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The Evaluation Map in Field Theory, Sigma-Models and Strings I

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Abstract. The rôle of the evaluation map in anomaly calculations for field theory, sigma-models and strings is investigated. In this paper, anomalies in field theory (with and without a backgrounds connection), are obtained as pull-backs of suitable forms via evaluation maps. The cohomology of the group of gauge transformations is computed in terms of the cohomology of the base manifold and of the cohomology of the structure group. This allows us to clarify the different "topological significance" of gauge and gravitational anomalies. The relation between "locality" and "universality" is discussed and "local cohomology" is linked to the cohomology of classifying spaces. The problem of combining the locality requirement and the index theorem approach to anomalies is also examined. Anomaly cancellation in field theories derived from superstrings is analyzed and the relevant geometrical constraints are discussed.

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Introduction

The classical results of Green and Schwarz that chiral anomalies are absent in field theories whose field content corresponds to the zero mass excitation spectrum of some superstring theories (briefly, superstring inspired field theories) has spurred new interest in the nature and origin of anomalies.

The original motivation of our work was to disentangle the series of problems connected with the Green and Schwarz mechanism and more generally with the cancellation of anomalies in higher dimensional space-times. As it turned out, anomaly cancellation is a very complex phenomenon. We may consider it from the point of view of field theory and then we are led to studying the precise meaning of locality and local cohomology in field theory and, subsequently, the relation between the locality requirement and the index theorem approach to anomalies. Or we can regard it from the point of view of sigma-models, where the locality requirement is replaced by a weaker one and (generalized) Wess-Zumino terms are introduced. In the two cases, the cancellation mechanism takes on radically different aspects.

In the course of the research the above two points of view, analyzed respectively in this paper and in the subsequent paper (which will be referred to as [1]), have grown up to systematic analyses of anomalies in field theories, sigma-models and strings. In both parts a central rôle is played by a basic mathematical tool: the evaluation map. One could claim that the evaluation map is, at the same time, one of the most "important" and one of the most "elementary" functions in mathematics and in mathematical physics.

If M and N are spaces (or manifolds), then the evaluation map ev is simply defined as follows:

$$\begin{aligned} \mathrm{ev}: M \times \mathrm{Map}(M,N) &\to N \\ (x,f) &\mapsto f(x). \end{aligned}$$

Instead of the full space of maps from M to N, one can consider, for instance, the imbeddings, the immersions, the diffeomorphisms (if $M = N$) etc.. The evaluation map allows us to construct cohomology classes on the spaces of maps by pulling back the cohomology of the "target" space.

This is a well-known fact, which has been used extensively in the past in order to compute the cohomology of spaces of maps. The pulled back cohomology seems to be the one which is relevant for field theory.

Another important ingredient strictly connected with locality is universality. By university, we mean that most objects relevant to perturbative field theory are pull-backs of forms defined on the universal bundle and of sections of bundles associated to it. In particular, as far as anomalies are concerned, local cochains are universal objects and it is possible to construct a natural framework in which the cohomology of classifying spaces is pulled back via the evaluation map. This is the cohomology which is relevant for computing anomalies.

Let us now recall a few basic facts about anomalies.

Chiral anomalies arise in field theory, sigma-models and strings theories when we are faced with the problem of evaluating determinants of elliptic operators. The latter represent the quantum version of a piece of the classical action bilinear in the relevant fields. Typical examples are the Dirac operators. These operators transform covariantly under some kind of transformations (gauge, diffeomorphisms). The problem arises as to whether it is possible to define a determinant with the corresponding type of invariance, that is, gauge- or diffeomorphism-invariant determinants.

Generally speaking, an obstruction to defining an invariant determinant is called an anomaly.

In order to reveal such an obstruction, one can modify the "classical" operator in question in such a way as to be able to compute a determinant: for example, in the perturbative calculations based on Feynman diagram techniques, one replaces the chiral Dirac operator $D = (1 - \gamma_5/2)(\not\partial + \not A)$, where A_μ is a gauge potential, by $D' = (\not\partial + (1 - \gamma_5/2)\not A)$. In this way, one is able to define a perturbative expansion and calcalate $\det D'$. However the result need not be gauge invariant. This can be revealed by taking the variation of $\det D'$ with respect to gauge transformations: one finds just what is known as the chiral anomaly (if it does not vanish identically). The anomaly here appears as an obstruction for the effective procedure to lead to a gauge invariant response.

A more systematic and rigorous method of calculating determinants and obstructions to their existence has been discussed by Atiyah and Singer [2] and by Quillen [3] who considered the determinant line bundle.

In any case, the cohomological nature of anomalies is clear: they are non-trivial cocycles of a suitable coboundary operator (for example, the BRS operator in perturbative field theory).

A cohomological analysis defines anomalies up to global multiplicative factors; in order to fix these factors we can use the family's index theorem. However the cohomological analysis is very powerful and allows us to calculate the anomaly's expression, to study uniqueness, "locality" and a good deal of problems connected with anomaly cancellation. The cohomological aspects of anomalies and their relation to the family's index theorem are thoroughly examined in this paper: the analysis is centered on the concept of locality, an outgrowth—it must be stressed—of the perturbative approach, and its relation to universality.

This paper is organized as follows.

Section 1 is dedicated to some mathematical preliminaries and to the setting of notation.

In Sect. 2, we consider the evaluation map defined by the group of all automorphisms of a principal bundle P acting on the bundle itself. All anomalies (gauge, gravitational, Lorentz) are generated by pulling back suitable forms on P via this evaluation map. The explicit expression of gauge and gravitational anomalies is then deduced in a simple way. Moreover, we compare the cohomology induced in this way with the cohomology of the Lie algebras of the group of gauge transformations and of the group of diffeomorphisms with coefficients in the functions on the space of connections $\mathscr{A}$.

In Sect. 3 we compute, via the evaluation map, the expression of gauge and gravitational anomalies with a background connection (see [4]). We discuss the relation between the above two ways of describing the anomalies (with and without a background connection).

In Sect. 4 we consider the "gauge interpretation" of gravitational anomalies, which is a key element for establishing the equivalence between gravitational and Lorentz anomalies. This gauge interpretation of gravitational anomalies is possible for a manifold which admits a flat linear connection.

In Sect. 5 we compute the cohomology, with real coefficients, of the group of gauge transformations. It is shown that this cohomology is independent of the isomorphism class of the given principal bundle (e.g. is independent of the instanton number). The topological significance of gauge anomalies is discussed in the general case, i.e. for any compact base manifold and any compact structure group. A remark on the topological significance of $U(1)$ anomalies is added.

Section 6 is devoted to the study of the relation between universality and locality: local objects are, essentially, the universal ones (i.e. "local" anomalies are generated by forms on P which are the pullback of forms defined on the total space of the universal bundle). We show that, in order to have the cancellation of anomalies in field theory, we have to require the forms on P, which generate the anomalies, to be derivatives of forms obtained by pullback of universal forms. We notice also that, if we consider, as in [5], the cohomology of the Lie algebras of the groups of gauge transformations and of diffeomorphisms defined in terms of cochains which decrease support (that is in terms of cochains represented, in any chart, by differential operators), then again the non-trivial cocycles are essentially given by universal objects.

In Sect. 7 we show that the identification of locality and universality naturally leads us to studying the evaluation map defined by considering any connection $A \in \mathscr{A}$ as a map from the principal bundle P into the corresponding universal bundle. This allows us to consider local (i.e. universal) forms on $P \times \mathscr{A}$ and on $P \times \mathscr{A}/\mathscr{G}$ provided that $\mathscr{G}$ is a subgroup of the group of automorphisms of P which acts freely on $\mathscr{A}$ (and acts in a suitable way on $P \times \mathscr{A}$, see [2]). The compatibility between locality and the family's index theorem approach to anomalies is then discussed.

Differences and analogies between gauge, gravitational and Lorentz anomalies are elucidated. In particular the peculiar rôle of Lorentz anomalies is discussed.

Section 8 is dedicated to anomaly cancellation in ten dimensional field theories. We see that in order to implement the Green–Schwarz mechanism in field theory, it is necessary to have an imbedding of the orthonormal frame bundle into the gauge bundle. Some features of a theory with such an imbedding are discussed. Moreover if also the gravitational anomalies have to be cancelled, then the gauge bundle must admit a lift of diffeomorphisms of the base and so is likely to be trivial. That is, there should be no higher dimensional analogs to instantons. Also there should not be instantons on any four-dimensional submanifold of the ten-dimensional manifold. Moreover the first Pontrjagin class of the ten-dimensional manifold must be trivial.

The Green–Schwarz mechanism will be discussed again in [1], in the framework of sigma-models.

In [1] we will also discuss the relation between the evaluation map and the structure of sigma-model anomalies and the rôle of (generalized) Wess–Zumino terms.

Particular attention will be given to the analysis of the conformal anomalies and the sigma-model anomalies of the string. Global anomalies in sigma-models and field theory will finally be discussed.

1. Some Basic Definitions and Notations

We denote by $P(M, G)$ a principal fibre bundle with base manifold M, total space P and structure group G. The base M will be supposed to be a compact, connected, orientable, n-dimensional spin manifold without boundary. The group G will be, in general, either a compact Lie group or the group $GL(n, \mathbb{R})$; in the latter case, P will be the bundle LM of linear frames. By $O_g M$ we will denote the $SO(n)$-principal bundle of frames which are orthonormal with respect to the Riemannian metric g. We denote by $\mathrm{Spin}_g M$ the Spin(n)-bundle of spin-frames which is a double covering of $O_g M$. If there is no ambiguity we will omit the symbol g in $O_g M$ and $\mathrm{Spin}_g M$.

The projection of P or of any other fiber bundle will be generally denoted by the symbol π.

A form ω on P is said to be basic if there exists a form $\bar{\omega}$ on M such that $\pi^* \bar{\omega} = \omega$. Hence any form on M can be considered also as a (basic) form on P.

a) *The Groups* $\mathrm{Aut}\, P$, $\mathrm{Aut}_v P$ *and* $\mathrm{Diff}\, M$. The group of all diffeomorphisms of M will be denoted by $\mathrm{Diff}\, M$. We consider also the group of automorphisms of P, $\mathrm{Aut}\, P$, defined as follows:

$$\mathrm{Aut}\, P = \{\psi \,|\, \psi \in \mathrm{Diff}\, P \quad \text{such that } \psi(ua) = \psi(u)a \quad \forall a \in G, \quad u \in P\}. \tag{1.1}$$

There is a group homomorphism $j: \mathrm{Aut}\, P \to \mathrm{Diff}\, M$ given by $(j\psi)(x) = \pi(\psi(u_x))$, where $x \in M$, π is the projection map in $P(M, G)$ and $u_x \in \pi^{-1}(x)$. The kernel of j is the group of vertical automorphisms, which will be denoted by $\mathrm{Aut}_v P$; this is the group of gauge transformations.

We have then the exact sequence:

$$1 \to \mathrm{Aut}_v P \to \mathrm{Aut}\, P \xrightarrow{j} \mathrm{Diff}\, M. \tag{1.2}$$

Together with $\mathrm{Aut}\, P$, $\mathrm{Aut}_v P$ and $\mathrm{Diff}\, M$, we consider their "Lie algebras" $\mathrm{aut}\, P$, $\mathrm{aut}_v P$, $\mathrm{diff}\, M$. As vector spaces they are defined as follows: $\mathrm{diff}\, M$ is the space of all vector fields on M, $\mathrm{aut}\, P$ and $\mathrm{aut}_v P$ are respectively the spaces of vector fields on P generated by one parameter subgroups of $\mathrm{Aut}\, P$ and $\mathrm{Aut}_v P$. The bracket of two elements of $\mathrm{aut}\, P$, $\mathrm{aut}_v P$ and $\mathrm{diff}\, M$ will be the opposite of their usual bracket as vector fields [6].

It is well known that $\mathrm{Aut}_v P$ and $\mathrm{aut}_v P$ are respectively isomorphic to the spaces of sections of $\mathrm{Ad}\, P$ and $\mathrm{ad}\, P$. Here $\mathrm{Ad}\, P$ is the bundle $P \times_G G$ associated to P through the action of G on itself given by conjugation, while $\mathrm{ad}\, P$ is the bundle $P \times_G \mathrm{Lie}\, G$ associated to P through the adjoint of G on its Lie algebra $\mathrm{Lie}\, G$. The product of sections of $\mathrm{Ad}\, P$ is defined by pointwise multiplication and also pointwise defined is the Lie bracket of sections of $\mathrm{ad}\, P$ [7]. Notice that vector fields on M can be lifted to P (via a fixed connection), so the image of the map j in (1.2) contains $\mathrm{Diff}_0 M$, the connected component of the identity in $\mathrm{Diff}\, M$.

Hence, if we set $\mathrm{Aut}_o P \equiv j^{-1}(\mathrm{Diff}_o M)$, we have also the exact sequence:

$$1 \to \mathrm{Aut}_v P \to \mathrm{Aut}_o P \to \mathrm{Diff}_o M \to 1. \tag{1.2'}$$

The question we would like to ask now is whether the exact sequence (1.2) (or (1.2′)) splits, i.e. whether there exists a group homomorphism $l{:}\mathrm{Diff}\, M \to \mathrm{Aut}\, P$ such that $j \circ l = \text{identity}$. When it exists, we call such a homomorphism a lift. There are two special cases in which such a lift exists.

The first is when $P = LM$. In this case, let $u = (x; X_1, \dots, X_n)$ be a frame at $x \in M$ (i.e. $X_i \in T_x M$ and the X_i are linearly independent), and let $\psi \in \mathrm{Diff}\, M$. We can then define a frame at $\psi(x) \in M$, as follows:

$$l(\psi)u = (\psi(x); \psi_* X_1, \dots, \psi_* X_n). \tag{1.3}$$

The automorphism $l(\psi)$ is called the natural lift of ψ. So we can represent $\mathrm{Aut}\, LM$ as a semidirect product of $\mathrm{Aut}_v LM$ and $\mathrm{Diff}\, M$, relative to the homomorphism (see [8]) $\sigma{:}\mathrm{Diff}\, M \to \mathrm{Aut}(\mathrm{Aut}_v LM)$ given by

$$\sigma_\psi(\varphi) = l(\psi) \circ \varphi \circ l(\psi^{-1}) \quad \varphi \in \mathrm{Aut}_v LM, \quad \psi \in \mathrm{Diff}\, M.$$

Correspondingly we have a Lie-algebra homomorphism $l{:}\mathrm{diff}\, M \to \mathrm{aut}\, LM$. For any vector field $X \in \mathrm{diff}\, M$, the lift $l(X)$ is the unique vector field on P which satisfies the following properties [9]:

a) $l(X)$ is invariant under right multiplication: $P \times G \to P$;
b) $L_{l(X)}\theta = 0$ where L is the Lie derivative and θ is the soldering (or canonical) form;
c) $\pi_* l(X) = X$.

The vector field $l(X)$ is called the natural lift of X. For the expression of $l(X)$ in local coordinates, see [10].

The second case we want to consider is when P is a trivial bundle. For simplicity we consider the product bundle $M \times G$. If $\varphi \in \mathrm{Diff}\, M$, then $l(\varphi)$ is defined by:

$$l(\varphi){:}M \times G \to M \times G, \quad (x, a) \mapsto (\varphi(x), a). \tag{1.4}$$

The group $\mathrm{Aut}\, P$ is again a semidirect product of $\mathrm{Diff}\, M$ and $\mathrm{Aut}_v P$. So any $\psi \in \mathrm{Aut}\, P$ can be represented as a pair (φ, ρ) with $\varphi \in \mathrm{Diff}\, M$ and $\rho \in \mathrm{Map}(M, G) \approx \mathrm{Aut}_v P$, as follows:

$$\psi(x, a) = (\varphi(x), \rho(\varphi(x)) \cdot a). \tag{1.5}$$

The corresponding lift for vector fields is the horizontal lift with respect to the canonical flat connection.

A splitting of the sequence (1.2) obviously occurs also when P is the natural principal bundle of k-jets on M, but there are no known cases where a lift exists for a principal bundle which is not natural or trivial [11]. In particular it is not known whether there exists a lift in a non-trivial principal bundle with a compact structure group.

We know only of a necessary condition for the existence of such a lift: the image of the Weil homomorphism (see below) must be contained in the ideal generated by the Pontrjagin classes of the manifold [12].

b) *Transgressions and Chern-Weil Homomorphism.* Let Q be an ad-invariant polynomial on Lie G with k-entries and let A be any connection on $P(M, G)$ with curvature $F = dA + \frac{1}{2}[A, A]$.

Following Cherm [13] we can consider the closed basic $2k$-form given by $Q(F, \ldots, F)$, where all the entries of Q are filled with the 2-form F.

If A_0 is another connection on $P(M, G)$ with curvature F_0, then we have:

$$Q(F, \ldots, F) - Q(F_0, \ldots, F_0) = dW_Q(A, A_0), \tag{1.6}$$

where we have set:

$$W_Q(A, A_0) = k \int_0^1 dt\, Q(A - A_0, \mathscr{F}_t, \ldots, \mathscr{F}_t); \tag{1.7}$$

here $\mathscr{F}_t$ is the curvature of the connection on P given by $(1-t)A_0 + tA$.

The form given by (1.7) is basic. Moreover Eq. (1.6) tells us that $Q(F, \ldots, F)$ gives a De Rham cohomology class of M which is independent of the connection A.

So we have an algebra homomorphism (called the Chern-Weil homomorphism or simply the Weil homomorphism):

$$w: I(G) \to H^*_{\text{deRham}}(M);$$

where $I(G)$ denotes the algebra of ad-invariant polynomials on Lie G.

The form $Q(F, \ldots, F)$ is always exact as a form on P.

