coordinates analogous to $z$ and $\theta$ when solving the equations. By expanding $w$, $\phi$, $\mu$, and $v$ in powers of the odd parameters as well as $\theta$ in Eqs. (5.5), we obtain a set of equations similar to (5.6), whose solutions are determined when known for soulless values of $z$ and the even parameters. These solutions have $G^\infty$ dependence on $z$, $\theta$, and all parameters.

For the applications below we will need to solve the super Beltrami equations in cases where the Beltrami coefficients are discontinuous. Since $w^0$ and $\phi^1$ are obtained by solving (inhomogeneous) Beltrami equations, they will be continuous even when the Beltrami coefficients are not [25]. However, $w^1$ and $\phi^0$ are computed from algebraic formulas and need not be continuous.

We will now use this machinery to exhibit the complex structure on super Teichmüller space. We follow a standard argument due to Bers which shows that ordinary Teichmüller space embeds as a bounded domain in a Banach space of quadratic differentials [24, 26]. This space has a natural complex structure. Our proof will show that the double covering supermanifold of the sheet of $ST_g(g > 1)$ corresponding to a particular spin structure embeds as a domain in a space of superconformal fields of weight 3/2, which is naturally a complex supermanifold.

Let $M$ and $M'$ be two metrizable SRS's with bodies of genus $g > 1$ and the same spin structure. We represent them by uniformizations $SU/G$, $SU/G'$ with a specific choice of generators for $G$ and $G'$. We regard $M$ as a fixed origin and $M'$ as a variable point in $ST_g$. For simplicity we will assume that $M$ is canonical, so that $G = G_0$. Although $M$ and $M'$ may be different as SRS's, they are equivalent as $G^\infty$ supermanifolds. This follows from Batchelor's theorem [27–29], according to which a $G^\infty$ supermanifold is completely characterized by its body and a vector bundle over the body. It can also be proven by an extension of the deformation theory of Sect. 4: since the transition functions $f^{\alpha\beta}(z_0^\beta)$ and $\psi^{\alpha\beta}(z_0^\beta)$ need no longer be holomorphic but merely smooth, the relevant cohomology groups are trivial for any genus [23]. Therefore any SRS can be reduced to canonical form by $G^\infty$ changes of coordinates. In particular there is a $G^\infty$ diffeomorphism $W: M \to M'$ which lifts to the covering space $SU$ as a $G^\infty$ map

$$W:(z,\theta) \to [w(z,\bar{z},\theta), \phi(z,\bar{z},\theta)], \tag{5.11}$$

which for the present we assume independent of $\bar{\theta}$. This map can be chosen to have $G^\infty$ dependence on the parameters of the group representing $M'$.

If we choose coordinates on $SU$ so that the metric of $M$ lifts to the standard form (3.3), then the pullback to $M$ via $W$ of the metric of $M'$ lifts to a Beltrami differential (5.3) which is $G$-invariant, and the map $W$ satisfies the super Beltrami equations. We fix $W$ up to the sign of $\phi$ by $w_0(0) = 0$, $w_0(1) = 1$, $w_0(\infty) = \infty$, and requiring the souls of $w^0$ and $\phi^1$ to vanish at $z = z_0 = 0, \infty$, which we will call standard boundary conditions. If $q$ is an element of $G$, the $G$-invariance of the Beltrami differential means that $Wq$ also solves the super Beltrami equations. There is an $SPL(2,R)$ element $p$ such that $p^{-1}Wq$ satisfies standard boundary conditions, so by uniqueness $p^{-1}Wq = W$ or $IW$. Then $WqW^{-1} = p$ or $pI$, so $WGW^{-1}$ is a Fuchsian supergroup which represents $M'$. Redefining $G'$ by conjugation if necessary, we get $G' = WGW^{-1}$.

