

Invariants for Smooth Conjugacy of Hyperbolic Dynamical Systems, III

José Manuel Marco* and Roberto Moriyón*

Universidad Autónoma de Madrid

Abstract. We give a characterization of Anosov diffeomorphisms smoothly conjugated to a toral automorphism in dimension two in term of the Lyapunov exponents of periodic points. We also give necessary and sufficient conditions for the regularity of solutions of the vector cohomology equations associated to an Anosov flow in three dimensions. This allows us to prove a corresponding conjugation theorem.

0. Introduction

In this paper we continue the study of smooth conjugacy problems for general Anosov systems, initiated in [MM, L1]. We are concerned with two questions. i) Given two Anosov diffeomorphisms on the 2-dimensional torus T^2 , when are they C^∞ conjugated? (This question is solved in [L1] when they are close enough in the C^1 topology). ii) Give necessary and sufficient conditions for the existence of a C^∞ conjugation between vector fields on a 3-dimensional compact manifold that generate Anosov flows (no result of this kind was previously known).

A motivation for these questions comes from inverse spectral problems in riemannian geometry (see [GK]), where one tries to construct isometries by smoothly conjugating the corresponding geodesic flows. Collet, Epstein and Gallavotti, [CEG], study them in relation with integrability properties of hamiltonian systems in a general sense. On the other hand, smooth conjugation problems are interesting by themselves, and they have given rise to important and beautiful Mathematics, as in [H].

In the case of diffeomorphisms of the torus, we prove that a C^∞ Anosov diffeomorphism is C^∞ conjugated to a toral automorphism if and only if the Lyapunov exponents of all periodic orbits are the same. In the second case we show that the periods and the Lyapunov exponents of periodic orbits are a complete set of invariants for the smooth conjugacy of one-parameter families of Anosov flows to

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one of them. This clarifies the role of Lyapunov exponents in this context, which was not clear in view of the example in [CEG, Appendix E].

Our approach in the case of flows is close to the original approach of Moser (see [Ma, Mo]) to the proof of Anosov’s stability theorem, and to the study in [LMM] of similar problems for hamiltonian flows. The conjugation problem is reduced to a linear problem for vector fields. We find necessary and sufficient conditions for the regularity of the solutions of vector cohomological equations ($L_X Y = Z$, or $f_* Y - Y = Z$ in the case of maps) that are interesting by themselves.

The main open problems related to these questions are the extension to higher dimensions, and proving similar results for isolated flows instead of parametric families of them. Finally, the study of the real analytic case is something that is missing in the whole theory.

1. Global Smooth Conjugacy of Toral Diffeomorphisms

In this section we prove that the Lyapunov exponents of periodic points characterize toral automorphisms up to smooth conjugacy, within the class of Anosov diffeomorphisms on T^2 . We also prove that the resulting conjugation is globally canonical if one of the diffeomorphisms is the composition of the other with a globally canonical map.

By a globally canonical map we mean an area preserving map on T^2 with vanishing Calabi invariant in the sense of [C, B]. See [MM] for a definition of the Lyapunov exponents of a periodic point of a diffeomorphism.

Theorem 1. *Let $f : T^2 \rightarrow T^2$ be a C^∞ Anosov diffeomorphism. Then f is C^∞ conjugate to a toral automorphism if and only if the Lyapunov exponents of all periodic orbits are the same. If f is real analytic, the conjugation is also real analytic. If f leaves the canonical volume form invariant, so does the conjugation. If moreover f is the composition of its corresponding toral automorphism with a globally canonical map, then the conjugation is globally canonical.*

Proof. If f is C^1 conjugate to a toral automorphism defined by a unimodular matrix A , it is obvious that the Lyapunov exponents of all periodic points are the logarithms of the absolute values of the eigenvalues of the matrix A . We shall assume now that the Lyapunov exponents are independent of the point, and we shall see that f is $C^{2+\delta}$ conjugate to a toral automorphism.

Let A be the 2×2 integer unimodular matrix obtained by transposition of the matrix of $f^\# : H^1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R})$ in the canonical basis. It is known (see [M]) that f is topologically (and even Hölder) conjugate to the automorphism defined by A . This implies that the stable and unstable foliations of f are globally defined by corresponding one-forms $\bar{\alpha}^s$ and $\bar{\alpha}^u$. In this situation we can repeat the proof of Lemma 4 in [MM] in a much simpler framework, since we do not have to care about smooth dependence on parameters, constructing two one-forms of class $C^{1+\delta}$, $\alpha^s = \phi^s \cdot \bar{\alpha}^s$ and $\alpha^u = \phi^u \cdot \bar{\alpha}^u$, satisfying

$$f_* \alpha^s = \lambda^s \alpha^s, \quad f_* \alpha^u = \lambda^u \alpha^u \tag{1.1}$$

with $0 < \lambda^s < 1 < \lambda^u$. (If λ^s or λ^u are not positive, we consider f^2 instead of f .)

Let us point out that as a consequence of the construction in [MM], α^s and α^u are regular along the corresponding invariant foliations, namely α^s is of class C_s^∞ in the sense that its components in any local system of coordinates are C_s^∞ according to Definition 2.2 in [LMM], and analogously for α^u . In case f is real analytic, α^s is of class C_s^ω and α^u is C_u^ω . These facts are a consequence of Lemma 2.2 in [LMM] for the C^∞ case, and estimate (2.16) and the comments that follow it in the same paper for the C^ω case.

The forms α^s, α^u must be closed, as a consequence of the following facts, since $\alpha^s \wedge \alpha^u$ is a volume element with $f^*(\alpha^s \wedge \alpha^u) = \lambda^s \lambda^u \alpha^s \wedge \alpha^u$, taking cohomology classes we see that $\lambda^s \lambda^u = 1$. Then, if X^s, X^u form the dual basis to α^s, α^u , we have

$$f_*[X^s, X^u] = [X^s, X^u]$$

and by the hyperbolicity of f_* on C^0 vector fields, we have $[X^s, X^u] = 0$, hence $d\alpha^s = d\alpha^u = 0$.

Notice that as a consequence of the regularity of α^s and α^u , we have $X^s \in C^{1+\delta} \cap C_s^\infty$ and $X^u \in C^{1+\delta} \cap C_u^\infty$. In case f is real analytic, ∞ can be substituted by ω .

Let α_0^s, α_0^u be the translation invariant one-forms in the same cohomology class of α^s and α^u respectively, and let ψ^s, ψ^u be $C^{2+\delta}$ functions on T^{2n} verifying $\alpha^s = \alpha_0^s + d\psi^s$, and $\alpha^u = \alpha_0^u + d\psi^u$. We define

$$\Psi : T^2 \rightarrow T^2, \quad x \mapsto \Psi(x) = x + \psi^s X_0^s + \psi^u X_0^u,$$

where X_0^s, X_0^u form the dual basis of α_0^s, α_0^u . Then

$$d\Psi = dx + d\psi^s \cdot X_0^s + d\psi^u \cdot X_0^u = dx + (\alpha^s - \alpha_0^s) X_0^s + (\alpha^u - \alpha_0^u) X_0^u = \alpha^s \cdot X_0^s + \alpha^u \cdot X_0^u,$$

so that

$$\Psi^* \alpha_0^s = \alpha^s, \quad \Psi^* \alpha_0^u = \alpha^u. \tag{1.2}$$

It is obvious then that Ψ is a local diffeomorphism. Its injectivity can be proved now as follows: on one hand, Ψ sends stable leaves for f into stable leaves for A , in a monotonous way (the leaves are one-dimensional). Since the leaves are not closed this implies first that different leaves are sent to different leaves, and then that different points are sent to different points. Thus, we have proved that Ψ is a diffeomorphism of class $C^{2+\delta}$. (The injectivity of Ψ can also be seen as a consequence of the strong expansivity properties of the lifting of f on the universal covering of T^{2n} , or it can even be shown by a simple argument using degree theory).