In fact, by setting $F_t \equiv t\,dA + (t^2/2)[A, A]$ and

$$TQ(A) \equiv k \int_0^1 dt\, Q(A, F_t, \ldots, F_t), \tag{1.8}$$

for any $Q \in I(G)$ with k entries, we have:

$$dTQ(A) = Q(F, \ldots, F).$$

The form (1.8) is called the Chern-transgression or simply the transgression form, relevant to the ad-invariant polynomial Q.

c) *Space of Connections.* We denote now by $\mathscr{A}$, the space of all connections on $P(M, G)$. The group $\operatorname{Aut} P$ (and a fortiori the group $\operatorname{Aut}_v P$) acts on the right on $\mathscr{A}$, as follows:

$$\mathscr{A} \times \operatorname{Aut} P \to \mathscr{A}, \quad (A, \psi) \mapsto (\psi^* A). \tag{1.9}$$

If $P = LM$, then also $\operatorname{Diff} M$ acts on $\mathscr{A}$, since the existence of the lift allows us to consider $\operatorname{Diff} M$ as a subgroup of $\operatorname{Aut} LM$. The same is true if P is trivial.

In general the action (1.9) is not free. In order to have a free action, one can consider the subgroup $\operatorname{Aut}_v^m P$ of $\operatorname{Aut}_v P$, consisting of all elements ψ of $\operatorname{Aut}_v P$ satisfying the condition:

$$\psi(u) = u, \quad \forall u \in \pi^{-1}(m). \tag{1.10}$$

Here m is a fixed point in M (the "point at infinity").

One can also consider the subgroup $\operatorname{Aut}^m P$ of $\operatorname{Aut} P$ consisting of all elements of $\operatorname{Aut} P$ satisfying Eq. (1.10), and the subgroup $\operatorname{Diff}^{m,1} M$ of $\operatorname{Diff} M$ consisting of all elements φ of $\operatorname{Diff} M$ satisfying the following condition for a fixed $m \in M$:

$$\varphi(m) = m \quad \text{and } \varphi_*|_m = \text{identity on } T_m M. \tag{1.11}$$

Obviously $l(\mathrm{Diff}^{m,1} M) \subset \mathrm{Aut}^m LM$. More precisely the semidirect product $\mathrm{Aut}_v^m LM \odot \mathrm{Diff}^{m,1} M$ is a proper subgroup of $\mathrm{Aut}^m LM$. In fact we have

$$\frac{\mathrm{Aut}^m LM}{\mathrm{Aut}_v^m LM \odot \mathrm{Diff}^{m,1} M} \approx \frac{\mathrm{Diff}^m M}{\mathrm{Diff}^{m,1} M} \approx LM\Big|_m \approx GL(n, \mathbb{R}),$$

where $\mathrm{Diff}^m M$ is the subgroup of $\mathrm{Diff}\, M$ consisting of diffeomorphisms which leave the point $m \in M$ fixed. The group $\mathrm{Diff}^{m,1} M$ acts freely on the space of all linear connections (see Lemma 4, p. 254 in [9]) and acts freely on the space $\mathscr{M}$ of all metrics on M [14]. Obviously it acts also freely on the space of all Levi–Civita connections. Let us recall, by the way, that the following commutative diagram holds:

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\psi} & \mathscr{M} \\ \downarrow \mathrm{LC} & & \downarrow \mathrm{LC} \\ \mathscr{A} & \xrightarrow{l(\psi)} & \mathscr{A} \end{array} \tag{1.12}$$

here $\psi \in \mathrm{Diff}\, M$ and LC is the map which assigns to each metric its Levi–Civita connection.

As far as the infinitesimal version of the action (1.9) is concerned, we would like to point out that $\forall Z \in \mathrm{aut}\, P$

$$L_Z A = d_A i_Z A + i_Z F(A). \tag{1.13}$$

Here L_Z, i_Z are respectively the Lie derivative and the interior product with respect to the vector field Z, d_A is the A-covariant derivative and $F(A)$ is the curvature form of A [9]. We denote respectively by $\mathrm{diff}^{m,0} M$ and by $\mathrm{aut}_v^m P$ the Lie algebras of $\mathrm{Diff}^{m,1} M$ and of $\mathrm{Aut}_v^m P$. For the differentiable structure of the various infinite dimensional manifolds we will consider in this paper, we refer to [14–19].

In particular we can consider the principal fibre bundle given by [17, 19],

$$\mathscr{A} \to \frac{\mathscr{A}}{\mathrm{Aut}_v^m P} \tag{1.14}$$

or the principal fibre bundle given by [14]

$$\mathscr{M} \to \frac{\mathscr{M}}{\mathrm{Diff}^{m,1} M}. \tag{1.15}$$

We can also notice that the space of all Levi–Civita connections $\mathscr{A}^{LC}$ is in a one-to-one correspondence with the space $\mathscr{M}_{(m)}$ of all metrics which, once restricted to $T_m M$, give rise to a fixed given inner product $g_m \in S^2 T_m M$.

In fact, let A be any torsionless linear connection on M such that its holonomy group is contained in $SO(n)$ (we assume M to be oriented).

By parallel transporting g_m, we obtain a section of $S^2 TM$, i.e. a metric whose Levi–Civita connection is, by construction, A.

Conversely, to any metric $g \in \mathscr{M}_{(m)}$, we can associate its Levi–Civita connection, which is such that g is obtained by parallel transporting g_m.

So we can give the space of all Levi–Civita connections $\mathscr{A}^{\mathrm{LC}}$ the differentiable structure induced by $\mathscr{M}_{(m)}$ and consider the principal fibre bundle:

$$\mathscr{A}^{\mathrm{LC}} \to \frac{\mathscr{A}^{\mathrm{LC}}}{\mathrm{Diff}^{m,1} M}. \tag{1.16}$$

More generally we could consider the space $\mathscr{A}^{\mathrm{metric}}$ of linear connections for M whose holonomy group is contained in $SO(n)$.

The space $\mathscr{A}^{\mathrm{metric}}$ can be identified with the cartesian product $\mathscr{A}^{\mathrm{LC}} \times \mathscr{T}$, where $\mathscr{T}$ is the space of torsion tensor fields, i.e. of tensor fields of type $(1,2)$, such that $T \in \mathscr{T} \Rightarrow T(X,Y) = -T(Y,X)\ \forall X, Y \in \mathrm{diff}\, M$.

The group $\mathrm{Diff}^{m,1} M$ acts freely and smoothly on $\mathscr{A}^{\mathrm{metric}}$.

Hence we can consider also the principal fibre bundle

$$\mathscr{A}^{\mathrm{metric}} \to \frac{\mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1} M}. \tag{1.17}$$

Notice that $\mathscr{A}^{\mathrm{metric}}$ is a contractible space, since $\mathscr{T}$ is a linear space and $\mathscr{A}^{\mathrm{LC}}$ is contractible. Moreover $\mathscr{A}^{\mathrm{metric}}/\mathrm{Diff}^{m,1} M$ is a vector bundle over $\mathscr{A}^{\mathrm{LC}}/\mathrm{Diff}^{m,1} M$ with fiber $\mathscr{T}$.

A last remark concerning notation: the space of k-differential forms on M will be denoted by $\Omega^k(M)$. As usual we set $\Omega^*(M) = \sum_k \Omega^k(M)$. Moreover, if N is another manifold, then we have: $\Omega^k(M \times N) = \sum_{r+s=k} \Omega^{r,s}(M \times N)$, where $\omega \in \Omega^{r,s}(M \times N)$ means that ω is a combination of forms like $\mathrm{pr}_1^* \omega^1 \wedge \mathrm{pr}_2^* \omega^2$. Here $\mathrm{pr}_1 : M \times N \to M$, and $\mathrm{pr}_2 : M \times N \to N$ are the projections, $\omega^1 \in \Omega^r(M)$ and $\omega^2 \in \Omega^s(N)$. For any form in $\Omega^k(M \times N)$, the corresponding element in $\Omega^{r,s}(M \times N)$ is also referred to as the (r,s)-component of the given form.

By the symbol $H^*(M)$ we will denote the De Rham cohomology of M. In considering the cohomology of manifolds with coefficients different from $\mathbb{R}$, we will explicitly write the coefficients (as e.g. $H^*(M, Z)$).

2. Gauge and Gravitational Anomalies and the Evaluation Map

Let $\mathscr{G}$ be any of the groups $\mathrm{Aut}\, P$, $\mathrm{Aut}_v\, P$ or $\mathrm{Diff}\, M$. If $\mathscr{G} = \mathrm{Diff}\, M$, we assume that the principal fibre bundle P is in fact LM, so that $\mathrm{Diff}\, M$ can be considered as a subgroup of $\mathrm{Aut}\, P$.

We can then consider the evaluation map:

$$\mathrm{ev} : P \times \mathscr{G} \to P, \quad (p, \psi) \mapsto \psi(p). \tag{2.1}$$

When G acts trivially on $\mathscr{G}$, $P \times \mathscr{G}$ is a principal G-bundle over $M \times \mathscr{G}$ and the evaluation map is a bundle homomorphism. So, for any connection A on P, $\mathrm{ev}^* A$ is a connection on $P \times \mathscr{G}$. Moreover if F is the curvature of A, then $\mathrm{ev}^* F$ is the curvature of $\mathrm{ev}^* A$.

In order to allow a better understanding of our notation, we would like to recall that vectors in $T_{\mathrm{id}}\mathscr{G} \approx \mathrm{Lie}\, \mathscr{G}$ are vector fields on P. If $Z \in T_{\mathrm{id}}\mathscr{G}$, then $\psi_* Z$ can be considered either as a vector field on P or as a vector in $T_\psi \mathscr{G}$. In fact we can think of ψ_* either as the tangent map of $\psi : P \to P$ or as the tangent map of the left multiplication by ψ in $\mathscr{G}$.

An easy calculation shows that, if X is a vector in T_pP and Y is a vector in $T_\psi\mathcal{G}$, then, by setting $Y = \psi_* Z$, we have:

$$(\mathrm{ev}^* A)_{p,\psi}(X, Y) = (\psi^* A)_p(X) + (i_Z \psi^* A)_p. \tag{2.2}$$

Here i_Z is the interior product for forms on P with respect to the vector field Z[1].

The second term of the right-hand side (2.2) is a 1-form on $\mathcal{G}$ which we denote $i_{(\cdot)}\psi^* A$ to indicate that is to be evaluated at each $\psi \in \mathcal{G}$, where it is defined by:

$$(i_{(\cdot)}\psi^* A)(Y) = i_{\psi_*^{-1}Y}\psi^* A = \psi^*(i_Y A) \quad Y \in T_\psi\mathcal{G}.$$

Notice that $i_Y A$ is a function on P. We will then write (2.2) in the following form:

$$\mathrm{ev}^* A = \psi^* A + i_{(\cdot)}\psi^* A.$$

If we limit ourselves to considering vectors in $\mathrm{Lie}\,\mathcal{G}$, then $i_{(\cdot)}$ represents simply the map $Z \mapsto i_Z$, $\forall Z \in \mathrm{Lie}\,\mathcal{G}$.

Notice that, in particular, when $\mathcal{G} = \mathrm{Aut}_v P$, then the curvature of $\mathrm{ev}^* A$ satisfies the following equation:

$$\mathrm{ev}^* F = \psi^* F.$$

When ψ = identity, the above equation has been called the "Russian formula" [20]. The connection $\mathrm{ev}^* A$ restricted to ψ = identity is also commonly written as $A + v$, where v is then called the ghost field (see e.g. [20, 21]), as we explain further below.

The map ev^* obviously transforms closed (exact) forms on P into closed (exact) forms on $P \times \mathcal{G}$. Moreover, if $\mathrm{ev}^* \chi$ is closed (exact) on $P \times \mathcal{G}$, then also χ is closed (exact) on P, because $\mathrm{ev}^* \chi = \psi^* \chi \otimes 1 + \cdots$ in $\Omega^q(P \times \mathcal{G}) \equiv \sum_{r+s=q} \Omega^{r,s}(P \times \mathcal{G})$.

From now on we assume that $n = \dim M$ is even.

Let Q be an ad-invariant polynomial on $\mathrm{Lie}\,G$ with $((n/2)+1)$-entries and let $TQ(A)$ be the relevant transgression form.

Then we have, using the same notation as in (2.2′):

$$\begin{aligned}\mathrm{ev}^* TQ(A) = TQ(\mathrm{ev}^* A) &= \psi^* TQ(A) + i_{(\cdot)}\psi^* TQ(A) + \\ &- i_{(\cdot)}i_{(\cdot)}\psi^* TQ(A) + \cdots + (-1)^{n(n+1)/2}\underbrace{i_{(\cdot)}\cdots i_{(\cdot)}}_{(n+1)\text{ terms}}\psi^* TQ(A)\end{aligned}$$

Here $\mathrm{ev}^* TQ(A) \in \Omega^{n+1}(P \times \mathcal{G})$ and $\underbrace{i_{(\cdot)}\cdots i_{(\cdot)}}_{k\text{ terms}}\psi^* TQ(A) \in \Omega^{n+1-k,k}(P \times \mathcal{G})$.

Since $d(TQ(A)) = Q(F, \ldots, F) = 0$, we have also:

$$(\hat{d} + \delta)\mathrm{ev}^* TQ(A) = 0. \tag{2.4}$$

Here $\delta: \Omega^{r,s} \to \Omega^{r,s+1}$ is the exterior derivative in $\mathcal{G}$, $d: \Omega^{r,s} \to \Omega^{r+1,s}$, is the exterior derivative in P and $\hat{d}\chi \equiv (-1)^p d\chi$, provided that χ is a p-form on $\mathcal{G}$.

Equation (2.4) is equivalent to what is known in the literature [5, 20–24] as a

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[1] Notice that if Y is the vector on $\mathcal{G}$ tangent at $t = 0$ to the curve ψ_t in $\mathcal{G}$ with $\psi_0 = \psi$, then $\psi^{-1}\psi_t$ is a curve in $\mathcal{G}$ whose tangent vector at $t = 0$ is Z. Z itself is the vector field on P such that $\forall p \in P$, Z_p is the vector tangent at $t = 0$ to the curve $\psi^{-1}\psi_t(p)$

descent equation. In fact if $\omega_k \equiv (-1)^{k(k-1)/2} \underbrace{i_{(\cdot)} \dots i_{(\cdot)}}_{k \text{ terms}} \psi^* TQ(A)$, then (2.4) implies: $\hat{d}\omega_{k+1} + \delta\omega_k = 0$.

The component of type $(n, 1)$ in $\mathrm{ev}^*\, TQ(A)$, when restricted to $\psi =$ identity, is to be identified with the chiral anomaly (respectively gauge anomaly if $\mathscr{G} = \mathrm{Aut}_v P$, gravitational anomaly if $P = LM$ and $\mathscr{G} = \mathrm{Diff}\, M$, Lorentz anomaly if $P = OM$ and $\mathscr{G} = \mathrm{Aut}_v OM$[2]).

In order to make contact with the usual definitions, we observe that, if we set:

$$\mathrm{Anom}(Z) \equiv i_Z TQ(A) + d\int_0^1 dt(t-1)\frac{n}{2}\left(\frac{n}{2}+1\right) Q(i_Z A, A, F_t, \dots, F_t), \tag{2.5}$$

then we have:

$$\begin{aligned}\mathrm{Anom}(Z) = &\int_0^1 dt \frac{n}{2}\left(\frac{n}{2}+1\right) tQ(i_Z F, A, F_t, \dots, F_t) \\ &+ \int_0^1 dt(t-1)\frac{n}{2}\left(\frac{n}{2}+1\right) Q(d i_Z A, A, F_t, \dots, F_t).\end{aligned} \tag{2.6}$$

If Z is vertical, then the first term of the right-handside of (2.6) disappears and $\mathrm{Anom}(Z)$ is the usual expression of the gauge anomaly. If Z is the natural lift of a vector field on M, then $\mathrm{Anom}(Z)$ is the usual expression of the gravitational (or Einstein) anomaly which is generally considered when the connection A is a Levi–Civita connection[3]. As we mentioned before, when $\mathscr{G}$ is the group $\mathrm{Aut}_v P$, the mapping $Z \mapsto i_Z A$ is called the ghost field.

If $\mathscr{G}$ is the group $\mathrm{Aut}_v P$, then, for a connection A, the mapping $Z \to i_Z A$, which is obviously independent of the choice of the connection A, gives the standard isomorphism between $\mathrm{aut}_v P$ and $\Gamma(\mathrm{ad}\, P)$, which is the space (Lie algebra) of sections of $\mathrm{ad}\, P$. In this case, by identifying $\mathrm{aut}_v P$ and $\Gamma(\mathrm{ad}\, P)$ the above mapping can be seen as the Maurer-Cartan form on $\mathscr{G}$ (i.e. as the identity map on $\mathrm{Lie}\,\mathscr{G}$) [5].

If $\mathscr{G}$ is the group $\mathrm{Aut}\, P$ or $\mathrm{Diff}\, M$, then the mapping $Z \to i_Z A$ is no longer independent of the choice of the connection A and does not represent the Maurer-Cartan form on $\mathscr{G}$.

