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Witten's Gauge Field Equations and an Infinite-Dimensional Grassmann Manifold

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Abstract. Witten's gauge fields are interpreted as motions on an infinite-dimensional Grassmann manifold. Unlike the case of self-dual Yang-Mills equations in Takasaki's work, the initial data must satisfy a system of differential equations since Witten's equations comprise a pair of spectral parameters. Solutions corresponding to (anti-) self-dual Yang-Mills fields are characterized in the space of initial data and in application, some Yang-Mills fields which are not self-dual, anti-self-dual nor abelian can be constructed.

0. Introduction

Consider a gauge field ∇ in the eight-dimensional complex space $\mathbb{C}^8$ satisfying

$$[\nabla_{y_\mu}, \nabla_{y_\nu}] = (1/2) \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \varepsilon_{\mu\nu\alpha\beta} [\nabla_{y_\alpha}, \nabla_{y_\beta}] \ ,$$

$$[\nabla_{z_\mu}, \nabla_{z_\nu}] = (-1/2) \sum_{\alpha=0}^{3} \sum_{\beta=0}^{3} \varepsilon_{\mu\nu\alpha\beta} [\nabla_{z_\alpha}, \nabla_{z_\beta}] \ ,$$

$$[\nabla_{y_\mu}, \nabla_{z_\nu}] = 0 \ , \quad (\mu, \nu = 0, 1, 2, 3) \ , \tag{0.1}$$

where $(y, z) = (y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3)$ are coordinates of $\mathbb{C}^8$, ∇_{y_μ} and ∇_{z_μ} are covariant derivatives, and $\varepsilon_{\mu\nu\alpha\beta}$ denotes the totally antisymmetric tensor such that $\varepsilon_{0123} = 1$.

Set $x = (y+z)/2$, $w = (y-z)/2$. Witten [9] pointed out that Eq. (0.1) imply the full Yang-Mills equations

$$\sum_{\mu=0}^{3} [\nabla_{x_\mu}, [\nabla_{x_\mu}, \nabla_{x_\nu}]] = 0 \qquad (\nu = 0, 1, 2, 3) \tag{0.2}$$

on the diagonal subspace $\Delta = \{(y, z) \in \mathbb{C}^8 | w = 0\}$, and further, that a gauge field on Δ satisfies (0.2) if and only if it can be extended to a neighborhood of Δ consistently to (0.1) mod $(w_0, w_1, w_2, w_3)^2$. Here $(w_0, w_1, w_2, w_3)^2$ denotes the square of the ideal generated by w_0, w_1, w_2, and w_3.

In this paper, we rewrite (0.1) in the language of Sato's soliton theory [4, 5] and investigate the structure of the solution space of (0.1) on the analogy of Takasaki's work on self-dual Yang-Mills fields [7, 8]: we solve an initial-value problem of differential equations with respect to functions with values in an infinite-dimensional Grassmann manifold (see Theorem 2).

In our case, there appear a pair of spectral parameters λ_1, λ_2. The main difference from the case of one spectral parameter is that the initial data must satisfy a system of differential equations if the problem is solvable (see Proposition 5 and cf. Takasaki [7, 8]).

Through the restriction to the diagonal space Δ, the totality of gauge fields satisfying (0.1) can be regarded as a class of Yang-Mills fields including all the self-dual or anti-self-dual fields. From our point of view, it is interesting to characterize self-dual or anti-self-dual fields in terms of initial data. In fact, a simple characterization is obtained (see Sect. 3) and in application, we shall construct an example of Yang-Mills fields which are neither self-dual nor anti-self-dual (see Sect. 4).

The announcement of our results [6] was already published in 1984. Ueno treated the same problem independently and gave it a cohomological formulation (unpublished).

Notations. We shall use the following standard notations: $\mathbb{N}$ denotes the set of non-negative integers. $\mathbb{Z}$ denotes the set of integers. $\mathbb{C}$ denotes the complex number field. $M_n(\mathbb{C})$ denotes the total matrix algebra. $\mathbb{1}$ denotes the unit matrix of size $n \times n$. Let R be a ring. Then we denote by $R[x]$ the ring of polynomials of x with coefficients in R, and denote by $R[[x]]$ the ring of formal power series of x with coefficients in R.

1. Linearization

Set $x_{11} = y_0 + \sqrt{-1}\,y_1$, $t_{11} = y_2 + \sqrt{-1}\,y_3$, $x_{12} = y_2 - \sqrt{-1}\,y_3$, $t_{12} = -y_0 + \sqrt{-1}\,y_1$, $x_{21} = z_0 + \sqrt{-1}\,z_1$, $t_{21} = z_2 - \sqrt{-1}\,z_3$, $x_{22} = z_2 + \sqrt{-1}\,z_3$, and $t_{22} = -z_0 + \sqrt{-1}\,z_1$. Then, introducing parameters λ_1, λ_2, we can rewrite (0.1) as follows:

$$[-\lambda_a \nabla_{x_{ab}} + \nabla_{t_{ab}}, -\lambda_c \nabla_{x_{cd}} + \nabla_{t_{cd}}] = 0 \quad (a, b, c, d = 1, 2) \ . \tag{1.1}$$

Throughout this paper we discuss in the category of formal power series. Hence the gauge potentials $A_{t_{ab}}$, $A_{x_{ab}}$ belong to the ring of formal power series with matrix coefficients $M_n(\mathbb{C})[[t, x]]$, where $\nabla_{t_{ab}} = \partial_{t_{ab}} + A_{t_{ab}}$, $\nabla_{x_{ab}} = \partial_{x_{ab}} + A_{x_{ab}}$, $t = (t_{11}, t_{12}, t_{21}, t_{22})$, and $x = (x_{11}, x_{12}, x_{21}, x_{22})$.

Now we "fix" the gauge, namely, restrict the freedom of gauge so that $A_{x_{ab}} = 0$ for $a, b = 1, 2$. (The gauge-fixing is analogous to that of Chau et al. [1] and Pohlmeyer [3] for self-dual Yang-Mills equations.) Then (1.1) reads

$$[-\lambda_a \partial_{x_{ab}} + \nabla_{t_{ab}}, -\lambda_c \partial_{x_{cd}} + \nabla_{t_{cd}}] = 0 \quad (a, b, c, d = 1, 2) \ . \tag{1.2}$$

More precisely, we have

Proposition 1. *For any ∇ satisfying* (0.1), *there exists a gauge transformation $\nabla \to \tilde{\nabla} = g^{-1} \nabla g, g \in M_n(\mathbb{C})[[t, x]]$, such that $\tilde{\nabla}_{x_{ab}} = g^{-1} \nabla_{x_{ab}} g = \partial_{x_{ab}}$ for $a, b = 1, 2$.*

Proof. Equations (1.1) imply that $[\nabla_{x_{ab}}, \nabla_{x_{cd}}]=0$ for $a,b,c,d=1,2$, which are the integrability conditions for the linear equations

$$\left(\frac{\partial}{\partial x_{ab}}+A_{x_{ab}}\right)g=0 \qquad (a,b=1,2)\ . \tag{1.3}$$

Thus for any $A_{x_{ab}}\in M_n(\mathbb{C})\,[[t,x]]$ $(a,b=1,2)$ satisfying (1.1) there exists a solution $g=\sum_{i,j,k,l\geqq 0} g_{ijkl}x_{11}^i x_{12}^j x_{21}^k x_{22}^l$ of (1.3) such that $g_{ijkl}\in M_n(\mathbb{C})\,[[t,x]]$ and $g_{0000}=\mathbb{1}$. This g is invertible in $M_n(\mathbb{C})\,[[t,x]]$ and satisfies $g^{-1}\nabla_{x_{ab}}g=g^{-1}(\partial_{x_{ab}}+A_{x_{ab}})g$ $=g^{-1}g\,\partial_{x_{ab}}+g^{-1}\left(\frac{\partial g}{\partial x_{ab}}+A_{x_{ab}}g\right)=\partial_{x_{ab}}$. q.e.d.

We shall investigate the structure of solutions to Eq. (1.2). First we note that the system of Eq. (1.2) is nothing but the integrability condition for the linear equations,

$$(-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}}+A_{t_{ab}})w(\lambda)=0 \qquad (a,b=1,2)\ . \tag{1.4}$$

Proposition 2. $A_{t_{ab}}\in M_n(\mathbb{C})\,[[t,x]]$ $(a,b=1,2)$ *are solutions of* (1.2) *if and only if there exists a solution* $w(\lambda)=\sum_{i,j\geqq 0} w_{ij}\lambda_1^{-i}\lambda_2^{-j}$ *of* (1.4) *such that* $w_{00}=\mathbb{1}$, *namely,* $w_{ij}\in M_n(\mathbb{C})\,[[t,x]]$ *which satisfy* $w_{00}=\mathbb{1}$, $w_{ij}=0$ *if* $i<0$ *or* $j<0$, *and*

$$\begin{aligned}&-\partial_{x_{1b}}w_{i+1,j}+(\partial_{t_{1b}}+A_{t_{1b}})w_{ij}=0\ ,\\ &-\partial_{x_{2b}}w_{i,j+1}+(\partial_{t_{2b}}+A_{t_{2b}})w_{ij}=0\ ,\end{aligned} \tag{1.5}$$

for any $i,j\in\mathbb{Z}$, $b=1,2$.

Proof of sufficiency. Suppose that there exists $w(\lambda)=\sum_{i,j\geqq 0} w_{ij}\lambda_1^{-i}\lambda_2^{-j}$ satisfying (1.4) such that $w_{00}=\mathbb{1}$. Equations (1.4) imply that

$$[-\lambda_a\partial_{x_{ab}}+\nabla_{t_{ab}},\ -\lambda_c\partial_{x_{cd}}+\nabla_{t_{cd}}]w(\lambda)=0 \quad (a,b,c,d=1,2)\ . \tag{1.6}$$

Note that the commutator is a differential operator of order 0, namely, an element of $M_n(\mathbb{C})\,[[t,x]]$. Multiplying both sides of Eqs. (1.6) by $w(\lambda)^{-1}$ from the right, we obtain (1.2).