In this situation, by (1.1) and (1.2),

$$(\Psi \circ f \circ \Psi^{-1})^* \alpha_0^s = \lambda^s \alpha_0^s \quad \text{and} \quad (\Psi \circ f \circ \Psi^{-1})^* \alpha_0^u = \lambda^u \alpha_0^u,$$

so that $\Psi \circ f \circ \Psi^{-1}$ differs from A by a translation, let us say T_v . An elementary computation shows then that

$$(T_w \circ \Psi) \circ f \circ (T_w \circ \Psi)^{-1} = A,$$

if $w = (A - I)^{-1} \cdot v$.

Let us see now that $h = T_w \circ \Psi$ is of class C^∞ if f is C^∞ , and it is real analytic if f is so. By (1.2),

$$h' \cdot X^s = X_0^s, \quad h' \cdot X^u = X_0^u. \tag{1.3}$$

Since X^s is of class C_s^∞ , by (1.3), the derivative of h along the stable manifold of f with respect to arc length in the same direction of X^s is also of class C_s^∞ , and this implies that h is of class C_s^∞ . In a similar way, one proves that h is of class C_u^∞ . This in turn implies that h^{-1} is of class $C_s^\infty \cap C_u^\infty$ with respect to the foliations defined by A . Since these foliations are of class C^∞ (they have actually straight lines as leaves), an elementary argument using Fourier series shows that h is of class C^∞ . The above argument can be used in the case of a real analytic f to show that h is then real analytic (see [LMM, p. 43]).

The statements concerning the area preserving case are simple to prove now: first of all, since A is an invertible matrix with integer coefficients, it preserves area up to sign; since it is the matrix corresponding to f^* on $H^1(T^2, \mathbb{Z})$, it is area preserving. Moreover,

$$f^*h^*\omega = h^*A^*\omega = h^*\omega .$$

Since $h^*\omega$ is equivalent to Lebesgue measure, it follows from the transitivity of f that it equals a constant times ω , so h is area preserving.

Assume now that f and A are cohomologous in the sense that $f \circ A^{-1}$ is globally canonical. Since they are isotopic there is a vector field $\tilde{f} \in C^\infty(T^2, \mathbb{R}^2)$ such that $f = A + \tilde{f}$. Then $f \circ A^{-1} = I + \tilde{f} \circ A^{-1}$ is cohomologous to the identity, so according to [CZ, Theorem 6], $\tilde{f} \circ A^{-1}$ (and so \tilde{f}) has vanishing average over T^2 .

Since $A = f^*$, the conjugation equation implies that $A \circ h^* = h^* \circ A$, and since h^* has integer coefficients this implies that h^* is a power of A ; substituting h by itself composed with the inverse of that power, the conjugation still holds for the new h , and h^* is the identity. Then $h = I + \tilde{h}$, where $\tilde{h} \in C^\infty(T^2, \mathbb{R}^2)$, and the conjugation equation says that

$$\tilde{f} + \tilde{h} \circ f = A \circ \tilde{h} .$$

Integrating both sides of this expression on T^2 , and using the fact that the integral of \tilde{f} vanishes, we see that

$$\int_{T^2} \tilde{h} = A \int_{T^2} \tilde{h} ,$$

and the hyperbolicity of A implies that the average of \tilde{h} vanishes. Again by Theorem 6 in [CZ], this means that h is cohomologous to the identity. This finishes the proof of Theorem 1.

2. The Cohomology Vector Equation, and Smooth Conjugacy of Anosov Flows

In this section we give necessary and sufficient conditions for the regularity of solutions of the vector cohomology equations $f_* Y - Y = Z$, and $L_X Y = Z$, where f is a C^∞ Anosov diffeomorphism on the 2-dimensional torus T^2 , and X is a C^∞ vector field that generates an Anosov flow. It is well known that the first equation always has a solution, and the same is true for the second equation modulo multiples of X . However, the solutions are in general only continuous. This fact is closely related to the fact that nearby Anosov systems are only C^0 conjugated in general (up to reparametrizations in the case of flows), according to Anosov's structural stability theorem.

The conditions for regularity that we find are of a cohomological character, since they are relations between sums or integrals along periodic orbits of expressions involving the values of coordinates of Z and their derivatives. Our results (Theorems 2 and 3) cannot be extended to higher dimensions, since the invariant foliations are not regular enough. However, there should be results analogous to Lemmas 1 and 1' that hold in higher dimensions.

We apply the described results to the study of smooth conjugacy problems for one-parameter families of Anosov flows on 3-dimensional manifolds in Theorem 4.

We introduce now some notation. For a fixed C^∞ Anosov diffeomorphism on T^2 we assume that X^s, X^u are global $C^{1+\alpha} \cap C_s^\infty$ (respectively $C^{1+\alpha} \cap C_u^\infty$) sections of the stable and unstable bundles respectively, as at the beginning of the proof of Theorem 1. We define the functions $\lambda_j^s \in C_s^\infty(M)$, $j \in \mathbb{Z}$, by $f_*^j X^s = \lambda_j^s X^s$. By the contractivity of f on stable directions, there are constants $C > 0, \lambda \in (0, 1)$ such that

$$|\lambda_j^s(x)| \leq C \lambda^j, \quad \forall x \in M, \quad \forall j \in \mathbb{N}. \tag{2.1}$$

Given a continuous stable vector field Y^s we define the continuous function Y_s by $Y^s = Y_s \cdot X^s$. In case Y^s is of class C_s^k , so is Y_s . We use the analogous notation for unstable vector bundles. Finally, if Y is a continuous vector field, we denote by Y^s, Y^u its components in the decomposition $TM = E^s \oplus E^u$. If Y is of class $C^{1+\alpha}$, so are Y^s and Y^u .

Theorem 2. *Let $f: T^2 \rightarrow T^2$ be a C^∞ Anosov diffeomorphism, and Y be a continuous vector field. If $Z = f_* Y - Y$ is of class C^∞ , then Y is of class C^∞ if and only if for any periodic orbit of period N ,*

$$\sum_{j=0}^{N-1} (X^s Z_s)(x_j) = \frac{1}{\lambda_N^s(x_0) - 1} \cdot \sum_{j=0}^{N-1} \lambda_k^s(x_j) \cdot Z_s(x_{j-k}) \cdot (X^s \lambda_1^s)(x_{j+1}), \tag{2.2}$$

and similarly for the unstable component Z_u of Z .