Now we extend the Beltrami coefficients from $SU$ to all of $SC$ in a different way by defining $\mu = 0$, $v = \theta$ in the region $SL$ above the lower half-plane $L$. Let $W^{\mu v}$ denote a solution to the super Beltrami equations with these new Beltrami coefficients which obeys standard boundary conditions. Then $W^{\mu v}$ is superconformal in $SL$ and superconformally related to $W$ in $SU; W = HW^{\mu v}|SU$ with $H$ superconformal. $W^{\mu v}GW^{\mu v-1}$ is a subgroup of $SPL(2,C)$ by the same uniqueness argument that showed that $WGW^{-1}$ was a subgroup of $SPL(2,R)$.

Since $W^{\mu v}|SL$ is superconformal, it is reasonable to compute its super Schwarzian derivative. This is defined for any superconformal map $\tilde{Z}(Z)$ by [7]

$$S(Z;\tilde{Z}) = \frac{D^4\tilde{\theta}}{D\tilde{\theta}} - 2\frac{D^3\tilde{\theta}D^2\tilde{\theta}}{(D\tilde{\theta})^2}. \tag{5.12}$$

It obeys the composition law

$$S(Z;\tilde{\tilde{Z}}) = S(Z;\tilde{Z}) + (D\tilde{\theta})^3 S(\tilde{Z};\tilde{\tilde{Z}}), \tag{5.13}$$

and it vanishes when $\tilde{Z}(Z)$ belongs to $SPL(2, C)$. These properties imply that $S(Z; W^{\mu\nu})(dz + \theta d\theta)^{3/2}$ defines a $G$-invariant odd superdifferential of weight 3/2 on $SL$. If we expand $S(Z; W^{\mu\nu}) = S^1(z) + \theta S^0(z)$, then the components $S^0(z_0)$ and $S^1(z_0)$ are $G_0$-invariant differentials on the lower half plane of orders 2 and 3/2 respectively. Using the Riemann–Roch theorem we conclude that the space of all such superdifferentials is naturally a complex supermanifold of dimension $(3g–3, 2g–2)$.

This construction does not quite succeed in associating a superdifferential of weight 3/2 with each Beltrami differential $(\mu, \nu)$, because of the nonuniqueness of the solution of the super Beltrami equations. There are in fact four solutions satisfying standard boundary conditions, because the sign of $\phi^0$ can be chosen independently in $SU$ and $SL$. If $W^{\mu\nu}$ is one solution, the others are: $W^{\mu\nu}$ in $SU$ and $W^{\mu\nu}I$ in $SL$; $IW^{\mu\nu}$; and $IW^{\mu\nu}$ in $SU$ and $IW^{\mu\nu}I$ in $SL$. Since multiplication by $I$ on the left does not affect the super Schwarzian derivative, these four solutions produce two distinct super Schwarzian derivatives, namely $S(Z; W^{\mu\nu}) = S^1 + \theta S^0$ and $S(Z; W^{\mu\nu}I) = -S^1 + \theta S^0$. Thus each Beltrami differential is associated to a pair of 3/2-superdifferentials differing in the sign of the odd component. As we will show, this association defines an embedding of the given sheet of $ST_g$ in the space of superdifferentials mod $Z_2$. The embedding is $G^\infty$ because the Beltrami coefficients can be chosen to have $G^\infty$ dependence on the parameters of the group representing $M'$, whereupon the solution $W^{\mu\nu}$ and its super Schwarzian derivative will also have such dependence. This embedding defines the complex super-orbifold structure of $ST_g$. The image of the embedding has as its body the bounded domain in $C^{3g-3}$ which represents the ordinary Teichmüller space. It is unbounded in the soul directions, since we already know that the double cover of $ST_g$ is a supermanifold with the DeWitt topology.

