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Absence of Divergences in Type II and Heterotic String Multi-Loop Amplitudes

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Abstract. A detailed analysis is given of the two main types of degeneration of Riemann surface of arbitrary genus by domain variational theory. Explicit estimates for first and third Abelian functions are given. These estimates are used to analyse the possible divergences of type II or heterotic superstring multi-loop amplitudes for the scattering of massless particles. They are all shown to be finite at arbitrary loop order.

I. Introduction

A vast mood of euphoria has swept over the theoretical high-energy physics community with the discovery of five superstring theories as putative candidates for a unified theory of the forces of nature. These are the $O(32)$ Chan-Paton model, equal to the even G-parity sector of the 1971 dual pion model for the bosonic sector [1], the $O(32)$ or $E(8)\times E(8)$ heterotic strings [2], and the chiral or non-chiral versions of the closed superstring [3]. It is supposed that these theories all give a viable quantum gravity, though their phenomonological features may single out the $E(8)\times E(8)$ heterotic string as most promising [4]. Yet the euphoric atmosphere is based only on results of finiteness of tree and one-loop amplitudes. There is almost no information on the finiteness properties of higher-loop super-string amplitudes, nor on the convergence or otherwise of the loop perturbation expansion. Indeed there are no specific evaluations of any higher loop amplitudes. However, on the basis of general arguments, using supersymmetry and/or functional methods, it has been claimed [2, 5] that all closed superstring amplitudes are finite at all loops. It is the purpose of this paper to analyse that situation.

Only heterotic and type II closed superstrings will be discussed here, and that by means of functional techniques in the light cone gauge. Closed superstring theories are chosen rather than the open case mentioned above since the former are much simpler to consider. In the case of open superstrings there are mathematical complexities arising from the fact that the corresponding world sheets Σ are open Riemann surfaces, and some difficult mathematical problems ensue (due to the continuous part of the spectrum of the Laplace-Beltrami operator). Functional techniques [6] will be used since they will allow deployment of powerful conformal mapping techniques pioneered by Mandelstam [7] to obtain an explicit form for closed superstring amplitudes.

The light-cone (LC) gauge will be chosen since only physical modes appear in that case [8]. Moreover it has proven possible to give a specific and complete construction of the second quantised field theory for type II superstrings in the LC gauge in a particularly useful form [9]. This can be reduced, by the methods of Kaku and Kikkawa [10], to a first quantised form. Thus functional methods can be used directly in that case. Such a construction can be extended to the heterotic string, since that for type II superstrings involved a separate construction for the left and right movers. Results obtained in this way should agree with those obtainable by covariant methods when those are available. Since the LC theory is the simplest available it seems appropriate to use it for the purpose at hand, viz divergence analysis.

Using these methods it has been possible to construct a closed form for the g-loop amplitude for any number of external massless particles [11]. These expressions involve solely the geometric quantities on the surface Σ, besides the external sources and momenta. They also involve an integration over parameters similar to those of Ahlfors [12] describing conformally inequivalent Riemann surfaces Σ. Besides divergences which may arise from integration over the external source variables there may also be difficulties from integration over the Ahlfors variables. The purpose of this paper will be to discuss, in detail, what such latter divergences might be. They correspond especially to degeneration of a compact Riemann surface Σ corresponding to the superstring world sheet.

The problem of divergences for bosonic string amplitudes were analysed in the 70's [13], though with inconclusive results as to their renormalizability. It was already appreciated then that divergences did not arise from the ultra-violet regime of the constituent string modes, but rather from infra-red divergences due to the presence of a tachyon and of a massless scalar particle (the dilaton) with non-zero vacuum expectation value. The divergences of the planar one-loop open bosonic string amplitude can be recognised as arising from these sources [14]. The one-loop finiteness of the $O(32)$ Chan-Paton model [1] has been argued [15] as being due to the absence of the tachyon and of dilaton tadpole-type divergences, due to supersymmetry annihilating the tachyon and causing the dilaton vacuum expectation value to vanish.

Attempts were made to give a detailed analysis of higher loop bosonic string amplitudes, especially in the last reference in ref. [13]. New developments, especially through the use of more sophisticated mathematical techniques, have allowed a more precise analysis to be made of divergences in the bosonic string multi-loop partition function. The tachyon and dilaton divergences are seen as arising from degenerations of the Riemann surface into one with the same number or fewer handles, with pinches at certain waists' or dividing geodesics or at the handles. Such degenerations produce new zero modes which cannot be removed from the laplacians on Σ. In particular the partition function $(\det' \Delta_0)^{-12}$ (in the bosonic case) is then divergent as a function of the length ℓ of a vanishing geodesic.

Initial analysis [16] of the bosonic string dilaton divergence by Selberg ζ-function techniques has more recently been superseded by use of the more powerful techniques of algebraic geometry [17]. These have allowed, in particular by the work of Belavin and Knizhnik [18], an elegant understanding of the dilaton and tachyon singularities in terms of a section of the appropriate holomorphic determinant line bundle on Σ. Poles in this section at points on the boundary of moduli space, obtained by adjoining stable curves with nodes, are the sources of the above-mentioned divergences in bosonic string amplitudes. Extension of the algebraic geometric approach to superstrings has been conjectured in the above Refs. [17, 18], but no results yet obtained on divergence analysis for any superstring case. In this article the explicitly constructed multi-loop amplitudes for type II and heterotic superstrings [11] will be analysed directly for such divergences (and any others); results from the bosonic analysis of refs. [17, 18] will be used where appropriate.

The multi-loop superstring amplitudes constructed in the LC gauge by functional methods [11] involve the first and third Abelian differentials and the period matrix for the relevant Riemann surfaces Σ. The manner in which these depend on the parameters ℓ etc. as these become zero near a degeneration of Σ can be determined by domain variational theory [19]. Such an approach leads to integral equations for the third Abelian differentials which may be solved by perturbation theory. Those solutions may then be used to analyse the possible singularities in the multi-loop amplitudes. This procedure had already been used for the bosonic string by Alessandrini and Amati [13] and by the authors earlier for the superstring [20]. It is proposed to give a more detailed analysis of the divergence question in this paper using the above techniques; this paper is to regarded as an amplification of the papers in ref. [20].

The paper begins with a description of domain variational theory using the notation of Lebowitz [21]. This is given both to outline the concepts and also to specify the functions needed. In the next section this description is given explicitly for the case of a handle degeneration, and in the following section for a dividing geodesic degeneration. The form of the multi-loop superstring amplitude for external bosonic states is specified in Sect. 4. The degeneration analysis is then applied to the type II superstring in Sect. 5 and to the heterotic string in the following section. The succeeding section considers degenerations arising from the coalescence of punctures (external sources) with the handles, so corresponding to another way of reducing the genus. The bosonic partition function is then analysed at such degenerations. The final Sect. 8 contains conclusions and discussion.

2. Handle Degeneration

Handle degeneration may be analysed by starting with a Riemann surface S, of genus $g-1$, and attaching a handle of small size to it to construct a new Riemann surface S^* of genus g. All quantities on the new surface are starred to distinguish them from the unstarred quantities on S. S^* is constructed from S by deleting the interiors of two parameter-discs on S, and identifying their boundaries.

To describe this in detail let (Γ, Δ) be a fixed one-dimensional homology basis on S, with $\Gamma=(\gamma_1,\ldots,\gamma_{g-1})$, $\Delta=(\delta_1,\ldots,\delta_{g-1})$ with only γ_i intersecting with δ_i. A basis of the first Abelian differentials on S is denoted by $(du_1,\ldots,du_{g-1})$, with normalisation $\int_{\gamma_i} du_j = \delta_{ij}$. The period matrix Π_{ij} is defined by $\Pi_{ij} = \int_{\delta_i} du_j = \Pi_{ji}$, and has $\mathrm{Im}\,\Pi > 0$. A normal third Abelian integral η_{XY} has poles in $d\eta_{XY}$ with residue -1 at X and $+1$ at Y and is normalised so that $\int_{\gamma_i} d\eta_{XY} = 0$. For any two points P_0 and Q_0 of S not on (Γ, Δ), the parameter disks D_{P_0}, D_{Q_0} are defined about P_0 and Q_0 with boundaries $C_{P_0}: \mathrm{Re}\,\eta_{P_0Q_0}(P) = \log\varepsilon$, $C_{Q_0}: \mathrm{Re}\,\eta_{P_0Q_0}(P) = -\log\varepsilon$, respectively. Points P', Q' on C_{P_0}, C_{Q_0} are identified if

$$\eta_{P_0Q_0}(P') - \eta_{P_0Q_0}(Q') = 2\log\varepsilon + 2i\alpha \ . \tag{2.1}$$

A Jordan arc is drawn from C_{P_0} to C_{Q_0} so as to have no intersection with (Γ, Δ) and $C_{P_0} = C_{Q_0}$ is denoted by δ. Then the surface S^* obtained by deleting the interiors of C_{P_0} and C_{Q_0} and taking $\Gamma^* = (\gamma_1, \ldots, \gamma_{g-1}, \gamma)$, $\Delta^* = (\delta_1, \ldots, \delta_{g-1}, \delta)$ is a Riemann surface of genus g.