The proof of Theorem 2 will be postponed until we prove an analogous result for flows (Theorem 3). Before stating the corresponding result for flows we introduce the appropriate notation.

By the stable foliation of an Anosov flow ϕ_t in a 3-dimensional manifold M we mean the foliation by (one-dimensional) contracting leaves. The corresponding foliation by (two-dimensional) stable manifolds of orbits will be called the center-stable foliation. A map or function will be said to be of class C_s^k , just as in [LMM] where these spaces were introduced, if its restriction to each center-stable manifold is of class C^k and the k -jets of these restrictions depend continuously on the point in M . Finally, it will be said to be of class C_X^k when it satisfies the corresponding properties on orbits of the vector field X that generates the flow.

We assume that the vector fields X^s, X^u are $C^\alpha \cap C_s^\infty$ (respectively $C^\alpha \cap C_u^\infty$) sections of the stable and unstable bundles respectively. Notice that as a consequence of the results in [P] we cannot expect $X^{s,u}$ to be of class C^1 . We define the function $l^s \in C_s(M)$ by $L_X X^s = l^s X^s$, and we set $L^s(r, x_0) = \int_0^r (l^s \circ \phi_{-v})(x_0) dv$,

$F^s(r, x_0) = \exp[-L^s(r, x_0)]$. Since the flow is Anosov, there exist $K \in \mathbb{R}$, $C > 0$ such that

$$L^s(r, x_0) \geq K + C \cdot r \quad (2.3)$$

Just as in the case of diffeomorphisms, given a continuous stable vector field Y^s we define the continuous function Y_s by $Y^s = Y_s \cdot X^s$. If Y^s is of class C_s^k , so is Y_s . We use the analogous notation for unstable vector fields. Finally, if Y is a continuous vector field, we denote by Y_0 , Y^s , Y^u its components in the decomposition $TM = \mathbb{R}X \oplus E^s \oplus E^u$. If Y is of class C^1 , Y^s is of class C_s^1 and Y^u is of class C_u^1 , since the center-stable and center-unstable foliations are of class $C^{1+\alpha}$ (this well-known fact is proved for Anosov diffeomorphisms in [HP, Theorem 6.3]. The proof can be trivially adapted to the case of Anosov flows).

Theorem 3. *Let X be a C^∞ vector field on a 3-dimensional compact manifold M ; assume X generates a transitive Anosov flow. Let Y be a vector field of class C_X^k . If $L_X Y = Z$ is of class C^∞ , then Y is of class C^∞ if and only if for any periodic orbit γ of period T ,*

$$\int_0^T (X^s Z_s)(\gamma(t)) dt = \frac{1}{F^s(T, \gamma(0)) - 1} \int_0^T \int_0^T (X^s l^s)(\gamma(t)) \cdot F^s(r, \gamma(t)) \cdot Z_s(\gamma(t-r)) dr dt \quad (2.4)$$

and similarly for the unstable component of Z .

Remark. For a fixed Anosov flow and a fixed vector field Z , all the functions appearing in (2.4), namely Z_s , F^s , l^s depend on the choice of a non-vanishing stable vector field X^s of class C_s^∞ . However, if (2.4) holds for one choice of X^s , then it also holds for any other choice. This is a consequence of the proof of Theorem 3 that will be given below, since we shall prove that if (2.4) holds for a specific choice of X^s , then Y is smooth, and we shall also prove that Y^s being smooth implies that (2.4) holds for any choice of X^s .

Proof of Theorem 3. The proof is based on the following two lemmas:

Lemma 1. *Let Y^s be a stable vector field of class C_X^1 . If $L_X Y^s = Z^s$ is of class C_s^k , $k \geq 1$, then Y^s is of class C_s^k if and only if for any periodic orbit γ of period T , (2.4) holds. The corresponding statement for unstable vector fields is also true.*

For the second lemma we need some extra notation. Given a C^∞ one-dimensional vector subbundle \hat{E}^u of TM , transversal to the center-stable subbundle E^s , and given any continuous vector field Y , we define $\hat{Y}^s \in E^s$, $\hat{Y}^u \in E^u$ by $Y = \hat{Y}^s + \hat{Y}^u$. If Y is of class C_s^∞ , then \hat{Y}^s , \hat{Y}^u are also of class C_s^∞ .

Lemma 2. *Let Y be a vector field of class C_X^1 . If $L_X Y = Z$ is of class C^∞ , then \hat{Y}^u is of class C_s^∞ .*

We show now how Theorem 3 is a consequence of these two lemmas, and than their proof will be given.

Assume that Y is a C^∞ vector field. By the remarks preceding Lemma 2, Y^s is of class C_s^1 ; moreover, $L_X Y^s = Z^s$ is also of class C_s^1 , so we can apply Lemma 1 and we see that (2.4) is satisfied. The corresponding formula for the unstable component is proved in the same way.

Assume now that $L_X Y = Z$ is a C^∞ vector field and it satisfies (2.4). Let \hat{E}^u be a smooth subbundle of TM as in Lemma 2. According to this lemma, \hat{Y}^u is of class C_s^∞ . Assume

$$\hat{Y}^s = \hat{Y}^s + \mu X \quad (2.5)$$

where $\hat{Y}^s \in E^s$. Then $L_X \hat{Y}^s$ is the stable component of the C_s^∞ center-stable vector field $Z - L_X \hat{Y}^u$. Since the splitting $E^s = E^s \oplus \mathbb{R}X$ is of class C_s^∞ , we see that $L_X \hat{Y}^s$ is a stable vector field of class C_s^∞ . Since $Y = \hat{Y}^s + \mu X + \hat{Y}^u$, we see that

$$Z = L_X \hat{Y}^s + (X\mu)X + L_X \hat{Y}^u \quad (2.6)$$

Taking the stable components in both sides we have

$$L_X \hat{Y}^s = Z^s - L_X(\hat{Y}^u)^s \quad .$$

Since both $(\hat{Y}^u)^s$ and $L_X(\hat{Y}^u)^s$ are of class C_s^1 , by Lemma 1, $(L_X \hat{Y}^u)^s$ satisfies the same condition Z_s does in (2.4); by our assumption on Z , and the linearity of (2.4) on Z_s , $\left(Z - L_X \int_0^n \hat{Y}^u\right)_s$ also satisfies that condition, so again by Lemma 1, \hat{Y}^s is a C_s^∞ vector field.

On the other hand, taking components along X in (2.6) we see that $X\mu$ is the component along X of $Z - L_X \hat{Y}^u$, so it is of class C_s^∞ . By Lemma 2.2 in [LMM], μ itself is of class C_s^∞ . This proves that $Y = \hat{Y}^s + \mu X + \hat{Y}^u$ is of class C_s^∞ .

A similar argument shows that if the unstable component of Z satisfies the condition that corresponds to (2.4), then Y is of class C_u^∞ . By Lemma 2.3 in [LMM] it follows that Y is of class C^∞ . This finishes the proof of Theorem 3.