First we claim that the same pair of super Schwarzian derivatives is associated to all Beltrami coefficients representing the same point of $ST_g$. If $W$ and $\hat{W}$ are two $G^\infty$ maps representing $M'$ as in (5.11), with corresponding Beltrami coefficients $(\mu, \nu)$ and $(\hat{\mu}, \hat{\nu})$, then we have

$$WGW^{-1} = G', \quad \hat{W}G\hat{W}^{-1} = qG'q^{-1} \tag{5.14}$$

for some $q$ in $SPL(2, R)$. Combining these equations yields

$$(W^{-1}q^{-1}\hat{W})G(\hat{W}^{-1}qW) = G, \tag{5.15}$$

so that conjugation by the map $W^{-1}q^{-1}\hat{W}$ takes $G$ into itself. Since the fixed points of the elements of the Fuchsian group $G = G_0$ are dense on the real axis $z = z_0 = \bar{z}_0$, $\theta = 0$, we know that the map $W^{-1}q^{-1}\hat{W}$ must fix every point on the real axis $R$. Therefore on $SR$ it takes the form,

$$W^{-1}q^{-1}\hat{W}|SR: \tilde{z} = z + \theta\zeta(z), \quad \tilde{\theta} = \theta g(z). \tag{5.16}$$

Since $W$, $\hat{W}$, and (5.16) all obey standard boundary conditions, so does $q$, so $q$ can only be the identity or $I$.

Now define a map $\omega$ on $SC$ by

$$\omega = W^{\mu\nu}W^{-1}q^{-1}\hat{W} \text{ in } SU, \quad \omega = W^{\mu\nu} \text{ in } SL. \tag{5.17}$$

Then $\omega$ obeys standard boundary conditions, is superconformal in $SL$, and is superconformally related to $\hat{W}$ in $SU$; and its $\theta=0$ components are continuous across the real axis. Therefore $\omega$ solves the super Beltrami equations defining $\hat{W}^{\hat{\mu}\hat{\nu}}$ and must be one of the four solutions of these equations with standard boundary conditions. This implies that the super Schwarzian derivatives of $W^{\mu\nu}$ and $\hat{W}^{\hat{\mu}\hat{\nu}}$ in $SL$ agree up to the sign of the odd components as claimed.

Next we must verify that the map from a sheet of $ST_g$ to super Schwarzian derivatives mod $Z_2$ is invertible. Given a function $S(z,\theta)$ one can reconstruct the superconformal map whose super Schwarzian derivative it is by solving the differential equation [14]

$$D^3 F = -SF. \tag{5.18}$$

This equation has two independent even solutions and one odd one. If $\varepsilon$ is the odd solution and $x$ an even one, then direct calculation shows that $\tilde{\theta}=\varepsilon/x$ is the odd part of a superconformal map with super Schwarzian derivative $S$. Thus, given $S(Z;W^{\mu\nu})$ in $SL$ one can reconstruct $W^{\mu\nu}|SL$ up to an $SPL(2,C)$ element which in turn is fixed up to composition with $I$ by the standard boundary conditions on $W^{\mu\nu}$. So let $W^{\mu\nu}$ and $\hat{W}^{\hat{\mu}\hat{\nu}}$ be such that $W^{\mu\nu}=q\hat{W}^{\hat{\mu}\hat{\nu}}$ in $SL$, with $q$ the identity or $I$. What can be inferred about the relation between $W$ and $\hat{W}$? Define a map $p$ on $SU$ by

$$p=(WW^{\mu\nu-1})q(\hat{W}^{\hat{\mu}\hat{\nu}}\hat{W}^{-1}). \tag{5.19}$$

Then $p$ is superconformal and obeys standard boundary conditions. It also fixes the superboundary, because it sends the curve $\hat{W}(R)$ in $SR$ to the curve $W(R)$ in $SR$, and a superconformal map is determined up to a choice of sign by its values on such a curve transverse to the soul fibers. So $p$ must be the identity or $I$. Now consider the rearranged equation

$$W^{-1}p\hat{W}=W^{\mu\nu-1}q\hat{W}^{\hat{\mu}\hat{\nu}} \tag{5.20}$$

on the real axis. The right side is the identity because $W^{\mu\nu}$ agrees with $q\hat{W}^{\hat{\mu}\hat{\nu}}$ there. Therefore $W=p\hat{W}$ on $R$. But the knowledge of these maps on the real axis is sufficient to determine the conjugations $WGW^{-1}$ and $\hat{W}G\hat{W}^{-1}$, since all the fixed points of elements of $G$ lie on $R$. Therefore these conjugations do produce equivalent groups $G'$ representing the same point of $ST_g$. Starting from a superdifferential of weight 3/2 we have succeeded in reconstructing uniquely a point of $ST_g$. This completes the proof.