The basic variational equation relating normal third Abelian integrals η_{XY} and η^*_{ZW}, in terms of the difference $\Delta\eta_{XY} = \eta^*_{XY} - \eta_{XY}$ is [21]

$$\Delta\eta_{XY}(Z) - \Delta\eta_{XY}(W) = \frac{1}{2\pi i} \int_{\partial S_0} \eta_{XY}\, d\eta^*_{ZW} \ , \tag{2.2}$$

where S_0 is the common domain of S and S^*, so is the whole of S outside C_{P_0} and C_{Q_0}. The integral Eq. (2.2) may be solved by expanding η_{XY} on the right-hand side of (2.2) about P_0 and keeping only the lowest order term. From (2.1), $[2\ln\varepsilon + 2i\alpha]^{-1}$ $\eta_{P_0Q_0}$ has the same discontinuity in W on S^* as $(2\pi i)^{-1} \int_\delta d\eta^*_{XY}$; the right-hand side of (2.2) thereby reduces to an expression purely in terms of η to give

$$d\eta^*_{XY}(Z) = d\eta_{XY}(Z) - [2\ln\varepsilon + 2i\alpha]^{-1}\,[\eta_{XY}(P_0) - \eta_{XY}(Q_0)]\,d\eta_{P_0Q_0}(Z) + O(\varepsilon) \ . \tag{2.3}$$

Moreover the new first Abelian differential is therefore [21]

$$du^*_g(Z) = [2\ln\varepsilon + 2i\alpha]^{-1}\, d\eta_{P_0Q_0}(Z) + 0(\varepsilon) \ . \tag{2.4}$$

The remaining first Abelian differentials are obviously $du^*_i = du_i$ $(i = 1, \ldots, g-1)$. Besides the first Abelian differentials the object on Σ of crucial importance for the construction of the multi-loop amplitudes [11] is the Greens function $\mathbf{G}$ defined from the first and third Abelian integrals. This is defined to have real part $G = \mathrm{Re}\,\mathbf{G}$ single-valued round (Γ, Δ), and may be given in terms of the real parameters α^*_i, β^*_i $(1 \leq i \leq g)$ specifying the interior string widths and twists. In terms of the α^*_i, there is the explicit formula [5]

$$\mathbf{G}^*_{P_1P_N}(P) = \eta^*_{P_1P_N}(P) + i \sum_{j=1}^{i} \alpha^*_j u^*_j(P) \ . \tag{2.5}$$

Then

$$\alpha^*_j = \mathrm{Im}\,[\mathbf{G}^*_{P_1P_N}(\gamma_i P) - \mathbf{G}^*_{P_1P_N}(P)] \ , \tag{2.6}$$

since η^* is single-valued and u^*_k has change δ_{jk} around γ_j. The string width variable β^*_i is then defined by

$$\beta^*_j = \mathrm{Im}\,[\mathbf{G}^*_{P_1P_N}(\delta_j P) - \mathbf{G}^*_{P_1P_N}(P)] \ . \tag{2.7}$$

From the residue theorem

$$\int_{\delta_j^*} d\eta^*_{XY} = 2\pi i [u_j^*(Y) - u_j^*(X)]$$

and the definition of the period matrix,

$$\beta_j^* = 2\pi \cdot \mathrm{Re}\,[u_j^*(P_N) - u_j^*(P_1)] + \sum_i \alpha_i^* \,\mathrm{Re}\,\Pi_{ij}^* \ . \tag{2.8}$$

The condition of single-valuedness of Re $\mathbf{G}^*_{P_1 P_N}$ on Σ leads directly to the condition

$$\alpha_j^* = 2\pi (\mathrm{Im}\,\Pi^*)^{-1}_{jk}\, \mathrm{Im}\,[u_k^*(P_1) - u_k^*(P_N)] \ . \tag{2.9}$$

Equations 2.8) and (2.9) may be combined together as

$$\beta_j^* = 2\pi [u_j^*(P_N) - u_j^*(P_1)] + \sum_i \alpha_i^* \Pi_{ij}^* \ . \tag{2.10}$$

The notation being used here is that of ref. [21], in which capital letters denote points on the surface and small latin letters denote their uniformizations. Then P_1 and P_N are usually denoted by the Koba-Nielsen source values z_1, z_N, and the mapping function for the string from the z to the ϱ-plane is usually [11] denoted $\varrho = \mathbf{G}_{z_1 z_N}(z)$. In the above notation

$$\varrho = F^*(P) = \mathbf{G}^*_{P_1 P_N}(P) \ . \tag{2.11}$$

Equations (2.3) and (2.4) may be used to give, to first non-trivial order, the functions and parameters on the surface S^* in terms of their values on S, provided that the period matrix Π^* is written explicitly in terms of Π. Using (2.4) and the definition of the period matrix this results in

$$\begin{aligned} &\Pi_{ij}^* = \Pi_{ij} \quad (1 \leqq i, j \leqq (g-1)) \ , \\ &\Pi_{gg}^* = 2\pi i [2 \ln \varepsilon + 2i\alpha)]^{-1} \ , \\ &\Pi_{gj}^* = 2\pi i [2 \ln \varepsilon + 2i\alpha]^{-1} [u_j(Q_0) - u_j(P_0)] \quad (1 \leqq j \leqq (g-1)) \end{aligned} \tag{2.12}$$

with the associated inverse

$$\begin{aligned} &(\mathrm{Im}\,\Pi^*)^{-1}_{ij} = (\mathrm{Im}\,\Pi^{-1})_{ij} \quad (1 \leqq i, j \leqq (g-1)) \ , \\ &(\mathrm{Im}\,\Pi^*)^{-1}_{gg} = \Pi^{-1} \ln \varepsilon \ , \\ &(\mathrm{Im}\,\Pi^*)^{-1}_{gj} = -\sum_{k=1}^{(g-1)} (\mathrm{Im}\,\Pi)^{-1}_{jk}\, \mathrm{Re}\,[u_k(Q_0) - u_k(P_0)] \ . \end{aligned} \tag{2.13}$$

Using (2.3), (2.4), (2.8), (2.9), (2.11), (2.12), and (2.13) leads to

$$\begin{aligned} dF^*(P) = d\eta_{P_1 P_N}(P) + i \sum_{j=1}^{(g-1)} \alpha_j du_j(P) + (2 \ln \varepsilon)^{-1} [i\alpha_g^* - \eta_{P_1 P_N}(P_0) \\ + \eta_{P_1 P_N}(Q_0)] d\eta_{P_0 Q_0}(P) \end{aligned} \tag{2.14}$$

with

$$\alpha_j^* = \alpha_j = 2\pi(\mathrm{Im}\,\Pi)_{jk}^{-1}\,\mathrm{Im}\,[u_k(P_1) - u_k(P_N)] \quad (j<g) \ , \tag{2.15}$$

$$\alpha_j^* = \mathrm{Im}\,\{[\eta_{P_1P_N}(P_0) - \eta_{P_1P_N}(Q_0) - 2\pi i((u_k(Q_0) - u_k(P_0))]\,[1 - i\alpha(\ln\varepsilon)^{-1}]\} \\ \cdot(\mathrm{Im}\,\Pi)_{kj}^{-1} \times \mathrm{Im}\,[u_j(P_1) - u_j(P_N)] \ , \tag{2.16}$$

$$\beta_j^* = \beta_j \quad (j<g) \ ,$$

$$\beta_g^* = +(\pi/\ln\varepsilon)\{\mathrm{Re}\,[(\eta_{P_1P_N}(Q_0) - \eta_{P_1P_N}(P_0))(1 - i\alpha/\ln\varepsilon)] \\ + \alpha\alpha_g^*(\ln\varepsilon)^{-1} - \sum_{j=1}^{g-1} \alpha_j\,\mathrm{Re}\,[(u_j(Q_0) - u_j(P_0))(1 - i\alpha(\ln\varepsilon)^{-1})]\} \ , \tag{2.17}$$

where terms of $O(\ln\varepsilon)^{-2}$ in (2.14) and $O(\ln\varepsilon)^{-1}$ in (2.16) have been dropped, though the terms $O(\alpha(\ln\varepsilon)^{-1})$ in the latter have been kept. The analysis of handle degeneration has now reached the point at which it can be applied to the specific analysis of the superstring amplitudes of ref. [11] by solving for the interaction points $\tilde{P}$ for which

$$dF^*(\tilde{P}) = 0 \ . \tag{2.18}$$

This will be considered after the degeneration of a dividing geodesic is analysed in the next section.

3. Dividing Geodesic Degeneration

Degeneration as the length ℓ of a dividing geodesic tends to zero has been considered in detail in ref. [21] and also by Fay [22]. It will turn out that there is further work to do beyond a simple application of these references, since the degeneration parameter (Σ of ref. [21] or t of ref. [22]) is not directly related to the length ℓ. Since this latter parameter enters the superstring amplitude in a crucial fashion, the relation between these parameters ℓ and ε (or t) must be determined; that will be discussed at the end of this section.

The principles of the degeneration anylysis of the last section can be used to go beyond the analysis of Lebowitz [21] to produce the equivalent to Eqs. (2.4), (2.12), (2.14), (2.15), and (2.16). The degenerating surface S^* is obtained by gluing together two surfaces S_1, S_2 of genus g_1 and g_2 respectively at points A_1 and B_2. This is achieved across the boundaries $C_{P_1}: t_1 = \exp[-\eta_{A_1B_1,1}(P)]$, $C_{Q_2}: t_2 = \exp[\eta_{A_2B_2,2}(Q)]$ on S_1 and S_2 respectively with $|t_1| = |t_2| = \varepsilon$, where A_i, B_i are points on S_i and $\eta_{A_iB_i,i}$ are normal third Abelian integrals on S_i $(i=1,2)$. The identification is by means of the condition

$$t_1 t_2 = t \ , \tag{3.2}$$

where t is a small parameter (in the notation of Fay [22]) equal to $\varepsilon^2\,\phi$ (in the notation of Lebowitz [21]) with $|\phi| = 1$. Following the arguments of ref. [21] it is possible to obtain the equivalent integral equation to (2.2), in terms of $\Delta\eta_{XY} = d\eta_{XY}^*$

$-\tilde{\eta}_{XY}$, when X and Y are both in the interior of $S_1 \cap S^*$, with

$$\tilde{\eta}_{XY} = \begin{cases} \eta_{XY1} & \text{on} \quad S_1 \cap S^* \\ 0 & \text{on} \quad S_2 \cap S^* \end{cases} . \tag{3.2}$$