Proof of Lemma 1. The fact that Z^s and Y^s being of class C_s^1 implies (2.4) has a local character for each periodic orbit γ ; it is actually a simple property of attracting periodic orbits of flows in two dimensions, since only the local stable manifold of γ is involved. The proof goes as follows:

By the definitions of l^s , Y_s and Z_s , the equality $L_X Y^s = Z^s$ is equivalent to

$$XY_s + l^s Y_s = Z_s \quad (2.7)$$

By (2.3), the previous expression implies that

$$Y^s(x_0) = \int_0^\infty F^s(r, x_0) \cdot (Z_s \circ \phi_{-r})(x_0) dr \quad (2.8)$$

Since $F^s(a+b, x) = F^s(a, \phi_{-b}(x)) \cdot F^s(b, x)$, we see that on γ ,

$$\begin{aligned} Y^s(x_0) &= \sum_{k=0}^\infty \int_0^T F^s(kT+r, x_0) \cdot z_s(\gamma(-r)) dr dt \\ &= \frac{1}{F^s(T, x_0) - 1} \int_0^T F^s(r, x_0) \cdot z_s(\gamma(-r)) dr \quad (2.9) \end{aligned}$$

If we differentiate along X^s in (2.7), taking into account the definition of l^s and (2.9),

we see that

$$\begin{aligned} [X(X^s Y_s)](\gamma(t)) &= (X^s Z_s)(\gamma(t)) - (X^s l^s)(\gamma(t)) \cdot Y_s(\gamma(t)) \\ &= (X^s Z_s)(\gamma(t)) - (X^s l^s)(\gamma(t)) \\ &\quad \cdot \frac{1}{F^s(t, \gamma(t)) - 1} \int_0^T F^s(r, \gamma(t)) \cdot z_s(\gamma(t-r)) dr . \end{aligned}$$

Integrating the above expression between $t=0$ and T , the term in the left cancels out and we get (2.4).

Let us assume now that Z_s satisfies (2.4). Consider the sequence of functions

$$Y_s^n(x_0) = \int_0^n F^s(r, x_0) \cdot (Z_s \circ \phi_{-r})(x_0) dr .$$

Then by (2.8), $Y_s = \lim_{n \rightarrow \infty} Y_s^n$ uniformly on M , each Y_s^n is of class C_s^∞ on M , and its derivative along X^s can be computed directly:

$$X^s Y_s^n(x_0) = \int_0^n [(X^s Z_n) \circ \phi_{-r}](x_0) dr + \int_0^n (X^s F^s)(r, x_0) \cdot (Z_s \circ \phi_{-r})(x_0) dr .$$

Adding and subtracting the integral of $[(X^s l^s) \cdot Y_s] \circ \phi_{-r}$, and using (2.8),

$$\begin{aligned} X^s Y_s^n(x_0) &= \int_0^n \{ [X^s Z_n - (X^s l^s) \cdot Y_s] \circ \phi_{-r} \} (x_0) dr \\ &\quad + \int_0^n [(X^s l^s) \circ \phi_{-r}](x_0) \int_0^\infty \exp \left[- \int_0^t (l^s \circ \phi_{-r-u})(x_0) du \right] \\ &\quad \cdot (Z_s \circ \phi_{-r-t})(x_0) dt dr \\ &\quad - \int_0^n F^s(r, x_0) \int_0^r \exp [L^s(s, x_0)] \cdot [(X^s l^s) \circ \phi_{-s}](x_0) ds \cdot (Z_s \circ \phi_{-r})(x_0) dr . \end{aligned} \tag{2.10}$$

In the second integral we make the change of variables $\tilde{r} = r$, $\tilde{t} = r + t$, and change the order of integration. In this way we get the sum of two integrals, one over $0 \leq \tilde{r} \leq \tilde{t} \leq n$, and the second over $0 \leq \tilde{r} \leq n \leq \tilde{t}$. The first of these two integrals cancels out with the last term in (2.10), so we have, using (2.8) again,

$$\begin{aligned} X^s Y_s^n(x_0) &= \int_0^n \{ [X^s Z_s - (X^s l^s) \cdot Y_s] \circ \phi_{-r} \} (x_0) dr \\ &\quad + \int_n^\infty \int_0^n \exp \left[- \int_r^t (l^s \circ \phi_{-v})(x_0) dv \right] \cdot [(X^s l^s) \circ \phi_{-r}](x_0) \cdot (Z_s \circ \phi_{-t})(x_0) dr dt \\ &= \int_0^n \{ [X^s Z_s - (X^s l^s) \cdot Y_s] \circ \phi_{-r} \} (x_0) dr \\ &\quad + (Y_s \circ \phi_{-n})(x_0) \cdot \int_0^n F^s(n-r, \phi_{-r}(x_0)) \cdot [(X^s l^s) \circ \phi_{-r}] dr . \end{aligned}$$

Since X^s is of class $C^{1+\alpha}$, l^s is of class C^α , so $X^s Z_s - (X^s l^s) \cdot Y_s$ is of class C^α . By (2.4) and Livsic's theorem (see [Li]), there exists a Hölder function H on M of class C_X^1 such that

$$XH = X^s Z_s - (X^s l^s) \cdot Y_s .$$

Notice that it is at this point of the proof where we need the transitivity of X . Substituting this in the above expression we see that

$$\begin{aligned} X^s Y_s^n(x_0) &= (H \circ \phi_{-n})(x_0) - H(x_0) \\ &\quad + (Y^s \circ \phi_{-n})(x_0) \int_0^n F^s(n-r, \phi_{-r}(x_0)) \cdot [(X^s l^s) \circ \phi_{-r}] dr . \end{aligned}$$

The first two terms are bounded in absolute value by a constant independent of n , and the same happens to $(Y^s \circ \phi_{-n})(x_0)$. Finally, by (2.3),

$$\left| \int_0^n F^s(n-r, \phi_{-r}(x_0)) \cdot [(X^s l^s) \circ \phi_{-r}](x_0) dr \right| \leq \bar{C} \int_0^n \exp[-K-C(n-r)] dr$$

is also bounded by a constant independent of n .

This implies that $Y_s = \lim_{n \rightarrow \infty} Y_s^n$ is uniformly Lipschitz along center-stable manifolds. Then it is differentiable along center-stable manifolds almost everywhere, and by (2.7), XY_s is also differentiable almost everywhere along center-stable manifolds. Since X, X^s are C^∞ vector fields along these manifolds, $XX^s Y_s$ exists in the sense of distributions on any center-stable manifold, it is a bounded function, and the same argument of the beginning of the proof of this lemma shows that

$$XX^s Y_s = X^s Z_s - (X^s l^s) Y_s .$$

By Livsic's theorem, $X^s Y_s$ is a Hölder function, so Y_s is of class C_s^1 . But this implies that the right-hand side in the above formula is of class C_s^1 ; the proof of Lemma 2.2 in [LMM] gives then that $X^s Y_s$ is of class C_s^1 , hence Y_s is of class C_s^2 . The bootstrap argument can be continued to prove Y_s is of class C_s^∞ . This finishes the proof of Lemma 1.

Proof of Lemma 2. Given $\tau \in \mathbb{R}$, consider the vector field $X_\tau = X + \tau Z$. By Anosov's persistence theorem, X_τ generates an Anosov flow if τ is close to 0.