Proof of necessity. For any $i,j\in\mathbb{Z}$, consider a system of four equations

$$\begin{aligned}(E_{ij})\qquad &-\partial_{x_{1b}}w_{ij}+(\partial_{t_{1b}}+A_{t_{1b}})w_{i-1,j}=0\ ,\\ &-\partial_{x_{2b}}w_{ij}+(\partial_{t_{2b}}+A_{t_{2b}})w_{i,j-1}=0 \quad (b=1,2)\ ,\end{aligned}$$

which is a part of the system of Eqs. (1.5). The integrability condition for the equations (E_{ij}) with w_{ij} as the unknown function is as follows:

$$(\partial_{x_{11}}\nabla_{t_{12}}-\partial_{x_{12}}\nabla_{t_{11}})w_{i-1,j}=0\ , \tag{1.7a}$$

$$(\partial_{x_{21}}\nabla_{t_{22}}-\partial_{x_{22}}\nabla_{t_{21}})w_{i,j-1}=0\ , \tag{1.7b}$$

$$\partial_{x_{1b}}\nabla_{t_{2d}}w_{i,j-1}-\partial_{x_{2d}}\nabla_{t_{1b}}w_{i-1,j}=0 \quad (b,d=1,2)\ . \tag{1.7c}$$

Now we define $w_{00}=\mathbb{1}$ and $w_{ij}=0$ for any $i,j,\in\mathbb{Z}$ such that $i<0$ or $j<0$. Then (E_{ij}) is trivially satisfied for $i=j=0$ and for any $i,j\in\mathbb{Z}$ such that $i<0$ or $j<0$. For

$i,j \in \mathbb{N}$, we define w_{ij} inductively. Assume that $\{w_{ij}\}_{i,j\in\mathbb{N}, i+j\leqq m}$ are defined to satisfy (E_{ij}) for any $i,j\in\mathbb{N}$ such that $i+j\leqq m$. (This assumption actually holds for $m=0$.) We shall prove that for any $i,j\in\mathbb{N}$ such that $i+j=m+1$, there exists w_{ij} which satisfies (E_{ij}). To prove this, it is sufficient to prove the integrability conditions (1.7a), (1.7b) and (1.7c).

Proof of (1.7a). (i) Equations (1.2) imply $[\nabla_{t_{12}}, \partial_{x_{11}}]-[\nabla_{t_{11}}, \partial_{x_{12}}]=0$.

(ii) By the assumption, $w_{i-1,j}$ satisfies $(E_{i-1,j})$. Especially,

$$\partial_{x_{1b}} w_{i-1,j} = \nabla_{t_{1b}} w_{i-2,j} \quad (b=1,2) \ .$$

(iii) Equations (1.2) imply $[\nabla_{t_{12}}, \nabla_{t_{11}}]=0$.

It follows from (i), (ii), and (iii) that

$$\begin{aligned}(\partial_{x_{11}}\nabla_{t_{12}}-\partial_{x_{12}}\nabla_{t_{11}})w_{i-1,j} &= (\nabla_{t_{12}}\partial_{x_{11}}-\nabla_{t_{11}}\partial_{x_{12}})w_{i-1,j}\\ &= (\nabla_{t_{12}}\nabla_{t_{11}}-\nabla_{t_{11}}\nabla_{t_{12}})w_{i-2,j}\\ &= [\nabla_{t_{12}}, \nabla_{t_{11}}]w_{i-2,j}\\ &= 0 \ .\end{aligned}$$

Equation (1.7b) *can be derived in the same way.*

Proof of (1.7c).

(i) Equations (1.2) imply $[\partial_{x_{1b}}, \nabla_{t_{2d}}]=[\partial_{x_{2d}}, \nabla_{t_{1b}}]=0$.

(ii) By the assumption, $w_{i,j-1}$ satisfies $(E_{i,j-1})$ and $w_{i-1,j}$ satisfies $(E_{i-1,j})$. Especially, we obtain

$$\partial_{x_{1b}} w_{i,j-1} = \nabla_{t_{1b}} w_{i-1,j-1} \ , \qquad \partial_{x_{2d}} w_{i-1,j} = \nabla_{t_{2d}} w_{i-1,j-1} \ .$$

(iii) Equations (1.2) imply $[\nabla_{t_{2d}}, \nabla_{t_{1b}}]=0$.

It follows from (i), (ii), and (iii) that

$$\begin{aligned}\partial_{x_{1b}}\nabla_{t_{2d}}w_{i,j-1}-\partial_{x_{2d}}\nabla_{t_{1b}}w_{i-1,j} &= \nabla_{t_{2d}}\partial_{x_{1b}}w_{i,j-1}-\nabla_{t_{1b}}\partial_{x_{2d}}w_{i-1,j}\\ &= \nabla_{t_{2d}}\nabla_{t_{1b}}w_{i-1,j-1}-\nabla_{t_{1b}}\nabla_{t_{2d}}w_{i-1,j-1}\\ &= [\nabla_{t_{2d}}, \nabla_{t_{1b}}]w_{i-1,j-1}\\ &= 0 \ .\end{aligned}$$

Thus we can obtain $\{w_{ij}\}_{i,j\in\mathbb{N}}$ satisfying (1.5) inductively (more precisely, by using Zorn's lemma). q.e.d.

When $i=j=0$, (1.5) reads

$$-\partial_{x_{1b}} w_{1,0} + A_{t_{1b}} = 0 \ , \qquad -\partial_{x_{2b}} w_{0,1} + A_{t_{2b}} = 0 \ . \tag{1.8}$$

Therefore, to solve Eqs. (1.2), it is sufficient to solve the equations

$$\begin{aligned}&-\partial_{x_{1b}} w_{i+1,j} + \partial_{t_{1b}} w_{ij} + (\partial_{x_{1b}} w_{1,0}) w_{ij} = 0 \ ,\\ &-\partial_{x_{2b}} w_{i,j+1} + \partial_{t_{2b}} w_{ij} + (\partial_{x_{2b}} w_{0,1}) w_{ij} = 0\end{aligned} \tag{1.9}$$

$(i,j\in\mathbb{Z}, b=1,2)$.

More precisely, we have

Proposition 3. *The relations* (1.8) *give a one-to-one correspondence between*

(i) *solutions* $A=(A_{t_{ab}})_{a,b=1,2}$ *to* (1.2)

and

(ii) *equivalence classes of solutions* $w(\lambda)=\sum_{i,j\geqq 0} w_{ij}\lambda_1^{-i}\lambda_2^{-j}$ *to* (1.9) *such that* $w_{00}=\mathbb{1}$ *modulo right-multiplication by* $v(\lambda)=\sum_{i,j\geqq 0} v_{ij}\lambda_1^{-i}\lambda_2^{-j}$ *satisfying* $v_{00}=\mathbb{1}$ *and*

$$(-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})v(\lambda)=0 \quad \textit{for} \quad a,b=1,2 \ . \tag{1.10}$$

Proof. A surjection {solutions $w(\lambda)$ of (1.9) such that $w_{00}=\mathbb{1}$}→{solutions A of (1.2)} is established by Proposition 2. Now let $w(\lambda)$ and $\tilde{w}(\lambda)$ be solutions of (1.9) both corresponding to $A=(A_{t_{ab}})_{a,b=1,2}$. Set $v(\lambda)=w(\lambda)^{-1}\tilde{w}(\lambda)$. Then we have

$$\begin{aligned}(-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})v(\lambda)&=((-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})w(\lambda)^{-1})\tilde{w}(\lambda)\\&\quad+w(\lambda)^{-1}(-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})\tilde{w}(\lambda)\\&=w(\lambda)^{-1}A_{t_{ab}}\tilde{w}(\lambda)-w(\lambda)^{-1}A_{t_{ab}}\tilde{w}(\lambda)\\&=0 \ .\end{aligned}$$

If $w(\lambda)$, $\tilde{w}(\lambda)\in M_n(\mathbb{C})[[t,x]][[\lambda_1^{-1},\lambda_2^{-1}]]$ and $w_{00}=\tilde{w}_{00}=\mathbb{1}$, then $v(\lambda)=w(\lambda)^{-1}\tilde{w}(\lambda)\in M_n(\mathbb{C})[[t,x]][[\lambda_1^{-1},\lambda_2^{-1}]]$ and $v_{00}=\mathbb{1}$.

Conversely, let $v(\lambda)\in M_n(\mathbb{C})[[t,x]][[\lambda_1^{-1},\lambda_2^{-1}]]$ be a solution of (1.10) such that $v_{00}=\mathbb{1}$, and let $w(\lambda)\in M_n(\mathbb{C})[[t,x]][[\lambda_1^{-1},\lambda_2^{-1}]]$ be a solution of (1.9) corresponding to A such that $w_{00}=\mathbb{1}$. Set $\tilde{w}(\lambda)=w(\lambda)v(\lambda)$. Then we obtain $w(\lambda)\in M_n(\mathbb{C})[[t,x]][[\lambda_1^{-1},\lambda_2^{-1}]]$, $w_{00}=\mathbb{1}$, and

$$\begin{aligned}(-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})\tilde{w}(\lambda)&=(-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})(w(\lambda)v(\lambda))\\&=\{(-\lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})w(\lambda)\}v(\lambda)\\&=-A_{t_{ab}}w(\lambda)v(\lambda)\\&=-A_{t_{ab}}\tilde{w}(\lambda) \ .\end{aligned}$$

Namely, $\tilde{w}(\lambda)$ is a solution of (1.9) corresponding to A. q.e.d.