There is one loophole in this argument: we have not been able to justify our assumption that $M'$ can be related to $M$ by a $G^\infty$ map (5.11) with no $\bar{\theta}$ dependence, although we believe this is true. Fortunately the argument can be repeated allowing for the possibility of such dependence. The invariant metric (5.3) is generalized to

$$\lambda(z,\bar{z},\theta,\bar{\theta})|dz+\mu(z,\bar{z},\theta,\bar{\theta})d\bar{z}+\nu(z,\bar{z},\theta,\bar{\theta})d\theta+\sigma(z,\bar{z},\theta,\bar{\theta})d\bar{\theta}|, \tag{5.21}$$

leading to the extended super Beltrami equations

$$\begin{aligned} w_{\bar{z}}+\phi\phi_{\bar{z}}&=\mu(w_z+\phi\phi_z),\\ -w_\theta+\phi\phi_\theta&=\nu(w_z+\phi\phi_z),\\ -w_{\bar{\theta}}+\phi\phi_{\bar{\theta}}&=\sigma(w_z+\phi\phi_z). \end{aligned} \tag{5.22}$$

These equations can be solved order by order in the $v_i$ just as above, and the entire proof goes through with no change. Note however that when Eqs. (5.22) are written out in components there will be 12 component equations for only 8 unknowns, namely the components of $w$ and $\phi$. This means that not every set of functions $\mu, v, \sigma$ can be the Beltrami coefficients of a SRS; they are constrained by 4 consistency conditions which allow the solutions of Eqs. (5.22). This redundancy in the description is our main reason for believing that a map with no $\bar{\theta}$ dependence can always be found.

The Schottky uniformization theorem for Riemann surfaces can also be proved using Beltrami differentials and will therefore extend to SRS's [30]. A Schottky supergroup is a subgroup of $SPL(2, C)$ having $g$ generators whose fundamental region in $SC^*$ has as its body $C^*$ minus the interiors of $2g$ nonoverlapping circles. This fundamental region represents the SRS after its body has been cut open along $g$ of the $2g$ generating curves of $\pi_1(M_0)$. If these $g$ cycles are fixed in advance, not all spin structures can be represented, since the $g$ generators of the Schottky supergroup allow only $g$ choices of signs rather than the $2g$ choices needed. However, if the choice of cycles can vary with the spin structure, then all spin structures and all SRS's of genus $g > 1$ can be represented by Schottky supergroups.

Using the ordinary Beltrami equation, one can define a universal Teichmüller space which contains the Teichmüller space for every genus $g > 1$ [24]. The relation between this space and the universal moduli space which has been proposed as a setting for string field theory [31] is unclear. Nevertheless, the analogous definition of universal super Teichmüller space may be relevant to the ultimate formulation of superstring field theory. It is the space of all quasisuperconformal maps (maps obeying the super Beltrami equations with some choice of Beltrami coefficients) of $SU$ to itself, fixing the superboundary, with two maps considered equivalent if they agree on the real axis. This infinite-dimensional space contains each sheet of $ST_g$ for every $g > 1$. We do not know whether it is a supermanifold or what other geometric structure it may possess.

## 6. Toward a Uniformization Theorem for Rogers SRS's

We have seen that the quotient of $SC$ or $SU$ by a Kleinian supergroup can be a SRS of Rogers type representing a boundary point of super moduli space. It is of interest to try to characterize the Rogers SRS's which can arise in this way. This should lead to an improved picture of the super moduli space and perhaps a more general uniformization theorem for SRS's. We will present a uniformization conjecture in this section and sketch some plausibility arguments for it. In this section the term SRS will be used in the general Rogers sense.