By the smooth dependence on parameters in Anosov's structural stability theorem (see Theorem A.1 in [LMM] for a detailed proof), there exists a C^∞ family of homeomorphisms h_τ , and a C^∞ family of continuous functions on M ξ_τ , such that h_τ conjugates the flows of X_0 and $\xi_\tau \cdot X_\tau$. Moreover, h_τ transforms the center-stable manifold of $x_0 \in M$ with respect to X_0 into the center-stable manifold of $h_\tau(x_0)$ with respect to X_τ .

Lemma 2 will be a consequence of this and the smooth dependence of the center-stable spaces on parameters, as stated in the following lemma:

Lemma 3. *Let X_τ be a C^∞ family of C^∞ vector fields that generate Anosov flows on the compact 3-dimensional manifold M . Then the map that sends each $(x, \tau) \in M \times I$ to the center-stable subspace $E_{x,\tau}^s$ of X_τ at x is of class C^1 as a section of the Grassmanian bundle over M . Moreover, its derivative with respect to t defines a C_s^∞ map. If T is a C^∞ vector field, the derivative of $E_{x,\tau}^s$ along T is also a C_s^∞ map.*

Taking Lemma 3 for granted, we shall finish the proof of Lemma 2. We restrict ourselves to a small neighbourhood of an arbitrary point $x_0 \in M$, where we assume \hat{E}^u is generated by $\partial/\partial x_1$ for some system of coordinates (x_1, x_2, x_3) . We can also assume that the local center-stable manifold of x_0 with respect to X_0 is given by $x_1=0$, $X_0 = \partial/\partial x_2$, and $E_{x,\tau}^s$ is generated by $\frac{\partial}{\partial x_j} + m_\tau^j(x) \cdot \frac{\partial}{\partial x_1}$, for $j=2, 3$, where m_τ^j , $\partial m_\tau^j/\partial x_1$ and $dm_\tau^j/d\tau$ are functions of class C_s^∞ for $\tau=0$, and they are of class C^1 jointly on x and τ .

Then the center-stable map of X_τ through $h_\tau(x_0)$ is given by a function $x_1 = \eta_\tau(x_2, x_3)$ satisfying

$$\frac{\partial \eta_\tau}{\partial x_j} = m_\tau^j(\eta_\tau, x_2, x_3) \quad , \quad j=2, 3$$

and

$$\eta_\tau(x_2^\tau, x_3^\tau) = x_1^\tau \quad ,$$

where $(x_1^\tau, x_2^\tau, x_3^\tau)$ are the coordinates of $h_\tau(0, x_2, x_3)$.

By the smooth dependence of solutions of ordinary differential equations on initial conditions and parameters, η_τ can be differentiated with respect to τ , and we can also differentiate the last two expressions with respect to τ . If we denote derivatives with respect to τ by $\dot{}$, and taking into account that $\eta_0 \equiv 0$, we get the following making $\tau=0$:

$$\frac{\partial \eta_0}{\partial x_j} = \dot{m}_0^j(0, x_2, x_3) + \frac{\partial m_0^j}{\partial x_1}(0, x_2, x_3) \cdot \dot{\eta}_0 \tag{2.11}$$

and

$$\dot{\eta}_0(x_2, x_3) = \dot{x}_0^1 \quad . \tag{2.12}$$

By (2.12), the component along \hat{E}^u of \dot{h} at points of the center-stable manifold of x_0 is just $\dot{\eta}_0$. By (2.11), this map is of class C^∞ along that center-stable manifold. Moreover, since \dot{m}_0^j , $\partial \dot{m}_0^j/\partial x_1$ depend continuously on the choice we made of the parametrization of the center-stable manifold, the component along \hat{E}^u of \dot{h} is of class C_s^∞ on M . We claim that this component is also the component of Y along \hat{E}^u , and this will finish the proof of Lemma 2. To see this we differentiate

$$h_\tau^* X_\tau = \xi_\tau \cdot X_0$$

with respect to τ at $\tau=0$, and we have

$$L_X \dot{h}_0 = Z + \dot{\xi}_0 X \quad .$$

Since $L_X Y = Z$, we see that $Y = \dot{h}_0 - \eta X$, where $X\eta = \dot{\xi}_0$. Taking components along \hat{E}^u on the center-stable manifold of x_0 our claim is proved, and this finishes the proof of Lemma 2.

Proof of Lemma 3. The fact that $E_{x,\tau}^s$ has C^1 dependence on x and τ is a direct consequence of the C^r section theorem (Theorem 3.5 in [HPS]). The corresponding result for Anosov diffeomorphisms on T^2 is proved in the first part of Lemma 4 in [MM], and the same proof can be extended immediately to our case.

Consider two vector subbundles of TM , \tilde{E}^s, \tilde{E}^u , and assume they are uniformly close enough to E_τ^s, E_τ^u for $|\tau| < \varepsilon_0$. Then the center stable space $E_{x,\tau}^s$ is the graph of a linear map $\sigma_\tau(x) \in L(\tilde{E}_{x,\tau}^s, \tilde{E}_{x,\tau}^u)$, uniformly close to 0. The invariance of $E_{x,\tau}^s$ under the flow implies that

$$\sigma_\tau(f_\tau(x)) \cdot (f_\tau^{11}(x) + f_\tau^{12}(x) \cdot \sigma_\tau(x)) = f_\tau^{21}(x) + f_\tau^{22}(x) \cdot \sigma_\tau(x) , \quad (2.13)$$

where f_τ is the time-(-1) flow of X_τ , and the tangent map $f'_\tau(x) : T_x M \rightarrow T_{f_\tau(x)} M$ defines corresponding linear maps $f_\tau^{ij}(x)$ under the splittings $T_x M = \tilde{E}_x^s \oplus \tilde{E}_x^u$ and $T_{f_\tau(x)} M = \tilde{E}_{f_\tau(x)}^s \oplus \tilde{E}_{f_\tau(x)}^u$.

We want to take derivatives with respect to τ in (2.13), but the dependence on τ of the fibers where each linear map is defined would make the resulting formula unuseful if we did it directly. Instead of that, we shall first compose with suitable fibered maps that send the fiber over $f_\tau(x)$ to the fiber over $f(x)$.

Given two points in M close enough, we consider the linear map $L_{x,y} : T_x M \rightarrow T_y M$ given by parallel translation along the shortest geodesic that joins x and y with respect to some fixed riemannian metric, and we set

$$\begin{aligned} \tilde{\sigma}_\tau(f_0(x)) &= L_{f_0(x), f_\tau(x)} \cdot \sigma_\tau(f_\tau(x)) \cdot L_{f_\tau(x), f_0(x)} \in L(T_{f_0(x)} M, T_{f_0(x)} M) , \\ \tilde{f}_\tau^{ij}(x) &= L_{f_0(x), f_\tau(x)} \cdot f_\tau^{ij}(x) \in L(T_x M, T_{f_0(x)} M) . \end{aligned}$$

Applying $L_{f_0(x), f_\tau(x)}$ to both sides of (2.13) we get

$$\tilde{\sigma}_\tau(f_0(x)) \cdot (\tilde{f}_\tau^{11}(x) + \tilde{f}_\tau^{12}(x) \cdot \sigma_\tau(x)) = \tilde{f}_\tau^{21}(x) + \tilde{f}_\tau^{22}(x) \cdot \sigma_\tau(x) . \quad (2.14)$$