2. Motions on an Infinite-Dimensional Grassmann Manifold

Let $Z=\mathbb{Z}\times\mathbb{Z}$, $N=\mathbb{N}\times\mathbb{N}$, $N^c=Z\setminus N$ and R be a ring with a unity $\mathbb{1}$. For any $w(\lambda)=\sum_{(i,j)\in Z} w_{ij}\lambda_1^{-i}\lambda_2^{-j}\in R[[\lambda_1^{-1},\lambda_2^{-1}]]$ such that $w_{00}=\mathbb{1}$ and $w_{ij}=0$ for $(i,j)\in N^c$, define a matrix of infinite size $\xi=(\xi_{kl}^{ij})_{(i,j),\in Z,(k,l)\in N^c}$ by the product of matrices $(w^*_{i-k,j-l})_{(i,j)\in Z,(k,l)\in N^c}$ and $(w_{i-k,j-l})_{(i,j)\in Z,(k,l)\in N^c}$, i.e. by $\xi_{kl}^{ij}=\sum_{(g,h)\in N^c} w^*_{i-g,j-h}w_{g-k,h-l}$, where w^*_{ij} are coefficients of w^{-1}, i.e. $w^{-1}=\sum_{(i,j)\in Z} w^*_{ij}\lambda_1^{-i}\lambda_2^{-j}$. Then we obtain $\xi_{kl}^{ij}=\delta_k^i\delta_l^j\mathbb{1}$ if $(i,j)\in N^c$, $\xi_{kl}^{ij}=0$ if $i<k$ or

$j<l$, and $\Lambda_a\xi=\xi C_a$ $(a=1,2)$, where

$$\Lambda_1=(\delta_k^{i+1}\delta_l^j \mathbb{1})_{(i,j)\in Z,(k,l)\in Z} ,$$

$$\Lambda_2=(\delta_k^i\delta_l^{j+1}\mathbb{1})_{(i,j)\in Z,(k,l)\in Z} ,$$

$$C_1=(\xi_{k,l}^{i+1,j})_{(i,j)\in N^c,(k,l)\in N^c} ,$$

$$C_2=(\xi_{k,l}^{i,j+1})_{(i,j)\in N^c,(k,l)\in N^c} .$$

Here δ_k^i denotes Kronecker's delta. Furthermore, the converse is true:

Proposition 4. *The above definition of* ξ *gives a one-to-one correspondence between*

i) $w(\lambda)\in R[[\lambda_1^{-1},\lambda_2^{-1}]]$ *such that* $w_{00}=\mathbb{1}$,

and

ii) $\xi=(\xi_{kl}^{ij})_{(i,j)\in Z,(k,l)\in N^c}$, $\xi_{kl}^{ij}\in R$, *satisfying the following conditions:*

$$\xi_{kl}^{ij}=\delta_k^i\delta_l^j\mathbb{1} \quad \text{if} \quad (i,j)\in N^c , \tag{2.1a}$$

$$\xi_{kl}^{ij}=0 \quad \text{if} \quad i<k\geqq 0 \quad \text{or} \quad j<l\geqq 0 , \tag{2.1b}$$

$$\Lambda_1\xi=\xi C_1 , \quad \Lambda_2\xi=\xi C_2 \quad \text{for some} \quad N^c\times N^c\text{-matrices} \quad C_1, C_2 . \tag{2.1c}$$

Here the inverse correspondence $\xi\to w(\lambda)$ *is defined by* $w_{ij}=-\xi_{-i,-j}^{0,0}$.

Proof.

1) *Proof of* (2.1b). If $i<k$, then $w_{i-g,j-h}^*w_{g-k,h-l}=0$ for any $g,h\in\mathbb{Z}$ because $i-g<0$ or $g-k<0$ holds for any $g\in\mathbb{Z}$. If $j<l$, then $w_{i-g,j-h}^*w_{g-h,h-l}=0$ for any $g,h\in\mathbb{Z}$ because $j-h<0$ or $h-l<0$ holds for any $h\in\mathbb{Z}$. Therefore $\xi_{kl}^{ij}=\sum_{(g,h)\in N^c} w_{i-g,j-h}^*w_{g-k,h-l}=0$ if $i<k$ or $j<l$.

2) *Proof of* (2.1a). By the definition of w_{ij}^*, we obtain $\sum_{i+k=g,j+l=h} w_{ij}^*w_{kl}=\delta_0^g\delta_0^h\mathbb{1}$. If $(i,j)\in N^c$, $i\geqq k$, and $j\geqq l$, then

$$\begin{aligned}\xi_{kl}^{ij}&=\sum_{(g,h)\in N^c} w_{i-g,j-h}^*w_{g-k,h-l}\\&=\sum_{g=k}^{i}\sum_{h=l}^{j} w_{i-g,j-h}^*w_{g-k,h-l}\\&=\sum_{g_1+g_2=i-k,h_1+h_2=j-l} w_{g_1h_1}^*w_{g_2h_2}\\&=\delta_0^{i-k}\delta_0^{j-l}\mathbb{1}\\&=\delta_k^i\delta_l^j\mathbb{1} .\end{aligned}$$

3) *Proof of* (2.1c). We denote $\Lambda=(\Lambda_1,\Lambda_2)$, $\xi_0=(\delta_k^i\delta_l^j\mathbb{1})_{(i,j)\in Z,(k,l)\in N^c}$, and $\Lambda_{a(-)}={}^t\xi_0\Lambda_a\xi_0$ for $a=1,2$. Note that $\Lambda_a\xi_0=\xi_0\Lambda_{a(-)}$ for $a=1,2$, and

$$\begin{aligned}\xi&=(w_{i-k,j-l}^*)_{(i,j)\in Z,(k,l)\in N^c}(w_{i-k,j-l})_{(i,j)\in N^c,(k,l)\in N^c}\\&=w(\Lambda)^{-1}\xi_0{}^t\xi_0w(\Lambda)\xi_0=w(\Lambda)^{-1}\xi_0w(\Lambda)_{(-)} ,\end{aligned}$$

where we denote $w(\Lambda)_{(-)} = {}^t\xi_0 w(\Lambda)\xi_0$. Then we obtain

$$\Lambda_a\xi = \Lambda_a w(\Lambda)^{-1}\xi_0 w(\Lambda)_{(-)} = w(\Lambda)^{-1}\Lambda_a\xi_0 w(\Lambda)_{(-)}$$
$$= w(\Lambda)^{-1}\xi_0\Lambda_{a(-)}{}^t\xi_0 w(\Lambda)\xi_0 = w(\Lambda)^{-1}\xi_0 w(\Lambda)_{(-)}C_a = \xi C_a \ ,$$

where $C_a = \{w(\Lambda)_{(-)}\}^{-1}\Lambda_{a(-)}w(\Lambda)_{(-)}$.

4) *Mapping* (i)→(ii)→(i) *is identity*. In fact, if $w(\lambda)\to\xi\to\tilde{w}(\lambda)$, then

$$\begin{aligned}\tilde{w}_{ij} &= -\xi^{0,0}_{-i,-j}\\ &= -\sum_{(g,h)\in N^c} w^*_{-g,-h}w_{g+i,h+j}\\ &= \sum_{(g,h)\in N} w^*_{-g,-h}w_{g+i,h+j}\\ &= w_{ij}\ .\end{aligned}$$

5) *Mapping* (ii)→(i)→(ii) *is identity*. $\Lambda_1\xi = \xi C_1$ means $\xi^{i+1,j}_{k,l} = \sum_{(g,h)\in N^c}\xi^{ij}_{gh}C^{gh}_{1kl}$, which reads $\xi^{i+1,j}_{k,l} = C^{ij}_{1kl}$ when $(i,j)\in N^c$ because $\xi^{ij}_{gh} = \delta^i_g\delta^j_h\mathbb{1}$ for $(i,j)\in N^c$. Similarly, $\Lambda_2\xi = \xi C_2$ implies $C^{ij}_{2kl} = \xi^{i,j+1}_{k,l}$ for (i,j), $(k,l)\in N^c$. Therefore, if $\Lambda_1\xi = \xi C_1$ and $\Lambda_2\xi = \xi C_2$ for some $N^c\times N^c$-matrices C_1, C_2, then

$$\begin{aligned}\xi^{i+1,j}_{k,l} &= \sum_{(g,h)\in N^c}\xi^{ij}_{gh}\xi^{g+1,h}_{k,l} = \xi^{i,j}_{k-1,l} + \sum_{h=0}^{j}\xi^{i,j}_{-1,h}\xi^{0h}_{kl}\ ,\\ \xi^{i,j+1}_{k,l} &= \sum_{(g,h)\in N^c}\xi^{ij}_{gh}\xi^{g,h+1}_{k,l} = \xi^{i,j}_{k,l-1} + \sum_{g=0}^{i}\xi^{i,j}_{g,-1}\xi^{g0}_{kl}\ .\end{aligned} \tag{2.2}$$

This means that for any $m\in\mathbb{N}$, $\{\xi^{ij}_{kl}\}_{(i,j)\in N,(k,l)\in N^c, i+j=m+1}$ are determined by $\{\xi^{ij}_{kl}\}_{(i,j)\in N,(k,l)\in N^c, i+j\leq m}$. Thus $\xi = (\xi^{ij}_{kl})_{(i,j)\in Z,(k,l)\in N^c}$ is uniquely determined by $\{\xi^{00}_{kl}\}_{(k,l)\in N^c}$, provided that ξ satisfies (2.1a), (2.1b), and (2.1c). Now we set $\xi\to w(\lambda)\to\tilde{\xi}$. *Then both* ξ *and* $\tilde{\xi}$ satisfy (2.1a), (2.1b), (2.1c) and $\xi^{00}_{kl} = -w_{-k,-l} = \tilde{\xi}^{00}_{kl}$, from which $\xi = \tilde{\xi}$ follows. q.e.d.

Remark. The matrix ξ can be regarded as an N^c-frame in the vector space R^Z, which represents a point in an infinite-dimensional Grassmann manifold. Then $\xi^{(+)} = (\xi^{ij}_{kl})_{(i,j)\in N,(k,l)\in N^c}$ is regarded as a local coordinate system for the Grassmann manifold. Equations (2.1b) and (2.1c) are the defining equations for the relevant submanifold.

Now we rewrite Eqs. (1.9):

Theorem 1. *Through the correspondence* $w(\lambda)\leftrightarrow\xi$, *Eqs.* (1.9) *are equivalent to the existence of* $N^c\times N^c$*-matrices* B_{ab} $(a,b=1,2)$ *such that*

$$(-\Lambda_a\partial_{x_{ab}} + \partial_{t_{ab}})\xi = \xi B_{ab}\quad (a,b=1,2)\ . \tag{2.3}$$

Here B_{ab} $(a,b=1,2)$ *are uniquely determined by* ξ *if they exist, and* (2.3) *can be regarded as non-linear equations for* ξ *as follows:*

$$\begin{aligned}-\partial_{x_{1b}}\xi^{i+1}_{k,l} + \partial_{t_{1b}}\xi^{ij}_{kl} &= -\sum_{h\geq 0}\xi^{i,j}_{-1,h}\partial_{x_{1b}}\xi^{0h}_{kl}\ ,\\ -\partial_{x_{2b}}\xi^{i,j+1}_{k,l} + \partial_{t_{2b}}\xi^{ij}_{kl} &= -\sum_{g\geq 0}\xi^{i,j}_{g,-1}\partial_{x_{2b}}\xi^{g0}_{kl}\ (b=1,2)\ .\end{aligned} \tag{2.4}$$

Proof.