The resulting integral equation is

$$\Delta\eta_{XY}(Z) - \Delta\eta_{XY}(W) = \frac{1}{2\pi i} \eta'_{XY,1}(A_1)\varepsilon^2 \phi \int_{C_{B_2}} t_2^{-1} d\eta^*_{ZW} + O(\varepsilon^4) , \tag{3.3}$$

where t_2 is the local parameter on C_{Q_2} given above. The standard relation

$$\frac{1}{2\pi i} \int_{\delta_i^*} d\eta^*_{XY} = [u_i^*(Y) - u_i^*(X)] \tag{3.4}$$

allows an evaluation of the right-hand side of (3.4) using the value of $d\eta^*_{XY}$ on the left-hand side given by (3.3), with that on the right-hand side of (3.3), $d\eta^*_{ZW} = O(\varepsilon^2)$ on S_2 if $Z, W \in S_1$, and $d\eta^*_{ZW} = [d\eta_{ZW,2} + O(\varepsilon^2)]$ on S_2 if $Z, W \in S_2$. Then for $\delta_i^* \in S_1$, (3.4) leads, for $X \in S_1$, to

$$du_i^*(X) = du_{i1}(X) + tc_1 d_x \eta'_{XY,1}(A_1) \quad (1 \leqq i \leqq g_1) \tag{3.5a}$$

with $c_1 = (2\pi i)^{-1} \int_{C_{B_2}} t_2^{-1} d\eta_{ZW,2}$ as $\varepsilon \sim 0$. Similarly if $\delta_i^* \in S_2$, then (3.3) and (3.4) lead, for $X \in S_1 \cap S^*$, by use of the residue theorem, to

$$du_i^*(X) = tu_i'(B_2) d_x \eta_{XY,1}(A_1) \quad ((g_1+1) \leqq i \leqq g_2) . \tag{3.5b}$$

A similar result follows for $X \in S_2 \cap S^*$, using the appropriate version of (3.3) with S_1 and S_2 interchanged, to give

$$du_i^*(X) = du_{i1}(X) + tc_2 d_x \eta'_{XY,2}(B_2) \tag{3.5c}$$

with $c_2 = (2\pi i)^{-1} \int_{C_{A_1}} t_1^{-1} d\eta_{ZW,1}$, for $\delta_i^* \in S_2$. Finally for $\delta_i^* \in S_1$ and $X \in S_2 \cap S^*$, then

$$du_i^*(X) = tu_i'(A_1) d_x \eta'_{XY,2}(B_2) . \tag{3.5d}$$

Equations (3.5a)–(3.5d) agree with the values obtained by Fay [22] by a different method. These values may then be used, together with the definition of the period matrix to lead to [22]

$$\Pi_{ij}^* = \Pi_{1ij} , \qquad 1 \leqq i,j \leqq g_1 \tag{3.6a}$$

$$= \Pi_{2ij} , \qquad (g_1+1) \leqq i,j \leqq (g_1+g_2) , \tag{3.6b}$$

$$= tu_i(A_1)u_j(B_2) \qquad 1 \leqq i \leqq g_1, (g_1+1) \leqq j \leqq (g_1+g_2) . \tag{3.6c}$$

It is finally possible to deduce from (3.3) that for $X, Y, Z \in S_1 \cap S^*$

$$d\eta^*_{XY}(Z) = d\eta_{XY,1}(Z) + O(t^2) \tag{3.7a}$$

and for $Z \in S_2 \cap S^*, X, Y \in S_1 \cap S^*$,

$$d\eta^*_{XY}(Z) = t\eta_{XY,1}(A_1) d_z \eta_{ZW,2}(B_2) . \tag{3.7b}$$

The construction of $d\eta^*_{XY}(Z)$ for $X, Z \in S_1 \cap S^*$, $Y \in S_2 \cap S^*$ is simple, since to $O(t)$, $d\eta^*_{XY}(Z)$ only has a simple pole at X, and so in this region

$$d\eta^*_{XY}(Z) = \partial/\partial_{Z_1} \eta_{Z_1 Z_2,1}(X)|_{Z_1=Z_2=Z} \ . \tag{3.7c}$$

Similarly for $X \in S_1 \cap S^*$, $Y, Z \in S_2 \cap S^*$,

$$d\eta^*_{XY}(Z) = -\partial/\partial_{Z_1} \eta_{Z_1 Z_2,2}(Y)|_{Z_1=Z_2=Z} \ . \tag{3.7d}$$

The above equations now lead to the values

$$dF^*(P) = d\eta_{P_1 P_N,1}(P) + i \sum_{j=1}^{g_1} \alpha_j du_j(P) \tag{3.8a}$$

for $P, P_1, P_N \in S_1 \cap S^*$, whilst for $P, P_1, P_N \in S_2 \cap S^*$,

$$dF^*(P) = d\eta_{P_1 P_N,2}(P) + i \sum_{j=g_1+1}^{g_1+g_2} \alpha_j du_j(P) \ . \tag{3.8b}$$

In the case $P_1, P_N \in S_1 \cap S^*$ and $P \in S_2 \cap S^*$,

$$\begin{aligned} dF^*(P) = {} & t\eta'_{P_1 P_N,1}(A_1) d_P \eta'_{PW,2}(B_2) + \sum_{i \leqq g_1} \alpha_i t u'_i(A_1) d_P \eta'_{PW,2}(B_2) \\ & + \sum_{g_1 < i \leqq g_1+g_2} \alpha^*_j du_j(P) \end{aligned} \tag{3.8c}$$

and for $P_1, P \in S_1 \cap S^*$, $P_N \in S_2 \cap S^*$,

$$dF^*(P) = \partial/\partial_{Z_1} \eta_{Z_1 Z_2,1}(P)|_{Z_1=Z_2=P} + i \sum_{j=1}^{g} \alpha_j du_j(P) \ . \tag{3.8d}$$

Moreover

$$\begin{aligned} (\operatorname{Im} \Pi^*)^{-1}_{ij} &= (\operatorname{Im} \Pi_1)^{-1}_{ij} && 1 \leqq i, j \leqq g_1 \\ &= (\operatorname{Im} \Pi_2)^{-1}_{ij} && g_1 < i, j \leqq g_1 + g_2 \\ &= -t \operatorname{Im} [u_i(A_1) u_j(B_2)] && 1 \leqq i, j \leqq g_1 ,\ g_1 < j \leqq g_1 + g_2 \ . \end{aligned} \tag{3.9}$$

For $P_1, P_N \in S_1 \cap S^*$,

$$\begin{aligned} \alpha^*_i &= \alpha_i \ , \\ \alpha^*_j &= -2\pi t \sum u_j(B_2) u_k(A_1) \operatorname{Im} [u_k(P_1) - u_k(P_N)] \\ &\quad + 2\pi t \sum_{k > g_1} (\operatorname{Im} \Pi_2)^{-1}_{jk} u'_k(B_2) d_{P_1} \eta'_{P_1 P_N,1}(A_1) \ , \end{aligned} \tag{3.10a}$$

whilst for $P_1 \in S_1$, $P_N \in S_2$,

$$\begin{aligned} \alpha^*_i &= 2\pi \sum_{j \leqq g_1} (\operatorname{Im} \Pi_1)^{-1}_{ij} \operatorname{Im} u_j(P_1) \quad (1 \leqq i \leqq g_1) \ , \\ \alpha^*_j &= -2\pi \sum_{k > g_1} (\operatorname{Im} \Pi_2)^{-1}_{jk} \operatorname{Im} u_k(P_1) \quad (g_1 < j \leqq g_1 + g_2) \ . \end{aligned} \tag{3.10b}$$

The values of β_j are accordingly

$$\beta_j^* = \beta_j \quad (1 \leqq j \leqq g_1\,;\, P_1, P_N \in S_1 \cap S^*) \tag{3.11a}$$

$$= 2\pi t u_i'(B_2)\,[d_{P_N}\eta'_{P_N P_1,1}(A_1) - d_{P_1}\eta'_{P_1 P_N,1}(A_1)]$$

$$+ \sum_{j=1}^{g_1} \alpha_j t \,\mathrm{Re}\,[u_i'(A_1)u_j'(B_2)] + \sum_{j=g_1+1}^{g_1+g_2} \alpha_j^* \,\mathrm{Re}\,\Pi_{2ij}$$

$$(g_1 < i \leqq g_1 + g_2\,;\, P_1, P_N \in S_1 \cap S^*) \tag{3.11b}$$

$$= -2\pi u_i(P_1) + \sum_{j,k \leqq g_1} 2\pi(\mathrm{Im}\,\Pi_1)_{jk}^{-1}\,\mathrm{Im}\,u_k(P_1)\,\mathrm{Re}\,\Pi_{1ij}$$

$$(1 \leqq i < g_1\,;\, P_1 \in S_1 \cap S^*,\, P_N \in S_2 \cap S^*) \tag{3.11c}$$

$$= 2\pi u_i(P_N) - \sum_{j,k > g_1} 2\pi(\mathrm{Im}\,\Pi_2)_{jk}^{-1}\,\mathrm{Im}\,u_k(P_N)\,\mathrm{Re}\,\Pi_{2ij}$$

$$(g_1 < i \leqq g_1 + g_2\,;\, P_1 \in S_1 \cap S^*,\, P_N \in S_2 \cap S^*)\ . \tag{3.11d}$$

This completes the analysis of the relevant functions on the surface associated with the dividing geodesic degeneration.

There is still the relationship between the vanishing parameters ℓ and t or ε. Upper and lower bounds on this relationship may be obtained by using the inequalities of Masur [23] relating the Poincare metric ϱ_* on the annulus $A_t^\delta : |t|(1-\delta)^{-1} < |z| < (1-\delta)$ around a uniformization of C_A, with the Poincaré metric ϱ on S^*. For small enough t, Masur showed that there is a constant C so that on A_t^δ,

$$\varrho_* \geqq \varrho \geqq C\varrho_*\ . \tag{3.12}$$

Moreover ϱ_* is given explicitly by

$$\varrho_*(z) = \pi\,[|z| \log|t| \sin(\pi \log|z|/\log|t|)]^{-1}\ . \tag{3.13}$$

As pointed out by Wolpert [24] the dividing geodesic has $|z| = |t|^{\frac{1}{2}}$ (by symmetry, as can be seen by uniformising in a strip by $w = \log z$). Then from (3.12) and (3.13) at $|z| = |t|^{\frac{1}{2}}$

$$2\pi|t|^{\frac{1}{2}}\varrho_*(|z| = |t|^{\frac{1}{2}}) \geqq \ell \geqq 2\pi C|t|^{\frac{1}{2}}\varrho_*(|z| = |t|^{\frac{1}{2}})$$

or

$$2\pi^2(\log|t|^{-1})^{-1} \geqq \ell \geqq 2\pi^2 C(\log|t|^{-1})^{-1}\ . \tag{3.14}$$

This [21] proves a conjecture made earlier by one of us (J.G.T.) that

$$\ell = O[(\log|t|)^{-1}] \tag{3.15}$$

in which also the left-hand inequality in (3.14) was proved. The Weyl-Petersson measure $\prod_{i=1}^{3g-3} d\tau_i \wedge d\ell_i$ [23] (where the τ_i are the Fenchel-Nielsen twists associated with the geodesic length parameters ℓ_i), near a degeneration $\ell \sim 0$ has ℓ-dependent part $\ell d\ell d(\arg\tau)$. Use of (3.14) allows this to be rewritten as $d(\arg t)|t|^{-1}(\ln|t|)^{-3}d|t|$. This measure will be used later in analysis of the dividing geodesic degeneration in the heterotic superstring amplitude.