If we differentiate (2.14) with respect to τ we have

$$\begin{aligned} \dot{\tilde{\sigma}}_\tau(f_0(x)) \cdot (\tilde{f}_\tau^{11}(x) + \tilde{f}_\tau^{12}(x) \cdot \sigma_\tau(x)) + \tilde{\sigma}_\tau(f(x)) \cdot (\dot{\tilde{f}}_\tau^{11}(x) \\ + \dot{\tilde{f}}_\tau^{12}(x) \cdot \sigma_\tau(x) + \tilde{f}_\tau^{12}(x) \cdot \dot{\sigma}_\tau(x)) = \dot{\tilde{f}}_\tau^{21}(x) + \dot{\tilde{f}}_\tau^{22}(x) \cdot \sigma_\tau(x) + \tilde{f}_\tau^{22}(x) \cdot \dot{\sigma}_\tau(x) . \end{aligned} \quad (2.15)$$

If we denote for a given section σ of $L(TM, TM)$,

$$F_\tau \sigma(f_0(x)) = L_{f_0(x), f_\tau(x)} \cdot \sigma(f_\tau(x)) \cdot L_{f_\tau(x), f_0(x)} \in L(T_{f_0(x)} M, T_{f_0(x)} M) ,$$

then we see that $\tilde{\sigma}_\tau = F_\tau \sigma_\tau$, so that since F_τ is linear, $\dot{\tilde{\sigma}}_\tau = \dot{F}_\tau \sigma_\tau + F_\tau \dot{\sigma}_\tau$. Using this and applying $L_{f_\tau(x), f_0(x)}$ to both sides of (2.15), we have

$$\begin{aligned} [\dot{\sigma}_\tau(f_\tau(x)) + L_{f_\tau(x), f_0(x)} (\dot{F}_\tau \sigma_\tau) L_{f_0(x), f_\tau(x)}] \cdot [f_\tau^{11}(x) + f_\tau^{12}(x) \cdot \sigma_\tau(x)] \\ + \sigma_\tau(f_\tau(x)) \cdot L_{f_0(x), f_\tau(x)} [\dot{\tilde{f}}_\tau^{11}(x) + \dot{\tilde{f}}_\tau^{12}(x) \cdot \sigma_\tau(x)] + \sigma_\tau(f_\tau(x)) \\ \cdot \tilde{f}_\tau^{12}(x) \cdot \dot{\sigma}_\tau(x) = L_{f_\tau(x), f_0(x)} [\dot{\tilde{f}}_\tau^{21}(x) + \dot{\tilde{f}}_\tau^{22}(x) \cdot \sigma_\tau(x)] + f_\tau^{22}(x) \cdot \dot{\sigma}_\tau(x) \end{aligned}$$

or, equivalently,

$$\begin{aligned} \dot{\sigma}_\tau(f_\tau(x)) = [f_\tau^{22}(x) - \sigma_\tau(f_\tau(x)) \cdot f_\tau^{12}(x)] \dot{\sigma}_\tau(x) [f_\tau^{11}(x) + f_\tau^{12}(x) \cdot \sigma_\tau(x)]^{-1} \\ + \{L_{f_\tau(x), f_0(x)} [\dot{\tilde{f}}_\tau^{21}(x) + \dot{\tilde{f}}_\tau^{22}(x) \cdot \sigma_\tau(x)] - \sigma_\tau(f(x)) \\ \cdot L_{f_0(x), f_\tau(x)} \cdot [\tilde{f}_\tau^{11}(x) + \tilde{f}_\tau^{12}(x) \cdot \sigma_\tau(x)]\} \cdot [f_\tau^{11}(x) + f_\tau^{12}(x) \cdot \sigma_\tau(x)]^{-1} \\ - L_{f_\tau(x), f_0(x)} \cdot (\dot{F}_\tau \sigma_\tau) \cdot L_{f_0(x), f_\tau(x)} . \end{aligned}$$

The right-hand side in this expression defines a Lipschitz fibered map on the linear bundle $L(TM, TM)$ over f_τ . This map involves σ_τ , which is not C^∞ , but only C_s^∞ , so it is of class $C_{\mathcal{F}_s}^\infty$, where \mathcal{F}_s is the foliation of $L(TM, TM)$ whose leaves are the unions of the fibers of all points in a center-stable manifold of f_τ . Moreover, it is a fiber-contracting map, since only the first term contributes to the Lipschitz constant along fibers, and both $f_\tau^{22}(x)$ and $(f_\tau^{11}(x))^{-1}$ are uniformly contracting, and the whole term involves small perturbations of them if \tilde{E}^s, \tilde{E}^u are close enough to E^s, E^u respectively. Then Lemma 3 will be a consequence of the following variation of the C^r section theorem:

Lemma 4. *Let $f: M \rightarrow M$ be a diffeomorphism, $\pi: E \rightarrow M$ a fiber bundle, and $F: E \rightarrow E$ a Lipschitz fiber contraction over f . Assume \mathcal{F} is C^0 foliation by C^∞ leaves, invariant with respect to f , and uniformly contracting. If F is of class $C_{\pi^*\mathcal{F}}^\infty$ then the invariant section is of class $C_{\mathcal{F}}^\infty$.*

The proof is identical to the usual proofs of the C^r case in [HP, Sh, HPS], and will be omitted.

Using Lemma 4 we see that $\hat{\sigma}_\tau$, and hence the derivative of $E_{x,\tau}^s$ with respect to τ , is C^∞ along stable (strictly contracting) directions. We shall prove now that it is also C^∞ along the flow of X (that is, it is in the class C_X^∞). These two facts imply that it is of class C_s^∞ .

The C_X^∞ regularity can be proved as follows: the subbundles E_τ^s of T_xM are the kernels of suitable $C^1 \cap C_s^\infty$ one-forms α_τ^s satisfying

$$L_{X_\tau} \alpha_\tau^s = \lambda_\tau^s \alpha_\tau^s$$

for some $C^1 \cap C_s^\infty$ functions λ_τ^s which show also C^1 dependence with respect to τ .

Differentiating the above expression with respect to τ we see that

$$L_{X_\tau} \dot{\alpha}_\tau + L_{\dot{X}_\tau} \alpha_\tau = \dot{\lambda}_\tau^s \alpha_\tau + \lambda_\tau^s \dot{\alpha}_\tau,$$

which can be seen as an ordinary differential equation for $\dot{\alpha}_\tau$ along the integral curves of X_τ with C_X^∞ coefficients. It follows that $\dot{\alpha}_\tau$, and hence $\dot{E}_{x,\tau}^s$, are of class C_X^∞ .

Finally, if T is a C^∞ vector field, we have to prove that the derivative of $E_{x,\tau}^s$ along T is a C_s^∞ map. This can be reduced to the previous case as follows. First, we can just look at the case $\tau=0$. If $\phi_t, t \in \mathbb{R}$, is the flow of T , we consider $\bar{X}_t = \phi_t^* X_0$. Let $\bar{E}_{x,t}^s$ be its corresponding center-stable spaces. Then $E_{\phi_t(x),0}^s = \phi_t' \bar{E}_{x,t}^s$, and by the previous case it can be differentiated explicitly with respect to t . This finishes the proof of Lemma 3.