1) *Proof that* (1.9) *implies* (2.3).

$$\begin{aligned}(-A_a\partial_{x_{ab}}+\partial_{t_{ab}})\xi &= (-A_a\partial_{x_{ab}}+\partial_{t_{ab}})(w(A)^{-1}\xi_0{}^t\xi_0 w(A)\xi_0)\\ &= \{(-A_a\partial_{x_{ab}}+\partial_{t_{ab}})(w(A)^{-1})\}\xi_0 w(A)_{(-)}\\ &\quad + w(A)^{-1}(-A_a\partial_{x_{ab}}+\partial_{t_{ab}})\xi_0 w(A)_{(-)}\\ &= w(A)^{-1}A_{t_{ab}}\xi_0 w(A)_{(-)}+w(A)^{-1}(-A_a\partial_{x_{ab}}+\partial_{t_{ab}})\xi_0 w(A)_{(-)}\\ &= w(A)^{-1}(-A_a\partial_{x_{ab}}+V_{t_{ab}})\xi_0 w(A)_{(-)}\\ &= w(A)^{-1}\xi_0(-A_{a(-)}\partial_{x_{ab}}+V_{t_{ab}})w(A)_{(-)}\\ &= w(A)^{-1}\xi_0 w(A)_{(-)}B_{ab}=\xi B_{ab}\ ,\end{aligned}$$

where $B_{ab}=\{w(A)_{(-)}\}^{-1}(-A_{a(-)}\partial_{x_{ab}}+V_{t_{ab}})w(A)_{(-)}$, for any $a,b=1,2$.

2) *Proof that* (2.3) *implies* (2.4). In terms of entries, the equation $(-A_1\partial_{x_{1b}}+\partial_{t_{1b}})\xi=\xi B_{1b}$ can be rewritten as

$$-\partial_{x_{1b}}\xi^{i+1,j}_{k,l}+\partial_{t_{1b}}\xi^{ij}_{kl}=\sum_{(g,h)\in N^c}\xi^{ij}_{gh}B^{gh}_{kl}\ , \tag{2.5}$$

which reads

$$-\partial_{x_{1b}}\xi^{i+1,j}_{k,l}+\partial_{t_{1b}}\xi^{ij}_{kl}=B^{ij}_{kl}$$

when $(i,j)\in N^c$. Substituting this into (2.5), we obtain

$$\begin{aligned}-\partial_{x_{1b}}\xi^{i+1,j}_{k,l}+\partial_{t_{1b}}\xi^{ij}_{kl} &= \sum_{(g,h)\in N^c}\xi^{ij}_{gh}(-\partial_{x_{1b}}\xi^{g+1,h}_{k,l}+\partial_{t_{1b}}\xi^{gh}_{kl})\\ &= -\sum_{h\geqq 0}\xi^{i,j}_{-1,h}\partial_{x_{1b}}\xi^{0h}_{kl}\end{aligned}$$

because $\xi^{gh}_{kl}=\delta^g_k\delta^h_l\mathbb{1}$ for $(g,h)\in N^c$. The second equation of (2.4) can be derived in the same way.

3) *Proof that* (2.4) *implies* (1.9). When $i=j=0$, Eqs. (2.4) read

$$\begin{aligned}-\partial_{x_{1b}}\xi^{10}_{kl}+\partial_{t_{1b}}\xi^{00}_{kl} &= -\xi^{0,0}_{-1,0}\partial_{t_{1b}}\xi^{00}_{kl}\ ,\\ -\partial_{x_{2b}}\xi^{01}_{kl}+\partial_{t_{2b}}\xi^{00}_{kl} &= -\xi^{0,0}_{0,-1}\partial_{t_{2b}}\xi^{00}_{kl}\ .\end{aligned}$$

Substituting $\xi^{10}_{kl}=-w_{1-k,-l}-w^*_{10}w_{-k,-l}$, $\xi^{00}_{kl}=-w_{-k,-l}$, and $\xi^{01}_{kl}=-w_{-k,1-l}-w^*_{01}w_{-k,-l}$ into the above, we obtain

$$\begin{aligned}-\partial_{x_{1b}}w_{1-k,-l}+\partial_{t_{1b}}w_{-k,-l}+(\partial_{x_{1b}}w_{10})w_{-k,-l}=0\ ,\\ -\partial_{x_{2b}}w_{-k,1-l}+\partial_{t_{2b}}w_{-k,-l}+(\partial_{x_{2b}}w_{01})w_{-k,-l}=0\ ,\end{aligned}$$

for any $(k,l)\in N^c$. Thus (2.4) implies all of Eqs. (1.9) except some trivial ones. q.e.d.

To investigate the structure of the solution space of (2.2), we consider an initial-value problem with respect to the subspace $t=0$. Unlike the case of self-dual Yang-Mills equations, we cannot solve it for arbitrary data; the data for which it is solvable must satisfy a system of differential equations. In fact, we have

Proposition 5. *The system of equations* (2.1a), (2.1b), (2.1c), *and* (2.3) *implies that*

$$\text{if}\quad k\geqq 0\quad \text{and}\quad p+q>i-k\ ,\qquad \text{then}\quad \partial^p_{x_{11}}\partial^q_{x_{12}}\xi^{ij}_{kl}=0\ \ ((i,j)\in Z,\ (k,l)\in N^c)\ ,$$
$$\text{if}\quad l\geqq 0\quad \text{and}\quad p+q>j-l\ ,\qquad \text{then}\quad \partial^p_{x_{21}}\partial^q_{x_{22}}\xi^{ij}_{kl}=0\ \ ((i,j)\in Z,\ (k,l)\in N^c)\ . \tag{2.6}$$

Under the conditions (2.1a), (2.1b), *and* (2.1c), *Eqs.* (2.6) *are equivalent to the following equations:*

$$\text{If}\quad p+q=i+1\ ,\qquad \text{then}\quad \partial^p_{x_{11}}\partial^q_{x_{12}}\xi^{i0}_{0l}=0\ ,\qquad (p,q,i\geqq 0,\ l<0)$$
and
$$\text{if}\quad p+q=j+1\ ,\qquad \text{then}\quad \partial^p_{x_{21}}\partial^q_{x_{22}}\xi^{0j}_{k0}=0\ ,\qquad (p,q,j\geqq 0,\ k<0) \tag{2.7}$$

Proof.

1) It is obvious that (2.6) implies (2.7)

2) Proof that (2.4) implies (2.6). It follows from (2.1b) that $\xi^{0h}_{kl}=0$ for $k>0$. Thus the first equation of (2.4) reads

$$-\partial_{x_{1b}}\xi^{i+1,j}_{k,l}+\partial_{t_{1b}}\xi^{ij}_{kl}=0 \tag{2.8}$$

for $k>0$. Thanks to this formula, the first equation of (2.6) can be proved by induction starting from the case $i-k=-1$ which is trivial. We cannot use the formula when $k=0$, but Eqs. (2.6) also hold for $k=0$ because $\xi^{ij}_{kl}=\xi^{i-k,j}_{0,l}$ for any $k\geqq 0$. The second equation of (2.6) can be derived similarly.

3) *Proof that* (2.7) *implies* (2.6). We shall prove the first equation only. (The second one can be proved similarly.) Since $\xi^{ij}_{kl}=\xi^{i-k,j}_{0,l}$ for $k\geqq 0$, it is sufficient to show that

$$\text{if}\quad p+q>i\ ,\qquad \text{then}\quad \partial^p_{x_{11}}\partial^q_{x_{12}}\xi^{ij}_{0l}=0\ . \tag{2.9}$$

We shall prove this by induction on j. The case $j=0$ is just (2.7). Assume that (2.9) holds for any $j\leqq m$. When $k=0$, the second equation of (2.2) reads

$$\xi^{i,m+1}_{0,l}=\xi^{i,m}_{0,l-1}+\sum_{g=0}^{i}\xi^{i,m}_{g,-1}\xi^{g0}_{0l}=\xi^{i,m}_{0,l-1}+\sum_{g=0}^{i}\xi^{i-g,m}_{0,-1}\xi^{g0}_{0l}\ .$$

Differentiating both sides, we obtain

$$\partial^p_{x_{11}}\partial^q_{x_{12}}\xi^{i,m+1}_{0,l}=\partial^p_{x_{11}}\partial^q_{x_{12}}\xi^{i,m}_{0,l-1}$$
$$+\sum_{g=0}^{i}\sum_{r=0}^{p}\sum_{s=0}^{q}\binom{p}{r}\binom{q}{s}\partial^{p-r}_{x_{11}}\partial^{q-s}_{x_{12}}\xi^{i-g,m}_{0,-1}\partial^r_{x_{11}}\partial^s_{x_{12}}\xi^{g0}_{0l}\ .$$

If $p+q>i$, then $\partial^p_{x_{11}}\partial^q_{x_{12}}\xi^{i,m}_{0,l-1}=0$ and either $\partial^{p-r}_{x_{11}}\partial^{q-s}_{x_{12}}\xi^{i-g,m}_{0,-1}=0$ or $\partial^r_{x_{11}}\partial^s_{x_{12}}\xi^{g0}_{0l}=0$ holds by the assumption of induction. Thus (2.9) holds for $j=m+1$. q.e.d.