4. The Form of the Multi-loop Amplitude

It is now proposed to apply the technology developed in the last two sections to superstring multi-loop amplitudes. The rules for constructing the latter were established in ref. [11]. They can be given in terms of string-Feynman diagram rules, which involve an amalgamation of string diagram factors and two-dimensional quantum field theoretic factors. The former correspond essentially to the Veneziano-Virasoro-Shapiro type expression $\exp[p_r p_s G(z_r, z_s)]$ associated with a set of N source points z_r $(1 \leqq r \leqq N)$ and contain the singular factors $|z_r - z_s|^{-p_r p_s}$ as $z_r \sim z_s$. The quantum field-theoretic factors involve a set of $2g$ interaction vertices $\tilde{z}_\alpha$ $(\alpha = 1, \ldots, 2g)$, which are the zeros of

$$dF(\tilde{z}_\alpha) = 0 \ , \tag{4.1}$$

where F is defined by (2.11). From now on the points P_r and their uniformisations z_r will be identified, as will $\tilde{P}_\alpha$ and $\tilde{z}_\alpha$. Then the field-theoretic factors arise from either (a) bosonic Feynman lines $2\partial_A \partial_B G(A, B)$, where A, B can be either a source point P_r or an interaction vertex P, with $G(A, B) = \mathrm{Re}\, \mathbf{G}_{BP_N}(A)$, (b) alternatively bosonic Feynman lines $\sum_{r=1}^{N-1} \partial_A G(A, P_r) P_r$, with P_r the external momenta, or (c) fermionic Feyman lines $\partial_{\tilde{z}_\alpha} G(\tilde{z}_\alpha, z_r)$ from an interaction vertex $\tilde{z}_\alpha$ to an external source z_r. There are at most two such lines from a given $\tilde{z}_\alpha$, and at most one to each z_r. There cannot be both a bosonic and a fermionic line between a $\tilde{z}_\alpha$ and a z_r. There must always be one bosonic line from each interaction vertex (IV), the latter having associated with it the factor $[F''(\tilde{P}_\alpha)]^{-3/2}$. There is also a tensor $T_{\mathbf{j}}(\{\tilde{P}\})$ when there are no fermionic lines, depending on the positions of the IV's and defined, by

$$T_{\mathbf{j}}(\{\tilde{P}\}) = \int \prod_{i=1}^{g} d^4\theta_i \prod_{\gamma=1}^{2g} \left(\sum_{i_\gamma=1}^{g} \theta_{i_\gamma} u'_{i_\gamma}(\tilde{z}_\gamma) \right)^{2 j_\gamma} \ , \tag{4.2}$$

where $\{\theta_i\}$ $(1 \leqq i \leqq g)$ form a set of Grassmann variables circulating around the loops of the string Riemann surface, and $\theta^{2j} = \varrho^j_{AB} \theta^A \theta^B$, with ϱ^j_{AB} being the $SU(4)$ Dirac matrices. Consider first the contribution with no fermionic lines. Let there be r lines of type (a) joining sources to IV's, s lines of type (a) joining IV's to each other, t lines of type (b) joining sources to IV's and u lines of type (b) joining sources to each other. Then the Feynman diagram factor is [11]

$$A^{(r,s,t,u)} = \prod_{A_r} \partial_{z_r} \partial_{\tilde{z}_{\alpha_r}} G(z_r, \tilde{z}_{\alpha_r}) \zeta_r^{\alpha_r} \prod_{A_s} \partial_{\tilde{z}_{\alpha_i}} \partial_{\tilde{z}_{\alpha'_i}} G(\tilde{z}_{\alpha_i}, \tilde{z}_{\alpha'_i}) \delta_{j_{\alpha_i} j_{\alpha'_i}}$$
$$\cdot \prod_{A_t} \left(\sum_{\ell=1}^{N-1} \partial \tilde{z}_{\alpha_\gamma} G(\tilde{z}_\gamma, z_\ell) p_\ell \right)^{j_\gamma} L_u T_{\mathbf{j}} \prod_{\alpha=1}^{2g} [F''(\tilde{z}_\alpha)]^{-3/2} \ . \tag{4.3}$$

In (4.3), A_r denotes the set of paired sources and IV's $(z_r, \tilde{z}_{\alpha_r})$ for type (a) lines, A_s the set of paired IV's $(\tilde{z}_{\alpha_i}, \tilde{z}_{\alpha'_i})$ for type (a) lines, and A_t the set of IV's $\tilde{z}_\gamma$ for type (b) lines. The remaining factor L_u in (4.3) is a function only of the external source points z_u, to which (b) lines are attached, and their polarisation vectors ζ^i. These latter occur either in scalar products with each other or with the external momenta. The function L_u was discussed in detail in the last ref. in [11]. When fermionic lines are present, it is easiest to consider solely the case $N = 4$, though the results can be generalised directly. The tensors $T_{\mathbf{j}}(\mathbf{z})$ of Eq. (4.2) are now replaced by multi-spinor tensors

$$T_{AB\,\mathbf{j}}(z_A, z_B; \tilde{\mathbf{z}}_{-\alpha}) = \int \prod_{i=1}^{g} d^4\theta_i \prod_{\beta \neq \alpha} \left[\sum_{i_\beta} \theta_{i_\beta} u'_{i_\beta}(\tilde{z}_\beta) \right]^{2j_\beta} \cdot [\theta_i u'_i(z_A)]^A [\theta_j u'_j(z_B)]^B \,, \tag{4.2a}$$

where $z_A = z_2$ or z_3, $z_B = \tilde{z}_\alpha$, $\mathbf{j} = \{j_\beta\}$, $\tilde{\mathbf{z}}_{-\alpha} = \{\tilde{z}_\beta\}_{\beta \neq \alpha}$, and by

$$T_{ABCD\,\mathbf{j}}(z_A, z_B, z_C, z_D; \tilde{\mathbf{z}}_{-\alpha-\beta}) = \int \prod_{i=1}^{g} d^4\theta_i \prod_{\gamma \neq \alpha,\beta} \left[\sum_{i_\gamma} \theta_{i_\gamma} u'_{i_\gamma}(\tilde{z}_\gamma) \right]^{2j_\gamma} \cdot \prod_{A,B,C,D} [\theta_{i_\alpha} u'_{i_\alpha}(z_A)]^A \,. \tag{4.2b}$$

The resulting Feynman diagram factors in these cases are, for one fermionic line

$$\begin{aligned}
&\{[\zeta_1\zeta_4\zeta_3 p_3]\,\mathrm{tr}\,[\not{p}_2 \not{\zeta}_2 \varrho^{j_\alpha} T_{\mathbf{j}}[z_2, \tilde{z}_\alpha; \tilde{\mathbf{z}}_{-\alpha})] G'(z_3, z_1) G'(\tilde{z}_\alpha, z_2) \\
&+ \mathrm{tr}\,[\not{p}_3 \not{\zeta}_3 \varrho^{j_\alpha} T_{\mathbf{j}}] (2\partial_{z_2} \overline{\times} + ---)^{j_2} \zeta_2^{j_2} (\zeta \cdot \zeta_4) G'(\tilde{z}_\alpha, z_3) G'(z_2, z_1) \\
&+ \mathrm{tr}\,[\not{p}_2 \not{\zeta}_2 \not{p}_3 \not{\zeta}_3 \varrho^{j_\alpha} T_{\mathbf{j}}) (\zeta_1 \zeta_4) G'(z_3, z_2) G'(\tilde{z}_\alpha, z_3)\} \\
&\cdot \prod_{\beta=1}^{2g} \left[2\partial_{\tilde{z}_\beta} \overline{\times} + \sum_{r=1}^{N-1} \mathbf{G}'(\tilde{z}_\beta, z_r) p_r \right]^{j_\beta}
\end{aligned} \tag{4.3a}$$

where $[p_1 p_2 p_3 p_4] = \mathrm{tr}\,(\varrho^{a_1}\varrho^{a_2}\varrho^{a_3}\varrho^{a_4}) p_1^{a_1} p_2^{a_2} p_3^{a_3} p_4^{a_4}$, and for two fermionic lines

$$\begin{aligned}
&\{(\not{p}_3 \not{\zeta}_3 \varrho^{j_\alpha})_{A_2 A_3} (\not{p}_2 \not{\zeta}_2 \varrho^{j_\beta})_{A_1 A_4} T_{A_1 A_2 A_3 A_4\,\mathbf{j}}(z_2, z_3, \tilde{z}_\alpha, \tilde{z}_\beta; \tilde{\mathbf{z}}_{-\alpha-\beta}) \\
&\cdot G'(\tilde{z}_\alpha, z_3) G'(\tilde{z}_\beta, z_2) + \mathrm{tr}\,[\not{p}_3 \not{\zeta}_3 \varrho^{j_\alpha} \not{\zeta}_2 \not{p}_2 T_{\mathbf{j}}(z_2, z_3, \tilde{z}_\alpha; \tilde{\mathbf{z}}_{-\alpha})] (\zeta_1 \zeta_4) \\
&\cdot G'(\tilde{z}_\alpha, z_2) G'(\tilde{z}_\alpha, z_3)\} \times \prod_{\gamma=1}^{2g} \left[2\partial_{\tilde{z}_\gamma} \overline{\times} + \sum_{r=1}^{N} \mathbf{G}'(\tilde{z}_\gamma, z_r) p_r \right]^{j_\gamma} \\
&\cdot \prod_{\delta} [F''(\tilde{z}_\delta)]^{-3/2}
\end{aligned} \tag{4.3b}$$

(and not all external factors have been included in (4.3a, b)). The total amplitudes for external bosons can now be written down by taking factors (4.3), multiplying by the Veneziano amplitude, and integrating over the interaction positions $\tilde{\varrho}_\alpha = F(\tilde{z}_\alpha)$ and source points z_r and the internal string loop widths α_i, β_i $(1 \leqq i \leqq g)$. These latter have been discussed fully in Sect. 2. Thus for type II superstrings the amplitude has contributions of form (4.3) from both left and right modes (with z and $\bar{z}$ dependence), and is therefore

$$\mathrm{Amplitude} = \int_{\mathscr{I}} \prod_{\alpha=2}^{2g} d^2\tilde{\varrho}_\alpha \prod_{i=1}^{g} d\alpha_i d\beta_i \prod_{r=2}^{N-1} d^2 z_r \exp\{p_r p_s G(z_r, z_s)\} \{\det \mathrm{Im}\, \Pi\}^{-4} \cdot \left\{ \sum_{(f,r,s,t,u)} A^{(f,r,s,t,u)} \right\} \left\{ \sum_{(g,r',s',t',u')} A^{(g,r',s',t',u')} \right\}^* \,, \tag{4.4}$$

where f, g denotes the number of fermionic lines. The symbol * applied to a L-polarization vector ζ takes it to R-polarisation vector $\tilde{\zeta}$.