Coming back to the evaluation map (2.1), we observe that for any closed $(n+1)$ – form χ, the $(n, 1)$-component of $\mathrm{ev}^*\chi$, i.e. $i_{(\cdot)}\psi^*\chi$, satisfies the consistency condition[4]:

$$\delta i_{(\cdot)}\psi^*\chi - d i_{(\cdot)} i_{(\cdot)}\psi^*\chi = 0, \tag{2.7}$$

and hence, when evaluated at $\psi = 1$, it is a possible anomaly. If the form χ on P is also exact, i.e. $\chi = d\eta$, then the $(n, 1)$-component of $\mathrm{ev}^*\chi$ satisfies the triviality

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[2] We can consider $\mathrm{Spin}\, M$ instead of OM. Also the case $P = LM$, $\mathscr{G} = \mathrm{Aut}_v LM$ will be considered. Generally speaking, we use the term "chiral anomaly" in a broader sense (more mathematical) than the one which is normally used in field theory. So we speak also of $\mathrm{Aut}\, P$-anomalies or of $\mathrm{Aut}_v LM$-anomalies, although $\mathrm{Aut}_v LM$ does not represent a physical symmetry

[3] In ref. [21] this anomaly is pulled back via a local section

[4] The usual consistency condition and triviality condition are Eq. (2.7) and (2.8) restricted to $\psi =$ identity

condition[4,5]:

$$i_{(\cdot)}\psi^*\chi = -di_{(\cdot)}\psi^*\eta + \delta\psi^*\eta. \tag{2.8}$$

Consequently, if the form $i_{(\cdot)}\psi^*\chi$ satisfies Eq. (2.8), then a simple calculation shows that:

$$i_{(\cdot)}\psi^*\chi = i_{(\cdot)}\psi^*\,d\eta,$$

so the anomaly $i_{(\cdot)}\chi$ is equal to the anomaly $i_{(\cdot)}\,d\eta$ generated by $d\eta$. Two anomalies $i_{(\cdot)}\chi$ and $i_{(\cdot)}\chi'$ are said to be equivalent if their difference satisfies the triviality condition, i.e. if $i_{(\cdot)}\psi^*(\chi - \chi')$ satisfies Eq. (2.8).

Hence two equivalent anomalies represent the same cohomology class of $P \times \mathscr{G}$. If no confusion arises, we will follow the usual convention and use the term anomaly also to denote an equivalence class of anomalies. So we might be tempted to claim provisionally that the non-trivial anomalies are in a one-to-one correspondence with non-trivial elements of $H^{n+1}_{\mathrm{deRham}}(P)$ [24].

These elements can in turn be represented by elements of the algebra generated by the forms $TQ_i(A)$, for ad-invariant polynomials Q_i, and by the basic forms on P (Chevalley's theorem [13]). We will be interested mainly in forms like

$$\sum_i TQ_i(A) \wedge q_i, \tag{2.9}$$

where q_i are both basic and closed. Here the number k_i of entries of Q_i and the order l_i of the forms q_i are such that $2k_i - 1 + l_i = n + 1$.

The real situation is in fact a little more complicated. Non-trivial anomalies do not necessarily derive from non-trivial elements of $H^{n+1}(P)$, as we shall see in Sect. 6, where the locality requirement will be taken into account and local anomalies will be considered. Also we will see that not every non-trivial element of $H^{n+1}(P)$ generates a local anomaly, i.e. an anomaly which is relevant for field theories.

In ref. [5] it has been pointed out that (integrated) anomalies are elements of the first cohomology group in the cohomology of Lie $\mathscr{G}$ with coefficients in $C^\infty(\mathscr{A})$.

If we limit ourselves to considering forms χ of $\Omega^*(P)$ depending on the connection $A \in \mathscr{A}$ in such a way that the following condition is satisfied:

$$\left.\frac{d}{dt}\chi(A + tL_Z A)\right|_{t=0} = L_Z\chi(A), \tag{2.10}$$

then the relation between the approach in ref. [5] and the present considerations is as follows.

Let χ and η be respectively forms on P of degree $n+1$ and n, satisfying the condition (2.10) and let c be any n-cycle in P. Consider the 1-form ω on $\mathscr{G}$ given by $\omega_\psi = \int_c i_{(\cdot)}\psi^*\chi$ and the 0-form θ on $\mathscr{G}$ given by $\theta_\psi = \int_c \psi^*\eta$.

The cochains ω_1 and θ_1 are cochains (respectively of degree one and zero) for the cohomology of Lie $\mathscr{G}$ with coefficients in $C^\infty(\mathscr{A})$. If $\delta_{\mathrm{Lie}\mathscr{G}}$ is the relevant coboundary operator we can prove:

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[5] We have to warn here that by "cancellation" of anomalies we mean two different phenomena: either the anomaly satisfies condition (2.8), or the coefficient of the anomaly, computed according to the family's index theorem (see Sect. 7), is zero

Theorem (2.11). *The form ω is δ-closed (respectively $=\delta\theta$) for all n-cycles c, if and only if ω_1 is δ_{Lie}-closed (respectively $=\delta_{\mathrm{Lie}\mathscr{G}}\theta_1$) for all n-cycles c.*

Proof. It is easy to verify that

$$\delta\omega|_{\psi=1}=\delta_{\mathrm{Lie}\mathscr{G}}\omega_1 \quad \text{and } \delta\theta|_{\psi=1}=\delta_{\mathrm{Lie}\mathscr{G}}\theta_1. \tag{2.12}$$

So if ω is δ-closed (respectively $=\delta\theta$) then ω_1 is $\delta_{\mathrm{Lie}\mathscr{G}}$-closed (respectively $=\delta_{\mathrm{Lie}\mathscr{G}}\theta_1$).

In order to prove the converse, first consider the following identity:

$$(\delta\int_c i_{(\cdot)}\psi^*\chi)(X_1,X_2)=(\delta_{\mathrm{Lie}\mathscr{G}}\int_{\psi(c)} i_{(\cdot)}\chi)(X_1,X_2). \tag{2.13}$$

Notice that on the left-hand side X_1 and X_2 are considered as elements of $T_\psi\mathscr{G}$, while in the right-hand side X_1 and X_2 are considered as vector fields on P. Both sides of (2.13) are equal to:

$$-(\int_c i_{(\cdot)}i_{(\cdot)}\psi^*\,d\chi)(X_1,X_2).$$

The identity (2.13) proves that if ω_1 is $\delta_{\mathrm{Lie}\mathscr{G}}$-closed with respect to any given n-cycle, then ω is δ-closed again with respect to any given n-cycle.

To prove that $\omega_1=\delta_{\mathrm{Lie}\mathscr{G}}\theta_1$ implies the δ-exactness of ω, notice that we have:

$$\begin{aligned}&\int_c i_X\chi=\int_c i_X d\eta, \quad \forall X\in\mathrm{Lie}\,\mathscr{G}, \quad \forall \ \text{cycle } c\\ &\qquad\Rightarrow\int_c i_X\psi^*\chi=\int_c i_X d\psi^*\eta \quad \forall X\in\mathrm{Lie}\,\mathscr{G} \quad \text{and} \quad \forall\psi\in\mathscr{G},\end{aligned} \tag{2.14}$$

since the first line in (2.14) implies:

$$\begin{aligned}\int_c i_X\psi^*\chi&=\int_c\psi\, i_{\psi_*X}\chi=\int_{\psi(c)} i_{\psi_*X}\chi\\ &=\int_c\psi^* i_{\psi_*X}d\eta=\int_c i_X d\psi^*\eta. \qquad \square\end{aligned} \tag{2.15}$$

We will consider in Appendix I of [1] an analogous situation for the evaluation map $M\times\mathrm{Diff}\,M\to M$.

Finally let us point out that the condition (2.10) suggests that the form χ is likely to be a polynomial function in $A, dA, [A,A], [A,dA]$, etc.[6] Condition (2.10) itself can be interpreted as a locality condition, as we shall see in Sect. 6.

3. Anomalies with a Background Connection

In Sect. 2 we expressed anomalies in terms of forms on the total space P of the principal fibre bundle $P(M,G)$. It is often instead convenient to express them in

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[6] Despite condition (2.10), we could admit also basic forms q dependent on or independent of $A\in\mathscr{A}$, provided that we assume that under an infinitesimal automorphism Z, q transforms as a "good form"; i.e. $q\to L_Zq$. For instance $\mathrm{Tr}(F\wedge *F)$ in four dimensions does not satisfy condition (2.10) unless $\pi_* Z$ is an infinitesimal conformal transformation. But we can think of Z acting also on the $*$-operator, so as to preserve condition (2.10)

terms of forms on the base manifold M. In order to do so, we need a fixed (background) connection A_0 on P [4]. This connection A_0 can in turn be interpreted as a connection on $P \times \mathscr{G}$ ($\mathscr{G}$ is again $\operatorname{Aut} P$ or $\operatorname{Aut}_v P$ or $\operatorname{Diff} M$). In fact by considering the projection map

$$P \times \mathscr{G} \xrightarrow{\mathrm{pr}_1} P, \tag{3.1}$$

we can introduce the connection $\mathrm{pr}_1^* A_0$ that we denote again by the same symbol A_0 in order not to have a cumbersome notation. So on $P \times \mathscr{G}$ we have two connections $\mathrm{ev}^* A$ and A_0, whose curvatures are respectively $\mathrm{ev}^* F$ and F_0. Here F is the curvature of A.

If Q is again an ad-invariant polynomial on Lie G with $(n/2+1)$-entries, then we have, for dimensional reasons, $\mathrm{ev}^* Q(F,\ldots,F) = Q(F_0,\ldots,F_0) = 0$.

Hence:

$$\begin{aligned} 0 &= Q(\mathrm{ev}^* F,\ldots,\mathrm{ev}^* F) - Q(F_0,\ldots,F_0) \\ &= (\hat{d}+\delta)\left(\frac{n}{2}+1\right)\int_0^1 dt\, Q(\mathrm{ev}^* A - A_0, \mathscr{F}_t,\ldots,\mathscr{F}_t) \\ &\equiv (\hat{d}+\delta)\, W_Q(\mathrm{ev}^* A, A_0). \end{aligned} \tag{3.2}$$

Here $\mathscr{F}_t$ is the curvature of the connection $t(\mathrm{ev}^* A) + (1-t)A_0$ and $t \in [0,1]$. The form $W_Q(\mathrm{ev}^* A, A_0)$ is basic (i.e. the pullback of a form on $M \times \mathscr{G}$).

It is easy to show that we have the following expression:

$$\begin{aligned} W_Q(\mathrm{ev}^* A, A_0) = {} & W_Q(\psi^* A, A_0) + j_{(\cdot)} W_Q(\psi^* A, A_0) - j_{(\cdot)} j_{(\cdot)} W_Q(\psi^* A, A_0) \\ & + \cdots + (-1)^{n(n+1)/2} \underbrace{j_{(\cdot)} \cdots j_{(\cdot)}}_{(n+1)\,\text{terms}} W_Q(\psi^* A, A_0). \end{aligned} \tag{3.3}$$

Here $\psi \in \mathscr{G}$ and the operator $j_{(\cdot)}$ is defined as follows: we first consider $W_Q(\psi^* A, A_0)$ as a formal polynomial in $\psi^* A, A_0$, $[\psi^* A, A_0]$, $d\psi^* A, dA_0, \psi^* F, F_0$; then, $\forall Z \in \operatorname{Lie} \mathscr{G}$, the formal antiderivation j_Z is given by:

$$\begin{aligned} & j_Z \psi^* A = i_Z \psi^* A; \quad j_Z A_0 = 0; \\ & j_Z \psi^* dA = i_Z \psi^* dA; \quad j_Z dA_0 = 0. \end{aligned} \tag{3.4}$$

Finally $j_{(\cdot)}$ denotes the map $Z \mapsto j_Z$. Hence the form $j_{(\cdot)} W_Q(\psi^* A, A_0)$ is explicitly given by:

$$j_{(\cdot)} W_Q(\psi^* A, A_0)(Y) = j_{\psi^{-1} Y} W_Q(\psi^* A, A_0) \quad \forall Y \in T_\psi \mathscr{G}.$$

Even though $W_Q(\psi^* A, A_0)$ is zero for dimensional reasons, $j_{(\cdot)} W_Q(\psi^* A, A_0)$ is not. Moreover $j_Z W_Q(\psi^* A, A_0)$ is a basic form on P, $\forall Z \in \operatorname{Lie} \mathscr{G}$ and $\forall \psi \in \mathscr{G}$.

The following consistency condition is satisfied:

$$\delta j_{(\cdot)} W_Q(\psi^* A, A_0) - d j_{(\cdot)} j_{(\cdot)} W_Q(\psi^* A, A_0) = 0. \tag{3.5}$$

The form $j_{(\cdot)} W_Q(\psi^* A, A_0)$, when calculated at $\psi =$ identity, is the expression of the anomaly with a background connection as defined in [4][7].

[7] We assume here and below that A is different from A_0. For the special case $A = A_0$, see appendix II of [1]

We want now to show the relation which exists between the two expressions of the anomaly, with and without the background connection.

Consider the following identity:

$$W_Q(\mathrm{ev}^* A, A_0) = TQ(\mathrm{ev}^* A) - TQ(A_0) + (\hat{d} + \delta) S_Q(\mathrm{ev}^* A, A_0). \tag{3.6}$$

Here for any two connections A' and A we have set:

$$S_Q(A', A) = \int_0^1 dk \frac{n}{2}\left(\frac{n}{2} + 1\right) \int_0^1 t dt Q(A' - A, A^{(k)}, F_t^{(k)}, \dots, F_t^{(k)}); \tag{3.7}$$

where $A^{(k)} = kA' + (1-k)A$, $k \in [0,1]$ and $F_t^{(k)} = t dA^{(k)} + (t^2/2)[A^{(k)}, A^{(k)}]$.

Hence, by taking the $(n,1)$-component of (3.7), we have:

$$j_{(\cdot)} W_Q(\psi^* A, A_0) = i_{(\cdot)} TQ(\psi^* A) - d j_{(\cdot)} S_Q(\psi^* A, A_0) + \delta S_Q(\psi^* A, A_0). \tag{3.8}$$

So $j_{(\cdot)} W_Q(\psi^* A, A_0)$ and $i_{(\cdot)} TQ(\psi^* A)$ are the $(n,1)$-components of two forms which represent the same cohomology class of $P \times \mathcal{G}$. Namely they are the $(n,1)$-components of the two forms $W_Q(\mathrm{ev}^* A, A_0)$ and $TQ(\mathrm{ev}^* A) - TQ(A_0)$.

It is specifically in this sense that the two expressions of the anomaly $j_{(\cdot)} W_Q(A, A_0)$ and $i_{(\cdot)} TQ(A)$ can be considered as equivalent[8].

We would like now to comment briefly on the "topological significance" of the expressions of the anomalies with the background connection, that is we want to make a few preliminary considerations on the element of $H^1(\mathcal{G})$ which is determined by integrating over M the form $W_Q(\mathrm{ev}^* A, A_0)$. A further discussion will be carried out in Sect. 5.

It is easy, first of all, to check that, if A_0 and A_0' are two different connections on P, then $j_{(\cdot)} W_Q(\psi^* A, A_0)$ and $j_{(\cdot)} W_Q(\psi^* A, A_0')$ are the $(n,1)$-components of two forms which represent the same cohomology class of $P \times \mathcal{G}$ or of $M \times \mathcal{G}$ [4]. Hence $\int_M j_{(\cdot)} W_Q(\psi^* A, A_0)$ and $\int_M j_{(\cdot)} W_Q(\psi^* A, A_0')$ represent the same cohomology class of $\mathcal{G}$.

In the special case when P is a trivial bundle, there exists a global section $\sigma: M \to P$ and a connection A_σ such that $\sigma^* A_\sigma = 0$.

The forms $\sigma^* i_{(\cdot)} \psi^* TQ(A)$ and $j_{(\cdot)} W_Q(\psi^* A, A_\sigma)$ are again the $(n,1)$-components of two forms which represent again the same cohomology class of $M \times \mathcal{G}$, as is easily verified. Hence $\int_M \sigma^* i_{(\cdot)} TQ(\psi^* A)$ represents a cohomology class of $\mathcal{G}$ which is independent of the choice of the section σ.

Notice that the expression $\int_M j_{(\cdot)} W_Q(A, A_0)$ defines, for a variable $A \in \mathcal{A}$ and a fixed A_0, a 1-cochain in the cohomology of Lie $\mathcal{G}$ with coefficients in $C^\infty(\mathcal{A})$.

It is also possible, by considering n-forms on M which are polynomial functions in $A - A_0, F(A), F(A_0), [A - A_0, A - A_0]$, etc, to prove a theorem analogous to Theorem (2.11).

Let us now denote by ev_1 and ev_2 the evaluation maps obtained respectively by

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[8] We will see though, in Sect. 5, that when P is a non-trivial bundle and Q is a reducible polynomial, then the "topological significance" (see below) of the two expressions of the anomaly (with and without the background connection) can differ from each other

identifying $\mathrm{Aut}_v P$ with the right-hand side of the following group isomorphisms:

$$\mathrm{Aut}_v P \approx \{\psi \in \mathrm{Map}(P, G) | \psi(pg) = g^{-1}\psi(p)g\}, \tag{3.9'}$$

$$\mathrm{Aut}_v P \approx \text{Sections of } \mathrm{Ad}\, P. \tag{3.9''}$$

As a final remark, we would like to point out that in the gauge case the expression of the anomaly with a background connection can be equivalently derived from the following diagram:

$$\begin{tikzcd} P \times \mathrm{Aut}_v P \arrow[r, "\widetilde{\mathrm{ev}}_1"] \arrow[d, "\pi \times \mathrm{id}"] & P \times G \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "r"'] \arrow[d, "\pi'"] & P \arrow[d, "\pi"] \\ M \times \mathrm{Aut}_v P \arrow[r, "\mathrm{ev}_2"] & \mathrm{Ad}\, P \arrow[r, "\mathrm{pr}"] & M. \end{tikzcd} \tag{3.10}$$

Here $\widetilde{\mathrm{ev}}_1$ is defined as the combination of the following maps:

$$P \times \mathrm{Aut}_v P \xrightarrow{\Delta \times \mathrm{id}} P \times P \times \mathrm{Aut}_v P \xrightarrow{\mathrm{id} \times \mathrm{ev}_1} P \times G,$$

where Δ is the diagonal map. Moreover the map π' is by definition the projection obtained by the right action of G on $P \times G$ given by the right multiplication on P and by conjugation on G, pr is the projection in the bundle $\mathrm{Ad}\, P$, $\mathrm{pr}_1 : P \times G \to P$ is the projection on the first factor and r is the right multiplication by elements of G.