There is no complete structure theory for SRS's, but elsewhere we have obtained several results in this direction [9, 10]. The major complication is that a SRS need not be a bundle over any associated Riemann surface; indeed a general SRS need not have a body at all. Instead of a bundle structure, a general SRS has an extensive nested set of foliations. The surfaces of constant $z_0$ in the charts fit together smoothly to give the leaves of a global foliation, called the soul foliation. Additional foliations are obtained by fixing additional coordinates, for example $z_0$

and $\theta_i$, but they will not be important here. The space of leaves of the soul foliation is a topological space, but not generally a smooth manifold or even Hausdorff. We call this space the body of the SRS only when it is a smooth manifold. We have shown that the universal covering space of any leaf of the soul foliation immerses in the vector space $C^k, k = 2^L - 1$.

An important notion is that of completeness of the leaves. Intuitively a complete leaf contains no holes. The most suitable rigorous definition seems to be that a leaf is complete iff any smooth path of finite coordinate length in the leaf can be smoothly extended. Because the transition functions of a SRS are polynomial along the leaves, "finite coordinate length" is defined independent of the choice of charts. The universal cover of a complete leaf must be all of $C^k$. If a SRS is obtained as the quotient of $SC$, $SC^*$, or $SU$ by a discrete supergroup, all its leaves will be complete. The strongest uniformization conjecture is simply the converse of this fact.

*Conjecture.* Any SRS $M$ whose leaves are all complete is covered by $SC$, $SC^*$, or $SU$.

**Corollary.** *The Teichmüller theory of Sects.* 3–5 *classifies all metrizable* SRS's *with complete leaves.*

We can only offer suggestions and plausibility arguments toward the proof of this conjecture. We immediately pass to the universal cover $\hat{M}$ of $M$, which is also a SRS with complete leaves. First one must prove that $\hat{M}$ has a body. We have no argument for this, but we know no example of a simply connected supermanifold with complete leaves which fails to have a body. Next it must be shown that the leaves of $\hat{M}$ are all diffeomorphic. Since each leaf is covered by $C^k$, its topology is completely characterized by its fundamental group $\pi_1$. One way in which the topology of the leaves can change is for a particular leaf to be diffeomorphic to a neighboring leaf minus one or more points. This possibility is ruled out by completeness. Another possibility involves limit cycles: a particular leaf may contain a circle representing a nontrivial element of $\pi_1$, with the corresponding curve on a neighboring leaf being trivial in $\pi_1$ and represented by a spiral asymptotic to the circle [32]. But this contradicts the existence of a body, since the space of leaves is not Hausdorff if one leaf is asymptotic to another. It seems plausible that all possibilities for topology change can be similarly ruled out using completeness, existence of a body, and complex analyticity. If a body $B$ exists and every leaf is diffeomorphic to a manifold $F$, then $\hat{M}$ should be a fiber bundle with fiber $F$ over $B$. The homotopy exact sequence of this fibration reads [33]

$$\cdots \to \pi_2(F) \to \pi_2(\hat{M}) \to \pi_2(B) \to \pi_1(F) \to \pi_1(\hat{M}) \to \pi_1(B) \to 0. \tag{6.1}$$

Since $\pi_1(\hat{M}) = 0$, $\pi_1(B) = 0$ also and $B$ is a simply connected Riemann surface. If $B$ is $C$ or $U$, then $\pi_2(B) = 0$ so $\pi_1(F) = 0$. Then $\hat{M}$ is a SRS whose body is $B$ and whose fiber is $C^k$, so $\hat{M}$ must be $SC$ or $SU$.