We give now a proof of Theorem 2. It follows closely that of Theorem 3. Only those parts of the proof which are not direct translations of the case of flows will be worked out in detail.

Just like Theorem 3, the proof of Theorem 2 is based on two lemmas, as follows:

Lemma 1'. *Let Y^s be a continuous stable vector field. If $Z^s = f_* Y^s - Y^s$ is of class C_s^k , $k \geq 1$, then Y^s is of class C_s^k if and only if for any periodic orbit of period N , (2.2) holds. The corresponding statement for unstable vector fields is also true.*

Assume now that \hat{E}^u is a C^∞ one-dimensional vector subbundle of TM , transversal to the stable subbundle E^s . Given any continuous vector field Y , define

$\hat{Y}^s \in E^s$, $\hat{Y}^u \in \hat{E}^u$ by imposing $Y = \hat{Y}^s + \hat{Y}^u$. If Y is of class C_s^∞ , then \hat{Y}^s , \hat{Y}^u are also of class C_s^∞ .

Lemma 2'. *If Y is a continuous vector field and $Z = f_* Y - Y$ is of class C^∞ , then \hat{Y}^u is of class C_s^∞ .*

The proof of Theorem 2 from the above two lemmas is identical to the case of flows and it is left to the reader. On the other hand, Lemma 2' is a consequence of Lemma 2: given Y and \hat{E}^u on M , if the corresponding suspended flow is defined on $\tilde{M} = M \times \mathbb{R} / \sim$, we can construct a corresponding vector field \tilde{Y} on \tilde{M} by

$$\tilde{Y}(x, t) = \sum_{j=-\infty}^{\infty} \chi(t-j) \cdot f_*^j Y,$$

where $\chi \in C_0^\infty(-1, 1)$ satisfies $\sum_{j=-\infty}^{\infty} \chi(t-j) = 1$ (notice that the above infinite sums have at most two non-vanishing terms), and then

$$L_{\partial/\partial t} \tilde{Y} = \sum_{j=-\infty}^{\infty} \chi'(t-j) \cdot f_*^j Y = \frac{1}{2} \sum_{j=-\infty}^{\infty} \chi'(t-j) \cdot (f_*^j Y - f_*^{j-1} Y)$$

is a C_s^∞ vector field on \tilde{M} if $Z = f_* Y - Y$ is of class C_s^∞ on M . Lemma 2, applied to \tilde{Y} and a smooth extension of \hat{E}^u to \tilde{M} gives the desired result.

The proof of Lemma 1' is similar to that of Lemma 1. The only place where the translation from the language of flows to diffeomorphisms is not obvious is at the proof of the sufficiency of (2.3) for the regularity of Y^s , so we give the details now.

Assuming that Z^s is of class C_s^∞ , we consider the sequence of C_s^∞ functions

$$Y_s^n = - \sum_{j=1}^n \lambda_j^s \cdot (Z_s \circ f^{-j}).$$

Differentiating along X^s , and manipulating as in the proof of Lemma 1,

$$\begin{aligned} X^s Y_s^n &= - \sum_{j=0}^n (X^s Z_s) \circ f^{-j} - \sum_{j=0}^n (X^s \lambda_j^s) \cdot (Z_s \circ f^{-j}) \\ &= - \sum_{j=0}^n [X^s Z_s - (X^s \lambda_1^s) \cdot (Y_s \circ f^{-1})] \circ f^{-j} \\ &\quad + \sum_{j=0}^n \sum_{k=0}^{\infty} [(X^s \lambda_1^s) \circ f^{-j}] \cdot (\lambda_k^s \circ f^{-j-1}) \cdot (Z_s \circ f^{-j-k-1}) \\ &\quad - \sum_{j=0}^n \sum_{k=0}^{j-1} [(X^s \lambda_1^s) \circ f^{-k}] \cdot (\lambda_{j-k-1}^s \circ f^{-k-1}) \cdot (Z_s \circ f^{-j}), \end{aligned}$$

where we have used that $\lambda_j^s = \prod_{k=0}^{j-1} (\lambda_1^s \circ f^{-k})$ in order to compute its derivative along X^s , and that $Y_s = - \sum_{k=0}^{\infty} \lambda_k^s \cdot (Z_s \circ f^{-k})$, since $Z = f_* Y - Y$.

Defining new indices $\tilde{j} = j$, $\tilde{k} = j + k$ in the first double sum, and interchanging the order of summation, we get two sums, one for $0 \leq \tilde{j} \leq \tilde{k} \leq n-1$ and another for $0 \leq \tilde{j} \leq n \leq \tilde{k}$. The first one cancels out with the last double sum in the previous

expression, so we have

$$\begin{aligned}
 X^s Y_s^n &= - \sum_{j=0}^n [X^s Z_s - (X^s \lambda_1^s) \cdot (Y_s \circ f^{-1})] \circ f^{-j} \\
 &+ \sum_{k=0}^{\infty} \sum_{j=0}^n [(X^s \lambda_1^s) \circ f^{-j}] \cdot (\lambda_{n+k-j}^s \circ f^{-j-1}) \cdot (Z_s \circ f^{-n-k-1}) .
 \end{aligned}$$

By (2.2) and Livsic’s theorem (see [Li]), there exists a Hölder function H on M such that

$$H \circ f - H = X^s Z_s - (X^s \lambda_1^s) \cdot (Y_s \circ f^{-1}) .$$

Substituting this in the above expression, and taking into account that $\lambda_{n+k-j}^s = \lambda_{n-j}^s \cdot (\lambda_k^s f^{j-n})$, we see that

$$X^s Y_s^n = H - H \circ f^{n+1} + \sum_{j=0}^n [(X^s \lambda_1^s) \circ f^{-j}] (\lambda_{n-j}^s \circ f^{-j-1}) \cdot (Y_s \circ f^{-n-1}) ,$$

and the absolute value of $X^s Y_s^n$ is bounded by a constant independent of n as a consequence of (2.1). From this one gets that $Y_s = \lim_{n \rightarrow \infty} Y_s^n$ is uniformly Lipschitz on stable manifolds. The remaining part of the proof of Lemma 1’ is completely similar to that of Lemma 1. The details are left to the reader. As a consequence of Theorem 3 we have the following result on smooth conjugacy of Anosov flows in 3 dimensions.

Theorem 4. *Let X_τ , $-a < \tau < a$, be a C^∞ family of C^∞ transitive Anosov vector fields on a compact 3-dimensional manifold. A necessary and sufficient condition for the existence of diffeomorphisms g_τ , $-a < \tau < a$, satisfying*

$$g_\tau^* X_\tau = X_0$$

is that for any smooth family γ_τ of periodic orbits of X_τ both the periods and the Lyapunov exponents of γ_τ are independent of τ . Moreover, in such a case the g_τ are a C^∞ family of C^∞ diffeomorphisms.

Remark. According to R. Mañé (talk at the workshop on Dynamical systems, CTIP, Trieste, June 1986) the periods of periodic orbits are by themselves a complete set of invariants for smooth conjugation of Anosov flows with C^1 invariant foliations (in particular geodesic flows). This is a consequence of a theorem by Feldman and Ornstein that gives C^1 conjugacy. A. Katok has announced a proof of the fact that the C^1 conjugacy implies C^∞ conjugacy, based on the techniques of [LMM]. The techniques of [L1] also give a proof of this fact.