Conversely, for any initial datum satisfying (2.6) (or (2.7)) we can solve the initial-value problem:

Theorem 2. *For any* $\xi^{(0)}=(\xi_{kl}^{(0)ij})_{(i,j)\in Z,(k,l)\in N^c}$, $\xi_{kl}^{(0)ij}\in M_n(\mathbb{C})[[x]]$ *satisfying* (2.1a), (2.1b), (2.1c), *and* (2.6) *(or* (2.7)*), there exists a unique solution* ξ *to the initial-value problem, i.e.* $\xi=(\xi_{kl}^{ij})_{(i,j)\in Z,(k,l)\in N^c}$, $\xi_{kl}^{ij}\in M_n(\mathbb{C})[[t,x]]$ *satisfying* (2.1a), (2.1b), (2.1c), (2.3), *and* $\xi|_{t=0}=\xi^{(0)}$. *The solution* ξ *has the following form:*

$$\xi=\tilde{\xi}(\tilde{\xi}^{(-)})^{-1}\ , \tag{2.10a}$$

where

$$\tilde{\xi}=\exp\left(\sum_{a=1}^{2}\sum_{b=1}^{2}t_{ab}\Lambda_a\partial_{x_{ab}}\right)\xi^{(0)}\ , \tag{2.10b}$$

$$\tilde{\xi}=\begin{pmatrix}\tilde{\xi}^{(-)}\\ \tilde{\xi}^{(+)}\end{pmatrix},\qquad \tilde{\xi}^{(-)}=(\tilde{\xi}_{kl}^{ij})_{(i,j)\in N^c,(k,l)\in N^c}\ ,\qquad \tilde{\xi}^{(+)}=(\tilde{\xi}_{kl}^{ij})_{(i,j)\in N,(k,l)\in N^c}\ . \tag{2.10c}$$

Proof.

Proof of the uniqueness. Set

$$w(\Lambda)=\sum_{i,j,k,l\geqq 0}w_{ijkl}(\lambda)t_{11}^{i}t_{12}^{j}t_{21}^{k}t_{22}^{l}$$

and

$$w_{ijkl}(\lambda)=\sum_{g,h\geqq 0}w_{gh;ijkl}\lambda_1^{-g}\lambda_2^{-h}\ .$$

Then (1.9) are recursion formulae for $w_{ijkl}(\lambda)$:

$$(i+1)w_{i+1,j,k,l}(\lambda)=\lambda_1\partial_{x_{11}}w_{ijkl}(\lambda)-\sum_{p=0}^{i}\sum_{q=0}^{j}\sum_{r=0}^{k}\sum_{s=0}^{l}(\partial_{x_{11}}w_{10;i-p,j-q,k-r,l-s})w_{pqrs}(\lambda)$$

etc. Thus $\{w_{ijkl}(\lambda)\}_{i,j,k,l\geqq 0}$ are uniquely determined by $w_{0000}(\lambda)$ if they exist. This completes the proof because of the one-to-one correspondence $w(\lambda)\leftrightarrow\xi$ in Proposition 4.

Proof of the solution formulae

1) Let

$\mathscr{R}=\{A=(A_{kl}^{ij})_{(i,j)\in N^c,(k,l)\in N^c}|A_{kl}^{ij}\in M_n(\mathbb{C})[[x]]$, there exists an integer m such that $A_{kl}^{ij}=0$ if $i-k\leqq m$ or $j-l\leqq m\}$,

$\mathscr{F}=\{\xi=(\xi_{kl}^{ij})_{(i,j)\in Z,(k,l)\in N^c}|\xi_{kl}^{ij}\in M_n(\mathbb{C})[[x]]$, there exists an integer m such that $\xi_{kl}^{ij}=0$ if $i-k\leqq m$ or $j-l\leqq m\}$,

and $\mathscr{F}[[t]]=\left\{\sum_{i\geqq 0}\xi_i t^i|\xi_i\in\mathscr{F}\right\}$.

$\mathscr{R}$ is a $\mathbb{C}$-algebra on which $\Lambda_{a(-)}\in\mathscr{R}\,(a=1,2)$ and $\partial_{x_{ab}}\,(a,b=1,2)$ act. $\mathscr{F}$ is a right $\mathscr{R}$-module with $\mathscr{R}$-action defined by multiplication as matrices. $\Lambda_a\,(a=1,2)$ and

$\partial_{x_{ab}}(a,b=1,2)$ act on $\mathscr{F}$ from the left. Since $\xi^{(0)}\in\mathscr{F}$,

$$\tilde{\xi}=\sum_{p\geqq 0}\frac{1}{p!}\left(\sum_{a=1}^{2}\sum_{b=1}^{2}t_{ab}\Lambda_a\partial_{x_{ab}}\right)^p\xi^{(0)}$$

is well-defined as an element of $\mathscr{F}[[t]]$.

The following proposition is important because it means that the system of Eqs. (2.1c) and (2.3) defines a motion on an infinite-dimensional Grassmann manifold:

Proposition 6. *The system of equations* (2.1c) *and* (2.3) *is invariant under change of frame: let* $\xi\in\mathscr{F}[[t]]$ *and* $C_a, B_{ab}\in\mathscr{R}[[t]]$ $(a,b=1.2)$ *satisfy* (2.1c) *and* (2.3). *For any invertible element* $P\in\mathscr{R}[[t]]$, *set* $\xi'=\xi P$, $C'_a=P^{-1}C_aP$, *and* $B'_{ab}=P^{-1}(B_{ab}-C_a\partial_{x_{ab}}+\partial_{t_{ab}})P$ *for* $a,b=1,2$. *Then* ξ', C'_a, *and* $B'_{ab}(a,b=1,2)$ *also satisfy* (2.1c) *and* (2.3).

Proof. The following calculation proves the proposition:

$$\Lambda_a\xi'=\Lambda_a\xi P=\xi C_aP=\xi PP^{-1}C_aP=\xi'C'_a(a=1,2)\ ,$$

$$\begin{aligned}(-\Lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})\xi'&=(-\Lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})(\xi P)\\&=((-\Lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})\xi)P-\Lambda_a\xi\partial_{x_{ab}}P+\xi\partial_{t_{ab}}P\\&=\xi B_{ab}P-\xi C_a\partial_{x_{ab}}P+\xi\partial_{t_{ab}}P\\&=\xi(B_{ab}-C_a\partial_{x_{ab}}+\partial_{t_{ab}})P\\&=\xi PP^{-1}(B_{ab}-C_a\partial_{x_{ab}}+\partial_{t_{ab}})P\\&=\xi'B'_{ab}(a,b=1,2)\ .\quad\text{q.e.d.}\end{aligned}$$

2) In terms of entries, the definition of $\tilde{\xi}$ can be written in the following form:

$$\tilde{\xi}^{ij}_{kl}=\sum_{p,q\geqq 0}\frac{1}{p!}(t_{11}\partial_{x_{11}}+t_{12}\partial_{x_{12}})^p\frac{1}{q!}(t_{21}\partial_{x_{21}}+t_{22}\partial_{x_{22}})^q\xi^{(0)i+p,j+q}_{k,l}\ . \tag{2.11}$$

Since $\xi^{(0)}$ satisfies (2.6), we obtain

$$\tilde{\xi}^{ij}_{kl}=0\quad\text{if}\quad i<k\geqq 0\quad\text{or}\quad j<l\geqq 0\ . \tag{2.12}$$

3) It follows immediately from the definition of $\tilde{\xi}$ that

$$(-\Lambda_a\partial_{x_{ab}}+\partial_{t_{ab}})\tilde{\xi}=0\quad(a,b=1,2)\ . \tag{2.13}$$

4) There exist $\tilde{C}_1$, $\tilde{C}_2\in\mathscr{R}[[t]]$ such that

$$\Lambda_a\tilde{\xi}=\tilde{\xi}\tilde{C}_a(a=1,2)\ . \tag{2.14}$$

This can be proved as follows: let $w^{(0)}(\lambda)\leftrightarrow\xi^{(0)}$ through the correspondence in Proposition 4. Then $\xi^{(0)}=w^{(0)}(\Lambda)^{-1}\xi_0w^{(0)}(\Lambda)_{(-)}$, where $w^{(0)}(\Lambda)_{(-)}={}^t\xi_0w^{(0)}(\Lambda)\xi_0$. Set $\Lambda_{(-)}=(\Lambda_{1(-)},\Lambda_{2(-)})$, $P(\Lambda)=\sum_{a=1}^{2}\sum_{b=1}^{2}t_{ab}\Lambda_a\partial_{x_{ab}}$, and $P(\Lambda_{(-)})=\sum_{a=1}^{2}\sum_{b=1}^{2}t_{ab}\Lambda_{a(-)}\partial_{x_{ab}}$. We note that $\Lambda_a\xi_0=\xi_0\Lambda_{a(-)}$ for $a=1,2$ and hence that

$P(A)\xi_0=\xi_0 P(A_{(-)})$. Then it follows from the definition of $\tilde{\xi}$ that

$$\begin{aligned}\tilde{\xi}&=\sum_{i\geqq 0}\frac{1}{i!}P(A)^i w^{(0)}(A)^{-1}\xi_0 w^{(0)}(A)_{(-)}\\&=\sum_{i\geqq 0}\sum_{j\geqq 0}\frac{1}{(i-j)!}P(A)^{i-j}w^{(0)}(A)^{-1}\frac{1}{j!}P(A)^j\xi_0 w^{(0)}(A)_{(-)}\\&=\sum_{k\geqq 0}\frac{1}{k!}P(A)^k w^{(0)}(A)^{-1}\xi_0\sum_{j\geqq 0}\frac{1}{j!}P(A_{(-)})^j w^{(0)}(A)_{(-)}\\&=\exp[P(A)]w^{(0)}(A)^{-1}\xi_0\exp[P(A_{(-)})]w^{(0)}(A)_{(-)}\ ,\end{aligned}$$

and that

$$\begin{aligned}A_a\tilde{\xi}&=A_a\exp[P(A)]w^{(0)}(A)^{-1}\xi_0\exp[P(A_{(-)})]w^{(0)}(A)_{(-)}\\&=\exp[P(A)]w^{(0)}(A)^{-1}\xi_0 A_{a(-)}\exp[P(A_{(-)})]w^{(0)}(A)_{(-)}\\&=\tilde{\xi}\tilde{C}_a\ ,\end{aligned}$$

where $\tilde{C}_a=\{\exp[P(A_{(-)})]w^{(0)}(A)_{(-)}\}^{-1}A_{a(-)}\exp[P(A_{(-)})]w^{(0)}(A)_{(-)}$. The invertibility of $\exp[P(A_{(-)})]w^{(0)}(A)_{(-)}$ in $\mathscr{R}[[t]]$ follows from the fact that $\exp[P(A_{(-)})]w^{(0)}(A)_{(-)}|_{t=0}=w^{(0)}(A)_{(-)}$ and that $w^{(0)}(A)_{(-)}=(w^{(0)}_{i-k,j-l})_{(i,j)\in N^c,(k,l)\in N^c}$ is invertible in $\mathscr{R}$.