A similar expression is valid [25] for a heterotic string amplitude [11], now only involving R-mode factors $A^{(f,r,s,t,u)}$, the L-mode factor being replaced by a bosonic partition function. For non-gauge bosons the expression is:

$$\text{Amplitude} = \int_{\mathscr{I}} \prod_{\alpha=2}^{2g} d^2\tilde{\varrho}_\alpha \prod_{i=1}^{g} d\alpha_i d\beta_i \prod_{r=2}^{N-1} d^2 z_r \exp\{p_r p_s G(z_r, z_s)\}$$

$$\cdot \{\det \operatorname{Im} \Pi\}^{-4} P(\Sigma) \prod_{\alpha=1}^{2g} [\bar{F}''(\tilde{z}_\alpha)]^{-\frac{1}{2}} \cdot \Theta(0|\Pi \otimes \Gamma)$$

$$\cdot \left\{ \sum_{(f,r,s,t,u)} A^{(f,r,s,t,u)} \right\}^* . \tag{4.5}$$

The symbol $P(\Sigma)$ denotes the partition function $(\det \partial_z)^{-12}$ in the z-plane, where $\partial\bar{\partial}$ is the Laplacian on Σ, and $\Theta(0|\Pi \otimes \Gamma)$ is the theta function at zero for the matrix $\Pi \otimes \Gamma, \Gamma$ being the root lattice of the gauge group. The R-mode factor in (4.5) has arisen from the integration over the inexact Grassmann variables $\tilde{\theta}_i^A$, so has modular transformation by multiplication by $[\det(C\bar{\Pi}+D)]^{-4}$ under the modular transformation with matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The L-mode factor transforms with $[\det(C\Pi+D)]^{-4}$, and the total expression (4.5) is seen to be modular invariant on inclusion of the factor $[\det \operatorname{Im} \Pi]^{-4}$. An equivalent version to (4.5) may be developed by using NRS version of the compactified bosons; it is expected to be identical to (4.5) and will not be considered further here.

The domain $\mathscr{I}$ denotes a fundamental domain of the modular group. This is known to coincide with a fundamental domain of the Siegel upper-half plane [27] when $g=2$ and 3, where the shape of the fundamental domain is known in detail for $g=2$ [28]. Such a coincidence does not occur for $g \geqq 4$, and in that case $\mathscr{I}$ is not known explicitly. The expressions (4.4) and (4.5) must now be analysed for possible divergences.

5. Handle Degeneration Divergences

The analysis of possible divergences from handle degeneration in either the type II or heterotic multi-loop amplitudes (4.4), (4.5) commences by solving the equation

$$dF^*(\tilde{z}) = 0 \ , \tag{5.1}$$

where dF^* is given by (2.14). The term $O(\ln \varepsilon^{-1})$ on the right-hand side of (2.14) may be neglected if $\tilde{P}$ is not close to P_0 or Q_0, so giving

$$dF(\tilde{z}) = 0 \ . \tag{5.2}$$

There will be $2(g-1)$ solutions of (5.2). These correspond to the $2g-2$ IV's of the degenerated surface in which the degenerated handle plays no role.

On the other hand $\tilde{P}$ may approach P_0 or Q_0 to $O(\ln \varepsilon^{-1})$. If P_0 and Q_0 are uniformised by p_+, p_- in the z-plane there will be two further solutions $\tilde{z}_\pm$ of (5.1) given by

$$\tilde{z}_\pm = p_\pm + a_\pm (\ln \varepsilon)^{-1} \ . \tag{5.3}$$

Since $d\eta_{P_0Q_0}(\tilde{z}_\pm) \sim \mp \ln \varepsilon \cdot (a_\pm)^{-1}$, then (5.1) will be satisfied provided

$$a_\pm = \pm [i\alpha_g^* + \eta_{P_1P_N}(Q_0) - \eta_{P_1P_N}(P_0)] \left[\eta'_{P_1P_N}(P_\pm) + i \sum_{j=1}^{g-1} \alpha_j u'_j(P_\pm) \right]^{-1} . \quad (5.4)$$

To the order of $(\ln \varepsilon)^{-1}$ under consideration it is possible to write, from (2.16) and (2.17), that

$$\alpha_g^* = \alpha_g + \alpha (\ln \varepsilon)^{-1} \alpha'_g ,$$
$$\beta_g^* = (\ln \varepsilon)^{-1} \beta_g + \alpha (\ln \varepsilon)^{-2} \beta'_g , \quad (5.5)$$

where α_g, α'_g, β'_g are independent of α and ε, and may be read off directly from (2.16) and (2.17). Then

$$\frac{\partial(\alpha_g^*, \beta_g^*)}{\partial(\alpha, \ln \varepsilon)} = \alpha'_g \beta_g (\ln \varepsilon)^{-3} . \quad (5.6)$$

It is clear from the analysis of Sect. 3 that all of the functions entering the amplitudes (4.4) or (4.5) have a finite, non-zero limit as $\varepsilon \to 0$ except for $F''(\tilde{z}_\pm) = O(\ln \varepsilon)$ and $\det(\operatorname{Im} \Pi^*) = O(\ln \varepsilon)^{-1}$. For the type II string there is a total power of $(\det \operatorname{Im} \Pi^*)^{-4}$. It is necessary to remove $(\det \operatorname{Im} \Pi^*)^{-3}$ from this to be combined with the external measures so as to make the latter modular invariant. Combined with the change of measure (5.6) to the variables α and $(\ln \varepsilon)$ results in the asymptotic value of the integrand of (4.4) as $\varepsilon \to 0$ given by (with $x = (\ln \varepsilon)^{-1}$)

$$\int_0 x^6 dx \int d\alpha , \quad (5.7)$$

which is clearly finite. On taking account also for the factor $(\det \operatorname{Im} \Pi^*)$ in $\det' \Delta_0$ a similar analysis for (4.5) gives the potentially dangerous term

$$\iint_0 x dx d\alpha , \quad (5.8)$$

which is again finite. A similar analysis for multiple handle degenerations is expected to give the same result.

It is clear that the degeneration of a handle (or any number of them) leads to a pair of interaction points $\tilde{z}_\pm$ converging to the resulting punctures P_0, Q_0 on the surface. Such a degeneration does not bring into play the potentially lethal Feynman ultra-violet divergences arising from the coincidence of two interaction points. Such a further degeneration can now be performed in a controlled manner by letting the points P_0, Q_0 approach each other. This control cannot be easily achieved without the handle degeneration occurring simultaneously. Thus $g > 1$ would seem to be required, since for $g = 1$ there is only one available modulus and two complex degrees of freedom are not available. However there is still the variable z_1, and a possible divergence from the above cause would also be expected here. Since one loop amplitudes are finite this potential divergence should not be present. This will be seen after the details of the singularity are determined.

The values of the parameters and functions given in Sect. 2 can now be analysed further as the uniformising parameter δ between P_0 and Q_0 in the z-plane goes to zero. For then some of the functions and parameters also vanish with δ, and a careful analysis must be given. If $Q_0 = P_0 - \delta$, then the string parameters α_g, α'_g, β_g, β'_g in (5.5) are all of order δ, with

$$\begin{aligned}
\alpha_g &= \operatorname{Im}[\delta\eta'_{P_1P_N}(P_0)] + 2\pi \operatorname{Re}[u'_k(P_0)\delta](\operatorname{Im}\Pi)^{-1}_{jk}\operatorname{Im}[u_j(P_1)-u_j(P_N)] \ , \\
\alpha'_g &= -\operatorname{Re}[\delta\eta'_{P_1P_N}(P_0)] - 2\pi \operatorname{Im}[u'_k(P_0)\delta](\operatorname{Im}\Pi)^{-1}_{jk}\operatorname{Im}[u_j(P_1)-u_j(P_N)] \ , \\
\beta_g &= \pi\left\{\operatorname{Re}[\delta\eta'_{P_1P_N}(P_0)] + \sum_{j=1}^{g-1}\alpha_j \operatorname{Re}[u'_j(P_0)\delta]\right\} \ , \\
\beta'_g &= \pi\left\{\operatorname{Im}[\delta\eta'_{P_1P_N}(P_0)] + \sum_{j=1}^{g-1}\alpha_j \operatorname{Im}[u'_j(P_0)\delta] + \alpha_g\right\} \ , \\
a_\pm &= \pm[i\alpha_g - \delta\eta'_{P_1P_N}(P_0)]\left[\eta'_{P_1P_N}(P_0) + i\sum_{j=1}^{g-1}\alpha_j u'_j(P_0)\right]^{-1} \ .
\end{aligned} \tag{5.9}$$

It is now possible to evaluate the singular functions

$$\begin{aligned}
&u'_g(\tilde z_\pm) \sim \mp(2a_\pm)^{-1} = O(\delta^{-1}) \ , \\
&F''(\tilde z_\pm) \sim \ln\varepsilon(2a_\pm^2)^{-1}[i\alpha_g - \eta'_{P_1P_N}(P_0)\delta] = O(\ln\varepsilon\cdot\delta^{-1}) \ , \\
&\partial_{\tilde z_+}\partial_{\tilde z_-}G(\tilde z_+,\tilde z_-) \sim [\delta + (a_+ - a_-)(\ln\varepsilon)^{-1}]^{-2} = O(\delta^{-2}) \ , \\
&\tilde\varrho_+ - \tilde\varrho_- \sim \delta\left[\eta'_{P_1P_N}(P_0) + i\sum_{j=1}^{g-1}\alpha_j u'_j(P_0)\right] = O(\delta) \ .
\end{aligned} \tag{5.10}$$

The Jacobian J of (5.6) now becomes of order $\delta^2(\ln\varepsilon)^{-3}$.