Notice that, given the evaluation map ev defined as in (2.1), we have

$$r \circ \widetilde{\mathrm{ev}}_1 = \mathrm{ev}.$$

We know that [25]

$$P \times G \to \mathrm{Ad}\, P \tag{3.11}$$

is a principal G-bundle.

It is easy to see that both r and pr_1 are G-bundle homomorphisms. Hence, if Q is an ad-invariant polynomial on Lie G with $(n/2 + 1)$-entries, then we can consider the $(n + 1)$-form on $\mathrm{Ad}\, P$ given by

$$W_Q(r^* A, \mathrm{pr}_1^* A_0). \tag{3.12}$$

The pullback of (3.12) via the evaluation map ev_2, is the form $W_Q(\mathrm{ev}^* A, A_0)$ considered before, i.e. produces the expression of the anomaly with the background connection.

We will use diagram (3.10) and the expression (3.12) in discussing the "topological meaning" of anomalies (Sect. 5).

Notice that there is no analog of (3.10) and (3.12) for the diffeomorphism case.

4. On the Gauge Interpretation of Gravitational Anomalies

Let again M be an n-dimensional manifold, let P be the bundle LM and let $\mathscr{G}$ be the group $\mathrm{Diff}\,M$ (or $\mathrm{Diff}^{m,1}\,M$). The gravitational anomaly is given by the map:

$$X \mapsto i_{l(X)} TQ(A) \quad X \in \mathrm{diff}\,M, \tag{4.1}$$

where l is the natural lift and Q is an ad-invariant polynomial with $(n/2+1)$-entries. Here A is supposed to be the Levi–Civita connection for a metric $g \in \mathscr{M}$.

We assume that $n \equiv 2 \pmod 4$ so $TQ(A)$ is not necessarily exact.

Let us suppose that there exists a flat connection A_0, that is, let us suppose that there exist sections $\sigma_k: U_k \subset M \to LM$ with $\cup U_k = M$ and

$$\sigma_k^* A_0 = 0 \quad \forall k. \tag{4.2}$$

Under these assumptions, if M is simply connected, then there exists a global section $\sigma_0: M \to LM$ with:

$$\sigma_0^* A_0 = 0. \tag{4.3}$$

Hence if M is a simply connected manifold with a flat linear connection, then LM is a trivial bundle and M itself is, by definition, parallelizable. Let us recall that parallelizability is a very strong requirement for manifolds. For instance, among the spheres S^n, only S^1, S^3 and S^7 are parallelizable, but the cartesian product of two or more spheres is a parallelizable manifold provided that at least one of the spheres has odd dimension [26, Vol. 4].

We have now the following:

Theorem (4.4). *Let M be a manifold with a flat connection A_0, let*

$$\sigma_k: U_k \subset M \to LM$$

satisfy Eq. (4.2) and let $Z \in \mathrm{aut}\,LM$.

Then we have:

$$\sigma_k^* i_Z TQ(A) = \sigma_k^* i_{Z^v} TQ(A), \tag{4.5}$$

where Z^v is the vertical component of Z with respect to A_0.

Proof. If Z^h is the horizontal component of Z with respect to A_0, then we have:

$$\sigma_k^* i_{Z^h} TQ(A) = i_{\pi_* Z} \sigma_k^* TQ(A) = 0$$

due to the fact that $TQ(A)$ is an $(n+1)$-form. $\square$

If Z is the natural lift of a vector field, then Theorem (4.4) tells us that the expression of the gravitational anomaly, pulled back via a parallel section with respect to a flat connection, depends only on the vertical component of Z.

Notice that if $\sigma: U \subset M \to LM$ is given by $\sigma(x) = (\partial/\partial x^1, \ldots, \partial/\partial x^n)$ and $\xi^k (1 \leqq k \leqq n)$ are the components of a vector field on M, then the expression of the anomaly (4.1) in local coordinates is given (up to a total derivative) by:

$$\int_0^1 dt(t-1)\frac{n}{2}\left(\frac{n}{2}+1\right) Q(d\Lambda, \sigma^* A, \sigma^* F_t, \ldots, \sigma^* F_t), \tag{4.6}$$

where $\Lambda = (\Lambda)^\alpha_\beta$ is the matrix given by $\partial_\beta \xi^\alpha$ (see formula (2.6)).

Formula (4.6) has been found (using only local expressions) in a seminal paper by Bardeen and Zumino [21].

The consequence of Theorem (4.4) and formula (4.6) is that, when the gravitational anomaly is pulled back to the base manifold, its "horizontal contribution" disappears, and only its "vertical contribution" survives:
a) on a global level if the manifold M admits a flat linear connection;
b) on each chart separately even if the manifold does not admit a flat linear connection; but in this case, boundary terms do not necessarily match.

The vanishing of the "horizontal contribution" of the gravitational anomaly allows us to have a correspondence between the gravitational anomalies and the "gauge anomalies" of the frame bundle. This correspondence in turn allows us to establish an equivalence between Lorentz and gravitational anomalies [21][9]. To understand better the situation in which we can have such an equivalence, we will use the expression for the anomaly with a background connection (Sect. 3).

Let X be a vector field on M and let Z be its natural lift: according to Sect. 3, the gravitational anomaly [27] is given by:

$$Z \mapsto j_Z W_Q(A, A_0), \tag{4.7}$$

but we have also:

$$j_Z W_Q(A, A_0) = i_Z W_Q(A, A_0) - j'_Z W_Q(A, A_0), \tag{4.8}$$

where j'_Z is defined as in (3.4) with A and A_0 interchanged.

For dimensional reasons $i_Z W_Q(A, A_0) = 0$, and so we can write:

$$j_Z W_Q(A, A_0) = -j'_{Z^v} W_Q(A, A_0) - j'_{Z^h} W_Q(A, A_0), \tag{4.9}$$

where Z^h and Z^v are respectively the horizontal and vertical component of Z with respect to A_0.

It is easy to check that $j'_{Z^h} W_Q(A, A_0)$ is zero if an only if $F(A_0) = 0$. In this case the gravitational anomaly depends only on the vertical components of the natural lift of vector fields on M. So we have the equivalence between gravitational and Lorentz-anomalies if the manifold M admits a flat linear connection.

Nevertheless one could hope to recover, in some sense, such an equivalence also for manifolds M which do not admit a flat linear connection, provided that the frame bundle LM is trivial over $M \setminus \{x_0\}$, for a fixed $x_0 \in M$ and provided that we limit ourselves to considering only vector fields on M which go to zero sufficiently fast at x_0. In this case one could choose a flat connection A_0 on $M \setminus \{x_0\}$ and see whether the following integral:

$$\int_{M \setminus \{x_0\}} j_{Z^v} W_Q(A, A_0)$$

converges.

Further comments on the relation between Lorentz and gravitational anomalies will appear at the end of Sect. 7.

5. Cohomology of the Gauge Group

In this section we want to compute the real cohomology of the gauge group, i.e. of

[9] The equivalence between gravitational and Lorentz anomalies is motivated by the existence of the Bardeen–Zumino counterterm [21]

the group $\mathrm{Aut}_v P$, for any principal bundle P with compact structure group G. The base manifold M is assumed to be, as usual, compact, oriented, Riemannian, without boundary, but we do not assume further limitations on M.

We first have:

Theorem (5.1). $H^*(\mathrm{Ad}\,P) \approx H^*(M) \otimes H^*(G)$.

Proof. Consider again diagram (3.10) and particularly the part of it given by:

$$\begin{array}{ccc} P \times G & \overset{\mathrm{pr}_1}{\underset{r}{\rightrightarrows}} & P \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\ \mathrm{Ad}\,P & \xrightarrow{\mathrm{pr}} & M. \end{array} \tag{5.2}$$

We recall that pr_1 and r are respectively given by the projection onto the first factor and the right multiplication.

Let Q be an irreducible ad-invariant polynomial on $\mathrm{Lie}\,G$ and let A and A_0 be two connections on P, with curvature F and F_0 respectively. The form

$$W_Q(r^*A, \mathrm{pr}_1^* A_0) - \mathrm{pr}_1^* W_Q(A, A_0) \tag{5.3}$$

is a closed form on $\mathrm{Ad}\,P$ (or a basic closed form on $P \times G$).

In fact we have on $P \times G$:

$$\begin{aligned} dW_Q(r^*A, \mathrm{pr}_1^* A_0) &= Q(r^*F, \dots, r^*F) - Q(\mathrm{pr}_1^* F_0, \dots, \mathrm{pr}_1^* F_0) \\ &= \mathrm{pr}_1^* Q(F, \dots, F) - \mathrm{pr}_1^* Q(F_0, \dots, F_0) \\ &= d\,\mathrm{pr}_1^* W_Q(A, A_0). \end{aligned}$$

Notice that, in deriving the above formula, we used the fact that, for any basic form ω on P, we have: $\mathrm{pr}_1^* \omega = r^* \omega$. Consider now the fiber imbedding

$$J : G \to \mathrm{Ad}\,P.$$

If $J_2 : G \to P \times G$ is defined as the inclusion in the second factor, we obviously have:

$$J = \pi' \circ J_2.$$

Taking into account that $r \circ J_2 : G \to P$ is the fiber imbedding while $\mathrm{pr}_1 \circ J_2$ is the constant map, we have also:

$$J_2^*[W_Q(r^*A, \mathrm{pr}_1^* A_0) - \mathrm{pr}_1^* W_Q(A, A_0)] = TQ(\theta),$$

where θ is the Maurer Cartan form on G and $TQ(\theta)$ is given by:

$$TQ(\theta) = k \int_0^1 dt\, Q\left(\theta, \frac{t^2 - t}{2}[\theta, \theta], \dots, \frac{t^2 - t}{2}[\theta, \theta]\right),$$

provided that k is the number of entries of Q.

Now primitive generators of $H^*(G)$ are exactly given by the forms $TQ_i(\theta)$ for ad-invariant irreducible polynomials Q_i. Hence by restricting to the fiber of $\mathrm{Ad}\,P$ all the closed forms given by the expression (5.3) we are able to generate freely the cohomology of G.

So we are in position to apply the theorem of Leray–Hirsch ([28], Theorem 5.11) and prove the theorem. □

We can then prove the following:

Theorem (5.4). *Let* $\overset{o}{\mathrm{Aut}}_v P$ *be the connected component of the identity of* $\mathrm{Aut}_v P$. *A set of primitive generators of* $H^*(\overset{o}{\mathrm{Aut}}_v P)$ *is given by the following set of forms:*

$$\Psi_{q,i,k,j} = \int_M \mathrm{ev}_2^* (\alpha_i^q \wedge W_{Q_j^k}(r^* A, \mathrm{pr}_1^* A_0)) \in \Omega^s(\overset{o}{\mathrm{Aut}}_v P), \tag{5.5}$$

where $\mathrm{ev}_2 : M \times \overset{o}{\mathrm{Aut}}_v P \to \mathrm{Ad}\, P$ *is the evaluation map, defined as in diagram (3.10),* Q_j^k *are irreducible polynomials with k-entries, the forms* α_i^q *are assumed to represent a basis of* $H^q(M)$ *and the indexes q and k are supposed to satisfy the following condition:* $q + 2k - 1 = n + s$ (*for* $n = \dim M$).

Proof. The theorem is a direct consequence of Theorem (5.1) and of the Sullivan model [29, 30] for calculating the cohomology of the space of sections of fibre bundles. In particular, the Sullivan calculation shows that when the cohomology of the total space of a bundle $\pi : E \to M$ with compact base and a group G as fiber is isomorphic to $H^*(M) \otimes H^*(G)$ then given the map:

$$\begin{aligned} \int_M \mathrm{ev}^* : H^*(M) \otimes H^*(G) \approx H^*(E) &\to H^*(\Gamma(E)) \\ [\psi] &\mapsto [\int_M \mathrm{ev}^* \psi] \end{aligned} \tag{5.6}$$

the cohomology $H^*(\Gamma(E))$ is freely generated by the image of (5.6). Here $\Gamma(E)$ denotes the space of sections of E and ev is the evaluation map: $M \times \Gamma(E) \to E$ □

Notice that the kernel of (5.6) is in turn generated by the products $TQ_1(\theta) \wedge \cdots \wedge TQ_k(\theta)$ which represent non-trivial elements of $H^*(G)$, so the image of (5.6) is isomorphic to $H^*(M)$ tensored with the vector space spanned by the forms $TQ_i(\theta)$ for irreducible Q_i. In order to understand better the significance of the kernel of the map (5.6), consider for instance the non-trivial element in $H^*(\mathrm{Ad}\, P)$ represented by:

$$\begin{aligned} \omega \equiv \alpha &\wedge (W_{Q_1}(r^* A, \mathrm{pr}_1^* A_0) - \mathrm{pr}^* W_{Q_1}(A, A_0)) \\ &\wedge (W_{Q_2}(r^* A, \mathrm{pr}_1^* A_0) - \mathrm{pr}^* W_{Q_2}(A, A_0)). \end{aligned} \tag{5.7}$$

Here we have assumed that $0 \neq [\alpha] \in H^j(M)$ and that the entries k_i of the polynomials Q_i are such that $2k_1 + 2k_2 - 3 + j = n \equiv \dim M$.

It is clear that the $(n, 1)$-component of $\mathrm{ev}_2^* \omega = \alpha \wedge (W_{Q_1}(\mathrm{ev}^* A, A_0) - W_{Q_1}(A, A_0)) \wedge (W_{Q_2}(\mathrm{ev}^* A, A_0) - W_{Q_2}(A, A_0))$ (see Sect. 3 for the notation) is zero.

Therefore one cannot construct any meaningful anomaly by considering the product of two or more forms like $W_{Q_i}(\mathrm{ev}^* A, A_0)$.

Corollary (5.8). *The cohomology of the connected* component of the identity of the gauge group $\overset{o}{\mathrm{Aut}}_v P$ *is independent of the isomorphism class of* P, i.e.

$$H^*(\overset{o}{\mathrm{Aut}}_v P) \approx H^*(\overset{o}{\mathrm{Map}}(M, G)).$$

Here the superscript o denotes the connected component of the identity. □

We would like now to consider the group $\mathrm{Aut}_v^m P$ and compute the first cohomology group of its identity-connected component $\overset{o}{\mathrm{Aut}}{}_v^m P$. We identify now the fiber $\mathrm{pr}^{-1}(m) \subset \mathrm{Ad}\, P$ with G and we denote by l_m the map defined by:

$$\begin{aligned} l_m : \overset{o}{\mathrm{Aut}}_v P &\to G \\ \psi \quad &\mapsto \psi(m). \end{aligned} \tag{5.9}$$

From the principal fibration

$$\overset{o}{\mathrm{Aut}}{}_v^m P \to \overset{o}{\mathrm{Aut}}_v P \xrightarrow{l_m} G, \tag{5.10}$$

we obtain the exact sequence:

$$0 = \pi_2(G) \to \pi_1(\overset{o}{\mathrm{Aut}}{}_v^m P) \to \pi_1(\overset{o}{\mathrm{Aut}}_v P) \to \pi_1(G). \tag{5.11}$$

Hence we have:

Theorem (5.12). *Let G be any compact Lie group with $\pi_1(G) = 0$. Then we have* $H^1(\overset{o}{\mathrm{Aut}}_v P) \approx H^1(\overset{o}{\mathrm{Aut}}{}_v^m P)$.

Proof. Immediate from (5.11). □

We consider now the special case when G is the product of $U(1)$ times a compact simply connected Lie group G'.

In this case we can consider $P' \equiv P/U(1)$, which is obviously a principal G'-bundle over M.

It is easy to see that $\mathrm{Aut}_v P$ and $\mathrm{Aut}_v^m P$ are isomorphic to the direct products of groups given by $\mathrm{Aut}_v P' \times \mathrm{Map}(M, U(1))$ and $\mathrm{Aut}_v^m P' \times \mathrm{Map}^m(M, U(1))$ respectively, where the last symbol denotes the space of the "pointed maps."

From the discussion above we have: $H^1(\overset{o}{\mathrm{Aut}}_v P') \approx H^1(\overset{o}{\mathrm{Aut}}{}_v^m P')$. Moreover we have also that the fibration:

$$\overset{o}{\mathrm{Map}}{}^m(M, U(1)) \to \overset{o}{\mathrm{Map}}(M, U(1)) \xrightarrow{l_m} U(1) \tag{5.13}$$

is trivialized by the map which assigns to each $g \in U(1)$ the constant map with value g. Here again the superscript o denotes the connected component of the identity.

From Theorem (5.4) and from the discussion above, we know that

$H^1(\overset{o}{\mathrm{Map}}(M, U(1)) \approx H^1(U(1)) = R^{10}$.

Hence we have[11]:

$$H^1(\overset{o}{\mathrm{Map}}{}^m(M, U(1)) = 0. \tag{5.14}$$

We will return shortly to the above equation, while discussing the abelian anomalies.

We are now able to study the topological significance of gauge anomalies for the general case.