5) It follows from (2.13) and (2.14) that $\tilde{\xi}$ is a solution of the system of (2.1c) and (2.3) for $B_{ab}=0\,(a,b=1,2)$. $\tilde{\xi}_{(-)}\in\mathscr{R}[[t]]$ follows from that $\tilde{\xi}\in\mathscr{F}[[t]]$, and $\tilde{\xi}_{(-)}$ is invertible in $\mathscr{R}[[t]]$ because $\tilde{\xi}_{(-)}|_{t=0}=(\delta^i_k\delta^j_l\mathbb{1})_{(i,j)\in N^c,(k,l)\in N^c}=1\in\mathscr{R}$. Then Proposition 6 says that $\xi=\tilde{\xi}(\tilde{\xi}_{(-)})^{-1}\in\mathscr{F}[[t]]$ satisfies (2.1c) and (2.3). Equation (2.1a) follows from the definition of ξ. Thus the last to prove is (2.1b). Let

$$\mathscr{R}_1=\{A\in\mathscr{R}\,|\,A^{ij}_{kl}=0\quad\text{if}\quad i<k\geqq 0\quad\text{or}\quad j<l\geqq 0\}\ ,$$

$$\mathscr{F}_1=\{\xi\in\mathscr{F}\,|\,\xi^{ij}_{kl}=0\quad\text{if}\quad i<k\geqq 0\quad\text{or}\quad j<l\geqq 0\}\ ,$$

and

$$\mathscr{F}_1[[t]]=\left\{\sum_{i\geqq 0}\xi_i t^i\,|\,\xi_i\in\mathscr{F}_1\right\}\ .$$

Then $\mathscr{R}_1$ is a subring of $\mathscr{R}$ and $\mathscr{F}_1$ is an $\mathscr{R}_1$-module. $\tilde{\xi}$ is an element of $\mathscr{F}_1[[t]]$ and $\tilde{\xi}_{(-)}$ is an invertible element of $\mathscr{R}_1[[t]]$ because of (2.12). Therefore $\xi=\tilde{\xi}(\tilde{\xi}_{(-)})^{-1}\in\mathscr{F}_1[[t]]$. This completes the proof of Theorem 2.

In summary, by choosing the proper frame, the time evolutions in the initial-value problem can be regarded as evolutions defined by *linear* differential equations, and the solution space of (1.9) is faithfully parametrized by the solution space of Eqs. (2.6)[or (2.7)] in the subspace $t=0$.

3. Relation to the Yang-Mills Fields

First we describe the procedure for obtaining Yang-Mills potentials from any solution of Eqs. (1.9) (or (2.4)):

Proposition 7. *Given any solution of Eqs.* (1.9) *(or* (2.4)*), set*

$$A_{x_0} = -(A_{t_{12}} + A_{t_{22}}) = -(\partial_{x_{12}} w_{10} + \partial_{x_{22}} w_{01}) = \partial_{x_{12}} \xi^{00}_{-1,0} + \partial_{x_{22}} \xi^{00}_{0,-1} ,$$

$$A_{x_1} = \sqrt{-1}(A_{t_{12}} + A_{t_{22}}) = \sqrt{-1}(\partial_{x_{12}} w_{10} + \partial_{x_{22}} w_{01}) = -\sqrt{-1}(\partial_{x_{12}} \xi^{00}_{-1,0} + \partial_{x_{22}} \xi^{00}_{0,-1}) ,$$

$$A_{x_2} = A_{t_{11}} + A_{t_{21}} = \partial_{x_{11}} w_{10} + \partial_{x21} w_{01} = -(\partial_{x_{11}} \xi^{00}_{-1,0} + \partial_{x_{21}} \xi^{00}_{0,-1})$$

and

$$A_{x_3} = \sqrt{-1}(A_{t_{11}} - A_{t_{21}}) = \sqrt{-1}(\partial_{x_{11}} w_{10} + \partial_{x_{21}} w_{01}) = -\sqrt{-1}(\partial_{x_{11}} \xi^{00}_{-1,0} + \partial_{x_{21}} \xi^{00}_{0,-1}) .$$

Substitute

$$x_{11} = x_{21} = x_0 + \sqrt{-1}\, x_1 , t_{11} = t_{22} = x_2 + \sqrt{-1}\, x_3 ,$$

$$t_{12} = t_{22} = -x_0 + \sqrt{-1}\, x_1 , x_{12} = t_{21} = x_2 - \sqrt{-1}\, x_3$$

into the above. Then $A = (A_{x_0}, A_{x_1}, A_{x_2}, A_{x_3})$ *gives a set of Yang-Mills potentials (i.e. a solution of the system* (0.2)*).*

Proposition 8. *Let* ∇ *and* ∇' *be gauge fields in* $\mathbb{C}^8$ *satisfying* (0.1). *If* ∇ *and* ∇' *are gauge-equivalent as Yang-Mills fields in the diagonal subspace* Δ, *then they are gauge-equivalent in* $\mathbb{C}^8$.

Proof. If ∇ and ∇' are gauge-equivalent as gauge fields on Δ, there exists $g = g(x) \in M_n(\mathbb{C})[[x_0, x_1, x_2, x_3]]$ such that $\nabla_{x_\mu} = g^{-1} \nabla'_{x_\mu} g$ on Δ for $\mu = 0, 1, 2, 3$. Set $\tilde{\nabla} = g^{-1} \nabla' g$ in $\mathbb{C}^8$. Then $\tilde{\nabla}$ is gauge-equivalent to ∇' by definition and $\tilde{\nabla}_{x_\mu} = \nabla_{x_\mu}$ on Δ for $\mu = 0, 1, 2, 3$. It is sufficient to prove that $\tilde{\nabla}$ and ∇ are gauge-equivalent.

First we note that Eqs. (0.1) are rewritten in terms of ∇_{x_μ} and ∇_{w_μ} as follows:

$$[\nabla_{w_\mu}, \nabla_{x_\nu}] = \sum_{\kappa=0}^{3} \sum_{\lambda=0}^{3} (1/2) \varepsilon_{\mu\nu\kappa\lambda} [\nabla_{x_\kappa}, \nabla_{x_\lambda}] ,$$
$$[\nabla_{w_\mu}, \nabla_{w_\nu}] = [\nabla_{x_\mu}, \nabla_{x_\nu}] \ (\mu, \nu = 0, 1, 2, 3) . \tag{3.1}$$

Expanding ∇ with respect to w as

$$\nabla_{x_\mu} = \partial_{x_\mu} + \sum_{\alpha \in \mathbb{N}^4} A^\alpha_{x_\mu} w^\alpha , \qquad A^\alpha_{x_\mu} = A^\alpha_{x_\mu}(x) \in M_n(\mathbb{C})[[x_0, x_1, x_2, x_3]] ,$$

$$\nabla_{w_\mu} = \partial_{w_\mu} + \sum_{\alpha \in \mathbb{N}^4} A^\alpha_{w_\mu} w^\alpha , \qquad A^\alpha_{w_\mu} = A^\alpha_{w_\mu}(x) \in M_n(\mathbb{C})[[x_0, x_1, x_2, x_3]] ,$$

and substituting this into (2.1), we obtain

$$(\alpha_\mu + 1) A^{\alpha + e_\mu}_{x_\nu} = \partial_{x_\nu} A^\alpha_{w_\mu} - \sum_\beta [A^{\alpha-\beta}_{w_\mu}, A^\beta_{x_\nu}]$$
$$+ \sum_{\kappa=0}^{3} \sum_{\lambda=0}^{3} (1/2) \varepsilon_{\mu\nu\kappa\lambda} (\partial_{x_\kappa} A^\alpha_{x_\lambda} - \partial_{x_\lambda} A^\alpha_{x_\kappa} + \sum_\beta [A^{\alpha-\beta}_{x_\kappa}, A^\beta_{x_\lambda}]) ,$$

$$(\alpha_\mu+1)A_{w_\nu}^{\alpha+e_\mu}(\alpha_\nu+1)A_{w_\mu}^{\alpha+e_\nu}=-\sum_\beta [A_{w_\mu}^{\alpha-\beta},A_{w_\nu}^{\beta}]$$
$$+\partial_{x_\mu}A_{x_\nu}^{\alpha}-\partial_{x_\nu}A_{x_\mu}^{\alpha}+\sum_\beta [A_{x_\mu}^{\alpha-\beta},A_{x_\nu}^{\beta}]) ,$$

where $e_0=(1,0,0,0)$, $e_1=(0,1,0,0)$, $e_2=(0,0,1,0)$, $e_3=(0,0,0,1)\in\mathbb{N}^4$. We may assume without loss of generality that $\sum_{\mu=0}^{3} A_{w_\mu}^{\alpha-e_\mu}=0$ for any $\alpha\in\mathbb{N}^4$ by virtue of gauge transformation. Then $\{A_{w_\mu}^{\alpha}\}_{\alpha\in\mathbb{N}^4,\mu=0,1,2,3}$ and $\{A_{x_\mu}^{\alpha}\}_{\alpha\in\mathbb{N}^4,\mu=0,1,2,3}$ are recursively and uniquely determined by $\{A_{x_\mu}^{0}\}_{\mu=0,1,2,3}$. Finally, we prove the existence of such gauge transformation. Let m be any positive integer and let $g_m=\mathbb{1}-\sum_{|\alpha|=m} g_m^\alpha w^\alpha$. Then $g_m^{-1}=\sum_{j\geq 0}\left(\sum_{|\alpha|=m} g_m^\alpha w^\alpha\right)^j$. Let $A_{w_\mu}\to\tilde{A}_{w_\mu}=g_m^{-1}A_{w_\mu}g_m+g_m^{-1}(\partial_{w_\mu}g_m)$. Then $\tilde{A}_{w_\mu}^{\alpha}=A_{w_\mu}^{\alpha}$ if $|\alpha|<m-1$, $\tilde{A}_{w_\mu}^{\alpha}=A_{w_\mu}^{\alpha}-(\alpha_\mu+1)g_m^{\alpha+e_\mu}$ if $|\alpha|=m-1$, and hence $\sum_{\mu=0}^{3}\tilde{A}_{w_\mu}^{\alpha-e_\mu}=\sum_{\mu=0}^{3}A_{w_\mu}^{\alpha-e_\mu}-mg_m^\alpha$ if $|\alpha|=m$. Thus for any given A, we define $\left\{g_m=\mathbb{1}-\sum_{|\alpha|=m} g_m^\alpha w^\alpha\right\}_{m\geq 1}$ inductively by

$$g_1^\alpha=\sum_{\mu=0}^{3} A_{w_\mu}^{\alpha-e_\mu} , \qquad A_{1,w_\mu}=g_1^{-1}A_{w_\mu}g_1+g_1^{-1}(\partial_{w_\mu}g_1) ,$$

$$g_m^\alpha=\sum_{\mu=0}^{3} A_{m-1,w_\mu}^{\alpha-e_\mu} , \qquad A_{m,w_\mu}=g_m^{-1}A_{m-1,w_\mu}g_m+g_m^{-1}(\partial_{w_\mu}g_m) ,$$

and set $g=\prod_{j\geq 1} g_j=g_1g_2g_3\dots$. q.e.d.