The case $B = C^*$ is more complicated. Since $\pi_2(C^*) = Z$, (6.1) implies that $\pi_1(F) = 0$, $Z$ or $Z_p$. $Z_p$ is ruled out because it has no fixed-point-free action on $C^k$, so the only exceptional case has $F$ homotopic to a circle and the fibration homotopic to the Hopf fibration. We have not been able to exclude this possibility or to construct an example of such a SRS. We think it likely that a proof of our conjecture can be constructed along these lines, possibly with the Hopf fibration as an exceptional case.

Finally we would like to discuss the physical relevance of SRS's of Rogers type. If our conjecture is true, then metrizable Rogers SRS's can be represented by Fuchsian supergroups and the super Teichmüller space can be enlarged to include them. Should such SRS's be included in the Polyakov path integral for superstrings? In the absence of a proof of our conjecture, or some alternative characterization, we cannot rule out this possibility, but we can argue against it. Typically a Rogers SRS differs from a DeWitt SRS in that some leaves of some foliations are either compact or dense. We have shown elsewhere that any $G^\infty$ function on a supermanifold must be constant along compact leaves [9, 10]. In particular, the $G^\infty$ map $X^\mu(z,\theta)$ of the superstring world sheet into the ten-dimensional bosonic superspace with coordinates $X^\mu$ must send such leaves to points. Similarly, at least the body of a $G^\infty$ function is constant on any dense leaf, hence it is constant on the entire SRS by continuity. In either case the map $X^\mu$ cannot be an immersion, so there is no sensible theory of a Rogers superstring moving in ten dimensions. This is the physical explanation for the choice of the DeWitt topology: to ensure that the SRS represents a superstring moving in spacetime.

## 7. Conclusions

In this paper we have provided a rigorous foundation for the mathematical theory of super Riemann surfaces and for their physical applications. All the standard results of Teichmüller theory were generalized to super Riemann surfaces: uniformization, representation by Fuchsian and Schottky groups and by Beltrami differentials, and the Bers embedding of $ST_g$ in a space of invariant superdifferentials which defines its complex structure. The super Beltrami equations in particular should provide a powerful tool for the deeper study of the geometry of super moduli space. A proof of our conjecture that the Beltrami coefficients can always be chosen independent of $\bar{\theta}$ would simplify this study. It should not be difficult to generalize such tools as Poincaré series, theta functions, and the Selberg trace formula to super Riemann surfaces. A proof of our uniformization conjecture for Rogers SRS's is also an interesting mathematical problem, but probably has little physical importance.

Throughout our work we made use of Rogers' theory of supermanifolds to make rigorous statements about topology and analysis in superspace. Many of our results have also been obtained in the supermanifold formalism of Berezin-Leites-Kostant (BLK) [34–36]. Although the two formalisms are mathematically equivalent for SRS's with the DeWitt topology, the translation between them is not straightforward, and Rogers' theory is closer to the physicist's notion of superspace. In particular, in the BLK theory there is no concept of a SRS with specific nonzero values of the odd supermoduli parameters. There is instead the notion of a family of SRS's with the odd supermoduli being coordinates on the parameter space of the family. Therefore some statements which apply to individual SRS's in Rogers' theory must be translated into statements about families of BLK SRS's.

Ultimately one wants to use super Riemann surfaces to discover the most elegant and transparent geometric formulation of superstrings. Of course, superstring theory can always be expressed in terms of ordinary Riemann surfaces, just as supergravity can be expressed in terms of component fields. But we have learned that the superspace formulation of supergravity simplifies calculations, exposes the geometric content of the theory, and reveals the origin of "miraculous" divergence cancellations. The same should be true of the superstring. The proof of finiteness of superstring amplitudes should be simplified by working directly with super Riemann surfaces. One obstacle to such a proof is that finiteness depends on cancellations between the contributions of different spin structures, which correspond to different sheets of super moduli space. It is clearly desirable to relate the geometries of the various sheets. This may involve a deeper understanding of the action of the modular group on the sheets and particularly in the neighborhood of the branch points.

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