One can now use the above estimates to determine the putative divergence character of the type II superstring. The relevant factor in (4.4) is

$$J\cdot|u'_i(\tilde z_+)u'_i(\tilde z_-)|^4|F''(\tilde z_+)F''(\tilde z_-)|^{-3}|\partial_{\tilde z_+}\partial_{\tilde z_-}G(\tilde z_+,\tilde z_-)|^2 \ , \tag{5.11}$$

which reduces, using (5.10), to have order

$$\delta^{-4} \tag{5.12}$$

(where the ε-dependence is not of relevance). Combining with the measure $d^2(\tilde\varrho_+ - \tilde\varrho_-)\alpha d^2\delta$ leads to the apparent singularity

$$\int_0 |\delta|^{-3}d|\delta| = \infty^2 \ . \tag{5.13}$$

Since such a divergence can arise from each handle but one, the maximum apparent divergence will be

$$\infty^{2(g-1)} \ . \tag{5.14}$$

On the other hand the factor for the heterotic string replacing (5.11) is

$$J[u'_i(\tilde z_+)u'_i(\tilde z_-)]^2|F''(\tilde z_+)F''(\tilde z_-)|^{-2}\cdot\partial_{\tilde z_+}\partial_{\tilde z_-}G(\tilde z_+,\tilde z_-) \ . \tag{5.15}$$

This now has order, instead of (5.12), equal to

$$\delta^0 \ , \tag{5.16}$$

and therefore (5.13) is replaced by the finite integral

$$\int_0 |\delta| d|\delta| \cdot < \infty \ . \tag{5.17}$$

The heterotic string therefore does not suffer from the apparent divergence afflicting the type II superstring. The ε-dependent factors in this case are also finite, being $(\ln \varepsilon)^{-4}$, so are the same as before taking the $\delta \to 0$ limit. The modular transformation properties of the R-moving factor discussed in Sect. 4 implies that this factor is finite at the degeneration $\det \operatorname{Im} \Pi \to \infty$ obtained by taking the modular transformation inverting the vanishing diagonal matrix element of Π. This fact will be used in the later divergence analysis involving the bosonic partition function in Sect. 7.

The amplitudes (4.4), (4.5) were defined by integration over a fundamental domain $\mathscr{I}$ of the modular group. Can the moduli in the degeneration of Sects. 2 or 3 be shown, in particular for genus 2, to lie in the fundamental domain $\mathscr{I}_2$ of Siegel [27]? This is an important question to answer for the type II superstring. For it is possible that a narrowing down of the volume of $\mathscr{I}_2$ near its boundary could remove the divergence discovered above. If all of the surfaces which possess the degenerations being discussed lie in $\mathscr{I}$, that cannot happen, and the divergences still remain.

In the case $g=2$ a fundamental domain $\mathscr{I}'$ in the Siegel upper half-plane G_2 is defined by the conditions on the period matrix $\Pi = X + iY$ as [27]

$$\text{(i)} \qquad |x_{ij}| \leqq \tfrac{1}{2} \ , \tag{5.18}$$

$$\text{(ii)} \qquad |\det(C\Pi + D)| \leqq 1 \tag{5.19}$$

for all $g \times 2g$ matrices (CD) which are the second row of a matrix in $Sp(2g, \mathbf{Z})$,

$$\text{(iii) (a)} \qquad \mathbf{g}_r^T Y \mathbf{g}_r \geqq \mathbf{e}_r^T Y \mathbf{e}_r \quad (r = 1, \ldots, g) \tag{5.20}$$

for all integer valued g-vectors $\mathbf{g}_r$, where $(\mathbf{e}_r, \ldots, \mathbf{e}_r)$ are the columns of the unit matrix, with $\mathbf{g}_r \neq \pm \mathbf{e}_r$, the last $(g - r + 1)$ entries of $\mathbf{g}_r$ being relatively prime,

$$\text{(b)} \qquad \mathbf{e}_1^T Y \mathbf{e}_r \geqq 0 \quad (r = 2, \ldots, g) \ . \tag{5.21}$$

By analogy with the case of $g=1$ it is necessary to transform to another fundamental domain $\mathscr{I}$ by means of the modular matrix with $a_{11}=0$, $a_{ij}=0$ $(i \neq j)$, $a_{ii}=1$ $(i=2, \ldots, g)$, $b_{11}=-1$ all other $b_{ij}=0$, $c_{11}=1$, all other $c_{ij}=0$. Then

$$\Pi'_{11} = -(\Pi_{11})^{-1} \ , \qquad \Pi'_{1i} = \Pi_{1i}/\Pi_{11} \ , \qquad \Pi'_{ij} = \Pi_{ij} \quad (\text{to } O((\ln \varepsilon)^{-1}) \tag{5.22}$$

so that $\Pi'_{22} = +i \ln \varepsilon / \Pi$, $\Pi'_{11} = \Pi_{11}$, $\Pi'_{1i} = O(1)$. Condition (i) can easily be satisfied without any restriction on Π'_{22}, as can (iii) (b). Moreover

$$\mathbf{g}_r^T Y \mathbf{g}_r = \Pi^{-1} g_{r1}^2 \ln \varepsilon + O(1) \ , \qquad \mathbf{e}_r^T Y \mathbf{e}_r = \Pi^{-1} \ln \varepsilon \delta_{r1} + O(1) \ .$$

Since $g_{r_1}^2 \geqq 1$ then (iii) (a) gives no restriction on Π'_{22}. Finally (5.19) is

$$\det(C\Pi' + D) = \ln \varepsilon \{\det C \cdot \Pi'_{22} + (c_{11} d_{22} - c_{21} d_{12})\} + O(1) \ ,$$

and

$$|\det(C\Pi' + D)| = |\ln \varepsilon| \{|\det C|^2 + (c_{11} d_{22} - c_{21} d_{12})^2\}^{1/2} + O(1) \ . \tag{5.23}$$

If both det $C=0$ and $c_{11}d_{22}=c_{21}d_{12}$, then condition (ii) imposes a constraint on the terms of $O(1)$ in Π; if det $C=0$ or $c_{11}d_{22}\neq c_{21}d_{12}$, then either has modulus at least one and (5.19) is satisfied for $\varepsilon<e^{-1}$. Yet again there is no constraint to $O(\ln\varepsilon)$, and there are only constraints on the $O(1)$ terms. It therefore does not appear likely that the requirement that integration of the moduli be only over $\mathscr{I}$ require an ε-dependent restriction on the other variables, and so bring about a reduction of the volume of integration. It is to be noted that modular invariance indicates that the singularity will also be present if $\mathscr{I}'$ was used directly.

It has already been remarked that for $g=1$ (1 loop) the apparent ∞^2 divergence is not present in the type II superstring amplitude [1, 2, 29]. There must therefore be some way of removing the divergence (5.13) in the $g=1$ amplitude. It is possible that such removal occurs by reduction of the "raw" amplitudes of Sect. 4 to expressions which do not involve the interaction points $\tilde{\varrho}$. Indeed this can be shown (last paper in ref. [11]) to occur at 1 loop to reproduce the known DRM results. A similar reduction is not presently known in the case $g>1$, so such a technique would not be of help.

An alternative approach [20] is to remove the divergence by partial integration [30]. This may be done in two stages in which integration by parts is followed by a direct analysis. The expression under consideration has the form

$$\lim_{\varepsilon\to 0}\int d^2\tilde{z}_+ d^2\tilde{z}_-\partial_{\tilde{z}_+}\partial_{\tilde{z}_-}\eta^*_{\tilde{z}_-z_N}(\tilde{z}_+)\partial_{\bar{\tilde{z}}_+}\partial_{\bar{\tilde{z}}_-}\bar{\eta}^*_{\bar{\tilde{z}}_-z_N}(\tilde{z}_+)F(\tilde{z}_+,\tilde{z}_-)\ . \tag{5.24}$$

In (5.24) F denotes the factors which do not produce the $|\delta|^{-4}$ singularity as $\tilde{z}_+\sim\tilde{z}_-$, and so F is finite in this limit. The integration region is over $|\tilde{z}_+-\tilde{z}_-|>\varepsilon$, and the limit $\varepsilon\to 0$ taken at the end. This limit in (5.24) corresponds to the usual definition of an infinite integral in terms of first integration outside a neighbourhood of infinity and secondly letting that neighbourhood contract to zero. The derivatives with respect to $\tilde{z}_+$ and $\tilde{z}_-$ may be integrated by parts with respect to these derivatives. Care must be taken in including the boundary contribution from $C:|\tilde{z}_+-\tilde{z}_-|=\varepsilon$ in (5.24) (other boundaries can be neglected since they would involve further degenerations not considered). Then (5.24) can be reduced to

$$-\int_C \eta\bar{\partial}_1\bar{\partial}_2\bar{\eta}\bar{\partial}_2 F+\int \eta\bar{\partial}_1\bar{\partial}_2\bar{\eta}\partial_1\partial_2 F\ , \tag{5.25}$$

where 1 and 2 denote $\tilde{z}_+,\tilde{z}_-$ and $\partial=dz\partial/\partial z$ as usual. The δ-functions arising say, from $\partial_1\bar{\partial}_1\eta$, have support outside the domain of integration, so give no contribution.