For any self-dual Yang-Mills field ∇, i.e. covariant derivatives $\nabla_{x_\mu}=\partial_{x_\mu}+A_{x_\mu}(x)$ satisfying

$$[\nabla_{x_\mu},\nabla_{x_\nu}]=(1/2)\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\varepsilon_{\mu\nu\alpha\beta}[\nabla_{x_\alpha},\nabla_{x_\beta}]\,(\mu,\nu=0,1,2,3) ,$$

define a gauge field $\tilde{\nabla}$ on $\mathbb{C}^8$ by

$$\begin{aligned}\tilde{\nabla}_{y_\mu}&=\partial_{y_\mu}+\tilde{A}_{y_\mu} , \quad \tilde{A}_{y_\mu}=A_{x_\mu}(y) ,\\ \tilde{\nabla}_{z_\mu}&=\partial_{z_\mu}+\tilde{A}_{z_\mu} , \quad \tilde{A}_{z_\mu}=0 \ (\mu=0,1,2,3) .\end{aligned}\tag{3.2}$$

Then we obtain

$$\begin{aligned}&[\tilde{\nabla}_{y_\mu},\tilde{\nabla}_{y_\nu}]=(1/2)\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\varepsilon_{\mu\nu\alpha\beta}[\tilde{\nabla}_{y_\alpha},\tilde{\nabla}_{y_\beta}] ,\\ &[\tilde{\nabla}_{z_\mu},\tilde{\nabla}_{z_\nu}]=0 ,\\ &[\tilde{\nabla}_{y_\mu},\tilde{\nabla}_{z_\nu}]=0 \ (\mu,\nu=0,1,2,3) ,\end{aligned}\tag{3.3}$$

which imply Eqs. (0.1). Thus all the self-dual fields belong to the class of Yang-Mills fields given by the restriction of Witten's gauge fields (0.1). Note that the trivial extension (3.2) is the unique one up to gauge equivalence by virtue of

Proposition 8 and that Eqs. (3.3) are gauge-invariant. Therefore, if any gauge field ∇ satisfies (0.1) and its restriction to the diagonal subspace Δ is self-dual, then ∇ satisfies (3.3). Conversely, suppose that a gauge field $\tilde{\nabla}$ satisfies (3.3). We may assume that $\tilde{A}_{z_\mu}=0$, because such a gauge can be taken by virtue of the equations $[\tilde{\nabla}_{z_\mu}, \tilde{\nabla}_{z_\nu}]=0$ $(\mu, \nu=0,1,2,3)$. Then $0=[\tilde{\nabla}_{y_\mu}, \tilde{\nabla}_{z_\nu}]=[\tilde{\nabla}_{y_\mu}, \partial_{z_\nu}]=-\partial_{z_\nu}A_{y_\mu}$, i.e., $\tilde{A}_{y_\mu}=\tilde{A}_{y_\mu}(y)$. Set $A_{x_\mu}=\tilde{A}_{y_\mu}(x)$ and $\nabla_{x_\mu}=\partial_{x_\mu}+A_{x_\mu}$. Then we obtain $[\nabla_{x_\mu}, \nabla_{x_\nu}]$ $=(1/2)\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3}\varepsilon_{\mu\nu\alpha\beta}[\nabla_{x_\alpha}, \nabla_{x_\beta}]$ and $\nabla_{x_\mu}=\tilde{\nabla}_{x_\mu}|_{w=0}$. Thus we have

Proposition 9. *The solutions* ∇ *of* (0.1) *which correspond to self-dual or anti-self-dual fields on* Δ *are characterized by*

$$[\nabla_{z_\mu}, \nabla_{z_\nu}]=0 \quad (\mu, \nu=0,1,2,3)$$

or

$$[\nabla_{y_\mu}, \nabla_{y_\nu}]=0 \quad (\mu, \nu=0,1,2,3) \tag{3.4}$$

respectively. All the self-dual or anti-self-dual Yang-Mills fields can be obtained in this way.

Rewriting (3.4) in terms of ξ, we obtain

Proposition 10. (i) *A solution* ξ *to the system of Eqs.* (2.1a), (2.1b), (2.1c), *and* (2.3) *corresponds to a self-dual field on* Δ *if and only if it satisfies*

$$\partial_{x_{21}}^2\xi_{0,-1}^{0,0}=\partial_{x_{21}}\partial_{x_{22}}\xi_{0,-1}^{0,0}=\partial_{x_{22}}^2\xi_{0,-1}^{0,0}=0 \ . \tag{3.5}$$

(ii) *A solution* ξ *to the system of Eqs.* (2.1a), (2.1b), (2.1c), *and* (2.3) *corresponds to an anti-self-dual field on* Δ *if and only if it satisfies*

$$\partial_{x_{11}}^2\xi_{-1,0}^{0,0}=\partial_{x_{11}}\partial_{x_{12}}\xi_{-1,0}^{0,0}=\partial_{x_{12}}^2\xi_{-1,0}^{0,0}=0 \ . \tag{3.6}$$

Proof. We prove (i) only. Noting that $(x_{21}, x_{22}, t_{21}, t_{22})$ are the coordinates of z-space, we can see that (3.4), the integrability in z-directions is equivalent to the following system:

$$[\nabla_{x_{21}}, \nabla_{x_{22}}]=[\nabla_{t_{21}}, \nabla_{t_{22}}]=0 \ ,$$

$$[\nabla_{x_{2a}}, \nabla_{t_{2b}}]=0 \quad (a,b=1,2) \ .$$

The first two equations are trivial since we assume that the gauge field satisfies (1.1). Substituting $\nabla_{x_{2b}}=\partial_{x_{2b}}$ and $\nabla_{t_{2b}}=\partial_{t_{2b}}-\partial_{x_{2b}}\xi_{0,-1}^{00}$ into the rest of them, we obtain (3.5). q.e.d.

Remark. (3.5) or (3.6) is *not* stable under the time evolutions.

In fact, we have

Proposition 11. *Suppose that* ξ *satisfies the system of Eqs.* (2.1a), (2.1b), (2.1c), *and* (2.3) *and corresponds to a self-dual (anti-self-dual) field on* Δ. *Then*

$$\begin{aligned} &\text{if} \quad p+q\geqq i-k,\ k\geqq 0,\ r+s>j-l, \quad \text{then} \quad \partial_{x_{11}}^p\partial_{x_{12}}^q\partial_{x_{11}}^r\partial_{x_{12}}^s\xi_{kl}^{ij}=0 \ , \\ &(\text{if} \quad p+q>i-k,\ r+s\geqq j-l,\ l\geqq 0, \quad \text{then} \quad \partial_{x_{11}}^p\partial_{x_{12}}^q\partial_{x_{11}}^r\partial_{x_{12}}^s\xi_{kl}^{ij}=0) \ . \end{aligned} \tag{3.7}$$

Under the conditions (2.1a), (2.1b), *and* (2.1c), *Eqs.* (3.7) *are equivalent to the following equations:*

$$\begin{aligned} &\textit{If}\quad p+q\geqq i \quad\textit{and}\quad r+s>-l, \quad\textit{then}\quad \partial^p_{x_{11}}\partial^q_{x_{12}}\partial^r_{x_{21}}\partial^s_{x_{22}}\xi^{i0}_{0l}=0\ . \\ &(\textit{If}\quad p+q>-k \quad\textit{and}\quad r+s\geqq j, \quad\textit{then}\quad \partial^p_{x_{11}}\partial^q_{x_{12}}\partial^r_{x_{21}}\partial^s_{x_{22}}\xi^{0j}_{k0}=0.) \end{aligned} \tag{3.8}$$

Proof. By using (2.8), the proposition can be reduced by induction to the case $p=q=i=k=0$. (Note that $\xi^{ij}_{kl}=\xi^{i-k,j}_{0,l}$ if $k\geqq 0$.) Moreover, by using (2.2), it can be reduced to the case $j=0$, i.e. $\partial^r_{x_{21}}\partial^s_{x_{22}}\xi^{00}_{0l}=0$ if $r+s>-l$, or $\partial^r_{x_{21}}\partial^s_{x_{22}}w_{0j}=0$ if $r+s>j$. By using the second equation of (1.9), it can be reduced to the case $j=1$ which is nothing but (3.6). This completes the proof. (The latter half of the statement can be proved as in Proposition 5.)

Theorem 3. *A solution* ξ *to the system of Eqs.* (2.1a), (2.1b), (2.1c), *and* (2.3) *corresponds to a self-dual (anti-self-dual) field on* Δ *if and only if its initial datum* $\xi^{(0)}$ *satisfies* (3.7).