Proof of Theorem 4. The necessity is obvious. We prove the sufficiency of the conditions, so we assume the periods and Lyapunov exponents of corresponding orbits are the same. Differentiating (2.16) with respect to τ we see that it suffices to show that there exists a C^∞ family of C^∞ vector fields H_τ satisfying

$$L_{X_\tau} H_\tau = \frac{dX_\tau}{d\tau} . \tag{2.17}$$

First we shall prove that there exists a continuous family of $C^1_{X_\tau}$ vector fields H_τ satisfying (2.17), and then we shall prove that $dX_\tau/d\tau$ satisfies the conditions of Theorem 3, so that the H_τ are actually of class C^∞ .

The existence of the H_τ is a consequence of the constancy of the periods, as we shall see now. By the version in [LMM, Appendix A], of Anosov's structural stability theorem, there is a C^∞ isotopy \tilde{h}_τ and a C^∞ family of C^∞ functions η_τ such that

$$\tilde{h}_\tau^* X_\tau = \eta_\tau X_0 .$$

The same proof of Anosov's theorem gives that both η_τ and $\tilde{h}_\tau^* X_\tau$ and their successive derivatives with respect to τ are Hölder functions of order α for some fixed $\alpha > 0$ depending only on X_τ (the only changes are in the functional setting, where one has to consider spaces of Hölder functions and sections of vector bundles; in the proof of Lemma A.7 one has to prove that all the functions and sections that appear are Hölder, but this is a consequence of the invariant subspaces of the flow being Hölder, see [Sh, Corollary 5.19]).

If we differentiate our last expression with respect to τ at $\tau=0$, taking into account that $\eta_0=1$ and $\tilde{h}_0 = \text{Id}$, we see that

$$L_{\tilde{H}_0} X_0 + \frac{dX_\tau}{d\tau} \Big|_{\tau=0} = \left(\frac{d\eta_\tau}{d\tau} \Big|_{\tau=0} \right) \cdot X_0 .$$

By the constancy of the periods of the periodic orbits, the integral of η_τ along each periodic orbit of X_0 is equal to the period of the orbit. This means that the average of $\eta_\tau - 1$ along any periodic orbit of X_0 vanishes, so by Livsic's theorem, since $\eta_\tau - 1$ and its successive derivatives with respect to τ are Hölder functions, there exists a C^∞ family ψ_τ of Hölder functions such that $\eta_\tau - 1 = X_0 \psi_\tau$. Then, by our last equality,

$$L_{X_0} \tilde{H}_0 = \frac{dX_\tau}{d\tau} \Big|_{\tau=0} - \left[X_0 \left(\frac{d\psi_\tau}{d\tau} \right) \right] \cdot X_0 = \frac{dX_\tau}{d\tau} \Big|_{\tau=0} - L_{X_0} \left[\frac{d\psi_\tau}{d\tau} \cdot X_0 \right] .$$

Since this can be done for any particular value of τ instead of $\tau=0$, our claim is proved. We shall see now that $dX_\tau/d\tau$ satisfies the conditions of Theorem 3.

As a consequence of our hypothesis about the periods of periodic orbits, given any periodic orbit γ of X_0 there exists a C^∞ family of C^∞ diffeomorphisms h_τ such that γ is a periodic orbit of $h_\tau^* X_\tau$ for every τ . By Theorem 1 in [F] applied to the flow defined by the family X_τ on $M \times \mathbb{R}$, we can make a further conjugation (we still call h_τ the resulting isotopy) in such a way that the previous property still holds and the local stable manifold of γ with respect to $X_\tau^1 = h_\tau^* X_\tau$ is independent of τ .

We consider coordinates $(\theta, x) \in S^1 \times I$ on these local stable manifolds in such a way that $X^{1\tau} = T^{-1} \partial/\partial\theta$ on $S^1 \times \{0\}$. We can also take them in such a way that $X^1_0 = T^{-1} \partial/\partial\theta - T\lambda \partial/\partial x$, where λ is the Lyapunov exponent of γ . This is a direct consequence of Sternberg's theorem, [St], on the linearization of contractive maps in the real line, which allows one to assume that the Poincaré map of X^1_0 along the stable manifold of a point in γ is linear.

Assume now that on the stable manifold of γ ,

$$X_\tau^1 = H_\tau(\theta, x) \frac{\partial}{\partial \theta} + V_\tau(\theta, x) \frac{\partial}{\partial x} .$$

The Lyapunov exponents of X_τ^1 along γ are the same as those of

$$\tilde{X}_\tau^1 = \frac{\partial}{\partial \theta} + V_\tau(\theta, x) \cdot H_\tau(\theta, x)^{-1} \cdot \frac{\partial}{\partial x} ,$$

which can be computed explicitly :

$$\lambda_\tau = \int_0^1 \frac{\partial}{\partial x} [V_\tau(\theta, x) \cdot H_\tau(\theta, x)^{-1}]|_{x=0} d\theta .$$

By the hypothesis about the Lyapunov exponents, the derivative with respect to τ of the above expression vanishes. Since

$$V_\tau(\theta, 0) = H_\tau(\theta, 0) = \frac{\partial V_\tau}{\partial \tau}(\theta, 0) = \frac{\partial H_\tau}{\partial \tau}(\theta, 0) = 0 ,$$

we see that

$$\int_0^1 \frac{\partial^2 V_\tau}{\partial \tau \partial x}(\theta, 0) d\theta = 0 .$$

For $\tau=0$, $\partial/\partial x$ is a stable vector field, as a consequence of the explicit expression of X_0 , and $\partial V_\tau/\partial \tau$ is the component of $dX_\tau^1/d\tau$ along the stable direction. Since the C^∞ stable vector field $\partial/\partial x$ can be extended from the local stable manifold of γ to a C_s^∞ non-vanishing stable vector field on M , we see that $dX_\tau^1/d\tau$ satisfies (2.4) when $\tau=0$ for a particular stable section and, by the remark made after the statement of Theorem 4, it also satisfies (2.4) for any global non-vanishing stable vector field X^s . Since $X_\tau = g_\tau^* X_\tau^1$ with $g_\tau = h_\tau^{-1}$ and $g_0 = \text{Id}$,

$$\left. \frac{dX_\tau}{d\tau} \right|_{\tau=0} = L_{G_0} X_0^1 + \left. \frac{dX_\tau^1}{d\tau} \right|_{\tau=0} ,$$

where G_τ is the vector field that generates the isotopy g_τ . By Theorem 3, $L_{G_0} X_0^1$ satisfies (2.4), and this proves that $dX_\tau/d\tau$ also satisfies (2.4) for $\tau=0$. By Theorem 3 again, the solution H_0 of (2.17) is of class C^∞ .

The above argument can be carried out for any τ , since $\tau=0$ did not play any particular role. This finishes the proof of Theorem 4, since the C^∞ dependence on the parameter τ holds independently of any assumption on periods or Lyapunov exponents (see [LMM, Appendix A]).

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