The second stage in the analysis of (5.24) is to analyse the integration over $\delta=\tilde{z}_+-\tilde{z}_-$ in (5.25). We may express (5.25), for $\varepsilon\sim 0$, as

$$\int_{|\delta|=\varepsilon} d\bar{\delta}\bar{\delta}^{-2}f_1(\delta_1,\bar{\delta})+\int_{|\delta|>\varepsilon} d^2\delta\bar{\delta}^{-2}f_2(\delta,\bar{\delta})\ , \tag{5.26}$$

where f_1 and f_2 may be written near $\delta=0$ as

$$a_i+O(\delta)\ , \tag{5.27}$$

where a_i are constants ($i=1,2$). Only the constant terms a_i could give any divergence in (5.26). Inserting the form (5.27) into (5.26) results in the expressions

$$-\frac{i}{\varepsilon}\int_0^{2\pi} d\theta e^{i\theta}(a_1+O(\varepsilon))+\int_\varepsilon \frac{dr}{r}\int_0^{2\pi} d\theta e^{2i\theta}(a_2+O(r)) \ . \tag{5.28}$$

The $O(\varepsilon)$ and $O(r)$ terms are finite as $\varepsilon\to 0$, whilst the terms proportional to a_1 and a_2 are both zero. Therefore the putative divergence (5.13) is not actually present in the type II superstring at any loop order. A similar analysis of lower divergences arising, say, from $G'\cdot G''$ factors for coincident interaction points is also clearly removeable by the same technique. Moreover this use of a phase factor to achieve annihilation of a putative divergence is also necessary for the heterotic string, as we shall see shortly. In this latter case such a putative divergence would be regarded as arising from tachyon or dilaton states. A similar interpretation of the type II putative divergence (5.13) is also to be expected. The removal of (5.13) by general supersymmetric features is loosely regarded as the general guarantee of finiteness. However the explicit removal noted here is not obviously related to supersymmetry, and a closer understanding of the relation is called for.

In the heterotic amplitude (4.5) there is still a potential divergence from the vanishing of the partition function $P(\Sigma)$ at a degeneration. This will be discussed separately in Sect. 7 after the effect of a dividing geodesic degeneration on the right-moving supersymmetric vertex factors has been analysed.

6. Dividing Geodesic Divergences

An analysis may now be performed of the possible divergences which could arise when a dividing geodesic length $\ell\to 0$ following the estimates of the various relevant functions in Sect. 3. It appears necessary to consider separately the cases

(a) $P_1\in S_1\cap S^*$, $P_N\in S_2\cap S^*$,

(b) $P_1, P_N\in S_1\cap S^*$.

But in fact case (a) cannot arise, since it is clear from Sect. 3 that none of the variables $\varrho^*, \alpha_i, \beta_i$ become of $O(t)$ in that case. In other words the value $t=0$ does not enter as a boundary point over the range of integration. Thus only the case (b) need be considered in detail.

The discussion of Sect. 3 showed that there are g_1 IIV's $\tilde{P}_\alpha$ on $S_2\cap S^*$. Moreover the following estimates result:

$$F^{*\prime\prime}(\tilde{P}_\alpha)=O(1) \qquad \tilde{P}_\alpha\in S_1\cap S^* \tag{6.1a}$$

$$=O(t) \qquad \tilde{P}_\alpha\in S_2\cap S^* \tag{6.1b}$$

$$\alpha_i^*, \beta_i^*=O(1) \qquad 1\leqq i\leqq g_1 \tag{6.1c}$$

$$=O(t) \qquad g_1<i\leqq g_1+g_2 \tag{6.1d}$$

$$\tilde{\varrho}_\alpha^*=O(1) \qquad \text{on} \quad S_1\cap S^* \tag{6.1e}$$

$$=O(t) \qquad \text{on} \quad S_2\cap S^* \tag{6.1f}$$

$$G^*_{z^1 z_N} z = O(1) \qquad z,\, z^1 \in S_1 \cap S^* \tag{6.1g}$$

$$= O(1) \qquad z \in S_1 \cap S^*,\; z^1 \in S_2 \tag{6.1h}$$

$$= O(t) \qquad z,\, z^1 \in S_2 \cap S^* \;. \tag{6.1j}$$

It is necessary to be more careful on the estimate (6.1h). In fact the results of Sect. 3 or of Fay [22] that the second Abelian differential

$$w(x,y) = O(t) \qquad x \in S_1 \cap S^*,\; y \in S_2 \cap S^* \tag{6.2}$$

shows that

$$\eta'_{XY}(z) = \int_Y^X w(x',z)dx' = \int_{A_1}^X w(x',z)dx' + O(t) \;. \tag{6.3}$$

Thus $\eta_{XY}(z)$ is independent of Y in the region (6.2). The line factors

$$\partial_{\tilde{z}_+} \partial_{\tilde{z}_-} G_{\tilde{z}_- z_N}(\tilde{z}_+) = O(t) \quad \text{if} \quad \tilde{z}_+ \in S_1 \cap S^*, \;, \qquad \tilde{z}_- \in S_2 \cap S^* \;. \tag{6.4}$$

Similarly the line factors

$$\sum_{s=1}^{N-1} G'_{z_N}(\tilde{z}_-, z_s) P_s = O(t) \quad \text{if all} \quad z_s \in S_1 \cap S^* \;; \tag{6.5}$$

The contributions to these line factors for $z_s \in S_2 \cap S^*$ are also $O(t)$, by (6.1j).

The more detailed estimates given by (6.1)–(6.5) can now be applied to the type II superstring amplitude (4.4). Then the crucial factors in that amplitude can be written schematically as

$$\prod_{i>g_1} |u'_i(\tilde{P})|^8 \prod_{\tilde{P}_\alpha \in S_2 \cap S^*} F''(\tilde{P}_\alpha)|^{-3} d^2\tilde{\varrho}^* \prod_{i>g_1} d\alpha_i d\beta_i \prod |\partial_+ \partial_- G|^2 \;. \tag{6.6}$$

The leading order in t contribution from the first factor is $O(1)$, from the second $O(t^{-6g_2})$ (where one $\tilde{\varrho} \in S_1 \cap S^*$ has been fixed by translation invariance). The net factor of t that results is therefore

$$t^{2g_2-1}\, dt \;. \tag{6.7}$$

Since $g_2 \geqq 1$ this is finite and implies that the supersymmetric vertex factors give a bounded contribution.

In the case of the heterotic string, (6.6) is to be replaced by

$$\prod_{i>g_1} (u'_i(\tilde{P}))^4 \prod_{\tilde{P}_\alpha \in S_2 \cap S^*} [F''(\tilde{P}_\alpha)]^{-2} d\varrho^*_\alpha \prod_{i>g_1} d\alpha_i d\beta_i \Pi \partial_+ \partial_- G \;. \tag{6.8}$$

The net factor of t arising from this factor is now

$$t^{4g_2-3}\, dt \;, \tag{6.9}$$

which is integrable, since $g_2 \geqq 1$. Thus the heterotic string is also finite from this degeneration.

7. The Bosonic Partition Function

In the preceding two sections a careful analysis has been given of the possible divergences which may arise from the right-moving vertex factor contributions to the heterotic raw amplitude on degenerating Riemann surfaces as well as the companion left-moving vertex factor contributions to the type II superstring raw amplitudes. Putative divergences were discovered in the latter, which were shown to be absent on use of integration by parts and phase factor integration. There is still further analysis to perform on the heterotic amplitudes before it can be concluded that they are finite. In particular it is still necessary to analyse the putative tachyonic pole in the left-moving bosonic partition function $P_L(\Sigma)$ of (4.5).

Removal of the non-holomorphic factor $[\det \operatorname{Im} \Pi]^{-6}$ from $(\det \partial)^{-12}$ has left this partition function as a holomorphic section F of the determinant line bundle λ^{-12}, where λ is the Hodge line bundle on the moduli space M_g. The extension of this line bundle to the stable compactification $\bar{M}_g$ [31] leads to a pole in $F(y)$ of order one as the explicit calculation in ref. [18] shows. Thus

$$F(y) \sim \frac{1}{y_1} \quad \text{as} \quad y_1 \sim 0 \ , \tag{7.1}$$

where $y_1, \ldots, y_{3g-3}$ are the complex co-ordinates on M_g in which locally the curve $y_1 \to 0$ is transversal to the boundary $D = \bar{M}_g - M_g$. It is possible to relate this degeneration to the one of Sect. 2 by taking, by a suitable modular transformation,

$$\Pi_{11} = \frac{1}{2\pi i} \ln y_1 \ . \tag{7.2}$$

The handle degeneration analysis of Sect. 2 can be related directly to (7.2) by the modular transformation of (5.22), as was shown in Sect. 5. The dividing geodesic degeneration of Sect. 3 cannot be so obtained, and will be discussed later.

The integration over the resulting singularity will then be of form

$$\int d^2 y_1 \, [y_1 |y_1|^2 (\ln |y_1|)^3]^{-1} f_L(y_1, \ln |y_1|) f_R(\bar{y}_1, \ln |y_1|) \ . \tag{7.3}$$

In (7.3), f_L, f_R are the contributions from the L and R moving modes in (4.5) at the degeneration, other than the y_1^{-1} arising from L-boson partition function. The measure $d^2 y_1 [|y_1|^2 \cdot (\ln |y_1|)^3]^{-1}$ comes from the identification $y_1 = (\varepsilon e^{i\alpha})^2$, and the original measure $d(\ln \varepsilon) d\alpha$ times suitable powers of $\ln \varepsilon$ from the partition function, as discussed earlier.

An alternative method of arriving at the measure in (7.3) multiplying f_L and f_R is by changing variables to the analytic ones, y_i, on Teichmüller space by means of the Jacobian, in the notation of ref. [18],

$$j = (\det \operatorname{Im} \Pi)(\det \Delta_{-1})(\det \Delta_0),$$

and use of the work of Bost and Jolicoeur [17] (already incorporated in (4.5))

$$(\det \partial_z)^{-12} = (\det \operatorname{Im} \Pi)^{-6} F_L(\bar{y}) \ .$$

A factor of $(\det \operatorname{Im} \Pi)^{-3}$ is then removed from the resulting heterotic amplitude non-holomorphic factor $(\det \operatorname{Im} \Pi)^{-5}$, to leave the measure equal to

$d\Omega(y)$ (det Im $\Pi)^{-2}$. Near the degeneration $y_1 \sim 0$, $d\Omega \sim d^2y(|y|^2 \ln|y|)^{-1}$, so that with (7.2) the value of the measure in (7.3) results.