We consider an ad-invariant polynomial Q on Lie G(G compact), with $(n/2+1)$-entries and the integral over M of the relevant anomaly i.e.:

$$\int_M j_{(\cdot)} W_Q(\psi^* A, A_0). \tag{5.15}$$

We have then the following:

Theorem (5.16). *The form (5.15) represents a non-trivial element of* $H^1(\overset{o}{\mathrm{Aut}}_v P)$ *if and only if either of the following cases is true:*

a) *Q is an irreducible polynomial.*
b) *Q is the product of irreducible polynomials Q_i and $w(Q_i)$ is different from zero for all Q_i but possibly one. Here w denotes the Weil homomorphism.*

Proof. The proof follows directly from Theorem (5.4) and from the fact that, if Q is, for instance, the product of two, not necessarily irreducible, polynomials Q_1 and Q_2, then we have:

$$\begin{aligned} j_{(\cdot)} W_Q(\psi^* A, A_0) = {} & j_{(\cdot)} W_{Q_1}(\psi^* A, A_0) \wedge Q_2(F_0, \dots, F_0) \\ & + j_{(\cdot)} W_{Q_2}(\psi^* A, A_0) \wedge Q_1(F, \dots, F) + \text{exact}. \end{aligned} \tag{5.17}$$

Here the exact form in the right-hand side is exact as a form on $M \times \mathscr{G}$. □

Notice that if we consider, instead of the anomaly with the background connection, the expression of the anomaly given by $i_{(\cdot)}\psi^* TQ(A)$, then there exist cycles c in P, such that the 1-form on $\mathrm{Aut}_v P$ given by:

$$\int_c i_{(\cdot)} \psi^* TQ(A) \tag{5.18}$$

---

[10] An explicit expression of a generator of $H^1(\mathrm{Map}(M, U(1))$ is given by $\int_M d\,\mathrm{vol}\ \mathrm{ev}^* \theta$, where θ is the Maurer-Cartan form on $U(1)$. The (0, 1)-component of $\mathrm{ev}^* \theta$, which is the only one which survives in the above integral, is to be identified with the "abelian ghost"

[11] It is easy more generally to show that $\mathrm{Map}^m(M, U(1))$ is a contractible space. In fact, by denoting by $\wedge$ the smash product, by $[\cdot,\cdot]_*$ the homotopy classes of pointed maps and by $\Omega_{(k)}$ the k-th loop space, we have for any $k \geqq 1$:

$$\begin{aligned} \pi^k(\mathrm{Map}^m(M, S^1)) &= [S^k, \mathrm{Map}^m(M, S^1)]_* \\ &= [S^k \wedge M, S^1]_* = [M, \Omega_{(k)} S^1]_* = 0. \end{aligned}$$

Notice that the last identity is due to the fact the $\Omega_{(k)} S^1$ is a contractible space for $k \geqq 2$ and it is the disjoint union of contractible spaces for $k = 1$

represents a non-trivial cohomology class if and only if either of the following cases is true:

a) Q is an irreducible polynomial

b) Q is the product of irreducible polynomials Q_i and $w(Q_i)$ is different from zero for all Q_i, with no exception.

Hence the "integrated anomalies" (5.15) and (5.18) have different "topological meaning"; this is due to the fact that M cannot be considered as a "cycle" in a non-trivial bundle P.

Finally let us consider the special case of abelian anomalies.

Let again P be a principal bundle with structure group $G \equiv U(1) \times G'$, where G' is a simply connected compact Lie group. We can then consider, as before, the G'-bundle $P' \equiv P/U(1)$ and the $U(1)$-bundle $P'' \equiv P/G'$. The bundle P itself is isomorphic to the "sum" of P' and P'' ([9]) and so, given any two connections A' and A'' with curvatures F' and F'', respectively on P' and on P'', we can consider on P the connection $A = A' + A''$ with curvature $F = F' + F''$.

Let Q' be an ad-invariant polynomial on $\mathrm{Lie}\, G'$ with $n/2$-entries. We can then consider the polynomial Q on $\mathrm{Lie}(U(1) \times G')$ given by:

$$\xi Q'(\xi_1, \dots, \xi_{n/2}) \quad \forall \xi \in U(1), \quad \xi_i \in \mathrm{Lie}\, G'.$$

The corresponding anomaly is given by:

$$j_{(\cdot)} W_Q(A, A_0) = i_{(\cdot)} A'' \wedge Q'(F', \dots, F') + j_{(\cdot)} W_{Q'}(A', A'_0) \wedge F''_0, \tag{5.19}$$

where $A \equiv A' + A''$ and $A_0 \equiv A'_0 + A''_0$ are connections on P. The first term in the right-hand side of (5.19) is called the abelian anomaly.

If we assume that P'' is a trivial bundle, then we can choose $F''_0 = 0$. In this situation the total anomaly is given by just the abelian part.

If $Q'(F', \dots, F')$ represents a non-trivial class in $H^n(M)$[12], then the abelian anomaly represents a non-trivial class in $H^1(\overset{o}{\mathrm{Aut}}_v P)$. But nevertheless, due to (5.14), it does not represent a non-trivial class in $H^1(\overset{o}{\mathrm{Aut}}{}^m_v P)$.

This is equivalent to saying that the topological significance of the abelian anomaly is determined by the rigid $U(1)$ symmetry.

All the above considerations are still true if instead of a simply connected compact Lie group G', we consider the abelian group $U(1)$. In this case $Q'(F', \dots, F')$ is to be replaced by

$$F' \wedge F' \wedge \cdots \wedge F', \tag{5.20}$$

and we obtain the classical axial anomaly, which represents a non-trivial element of $H^1(\overset{o}{\mathrm{Aut}}_v P)$ only if there are magnetic monopoles (i.e. if F' is not exact as a form on M).

A final remark is in order; the abelian anomaly, due to the above considerations,

[12] In four dimensions the integral of $Q'(F', F')$ gives the instanton number

does not represent a topological obstruction to the definition of the logarithm of a regularized gauge invariant determinant of the Dirac operator as do the other gauge anomalies (see Sect. 7 and see also Sect. 4 of [1]).

For a previous study on the cohomology of the group of gauge transformations, see [49].

6. Locality and Universality

We start by considering a manifold M and a principal fibre bundle $P(M,G)$, with structure group G semisimple and compact. Let Q_1 and Q_2 be ad-invariant polynomials on Lie G such that $\chi \equiv TQ_1(A) \wedge Q_2(F,\dots,F)$ is a form on P which is closed.

The corresponding anomaly in gauge theories, i.e. the $(n,1)$-component of $\mathrm{ev}^*\chi$ is a true anomaly even when $Q_2(F,\dots,F)$ is exact as a form on M, i.e. even when χ, and hence $\mathrm{ev}^*\chi$ are exacts as forms on P and $P\times\mathscr{G}$ respectively.

Analogous considerations could be made for the expression of the anomalies with a background connection.

To understand this fact, we recall that for any Lie group there exists a universal principal bundle $EG(BG,G)$ such that any other principal G-bundle over any manifold M is the pullback of it [31]. That is, for any principal G-bundle P, there exists a G-bundle morphism $(\hat{f},f)$ described by the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\hat{f}} & EG \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & BG \end{array}, \tag{6.1}$$

where f is determined up to homotopy. The total space EG is contractible.

For instance if $G=SU(N)$, then we have:

$$EG=\lim_{l\to\infty}\frac{U(l)}{U(l-N)},\quad BG=\lim_{l\to\infty}\frac{U(l)}{U(l-N)\times SU(N)}. \tag{6.2}$$

Generally EG and BG are infinite dimensional spaces, but we can consider them as finite dimensional manifolds if we restrict ourselves to bundles $P(M,G)$ with $\dim M$ not exceeding a fixed value k. This corresponds in (6.2) to taking l large enough, instead of considering $\lim_{l\to\infty}$. In this case BG will be called a k-classifying space.

In EG there is a universal connection ξ [31] such that for any connection A on P, there exists a G-bundle morphism $(\hat{f},f)$ as in (5.1) with:

$$A=\hat{f}^*\xi. \tag{6.3}$$

A form χ on P depending on the space of connections $\mathscr{A}$ (i.e. a function $\chi:\mathscr{A}\to\Omega^q(P)$) will be called, with an abuse of language, universal if it is the pullback of a

[13] It is immediate to verify that condition (6.4) implies condition (2.10). In fact the finite version of (2.10) is $\chi(\psi_t^*A)=\psi_t^*\chi(A)$, where ψ_t generates the infinitesimal automorphism Z. So any universal form satisfies the identity (2.10)

form χ_ξ on EG constructed out of the universal connection ξ; that is if[13]:

$$\chi(A) = \chi(\hat{f}^* \xi) = \hat{f}^* \chi_\xi \quad \forall \quad \text{bundle map } \hat{f}. \tag{6.4}$$

Obviously forms like $TQ(A)$ and $Q(F, \ldots, F)$ are universal. We can consider now the pullback through the evaluation map of universal forms, i.e. we can consider the pullback of forms on EG through the following combination of maps:

$$P \times \mathscr{G} \xrightarrow{\text{ev}} P \xrightarrow{\hat{f}} EG. \tag{6.5}$$

In Sect. 2 we mentioned a possible relation between non-trivial elements of $H^{n+1}(P)$ and non-trivial anomalies. Here we would like to be more specific. Consider a closed $(n+1)$-form χ on P which is universal. Its pullback through the evaluation map has a $(n, 1)$-component which satisfies the consistency condition. We call it a local anomaly and in the future by anomaly we will simply mean "local anomaly."

We say that the anomaly is cancelled (is trivial) if χ is the differential of a universal form η. In this case the $(n, 1)$-component of $\text{ev}^* \chi$ satisfies the cancellation condition (2.8). But obviously, there are local anomalies which satisfy Eq. (2.8) only with a non-universal η. In this case we say that these anomalies are non-trivial. We can then establish a one-to-one correspondence between local non-trivial anomalies and the elements of the quotient:

$$\frac{\text{universal closed } (n+1)\text{-forms on } P}{\text{derivatives of universal } n\text{-forms on } P}.$$

So the form $TQ_1(A) \wedge Q_2(F, \ldots, F)$ we considered at the beginning of this section, gives a non-trivial anomaly since $Q_2(F, \ldots, F)$ is not the exterior derivative of any universal basic form. This is due to the fact that the Weil homomorphism is an isomorphism in the classifying space BG, provided that G is compact.

As far as mixed local anomalies are concerned, we would like only to say that they are generated, via ev^*, from $(n+1)$-forms on P which are products of EG-universal forms and EH-universal basic forms. Here H is another group and a EH-universal basic form is a form on M which is the pullback of a form on BH. It should be pointed out that mixed local anomalies can be alternatively seen as the anomalies generated via ev^* from $(n+1)$-forms on $P + Q$, where Q is an H-principal bundle over M and $P + Q$ is defined as in [9, Sect. II.6].

In order to understand how the "universality" (or "locality") requirement enters anomaly calculations which are performed with the use of the background connection, let us remark that together with the combination of maps given by (6.5) we can consider also the following one:

$$P \times \mathscr{G} \xrightarrow{\text{pr}_1} P \xrightarrow{\hat{f}'} EG, \tag{6.6}$$

where pr_1 is the projection map onto the first factor and $\hat{f}'$ is the bundle homomorphism which induces the background connection from the universal one.

In this framework, a universal form on $P \times \mathscr{G}$ will be a basic form (i.e. a form on $M \times \mathscr{G}$) constructed out of polynomial expressions in $(f \circ \text{ev})^* \xi$ and $(\hat{f}' \circ \text{pr}_1)^* \xi$.

We are now in position to identify "locality" and "universality."

This identification is consistent with what is meant by "locality" in field theory.

For instance, the form $TQ_1(A) \wedge Q_2(F, \ldots, F)$ considered at the beginning of this section can certainly be exact, but it cannot be represented as the derivative of a form which is a "nice" (i.e. local) expression of the "fields" A and F. The differential forms you are allowed to consider in anomaly cancellation are "nice" functions of these "fields" (i.e. forms whose expression in local coordinates is constructed with differential operators applied to the fields).

What is relevant in field theory is not whether such differential forms are exact or not, but whether such differential forms are the derivatives or not of other forms which are again "nice" functions of the same fields.

Another way of looking at the problem is the following one.

As far as the anomalies are concerned, the results of perturbative field theoretical calculations are independent of the specific choice of the manifold M and depend only on its dimension.

This should explain why universality is intrinsic to perturbative field theory.

Moreover, by the same argument, it should be clear why perturbative calculations may be able to detect non-trivial cohomology classes of the space of fields, or of the (local) symmetry groups (i.e. the anomalies), while ignoring the cohomology of the space time.

To be more specific, we consider the case of gauge theories. If we want the results to be "reasonably" independent of the "choice" of the specific manifold M, we have to select the non-trivial cohomology classes of the appropriate spaces of fields, or of the (local) symmetry groups, which are completely independent of the cohomology of M.

So, in gauge theories, we have to:

a) disregard the non-trivial cohomology classes of the space of fields, or of the symmetry groups, which depend explicitly on non-trivial elements of $H^*(M)$. By this we mean that we do not consider cohomology classes represented by forms like (5.5), when non-universal forms α_i^q are included;

b) consider all the closed forms on the space of fields, or on the symmetry groups, which do not depend explicitly on the cohomology of M and either are always non-exact or are non-exact for some manifold M, where they "hit" non-trivial elements of $H^*(M)$. As we said before, in perturbative field theory, we want to avoid making any explicit assumption about the cohomology of M or of the principal bundle(s) over M: only the dimension of M is relevant. This means, for instance, that we cannot exclude any 1-form on $\mathscr{G}$ given by (5.15), irrespectively of whether the Weil homomorphism, applied to the relevant ad-invariant polynomials, gives zero or not (see Theorem (5.16)).

We want to stress again that the relevance of gauge anomalies in field theory is, in the above sense, independent of their "topological significance," which has been discussed, in Sect. 5.

This is even more evident in the case of gravitational anomalies.

In fact when $P = LM$ and $\mathscr{G} = \operatorname{Diff} M$, we have to consider the following closed forms on $\operatorname{Diff} M$:

$$\int_c i_{(\cdot)} \psi^* TQ(A) \tag{6.7}$$

$$\int_M j_{(\cdot)} W_Q(\psi^* A, A_0). \tag{6.8}$$

Q is the product of two polynomials Q_1 and Q_2 and at least one of the two forms $Q_1(F,\ldots,F)$ and $Q_2(F,\ldots,F)$ is exact on M, then $TQ(A)$ is exact and (6.7) represents the trivial element of $H^1(\operatorname{Diff} M)$. The same is true for (6.8) when Q is the product of two polynomials Q_1 and Q_2 and both $Q_1(F,\ldots,F)$ and $Q_2(F,\ldots,F)$ are exact on M.

But there is no known general relation between the characteristics of gravitational anomalies and the cohomology of Diff M apart from the trivial cases considered before.

In any case, formula (6.8) (or (6.7)) gives a true field theory anomaly, independently of its "topological" significance.

Finally we are going to make a further comment on the identification we made between universality and locality.

We can consider, as in [5], l-cochains on Lie $\mathscr{G}$ ($\mathscr{G} = \operatorname{Aut}_v P$ or Diff M) with values in k-forms on the base manifolds M. These cochains, denoted by ψ_l^k, are supposed to satisfy the following properties[14]:

We fix a background connection A_0 and we assume that ψ_l^k is local, in the sense that it depends on $\eta(A) \equiv A - A_0$ and $Z_i \in \operatorname{Lie}\mathscr{G}$ through differential operators or, equivalently, that ψ_l^k is such that it decreases supports [33]. A dependence on $F(A_0)$ is also allowed in ψ_l^k. We assume that ψ_1^k satisfies a consistency condition, i.e. we assume that there exists a form ψ_2^{n-1} with:

$$\delta_{\operatorname{Lie}\mathscr{G}} \psi_1^n + d\psi_2^{n-1} = 0, \tag{6.9}$$

where $\delta_{\operatorname{Lie}\mathscr{G}}$ is the coboundary of the cohomology of Lie $\mathscr{G}$ (with coefficients in the space of forms on M, depending on $A \in \mathscr{A}$, in the way described above) and d is the derivative for forms on M. Then we can construct a descent equation and prove [5, Theorem 4.1][15], that there exists a 0-form ψ_0^{n+1} with:

$$\delta_{\operatorname{Lie}\mathscr{G}} \psi_0^{n+1} = 0. \tag{6.10}$$

If ψ_1^n satisfies the triviality condition:

$$\psi_1^n + \delta_{\operatorname{Lie}\mathscr{G}} \psi_0^n + d\psi_1^{n-1} = 0 \tag{6.11}$$

for suitable ψ_0^n and ψ_1^{n-1}, then ψ_0^{n+1} is $\delta_{\operatorname{Lie}\mathscr{G}}$-exact. So we are interested in $(n+1)$-cochains ψ_0^{n+1} which are $\delta_{\operatorname{Lie}\mathscr{G}}$-closed and not $\delta_{\operatorname{Lie}\mathscr{G}}$-exact. Now if $\mathscr{G} = \operatorname{Diff} M$ and $P = LM$, then M. De Wilde and P. Lecomte [34] (see also [35]) prove that all local ψ_0^{n+1} which are $\delta_{\operatorname{Lie}\mathscr{G}}$-exact are given by:

$$\underbrace{i_{(\cdot)}\cdots i_{(\cdot)}}_{(n+1)\text{ terms}}\Big(\sum_i TQ_i(A) \wedge q_i\Big).$$

Here Q_i are, as usual, ad-invariant polynomials and q_i represent non-trivial elements of $H^*_{\text{deRham}}(M)$.

So we have recovered an expression like the last term of the right-hand side of (3.3) (or of (2.3)). Presumably an analogous result can be proved for $\mathscr{G} = \operatorname{Aut}_v P$. The moral we can draw from the story is that, if we look for local cohomology, defined in terms of cochains which decrease supports, we end up with cohomology classes

[14] We are considering here what can be called B.R.S. cohomology. See also [32] for a different approach

[15] In [5] only the case $\mathscr{G} = \operatorname{Aut}_v P$ has been considered

constructed out of universal forms. What is suggested nere is the fact that there is probably no support-decreasing cochain, with arbitrarily complicated differential operators, satisfying the consistency condition, which is different from the chiral anomalies constructed form the transgressions $TQ_i(A)$.