Proof. Suppose that $\xi^{(0)}$ satisfies (3.7). Differentiating both sides of (2.11), we can see that $\tilde{\xi}$ also satisfies (3.7):

$$\text{If}\quad p+q\geqq i-k,\ k\geqq 0,\ r+s>j-l, \quad\text{then}\quad \partial^p_{x_{11}}\partial^q_{x_{12}}\partial^r_{x_{21}}\partial^s_{x_{22}}\tilde{\xi}^{ij}_{kl}=0\ . \tag{3.9}$$

Set

$$\mathscr{R}_3=\{A\in\mathscr{R}_2|A \text{ satisfies (3.7)}\}\ ,$$

$$\mathscr{F}_3=\{\xi\in\mathscr{F}_2|\xi \text{ satisfies (3.7)}\}$$

and

$$\mathscr{F}_3[[t]]=\left\{\sum_{i\geqq 0}\xi_i t^i|\xi_i\in\mathscr{F}_3\right\}\ .$$

Then $\mathscr{R}_3$ is a subring of $\mathscr{R}_2$ and $\mathscr{F}_3$ is an $\mathscr{R}_3$-module. It follows from (3.9) that $\tilde{\xi}\in\mathscr{F}_3[[t]]$, $\tilde{\xi}_{(-)}, \tilde{\xi}^{-1}_{(-)}\in\mathscr{R}_3[[t]]$. Therefore $\xi=\tilde{\xi}\cdot\tilde{\xi}^{-1}_{(-)}\in\mathscr{F}_3[[t]]$. q.e.d.

4. Special Solutions

Proposition 12. *Let* $w(\lambda)\leftrightarrow\xi$ *through the correspondence in Proposition* 4, $w^{(0)}(\lambda)=w(\lambda)|_{t=0}$ *and* $\xi^{(0)}=\xi|_{t=0}$. *Suppose that* $w(\lambda)$ *satisfies* (1.9). *Then the following* (i), (ii), (iii), *and* (iv) *are equivalent one another for any* $p,q\in\mathbb{N}$:

(i) *If* $i>p$ *or* $j>q$, *then* $w^{(0)}_{ij}=0$.

(ii) *If* $i>p$ *or* $j>q$, *then* $w_{ij}=0$.

(iii) *If* $k<-p$ *or* $l<-q$, *then* $\xi^{(0)ij}_{kl}=\delta^i_k\delta^j_l\mathbb{1}$ *for any* $(i,j)\in Z$.

(iv) *If* $k<-p$ *or* $l<-q$, *then* $\xi^{ij}_{kl}=\delta^i_k\delta^j_l\mathbb{1}$ *for any* $(i,j)\in Z$.

Proof.

Proof that (ii) *implies* (iv). When $(i,j)\in N^c$, (iv) is trivially satisfied because of (2.1a). Therefore we assume that $(i,j)\in N$. If $k<-p$, then $\xi^{ij}_{kl}=-\sum_{(g,h)\in N} w^*_{i-g,j-h}w_{g-k,h-l}=0$ because $g-k\geqq -k>p$ for $g\geqq 0$. If $l<-q$, then $\xi^{ij}_{kl}=-\sum_{(g,h)\in N} w^*_{i-g,j-h}w_{g-k,h-l}=0$ because $h-l\geqq l>q$ for $h\geqq 0$.

Proof that (iv) *implies* (ii). If $i>p$ or $j>q$, then $w_{ij}=-\xi^{0,0}_{-i,-j}=0$ because $-i<-p$ or $-j<-q$.

Equivalence of (i) *to* (iii). This can be proved similarly.

Equivalence of (iii) *to* (iv). It is obvious that (iv) implies (iii). We shall prove that (iii) implies (iv). Let

$$\mathscr{R}_4=\{A\in\mathscr{R}_1 \mid A \text{ satisfies (2.6)} \quad \text{and}$$

$$A^{ij}_{kl}=\delta^i_k\delta^j_l\mathbb{1} \quad \text{if} \quad i\geqq k<-p \quad \text{or} \quad j\geqq l<-q\} ,$$

$$\mathscr{F}_4=\{\xi\in\mathscr{F}_1 \mid \xi \text{ satisfies (2.6)} \quad \text{and}$$

$$\xi^{ij}_{kl}=\delta^i_k\delta^j_l\mathbb{1} \quad \text{if} \quad i\geqq k<-p \quad \text{or} \quad j\geqq l<-q\}$$

and

$$\mathscr{F}_4[[t]]=\left\{\sum_{i\geqq 0}\xi_i t^i \mid \xi_i\in\mathscr{F}_4\right\} .$$

Then $\mathscr{R}_4$ is a subring of $\mathscr{R}_1$ and $\mathscr{F}_4$ is a right $\mathscr{R}_4$-module on which $A_a\partial_{x_{ab}}$ $(a,b=1,2)$ act. If $\xi^{(0)}\in\mathscr{F}_4$, then $\tilde{\xi}=\sum_{p\geqq 0}\frac{1}{p!}\left(\sum_{a=1}^{2}\sum_{b=1}^{2}t_{ab}A_a\partial_{x_{ab}}\right)^p\xi^{(0)}\in\mathscr{F}_4[[t]]$, and hence $\tilde{\xi}_{(-)}\in\mathscr{R}_4[[t]]$. Therefore $\xi=\tilde{\xi}(\tilde{\xi}_{(-)})^{-1}\in\mathscr{F}_4[[t]]$. This completes the proof.

Thus, starting from an initial value $w^{(0)}(\lambda)$ which is a polynomial of λ_1^{-1}, λ_2^{-1}, we obtain such a solution.

Now we shall illustrate a simplest non-trivial example. Let

$$w^{(0)}(\lambda)=\mathbb{1}+w^{(0)}_{10}\lambda_1^{-1}+w^{(0)}_{01}\lambda_2^{-1} .$$

Then $\xi^{(0)i0}_{0l}=-\sum_{g=0}^{i}w^{(0)*}_{i-g,0}w^{(0)}_{g,-l}=-w^{(0)*}_{i0}w^{(0)}_{01}$ if $i\geqq 0$ and $l=-1$, and $\xi^{(0)i0}_{0l}=0$ otherwise. $\xi^{(0)0j}_{k0}=-\sum_{h=0}^{j}w^{(0)*}_{0,j-h}w^{(0)}_{-k,h}=-w^{(0)*}_{0j}w^{(0)}_{10}$ if $j\geqq 0$ and $k=-1$, and $\xi^{(0)0j}_{k0}=0$ otherwise. We can see that $w^{(0)*}_{i0}=(-w^{(0)}_{10})^i$ and that $w^{(0)*}_{0j}=(-w^{(0)}_{01})^j$ because $w(\lambda)^{-1}=\sum_{i,j\geqq 0}\binom{i+j}{i}(-w^{(0)}_{10})^i(-w^{(0)}_{01})^j\lambda_1^{-i}\lambda_2^{-j}$. Therefore (2.7) is written as

$$\begin{aligned}&\partial^p_{x_{11}}\partial^{i+1-p}_{x_{12}}\{(w^{(0)}_{10})^i w^{(0)}_{01}\}=0 \quad \text{for} \quad p=0,1,\ldots,i+1 ,\\ &\partial^q_{x_{21}}\partial^{j+1-q}_{x_{22}}\{(w^{(0)}_{01})^j w^{(0)}_{10}\}=0 \quad \text{for} \quad q=0,1,\ldots,j+1 .\end{aligned} \tag{4.1}$$

Since (4.1) reads $\partial_{x_{11}}w^{(0)}_{01}=\partial_{x_{12}}w^{(0)}_{01}=\partial_{x_{21}}w^{(0)}_{10}=\partial_{x_{22}}w^{(0)}_{10}=0$ when $i=j=0$, the gauge field corresponds to a self-dual Yang-Mills field on Δ if and only if

$$\partial^2_{x_{21}}w^{(0)}_{01}=\partial_{x_{21}}\partial_{x_{22}}w^{(0)}_{01}=\partial^2_{x_{22}}w^{(0)}_{01}=0 , \tag{4.2}$$

and corresponds to an anti-self-dual Yang-Mills field if and only if

$$\partial^2_{x_{11}}w^{(0)}_{10}=\partial_{x_{11}}\partial_{x_{12}}w^{(0)}_{10}=\partial^2_{x_{12}}w^{(0)}_{10}=0 . \tag{4.3}$$

Now set $w^{(0)}_{10}=c_{11}x^2_{11}+c_{12}x^2_{12}$, $w^{(0)}_{01}=c_{21}x^2_{21}+c_{22}x^2_{22}$, $c_{ab}\in M_n(\mathbb{C})$ for $a,b=1,2$. Then (4.1) is satisfied for $i=j=0$. Equation (4.2) is equivalent to $c_{11}=c_{12}=0$ and (4.3) is equivalent to $c_{21}=c_{22}=0$. On the other hand, noting that $A^{(0)}_{t_{ab}}=2c_{ab}x_{ab}$ for

$a, b=1, 2$, we see that if the gauge field ∇ is abelian, then $[c_{ab}, c_{de}]=0$ for $a, b, d, e=1, 2$. Note that the gauge field ∇ corresponds to an abelian Yang-Mills field on Δ if and only if ∇ is abelian itself. Now set, for example,

$$c_{11}=E_{21}=(\delta_{i2}\delta_{j1})_{i=1,\dots,n,\, j=1,\dots,n} ,$$

$$c_{12}=E_{32}=(\delta_{i3}\delta_{j2})_{i=1,\dots,n,\, j=1,\dots,n} ,$$

$$c_{22}=E_{54}=(\delta_{i5}\delta_{j4})_{i=1,\dots,n,\, j=1,\dots,n} ,$$

$$c_{22}=E_{65}=(\delta_{i6}\delta_{j5})_{i=1,\dots,n,\, j=1,\dots,n} .$$

Then $w_{01}^{(0)}w_{10}^{(0)}=w_{10}^{(0)}w_{01}^{(0)}=0$ and especially (4.1) is satisfied for $i\geqq 0$ and $j\geqq 0$. Neither (4.2) nor (4.3) holds because $c_{ab}\neq 0$ for $a, b=1, 2$. The gauge field is not abelian because $[c_{11}, c_{12}]$, $[c_{21}, c_{22}]\neq 0$. Thus we obtain a gauge field corresponding to a Yang-Mills field on Δ which is not abelian, self-dual nor anti-self-dual.

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