The analysis of Sect. 5 showed that $f_R(y_1, \ln|y_1|)$ is bounded as $y_1 \sim 0$ (and even as y_1, $y_2 \sim 0$ jointly with coinciding internal interaction vertices). Moreover the construction of f_R from various products of derivatives of Green's function ensures that f_R is a function of y or $\ln|y|$, but not explicitly of arg y other than as in powers of y. This result is crucial to the analysis of (7.3), and follows from various features (i) all Feynman line contributions can be reduced to products of sums of third Abelian differentials $d\eta$ or of first Abelian terms $du_i(\text{Im}\,\Pi)^{-1}_{ij}\,\text{Im}\,u_j$ or $du_i(\text{Im}\,\Pi)^{-1}_{ij}\,du_j$. (ii) First and third Abelian differentials are analytic in the modular variables, and the first Abelian integrals are also analytic, at a degeneration $y_1 \sim 0$. This can be seen, for example, by use of the explicit Burnside expansions [32], or by general arguments [12, 33]. Thus choosing y_1 to be the multiplier of the first generator T_1, with fixed points 0 and ∞, the possible singularities in $du_i(z)$ can be seen explicitly not to be of form log w but always involve terms of $O(w)$. (iii) Only $\ln|y|$ enters in $(\text{Im}\,\Pi)^{-1}$, so that arg y does not enter. (iv) Factors F'' are constructed only from the third Abelian differentials $d\eta$. The fact that f_R is a function only of y_1 or $\ln|y_1|$ then follows from the explicit expressions for the R-moving Feynman line contributions, together with (i)–(iv). The dependence of f_L solely on y_1 and $\ln|y_1|$ as $y_1 \sim 0$ is also discernable by inspection. This is immediate for the Θ-function since it is constructed using Π and not Im Π, and also in terms of the first Abelian differentials.

If $y_1 = re^{i\theta}$ the integral (7.3) can be written as

$$\int_0 dr\,[r^2(\ln r)^2]^{-2} \int_0^{2\pi} e^{-i\theta} d\theta f_L(re^{i\theta}, \ln r) f_R(re^{-i\theta}, \ln r) \;, \tag{7.4}$$

where as $r \sim 0$

$$\begin{aligned} f_L &\sim c_L + c'_L re^{i\theta} + O(r^2) \;, \\ f_R &\sim c_R + c'_R re^{-i\theta} + O(r^2) \;, \end{aligned} \tag{7.5}$$

where the expressions on the right-hand side of (7.5) are to within powers of log r. Thus c_L, c'_L, c_R, c'_R are all functions of $\ln r$. The function c'_L, c'_R when regarded as depending on $\ln r$, are bounded as $r \to 0$, since they both arise from powers $(\text{Im}\,\Pi)^{-1} \sim (\ln r)^{-1}$ associated with the first Abelian factors discussed in (i) above. The only possible singular term in (7.4), using (7.5), and integrating over the phase θ, is

$$\int_0 c'_L c'_R (\ln r)^{-2} d(\ln r) \;. \tag{7.6}$$

The properties of c'_L, c'_R noted above ensure that this integral is convergent. It is to be noted that such a detailed analysis is already necessary at 1 loop where the integral over the phase is crucial to remove the tachyon pole divergence; this completes the analysis of the divergences in ref. [2] and [29]. It disagrees with the remarks of Martinec [5], whose claim to be able to avoid the putative divergence by transforming to a different fundamental domain in Teichmüller space is clearly wrong. For any divergence in a particular fundamental domain is carried by the modular transformation to the new domain. However we would agree that the use of the degeneration domain $\operatorname{Im} \Pi_{11} \to \infty$ does seem to avoid the difficulty of detailed analysis of the volume reduction factor at the degeneration point $\operatorname{Im} \Pi_{11} \to 0$.

The degeneration of a dividing geodesic seems easier to handle, since there is no singularity in $\operatorname{Im} \Pi$, so it is effectively constant near the degeneration $y_1 \to 0$. All of the other factors in f_R are bounded near this point, as shown in the previous section, and are analytic as $y_1 \sim 0$ following the argument earlier for the handle geodesic degeneration but without powers of $\ln |y_1|$ entering. A similar feature occurs for $f_L(y)$. The integral (7.3) still arises, but is now immediately seen to be finite without the additional remarks associated with c_R, etc. needed (since the quantities c_R, c'_R, c_L, c'_L of (7.5) are now constants).

8. Further Possible Divergences

The topology changes considered so far have not included the case when sources z_r come arbitrarily close to some IIV $\tilde{\varrho}_\alpha$. Such a possibility does not appear necessarily preventable since the z_r's can move freely on Σ. Thus this possible source of divergence must also be considered. The form of the divergent factors are either

$$\begin{aligned} &\text{(a)} \qquad \partial_{z_r} \partial_{\tilde{z}_\alpha} G_{z_r z_N}(\tilde{z}_\alpha) \ , \\ &\text{(b)} \qquad \partial_{\tilde{z}_\alpha} G_{z_r, z_N}(\tilde{z}_\alpha) \ . \end{aligned} \tag{8.1}$$

These factors occur as the only possible source of divergence in the heterotic string or multiplied by the complex conjugate for the type II superstring.

In the heterotic string case it is possible to integrate (a) in (8.1) by parts with respect to z_r so as to leave a simple pole in $(\tilde{z}_\alpha - z_r)$, as occurs also in (b) in (8.1); both of these cases are integrable singularities in the variable $(z_r \sim \tilde{z}_\alpha)$. Repeated factors like (8.1) can be treated separately in the same manner, since at each I.I.V. only one factor of type (a) or (b) can occur. Thus the heterotic string has no divergences from this source either.

For the type II superstring the method of integration by parts may be used to remove the apparently non-integrable singularity $|\tilde{z}_\alpha - z_r|^{-4}$ which can arise in case (a) or $|\tilde{z}_\alpha - z_r|^{-2}$ if case (b) in (8.1). Thus yet again the type II superstring appears to have no divergence after integration by parts has been performed.

There remains still the possible divergences which can arise when two loops coalesce. Using familiar arguments on modular transforms [21, 34], and already used at the one-loop level [35], such a degeneration will be the modular transform of the one we have considered in Sect. 5. Since this new degeneration will be of the same form as that considered there, it will not cause any divergence.

9. Discussion

It has been shown in this paper that the matrix elements for the scattering of massless states in type II or heterotic superstring theories are finite at each string loop order. This result was proven here only explicitly for bosonic states, although the remarks in the last paper of ref. [1] indicate that the same result is clearly true when spinors are included. For the amplitudes in the latter case are very similar to those in the former. It would be of interest to extend the analysis to the case of massive states; this will be reported on elsewhere.

There is an argument from unitarity that indicates that such amplitudes must be finite if the massless ones are. An independent evaluation of the latter amplitudes is possible along the lines of ref. [11]; that and the ensuing divergence analysis will be reported on elsewhere. Another aspect of the above discussion which requires further consideration is that of repeating the divergence analysis in terms of "completely reduced" amplitudes instead of the raw amplitudes of Sect. 4. By completely reduced we mean those resulting on all possible use of analytic properties of the integrands on moduli space. The raw amplitudes may involve considerably more superstring Feynman diagrams than do the reduced amplitudes. Thus for $g=1$ there are 6 classes of raw superstring Feynman diagrams, each with various permuted terms, as compared to the single completely reduced one, which is also in much simpler form [2, 35]. Indeed the concept of superstring Feynman diagram does not even apply in the one-loop case, as was noted in the last paper of reference [11]. It may be the case that for $g>1$ the same situation prevails, and there are no Feynman lines for the completely reduced amplitudes. If so the above divergence analysis would be considerably simplified. It is clear that the discussion of Sect. 7 could not be dispensed with even in that case, since it is required for the one-loop completely reduced amplitude.

Even if type II and heterotic string amplitudes are finite order by order it is still necessary to determine the radius of convergence of the perturbation expansion in the over-all coupling constant G. A very rough (and possibly very misleading) estimate of this may be given in terms of the order of increase of the number of superstring Feynman diagrams with genus g. Using the Feynman rules of Sect. 4 (or see last ref. in [11] for a more complete version of them) it is clear that there are at least $O(2^g)$ different Feynman diagrams. These arise by considering the choices of a bosonic line from a IIV to an EIV to be either G'' or G'. Similarly bounds arise from the fermionic line possibilities. If this estimate is also an upper bound (after reduction of the amplitude) then the series will have a finite radius of convergence in G. This is a very important question which will be possible to analyse in more detail when complete reduction has been achieved.

The initial impetus to the search for proof of finiteness in superstring theory was the 1-loop finiteness results for the type I $SO(32)$ superstring [15]. It is therefore relevant to attempt to extend the above finiteness results to this case. The problem is made more difficult by the need to regularise separate 1-loop diagrams in a special way in order to obtain the $SO(32)$ cancellation. A recent suggestion [36] that such regularisation follows automatically from construction of the different diagrams from the same closed surface is encouraging since this may lead to testing if a similar technique will succeed at higher order. The analysis given in this paper will be of relevance to such an attempt, as will the construction of type I multi-loop amplitudes by L.C. gauge techniques [37].

Another question of great physical import is as to whether or not the above results extend to a non-trivial background space-time. It may be that criteria on the background in order to have finiteness at any loop order could be deduced so as to give even stronger conditions than those obtained by β-function conditions. It is hoped to report also on this question elsewhere.

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Note added in proof. In order to preserve stability (and hence supersymmetry) of the L. C. gauge field theory of ref. [9] it is necessary to add quartic contact terms to the cubic ones used in the above analysis [38]. These terms have been constructed in detail [39] and modify the analysis of Type II divergences of Sect. 5. In particular the boundary terms in (5.25, 5.26), which are themselves of contact form, must be modified. It is expected that the total set of these contact terms will be zero (by supersymmetry). The new contact terms do not appear to modify the results on heterotic finiteness of Sects. 6 and 7. A more complete analysis of this will be presented elsewhere [40].