There is though an important difference between universal cochains and cochains which decrease supports. This difference concerns the forms q_i in (6.12). It should be pointed out that, even if such q_i are exact, i.e. $q_i = d\tau_i$, still we have to require that also the forms τ_i (like q_i) depend nicely on the fields, in order to have the cancellation of anomalies. Hence we may have anomalies corresponding to $(n+1)$-forms on P, which are exact and are not the differential of universal forms, which "cannot be seen" if one studies the local cohomology, defined in terms of cochains which decrease supports. So in considering field theory anomalies, the identification of locality and universality seems the best approach.

7. Locality and the Index Theorem

Let $EG(BG, G)$ be a universal bundle for G and let $\mathrm{Hom}(P, EG)$ be the space of all G-bundle morphisms from P to EG.

Notice that for universal we mean generally k-universal for k large enough [36].

We can consider the following evaluation map:

$$\widetilde{\mathrm{ev}}: P \times \mathrm{Hom}(P, EG) \to EG, \tag{7.1}$$

and the following diagram:

$$\begin{array}{ccc} P \times \mathrm{Hom}(P, EG) & \xrightarrow{\widetilde{\mathrm{ev}}} & EG \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M \times \mathrm{Hom}(P, EG) & \xrightarrow{\underline{\widetilde{\mathrm{ev}}}} & BG. \end{array} \tag{7.2}$$

Here for any $x \in M$ and $\hat{f} \in \mathrm{Hom}(P, EG)$, $\underline{\widetilde{\mathrm{ev}}}$ is defined by: $\underline{\widetilde{\mathrm{ev}}}(x, \hat{f}) = f(x)$ provided that $\hat{f}$ covers $f: M \to BG$.

It is obvious that $\widetilde{\mathrm{ev}}$ is a G-bundle homomorphism, so for any (universal) connection ξ on EG, $\widetilde{\mathrm{ev}}^* \xi$ is a connection on $P \times \mathrm{Hom}(P, EG)$.

The group $\mathrm{Aut}\, P$ acts on the right on $\mathrm{Hom}(P, EG)$, the action being simply given by the composition of maps.

If $\hat{f}$ covers f, then $\forall \psi \in \mathrm{Aut}\, P$, $\hat{f} \circ \psi$ covers $f \circ j(\psi)$, where $j: \mathrm{Aut}\, P \to \mathrm{Diff}\, M$ is the projection as in (1.2).

Let now $\mathscr{G}$ be any subgroup of $\mathrm{Aut}\, P$ (possibly the group $\mathrm{Aut}\, P$ itself) and consider again diagram (7.2).

If χ is any $(n+1)$-form on EG, then $\widetilde{\mathrm{ev}}^* \chi$ is an $(n+1)$-form on $P \times \mathrm{Hom}(P, EG)$ which cannot be closed and non-cohomologous to zero. In fact if χ is closed, it is also exact, since EG is k-connected. If χ is not closed, them $\widetilde{\mathrm{ev}}^* \chi$ cannot be closed since:

$$d\, \widetilde{\mathrm{ev}}^* \chi = \widetilde{\mathrm{ev}}^* d\chi = 0 \Rightarrow d\chi = 0.$$

But we can consider $\hat{f} \in \mathrm{Hom}(P, EG)$ and a $\mathscr{G}$-orbit $\mathscr{G}_{\hat{f}}$ in $\mathrm{Hom}(P, EG)$.

It is now possible for a non-closed form χ, that the restriction of $\widetilde{\mathrm{ev}}^* \chi$ to $P \times \mathscr{G}_{\hat{f}}$ is a closed form.

This is in fact exactly the way in which anomalies are generated.

To understand the statement, notice that the restriction of $\widetilde{\mathrm{ev}}^* \chi$ to $P \times \mathscr{G}_{\hat{f}}$ is simply given by the form $\mathrm{ev}^* (\hat{f}^* \chi) = (\hat{f} \circ \mathrm{ev})^* \chi$, where $\hat{f} \circ \mathrm{ev}$ is the combination of maps as in (6.5). Notice moreover that:

a) If $d\chi$ is a non-basic $(n+2)$-form on EG, then $\hat{f}^* \chi$ cannot be closed for every bundle map $\hat{f}$[16],

b) if $d\chi$ is a basic $(n+2)$-form on EG which is also the differential of a basic $(n+1)$-form, ϕ, then $\chi - \phi = d\psi$ on EG and $\hat{f}^* \chi = d \hat{f}^* \psi$,

c) if $d\chi$ is a basic $(n+2)$-form on EG which is not the differential of a basic $(n+1)$-form on EG, then $\hat{f}^* \chi$ is closed for each bundle map $\hat{f}$, but there does not exist any n-form ψ on EG, such that: $\hat{f}^* \chi = d \hat{f}^* \psi$.

Taking into account the considerations of Sect. 6, we see that non-trivial anomalies are generated by forms which satisfy condition (c). We saw that any such form χ determines a non-trivial cohomology class on BG, namely $[d\chi]$. Conversely any $(n+2)$-form which represents a non-trivial cohomology class on BG, is exact on EG, i.e. is the differential of a form χ satisfying condition (c).

So we are able to identify the local cohomology with the cohomology of the classifying space.

Let us define an action of $\mathscr{G}$ on $P \times \mathrm{Hom}(P, EG)$ as follows:

$$\begin{aligned} P \times \mathrm{Hom}(P, EG) \times \mathscr{G} &\to P \times \mathrm{Hom}(P, EG), \\ ((p, \hat{f}), \psi) &\mapsto (\psi^{-1}(p), \hat{f} \circ \psi). \end{aligned} \tag{7.3}$$

Obviously we have:

$$\widetilde{\mathrm{ev}}(p, \hat{f}) = \widetilde{\mathrm{ev}}(\psi^{-1}(p), \hat{f} \circ \psi). \tag{7.4}$$

We want now to assume that $\mathscr{G}$ acts freely on $P \times \mathrm{Hom}(P, EG)$. This is, for instance, the case when $\mathscr{G}$ is the group $\mathrm{Aut}_v P$, which acts freely on $\mathrm{Hom}(P, EG)$.

The map $\widetilde{\mathrm{ev}}$ descends to a G-bundle morphism $\widetilde{\mathrm{Ev}}: (P \times \mathrm{Hom}(P, EG))/\mathscr{G} \to EG$, and we can consider the diagram:

$$\begin{array}{ccc} P \times \mathrm{Hom}(P, EG) & \xrightarrow{\widetilde{\mathrm{ev}}} & EG \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \dfrac{P \times \mathrm{Hom}(P, EG)}{\mathscr{G}} & \xrightarrow{\widetilde{\mathrm{Ev}}} & EG \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \dfrac{M \times \mathrm{Hom}(P, EG)}{\mathscr{G}} & \xrightarrow{\underline{\widetilde{\mathrm{Ev}}}} & BG, \end{array} \tag{7.5}$$

[16] We assume here $\dim P \geq (n+2)$. For the case $G = U(1)$ we are still able to identify the local cohomology with the cohomology of the classifying space (as we are going to do for the general case), since $(n+1)$-forms on P which are not universal, in the sense of Sect. 6, do not generate local anomalies

where $\mathscr{G}$ acts on $M \times \operatorname{Hom}(P, EG)$ as follows:

$$(x, \hat{f})\psi \equiv (j(\psi)^{-1}(x), \hat{f} \circ \psi),$$

and $\widetilde{\underline{\mathrm{Ev}}}$ is given by:

$$\widetilde{\underline{\mathrm{Ev}}}(x, \hat{f}) = f(x),$$

provided that $f: M \to BG$ is covered by $\hat{f}$.

Let us now consider forms χ_ξ on BG such that

$$0 \neq [\chi_\xi] \in H^{n+2}(BG). \tag{7.6}$$

i.e.:

$$[\chi_\xi] = [Q(F_\xi, \ldots, F_\xi)] \tag{7.7}$$

for some ad-invariant polynomial on Lie G with $(n/2+1)$-entries. If $G = GL(n, \mathbb{R})$, then Q is required to be a non-zero ad-invariant polynomial on Lie $O(n)$.

If $\mathscr{G}$ acts freely on $\operatorname{Hom}(P, EG)$, then we can fiber-integrate (over M) the $(n, 2)$-component of $\widetilde{\underline{\mathrm{Ev}}}^* \chi_\xi$, and obtain a 2-form on $\operatorname{Hom}(P, EG)/\mathscr{G}$, denoted by χ_2, which can be "antitransgressed" to a 1-form $\chi_1^{\mathscr{G}}$ on $\mathscr{G}$ [2, 37].

By "antitransgression" we mean what follows. If $\pi: \operatorname{Hom}(P, EG) \to \operatorname{Hom}(P, EG)/\mathscr{G}$ is the projection, then $d\pi^* \chi_2 = 0$ and moreover, since the lowest homotopy groups of $\operatorname{Hom}(P, EG)$ are zero[17], $\pi^* \chi_2 = d\chi_1$. We can consider now $\chi_1^{\mathscr{G}} \equiv J^* \chi_1$, when J is the fiber imbedding $J: \mathscr{G} \to \operatorname{Hom}(P, EG)$. Since $\pi^* \chi_2$ is basic by construction, $\chi_1^{\mathscr{G}}$ is closed. The map $\chi_2 \mapsto \chi_1^{\mathscr{G}}$, which gives in fact a homomorphism: $H^2(\operatorname{Hom}(P, EG)/\mathscr{G}) \to H^1(\mathscr{G})$, can be called "antitransgression" (also known as "suspension").

Diagram (7.5) tells us that, if χ_ξ is given by (7.7), then we have:

$$[\chi_1^{\mathscr{G}}] = [\int_M j_{(\cdot)} W_Q(\psi^* A, A_0)]. \tag{7.8}$$

In this formula, $[\chi_1^{\mathscr{G}}]$ is a cohomology class of the fiber through $\hat{f} \in \operatorname{Hom}(P, EG)$ and the connection A is given by $\hat{f}^* \xi$. (See Sect. 3).

Notice that $[\chi_\xi] \neq 0$ does not imply $[\chi_1^{\mathscr{G}}] \neq 0$. For instance, when $\mathscr{G}$ is $\operatorname{Aut}_v P$ and when moreover P is trivial and Q in (7.7) is a reducible polynomial, then we know that $[\chi_\xi] \neq 0$, while we have:

$$0 = [\chi_2] \in H^2\left(\frac{\operatorname{Hom}(P, EG)}{\mathscr{G}}\right) \quad \text{and } 0 = [\chi_1^{\mathscr{G}}] \in H^1(\mathscr{G}).$$

In order to take the locality requirements into account, we have to consider the pullback of all the characteristic classes of BG (defined in terms of the universal connection), irrespectively of whether the corresponding forms we obtain on $(M \times \operatorname{Hom}(P, EG))/\mathscr{G}$ once integrated over M, give forms on $\operatorname{Hom}(P, EG)/\mathscr{G}$ which are cohomologous to zero or not.

We notice now that anomalies can be certainly calculated by using $\operatorname{Hom}(P, EG)$

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[17] It can be proved (by obstruction theory) that if BG is $(n+h)$-classifying, then $\operatorname{Hom}(P, EG)$ is h-connected

instead of the space of connections $\mathscr{A}$. At least in the spirit of current algebra, there is no practical difference, since anomalies considered as 1-forms on $\mathscr{G}$, depending on $\mathrm{Hom}(P, EG)$, descend uniquely to 1-forms on $\mathscr{G}$, depending on $\mathscr{A}$.

However it is desirable to have a direct computation over $\mathscr{A}$. This would be immediate if we were to replace in (7.5) the space $\mathrm{Hom}(P, EG)$ with the space of connections $\mathscr{A}$.

This would be in turn possible, if we were able to find a C^∞ map

$$\sigma_\xi : \mathscr{A} \to \mathrm{Hom}(P, EG)$$

such that $\sigma_\xi(A)^* \xi = A$ and $\sigma_\xi(\psi^* A) = \sigma_\xi(A) \circ \psi$, $\forall \psi \in \mathscr{G}$ (here $\mathscr{G}$ is supposed to act freely on $\mathscr{A}$).

Since we are not able, in general, to construct such a map σ_ξ, we modify slightly our approach and proceeds as follows.

First of all we assume that $\mathscr{G}$ is a subgroup of $\mathrm{Aut}\, P$ which acts freely on $\mathscr{A}$.

Two cases are of interest:

a) $\mathscr{G} = \mathrm{Aut}_v^m P$ [2], [38],

b) $\mathscr{G} = \mathrm{Diff}^{m,1} M$ when $P = LM$ [37].

As far as case (b) is concerned, in this paper we consider instead of the space of all connections, either the space $\mathscr{A}^{\mathrm{metric}}$, namely the sapce of all linear connections whose holonomy group is contained in $SO(n)$, or the space $\mathscr{A}^{\mathrm{LC}}$ of all Levi–Civita connections. Since we want to discuss, for a while, cases (a) and (b) simultaneously, we advice the reader that, when, in the following arguments, we are considering case (b), i.e. when $\mathscr{G}$ is the group $\mathrm{Diff}^{m,1} M$, then by the symbol $\mathscr{A}$ we will mean either $\mathscr{A}^{\mathrm{metric}}$ or $\mathscr{A}^{\mathrm{LC}}$.

Having said that, we are able to consider for both cases (a) and (b) the principal fibre bundle $\mathscr{A} \to \mathscr{A}/\mathscr{G}$ with connection ω and the principal G-bundle

$$\frac{P \times \mathscr{A}}{\mathscr{G}} \to \frac{M \times \mathscr{A}}{\mathscr{G}}. \tag{7.9}$$

Notice that $(M \times \mathscr{A})/\mathscr{G}$ is a fiber bundle over $\mathscr{A}/\mathscr{G}$ with fiber M.

What we intend to do now, is to define a connection on the bundle (7.9), which, via the classification theorem of Narasimhan and Ramanan [31], will allow us to construct for the bundle $(P \times \mathscr{A})/\mathscr{G} \to (M \times \mathscr{A})/\mathscr{G}$ a diagram playing the rôle of diagram (7.5).

A connection on (7.9) is given as follows: we consider the G-bundle

$$P \times \mathscr{A} \to M \times \mathscr{A} \tag{7.10}$$

with connection η given by:

$$\eta_{p,A}(X_1, X_2) \equiv A(X_1) + A(\omega(X_2))_p, \tag{7.11}$$

where $X_1 \in T_p P$, $X_2 \in T_A \mathscr{A}$ and ω is the connection on $\mathscr{A}$ considered before.

Then we verify immediately that this connection descends to a connection η' on (7.9).

Remark that if $\mathscr{G}$ is the trivial group, then (7.11) becomes the so-called tautological connection [39]. We would like now to find a G-bundle morphism:

$$\begin{tikzcd} \dfrac{P \times \mathscr{A}}{\mathscr{G}} \arrow[r, "\mathrm{Ev}"] \arrow[d, "\pi"] & EG \arrow[d, "\pi"] \\ \dfrac{M \times \mathscr{A}}{\mathscr{G}} \arrow[r, "\underline{\mathrm{Ev}}"] & BG. \end{tikzcd} \tag{7.12}$$

such that $\mathrm{Ev}^* \xi = \eta'$.

Associated with (7.12) we would have also a bundle morphism:

$$\hat{\mathrm{ev}}: P \times \mathscr{A} \to EG$$

such that $\hat{\mathrm{ev}}^* \xi = \eta$.

Let N be an integer and let $(\mathscr{A}/\mathscr{G})^N$ be an N-skeleton, i.e. an N-dimensional subspace of $\mathscr{A}/\mathscr{G}$ which has cohomology isomorphic (by inclusion) to that of $\mathscr{A}/\mathscr{G}$ through dimension $(N-1)$.

We denote by the symbol p the projection $(M \times \mathscr{A})/\mathscr{G} \to \mathscr{A}/\mathscr{G}$.

By the classification theorem of Narasimhan and Ramanan [31] (see also [40]), we know that there exists a bundle morphism:

$$\begin{tikzcd} & \pi^{-1}\left(p^{-1}\left(\dfrac{\mathscr{A}}{\mathscr{G}}\right)^N\right) \arrow[r] \arrow[d, "\pi"] & EG \arrow[d, "\pi"] \\ \dfrac{M \times \mathscr{A}}{\mathscr{G}} \supset & p^{-1}\left(\dfrac{\mathscr{A}}{\mathscr{G}}\right)^N \arrow[r] & BG \end{tikzcd} \tag{7.13}$$

such that the pullback of the universal connection ξ is the restriction of η' over $p^{-1}(\mathscr{A}/\mathscr{G})^N$.

Since, on one side, we can consistently increase N and extend the relevant bundle morphism, and since, on the other side, we are only interested in computing the cohomology of $(M \times \mathscr{A})/\mathscr{G}$, up to a given finite order, we will rather improperly assume the existence of the bundle morphism (7.12), even though considering infinite dimensional classifying spaces and passing to the limit $N \to \infty$ would be not at all a trivial matter.

Let us be more precise: in the following we will write again for simplicity diagram (7.12) but, in reality, we will work with diagram (7.13).

By the same argument, we will consider also the connection $\mathrm{Ev}^* \xi$, together with its curvature $\mathrm{Ev}^* F_\xi$, meaning simply that we are considering the restriction of the connection η' to $\pi^{-1}(p^{-1}(\mathscr{A}/\mathscr{G})^N)$, together with the relevant curvature.

It is worth pointing out that anomaly calculations are independent of the choice of the connection ω which appears in (7.11). Moreover, if ω and $\bar{\omega}$ are two connections on $\mathscr{A}$ and if η and $\bar{\eta}$ are the relevant connections on (7.10) constructed as in (7.11), then

$$\eta = \bar{\eta} \Rightarrow \omega = \bar{\omega}.$$

It is now time to consider separately gauge, gravitational and Lorentz anomalies.

A) *Gauge Anomalies.* If $\mathscr{G} = \mathrm{Aut}_v^m P$, then $(M \times \mathscr{A})/\mathscr{G} = M \times \mathscr{A}/\mathscr{G}$ and the diagram (7.12) yields the following diagram:

$$\begin{array}{ccc} \dfrac{P \times \mathscr{A}}{\mathscr{G}} & \xrightarrow{\mathrm{Ev}} & EG \\ \Big\downarrow \pi & & \Big\downarrow \pi \\ M \times \dfrac{\mathscr{A}}{\mathscr{G}} & \xrightarrow{\underline{\mathrm{Ev}}} & BG. \end{array} \tag{7.14}$$

Let us now consider a representation ρ of G on a complex vector space V. Then we have:

a) the associated universal bundle [41] $\tilde{V} = EG \times_G V$;
b) the associated bundle $\hat{V} = P \times_G V$[18];
c) the associated bundle $\bar{V} = (P \times \mathscr{A})/\mathscr{G} \times_G V$.

We can then consider the Chern character $\mathrm{ch}(\tilde{V})$ as represented by a form constructed from the curvature of a universal connection ξ.

The components of the Chern character $\mathrm{ch}_i\,\bar{V} = \underline{\mathrm{Ev}}^*\,\mathrm{ch}_i\,\tilde{V}$ can consequently be represented by forms depending on (the curvature of) the connection η' determined by (7.10). We are interested in exactly this form, not only in its cohomology class.

We know that the obstruction to the definition of a gauge covariant propagator or to the existence of a gauge invariant (regularized) "determinant" of the chiral Dirac operators is given by the class [2, 37, 38]

$$\mathrm{ch}_1(\mathrm{Ind}) = \int_M \hat{A}(M)\,\mathrm{ch}(\bar{V})|_{n,2}. \tag{7.15}$$

Here, ch_1 represents the 2-form contained in the Chern character, Ind denotes the index bundle [2, 42], $\hat{A}(M)$ is the $\hat{A}$-class of M, which is supposed to be represented by an (inhomogeneous) form depending on the Levi–Civita connection of M and $\hat{A}(M)\,\mathrm{ch}(\bar{V})|_{n,2}$ is the $(n,2)$-component of $\hat{A}(M)\,\mathrm{ch}(\bar{V})$. The forms $\hat{A}(M)$ and $\mathrm{ch}(\bar{V})$ can be expressed in terms of a series of irreducible invariant polynomials in Lie Spin (n) and Lie G, respectively. So $\hat{A}(M)\,\mathrm{ch}(\bar{V})|_{n,2}$ will consist of a finite sum of products of polynomials coming from $\hat{A}(M)$ and $\mathrm{ch}(\bar{V})$.

In particular the term $1 \times \mathrm{ch}(\bar{V})|_{n,2}$ will contain only pure gauge anomalies, while the other terms will give rise to mixed gauge anomalies and a self-contained discussion must include also gravitational anomalies (see below).

Considering, for the time being, only pure gauge anomalies, we know that each one of the independent polynomials Q_i in the expansion of the $\mathrm{ch}(\bar{V})|_{n,2}$ will give rise, through antitransgression, to a definite anomaly with the relevant coefficient, which depends on the given representation of G.

It is entirely possible that by antitransgressing the integral over M of $Q_i(\mathrm{Ev}^* F_\xi, \ldots, \mathrm{Ev}^* F_\xi)$ (which is a 2-form on $\mathscr{A}/\mathscr{G}$) one obtains an exact 1-form on $\mathscr{G}$.

Such a 1-form on $\mathscr{G}$ does not represent a topological obstruction to defining a

[18] Section of $\hat{V}$ (i.e. matter fields) can be thought of as ρ-equivariant maps: $P \to V$. There exists a universal section of $\tilde{V}$, i.e. a ρ-equivariant map $\tilde{\varphi}: EG \to V$ such that any section of $\hat{V}$ given by $\psi: P \to V$ can be written as $\psi = \tilde{\varphi} \circ \hat{f}$, where $\hat{f}$ is a bundle map as in (6.1) [31]. So matter fields also are universal objects

gauge invariant Dirac determinant. However the corresponding anomaly is non-trivial from a field theory (locality) point of view, as we have seen in the previous section. We can interpret it as an obstruction to defining a local invariant expression for the logarithm of the (regularized) Dirac determinant. The index theorem, Eq. (7.15), gives us the right coefficient for this anomaly (i.e. in agreement with perturbative calculations).

Summarizing, by using the index theorem, gauge (and gravitational) anomalies are determined uniquely when we are given a manifold M, a group G and a representation ρ of G. More precisely the essential ingredients are the dimension of M, the multiplicative sequence of the A-class, the coefficients of the Chern character, the representation ρ of G.

At first sight it may appear surprising that the index theorem gives coefficients for the anomalies which are in agreement with the perturbative calculations, even when (7.15) gives trivial cohomology classes on $\mathscr{A}/\mathscr{G}$, in which case the coefficients themselves appear "meaningless."

This is exactly connected with the locality requirement.

In fact, local anomalies in field theory are independent of the topology (isomorphism class) of $P(M, G)$ and of the topology of M. Their expressions and their coefficients must be the same in each different "topological configuration" once we are given the above ingredients. Actually for any given anomaly we can envisage an n-dimensional manifold M and a bundle $P(M, G)$ such that the given anomaly is of topological origin. In this case the index theorem gives non-trivial cohomology classes and hence determines the coefficients in a "meaningful" way. Since these coefficients are independent of the topological configuration, they turn out to be uniquely determined in general.

B) *Gravitational Anomalies.* If $P = LM^+$ (the bundle of oriented frames) and $\mathscr{G} = \mathrm{Diff}_*^{m,1} M$ (that is the subgroup of orientation preserving elements of $\mathrm{Diff}^{m,1} M$), then $\mathscr{G}$ acts freely on the space $\mathscr{A}^{\mathrm{metric}}$ of all metric connections (for a given metric in $\mathscr{M}$) and we have the diagram:

$$\begin{array}{ccc} \dfrac{LM^+ \times \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}_*^{m,1} M} & \xrightarrow{\mathrm{Ev}} & EGL(n, \mathbb{R})^+ \\ \Big\downarrow & & \Big\downarrow \\ \dfrac{M \times \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}_*^{m,1} M} & \xrightarrow{\mathrm{Ev}} & BGL(n, \mathbb{R})^+ . \end{array} \tag{7.16}$$

We can also consider the double covering $\widetilde{GL}(n, \mathbb{R})$ of the group $GL(n, \mathbb{R})^+$ and the $\widetilde{GL}(n, \mathbb{R})$-principal bundle LM_{Spin} of spin frames [43][19].

We denote now by $\widetilde{\mathrm{Diff}}^{m,1} M$ the subgroup of $\mathrm{Diff}_*^{m,1} M$ whose elements are

[19] More precisely we suppose that [43]:

(i) LM_{Spin} is a double covering of LM^+.

(ii) There exist a bundle morphism

$$h: LM_{\mathrm{Spin}} \mapsto LM^+$$

which induces the standard homomorphism $\widetilde{GL}(n, \mathbb{R}) \to GL(n, \mathbb{R})^+$.

diffeomorphisms which lift (in two ways) to LM_{Spin}, i.e. which leave the spin structure invariant and by $\mathscr{A}_{\mathrm{Spin}}$ the space of all spin-connections for metrics in $\mathscr{M}$. The spin structure for each metric is the one induced by LM_{Spin}[20]. Obviously $\widetilde{\mathrm{Diff}}^{m,1} M$ acts freely on $\mathscr{A}_{\mathrm{Spin}}$. So we could also consider the following diagram:

$$\begin{array}{ccc}
\dfrac{LM_{\mathrm{Spin}} \times \mathscr{A}_{\mathrm{Spin}}}{\widetilde{\mathrm{Diff}}^{m,1}_{\mathrm{Spin}} M} \approx \dfrac{LM^{+} \times \mathscr{A}^{\mathrm{metric}}}{\widetilde{\mathrm{Diff}}^{m,1} M} & \xrightarrow{\mathrm{Ev}} & EGL(n, \mathbb{R})^{+} \\
\downarrow & & \downarrow \\
\dfrac{M \times \mathscr{A}_{\mathrm{Spin}}}{\widetilde{\mathrm{Diff}}^{m,1} M} & \xrightarrow{\underline{\mathrm{Ev}}} & BGL(n, \mathbb{R})^{+},
\end{array} \tag{7.17}$$

where $\widetilde{\mathrm{Diff}}^{m,1}_{\mathrm{Spin}} M$ is the double covering of $\widetilde{\mathrm{Diff}}^{m,1} M$ which acts on LM_{Spin}.

By comparing diagrams (7.16) and (7.17), it is apparent that the $\widetilde{\mathrm{Diff}}^{m,1}_{\mathrm{Spin}} M$-anomalies are all obtained from the ordinary diffeomorphism anomalies.

We want now to define the following objects:

$\mathcal{O}_M \ltimes \mathscr{A}^{\mathrm{metric}}$ is the space of all orthogonal frames paired with the corresponding metric connections. (7.18)

$\mathcal{O}_M \ltimes \mathscr{A}^{\mathrm{LC}}$ is defined as before, but only Levi–Civita connections are considered. Obviously $\mathcal{O}_M \ltimes \mathscr{A}^{\mathrm{LC}} \cong \mathcal{O}_M \ltimes \mathscr{M}_{(m)}$, where in the last expression orthonormal frames are paired with their metric [37]. (7.19)

$\mathrm{Spin}_M \ltimes \mathscr{A}_{\mathrm{Spin}}$. It is defined analogously to (7.18). (7.20)

$\mathrm{Spin}_M \ltimes \mathscr{A}^{\mathrm{LC}}_{\mathrm{Spin}}$. It is defined analogously to (7.19) (7.21)

We can then consider alternatively to diagram (7.16) the following diagram:

$$\begin{array}{ccc}
\dfrac{\mathcal{O}_M \ltimes \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1}_{*} M} & \xrightarrow{\mathrm{Ev}} & ESO(n) \\
\downarrow \pi & & \downarrow \pi \\
\dfrac{M \ltimes \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1}_{*} M} & \xrightarrow{\underline{\mathrm{Ev}}} & BSO(n).
\end{array} \tag{7.22}$$

Footnote 19 (*Contd.*)

So for any metric g, LM_{Spin} determines a unique spin structure, i.e. a bundle $\mathrm{Spin}_g(M)$. Two bundles LM_{Spin} and LM'_{Spin} are said to be equivalent if there exists a bundle isomorphism $\psi: LM_{\mathrm{Spin}} \to LM'_{\mathrm{Spin}}$ with $h' \circ \psi = h$.

One can construct as many inequivalent bundles as the number of inequivalent spin structures for a fixed metric g

[20] Let h be the bundle morphism considered in footnote (19). If A is a connection on LM^{+} then A determines a unique connection on LM_{Spin} given by $h_{-}^{-1} h^{*} A$, where $h_{-}: \mathrm{Lie}\,\widetilde{GL}(n, \mathbb{R}) \to \mathrm{Lie}\, GL(n, \mathbb{R})^{+}$. Vice versa if B is a connection on LM_{Spin}, whose horizontal space at u is H^B_u, then we can define a connection on LM^{+} whose horizontal space at $h(u)$ is $h_{*} H^B_u$ ([9], Theorem 5.1). So the space of connections of LM^{+} and the space of connections of LM_{Spin} are isomorphic and $\mathscr{A}_{\mathrm{Spin}} \cong \mathscr{A}^{\mathrm{metric}}$

Analogous diagrams can be constructed for $\mathcal{O}_M \ltimes \mathscr{A}^{\mathrm{LC}}$, $\mathrm{Spin}_M \ltimes \mathscr{A}_{\mathrm{Spin}}$ and for $\mathrm{Spin}_M \ltimes \mathscr{A}^{\mathrm{LC}}_{\mathrm{Spin}}$.

It should be noticed that, when we use diagram (7.22), or the analogous diagram for the Spin case, we can not antitransgress, as in (7.8), the forms given by the family's index theorem, by using the background connection formalism.

In fact there is not a "background connection" on (7.22), because $\mathcal{O}_M \ltimes \mathscr{A}^{\mathrm{metric}}$ is not the product of a principal G-bundle over M times $\mathscr{A}^{\mathrm{metric}}$.

This shows that we can use the background connection formalism, only in the framework given by diagrams (7.16) or (7.17). Notice, by the way, that this is the reason why the background connection considered in Sect. 4 is a generic linear connection.

Gravitational anomalies and their coefficients are obtained by pulling back via $\underline{\mathrm{Ev}}$ characteristic polynomials of (universal) associated vector bundles, i.e. bundles associated to $ESO(n)$ or $E\,\mathrm{Spin}(n)$. The correct choice is obviously to consider $E\,\mathrm{Spin}(n)$ and vector bundles associated to it.

As far as the coefficients of the anomalies are concerned, we observe that in formula (7.15) the form $\hat{A}(M)$ which is by definition $\hat{A}(TM)$, can be identified with the $\hat{A}$-class of the tangent bundle along the fibers relevant to the following trivial bundle:

$$M \times \frac{\mathscr{A}}{\mathscr{G}} \to \frac{\mathscr{A}}{\mathscr{G}}. \tag{7.23}$$

In the gravitational case, instead of (7.23), we have the bundle (see [37]):

$$Z = \frac{M \times \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1}_{*} M} \xrightarrow{\pi} \frac{\mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1}_{*} M}. \tag{7.24}$$

Its tangent bundle along the fibers is in turn given by:

$$Q \equiv \frac{TM \times \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1}_{*} M} \to \frac{M \times \mathscr{A}^{\mathrm{metric}}}{\mathrm{Diff}^{m,1}_{*} M}. \tag{7.25}$$

Notice that we must, as we did before for the gauge case, consider a (compact) submanifold (N-skeleton) of $\mathscr{A}^{\mathrm{metric}}/\mathrm{Diff}^{m,1}_{*} M$, restrict the bundle (7.24) to it, and consider the ensuing restriction of (7.25).

In order to take the locality requirement into account, we have to consider the classes $\hat{A}(Q)$ as represented by forms constructed with the connection induced by the connection $\mathrm{Ev}^* \xi$ (see diagrams (7.16) and (7.17)).

Now the discussion about the coefficients of the anomalies can be done similarly to the gauge case.

It is worth noticing, though, that in the gravitational case we are not any more guaranteed that, given any "local anomaly," we are able to exhibit a manifold M, such that the given anomaly is of topological origin, i.e. determines a non-trivial element of H^1 ($\mathrm{Diff}^{m,1} M$). Hence the use of the family's index theorem in the calculation of gravitational anomalies may need further clarification.

A final remark concerns mixed (gravitational + gauge) anomalies. These anomalies correspond to the "combination" of diagrams (7.16) and (7.14) where the bundle $LM^+ + P$ is considered [9]. Alternatively one can consider the bundle $O_g M$

$+ P$ (or $\mathrm{Spin}_g M + P$) and "combine" diagrams (7.14) and (7.22). If P does not admit a lift for $\mathrm{Diff}\, M$ (see Sect. 1), then the above "combinations" are possible provided that we consider the group $\mathrm{Aut}^m P$ which acts both on P and on LM^+. In fact if j: $\mathrm{Aut}^m P \to \mathrm{Diff}\, M$ is the projection defined in (1.2), then $\forall \psi \in \mathrm{Aut}\, P$, then we have that $l(j(\psi)) \in \mathrm{Aut}\, LM$. So the right invariance group to be considered is the group $J^{-1}(\mathrm{Diff}_*^{m,1} M) \subset \mathrm{Aut}^m P$, which acts freely on $\mathscr{A}_{LM^+ + P} \approx \mathscr{A}_{LM^+} \times \mathscr{A}_P$. By considering the bundle $LM^+ + P$ we are in fact able to obtain all the anomalies (gauge, gravitational, mixed).

C) *Lorentz Anomalies.* Lorentz anomalies could be considered on the same ground as gauge anomalies. They could be seen in fact as the gauge anomalies for the bundle $O_g M$ for a fixed metric g.

By doing so, we would be led to consider the expression:

$$\hat{A}(M)\,\mathrm{ch}(\bar{V}), \tag{7.26}$$

where two metric connections on M are used, namely the Levi–Civita connection which enters $\hat{A}(M)$ and a "variable" metric connection which enters $\mathrm{ch}(\bar{V})$. This is unnatural since a "Lorentz" (i.e. an "orthonormal frame") rotation should affect also the Levi–Civita connection.

As an example consider the Lorentz anomalies, meant as gauge anomalies, for the Spin $\frac{1}{2}$ case. In this case formula (7.26) reduces simply to $\hat{A}(M)$, which is a form on M, i.e. a form on $M \times \mathscr{A}/\mathscr{G}$ whose $(n,2)$-component is zero. So there would be no anomaly. The results of [44] and the locality requirements are rather compatible with the following scheme. We first consider the vector bundle

$$\frac{TM \times \mathscr{A}}{\mathrm{Aut}_v^m O_g M} \to M \times \frac{\mathscr{A}}{\mathrm{Aut}_v^m O_g M}, \tag{7.27}$$

where $\mathscr{A}$ is the space of connections on $O_g M$. The bundle (7.27) is associated to the principal $SO(n)$-bundle:

$$\frac{O_g M \times \mathscr{A}}{\mathrm{Aut}_v^m O_g M} \to M \times \frac{\mathscr{A}}{\mathrm{Aut}_v^m O_g M}, \tag{7.28}$$

which has a connection defined as in (7.11).

We can now construct the diagram:

$$\begin{array}{ccc} \dfrac{O_g M \times \mathscr{A}}{\mathrm{Aut}_v^m O_g M} & \xrightarrow{\mathrm{Ev}} & ESO(n) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ M \times \dfrac{\mathscr{A}}{\mathrm{Aut}_v^m O_g M} & \xrightarrow{\mathrm{Ev}} & BSO(n) \end{array} \tag{7.29}$$

which allows us to pull back cohomology classes of $BSO(n)$ represented by forms depending on the given connection on (7.28) (again the N-skeleton argument is to be considered).

Now we can represent the $\hat{A}$-class of the bundle (7.27) and obtain the correct expression for the Lorentz anomalies with their coefficients.

But it should be pointed out that the bundle (7.27) is not the tangent bundle along the fibers relevant to the bundle $M \times \mathscr{A}/\mathrm{Aut}_v^m O_g M \to \mathscr{A}/\mathrm{Aut}_v^m O_g M$. So care must be exercised in combining the locality requirements with the family's index theorem approach to Lorentz anomalies.

In fact, one should first consider the space $\mathscr{A}$ as the parameter space for the family of chiral Dirac operators. Next, by pulling back forms on $BSO(n)$ via the evaluation map, one can obtain forms representing characteristic classes of the bundle $TM \times \mathscr{A}$ which descend to forms on the bundle (7.27). Notice that, in this case, the tangent bundle along the fibers to $M \times \mathscr{A}$ is of course $TM \times \mathscr{A}$.

The same considerations could be made for the "$\mathrm{Aut}_v LM$-anomalies."

We will come back later to discussing the relation between the Lorentz and the gravitational anomaly.

Both these anomalies are originated by the same cohomology classes of $BSO(n)$. Moreover their coefficients are the same, due to the considerations discussed in the present section.

The difference between them is that Lorentz and gravitational anomalies are obtained by pulling back the same cohomology class(es) over different manifolds. Consequently the antitransgression yields 1-forms defined over different groups.

These observations tell us that if the coefficient of a given Lorentz anomaly is zero, then also the corresponding coefficient of the corresponding gravitational anomaly is zero and vice versa. But the cancellation of the Lorentz anomaly by local counterterms, for a given non-parallelizable manifold, is something different from the cancellation of the gravitational anomaly, as is shown is Sect. 4.

8. Anomaly Cancellation in Ten Dimensions

In their seminal 1984 paper [45], Green and Schwarz discovered, by computing the coefficients of the relevant anomaly (see Sect. 7), that certain field theories formed by coupling a supersymmetric Yang–Mills theory to supergravity in ten dimensions are anomaly free provided the gauge group G is $SO(32)$ or $E_8 \times E_8$.

The field content of these theories coincides with the zero mass spectrum of certain superstring theories, so this fact is assumed to be a signal that the corresponding superstring theories are anomaly free and probably finite.

Actually Green and Schwarz proved this to be the case in some instances at one-loop order in type I superstring theories (e.g. as far as pure gauge anomalies are concerned [45]). In this section we will not address the problem of anomaly cancellation in superstring theories but only in the limiting field theories.

There are other interesting problems concerning such effective theories: can they be formulated in a supersymmetric way? are they ghost free? We will consider here only the anomaly problem: the precise sense in which they are anomaly free will help us specify what the word effective hides more than reveals.

As we will eventually see, if considered as a superstring phenomenon, the Green–Schwarz cancellation scheme contains at least two different mechanisms: one is proper to field theory, the other to sigma-models. In order to disentangle them, it is important to carry out the analysis of anomaly cancellation while bearing in mind a clear distinction between the two cases.

In this section we will study the Green–Schwarz cancellation mechanism in the "category," so to speak, of field theories, that is taking into account the locality (i.e. universality) requirement. The latter renders the theory extremely rigid, the only ordinary method for cancelling anomalies being the subtraction of universal cochains.

In the present case only an extraordinary, very drastic, method is permitted: the imbedding (see also [46]). The theory in such a configuration gives us topological information, especially when we take into account, as we have to, the cancellation of gravitational anomalies.

In the ten dimensional supersymmetric Yang–Mills theory with gauge group $E_8 \times E_8$ or $SO(32)$ coupled to supergravity, the expression for the mixed chiral gauge and Lorentz anomaly[21] simplifies considerably [45]:

$$\mathrm{Anom} = (\mathrm{Anom}_{2G} - \mathrm{Anom}_{2L}) \wedge q, \tag{8.1}$$

where $\mathrm{Anom}_{2G}(\mathrm{Anom}_{2L})$ is the expression for the two dimensional gauge (Lorentz) anomaly (see (2.6)), while q is constructed out of $ESO(n)$- and EG-universal basic forms.

By modifying the Chapline–Manton model [47], Green and Schwarz introduced a 2-form field B and its 3-form "strength" H satisfying

$$H + dB = TK(A) - TK(\omega), \tag{8.2}$$

where A and ω are respectively the gauge and the spin connection; K is the ad-invariant polynomial with 2-entries normalized in such a way that $K(\Gamma^a, \Gamma^b) = 2\delta^{ab}$, where Γ^a are the generators of the vector representation of Lie $SO(n)$[22]. We do not label K with any group label. From the argument it will be clear what group we refer to.

Recall that if δ_G and δ_L are the coboundary operators respectively of $\mathrm{Aut}_v P$ and $\mathrm{Aut}_v OM$, then $\delta_G TK(A) + d\,\mathrm{Anom}_{2G} = 0$ and $\delta_L TK(A) + d\,\mathrm{Anom}_{2L} = 0$. Now H is supposed to be both gauge and Lorentz invariant so if δ is an operator which represents a "global" (i.e. gauge + Lorentz) variation, then the following equation is supposed to be satisfied:

$$\delta B = \mathrm{Anom}_{2L} - \mathrm{Anom}_{2G}. \tag{8.3}$$

Hence the anomaly (8.1) would turn out to be trivial, since $\delta(B \wedge q) = -\mathrm{Anom}$.

If we want an intrinsic mechanism to take care of this cancellation, we have to be careful about (8.2) and (8.3). It was shown in ref. [48] that these equations are meaningless if we do not make any further specification. This follows from the fact

---

[21] In this section we will work mainly with the bundle $O_g M$, instead of the bundle $\mathrm{Spin}_g M$. As far as the cancellation of anomalies is concerned, this does not matter, since $\mathscr{A}^{\mathrm{metric}} = \mathscr{A}_{\mathrm{Spin}}$ (see footnote (20)). In this section we will generally assume that $n = \dim M$ is equal to ten

[22] In the case of $E_8 \times E_8$, Eq. (8.2) should be slightly modified [45]. However we disregard these details here

that $TK(A)$ (respectively $TK(\omega)$) restricted to the fiber of P (respectively $O_g M$), reduces to a non-trivial generator of the third cohomology group of G (respectively $SO(n)$)[23]. Therefore Eq. (8.2) is inconsistent unless the two generators contained in the right-hand side of Eq. (8.2) annihilate each other. The latter possibility can be implemented in the following scheme.

Let M denote our 10-dimensional manifold and let $P(M, G)$ denote, as before, the gauge principal bundle.

We assume that $O_g M$ (for a fixed $g \in \mathcal{M}$) is imbedded into $P(M, G)$[24], that is, we assume that there exists a bundle morphism

$$i : O_g M \to P. \tag{8.4}$$

Let now A_ω denote a connection on P which is reducible to ω (i.e. such that $i^* A_\omega = \omega$).

We can now easily verify that

$$\begin{aligned} TK(A) - TK(A_\omega) &= K(\alpha, F(A)) + K(\alpha, F(A_\omega)) \\ &\quad - \tfrac{1}{6} K(\alpha, [\alpha, \alpha]) + dK(\alpha, A_\omega), \end{aligned} \tag{8.5}$$

where $\alpha \equiv A - A_\omega$. Hence Eq. (8.2) can be satisfied as follows. Let us set

$$\begin{aligned} \tilde{H}(A, A_\omega) &\equiv K(\alpha, F(A)) + K(\alpha, F(A_\omega)) - \tfrac{1}{6} K(\alpha, [\alpha, \alpha]), \\ \tilde{B}(A, A_\omega) &\equiv K(\alpha, A_\omega). \end{aligned} \tag{8.6}$$

Then Eq. (8.5) reads $\tilde{H} + d\tilde{B} = TK(A) - TK(A_\omega)$. Moreover if δ_P is the coboundary operator corresponding to $\mathrm{Aut}_v P$, then we have:

$$\begin{aligned} \delta_P \tilde{H} &= 0, \\ \delta_P \tilde{B} &= K(A, di_{(\cdot)} A) - K(A_\omega, di_{(\cdot)} A). \end{aligned} \tag{8.7}$$

The last equation in (8.7) is the correct interpretation of (8.3). Notice that in (8.7) we are considering only the "gauge ghost field." Remark though, that, due to the imbedding, any infinitesimal Lorentz transformation induces an infinitesimal gauge transformation so, in this framework, the "Lorentz ghost field" is a special kind of "gauge ghost field." Moreover recall that on P we have: $i_{(\cdot)} A = i_{(\cdot)} A_\omega$[25].

Observe that we need not introduce any new field B and H in order to cancel anomalies. However we can also satisfy Eq. (8.2) by introducing the following forms:

$$H \equiv \tilde{H} - dB', \quad B \equiv \tilde{B} + B',$$

where B' is supposed to satisfy the condition $\delta_P B' = 0$. The field B' could be identified

---

[23] In other words, an equation like (8.2) in the bundle $P + O_g M$ would imply that both $TK(A)$ and $TK(\omega)$, restricted to the fibers, are exact, which is clearly impossible

[24] An analogous assumption could be made for $\mathrm{Spin}_g M$ instead of $O_g M$

[25] We may ask whether, by using the background-connection formalism, it is possible to define a $\tilde{B}$ which is basic. The answer is yes, provided that we introduce two background connections A_0 and ω_0 (or A_0 and A_{ω_0}). Recall:

$$TK(A) - TK(A_\omega) = \tilde{H}(A, A_\omega) + dK(A, A_\omega).$$

with the second order antisymmetric tensor field of supergravity or string theory. If M is a complex manifold, in some instances B' could in turn be perhaps identified with the second fundamental form [9, Chap. IX] so that the condition $dB = 0$ is equivalent to saying that M is a Kaehler manifold.

We could also think, in principle, of a different imbedding scheme, in which G and $SO(n)$ are subgroups of a larger group H and $P(M, G)$ and $O_g M$ are reduced bundles of a larger bundle with structure group H. What we have said above would work equally well with the provision that A and ω are both thought as reducible connections on the larger bundle. However there does not exist any simple and natural realization of this more general imbedding scheme[26].

Another possibility would be to consider a group H larger than the holonomy group of (the Levi–Civita connection of) M and contained in $SO(n)$. If we assume that there exists a principal H-bundle Q imbedded both in P and in $O_g M$, then Q would also be imbedded in $P + O_g M$, and we could consider connections on such a

---

Footnote 25 (*Contd.*)
Then,

$$
\begin{aligned}
W_K(A, A_0) &- W_K(A_\omega, A_{\omega 0}) \\
&= TK(A) - TK(A_0) - dK(A, A_0) - TK(A_\omega) + TK(A_{\omega 0}) + dK(A_\omega, A_{\omega 0}) \\
&= \tilde{H}(A, A_\omega) - \tilde{H}(A_0, A_{\omega 0}) + d[K(A - A_{\omega 0}, A_\omega - A_0)] \\
&= \hat{H} + d\hat{B}.
\end{aligned}
$$

Hence we have that:

$$\hat{H} = \tilde{H}(A, A_\omega) - \tilde{H}(A_0, A_{\omega 0})$$

is basic and δ-invariant, while

$$\hat{B} = K(A - A_{\omega 0}, A_\omega - A_0)$$

is basic and generates the anomaly, i.e.

$$\delta[W_K(A, A_0) - W_K(A_\omega, A_{\omega 0})] + d\,\text{Anom} = \delta\hat{B} + d\,\text{Anom} = 0$$

[26] The simplest realization would be $K = G \times SO(n)$ with the natural imbedding, and the large bundle given by $P + O_g M$. However P and $O_g M$ are reduced subbundles of $P + O_g M$ only if there exists a bundle homomorphism from P to $O_g M$ which covers the identity map on M and vice versa. This can be easily arranged only if both P and $O_g M$ are trivial, which is not the case of interest.

It should be pointed out that the bundle $P + O_g M$ is considered in other situations where anomaly cancellation is required. For instance [45, Eq. 22–23] you can have an anomaly of the form

$$(*) \qquad i_{(\cdot)} TQ_1(A) \wedge Q_2(F_\omega, \ldots, F_\omega) - i_{(\cdot)} TQ_2(\omega) \wedge Q_1(F_A, \ldots, F_A),$$

where A, $F(A)$ and ω, $F(\omega)$ are respectively connections (curvatures) on P and $O_g M$, Q_1 and Q_2 are ad-invariant polynomials respectively on G and $SO(n)$ and in the first term (second term) of $(*)$ vectors in $\mathrm{aut}_v P$, $(\mathrm{aut}_v O_g M)$ are considered. By recalling that $\mathrm{aut}_v(P + O_g M) = \mathrm{aut}_v P + \mathrm{aut}_v O_g M$ we notice that the anomaly $(*)$ is cancelled in $P + O_g M$, since it can be written as

$$i_{(\cdot)} d(TQ_1(A) \wedge TQ_2(\omega)).$$

So in order to cancel the anomaly $(*)$, it is enough to consider the bundle $P + O_g M$ and no imbedding is required. But, as is obvious, the anomaly $(*)$ is also cancelled if $O_g M$ is imbedded in P and if ω is replaced by A_ω. So the imbedding is a framework in which the anomaly $(*)$ can be cancelled and Eqs. (8.2), (8.3) are meaningful bundle, obtained from connections on $O_g M$ and from connections on P, which are reducible to connections on Q.

So we could interpret Eq. (8.2) as in equation on $P + O_g M$, provided that we limit ourselves to considering reducible gauge connections. This looks like too strong a constraint.

For these reasons, from now on we will discuss only the imbedding scheme outlined above.

Let us examine its features more closely.

(i) If $i: O_g M \to P$ is an imbedding (for a given $g \in \mathscr{M}$), then there exists an imbedding $i': O_{g'} M \to P$ for any $g' \in \mathscr{M}$. It is enough to observe that $O_g M$ and $O_{g'} M$ are isomorphic; so, if $\rho: O_{g'} M \to O_g M$ is such an isomorphism, then we can set $i' = i \circ \rho$.

(ii) Two imbeddings $i, i': O_g M \to P$ are said to be equivalent if there exists $\psi \in \mathrm{Aut}_v\, O_g M$ with $i = i' \circ \psi$. The equivalence classes of imbeddings are represented by the elements of the homogeneous space: $\mathrm{Aut}_v\, P / \mathrm{Aut}_v\, O_g M$. To see this, let us first identify $\mathrm{Aut}_v\, O_g M$, via a fixed imbedding i_0, with a subgroup of $\mathrm{Aut}_v P$. Next, to any $\psi \in \mathrm{Aut}_v P$, we can associated the imbedding $i = \psi \circ i_0$. Vice versa, to any imbedding i we associate the only element $\psi \in \mathrm{Aut}_v P$, such that, for any $q \in O_g M$, we have: $\psi(i(q)) = i_0(q)$(ψ is defined on $i(O_g M)$ and is extended to P by equivariance);

(iii) Let $\mathscr{A}$ be the space of all connections on P, let $\mathscr{A}_i^{\mathrm{red}} \subset \mathscr{A}$ be the space of all connections on P which are reducible to connections on $i(O_g M)$ and let $\mathscr{A}^{\mathrm{red}} \equiv \bigcup_i \mathscr{A}_i^{\mathrm{red}}$. Obviously $\psi \in \mathrm{Aut}_v P$ and $A \in \mathscr{A}_i^{\mathrm{red}}$ imply $\psi^* A \in \mathscr{A}^{\mathrm{red}}_{\psi^{-1} \circ i}$. So $\mathrm{Aut}_v P$ maps $\mathscr{A}^{\mathrm{red}}$ into itself.

(iv) Let $\hat{\varphi}$ be a G-equivariant function on P with respect to a representation of G, which restricted to $i(O_g M)$ is an $SO(n)$-equivariant function with respect to a given compatible representation of $SO(n)$. Then $\varphi = i^* \hat{\varphi} = \hat{\varphi} \circ i$ is an $SO(n)$-equivariant function in $O_g M$ which corresponds to a section in an associated bundle of $O_g M$. With the same arguments as in (ii), one sees that if $\psi \in \mathrm{Aut}_v P$, then $\psi^{-1*} \hat{\varphi}$ is an equivariant function on $\psi \circ i(O_g M)$.

In a field theory with imbedding, we have the space of gauge connections $\mathscr{A}$, the space of reducible connections $\mathscr{A}^{\mathrm{red}}$ and the space of Lorentz connections $\mathscr{A}^{\mathrm{Lorentz}}$. In this way, apparently we have introduced new degrees of freedom, the imbeddings. However, due to (ii) above, the latter are pure gauge degrees of freedom. So in the theory just outlined, due to (i), (ii), (iii) and (iv) we have two symmetry groups $\mathrm{Aut}_v P$ and $\mathrm{Aut}_v\, O_g M$ which act both on $\mathscr{A}$ and $\mathscr{A}^{\mathrm{red}}$. The anomalies corresponding to these symmetry groups cancel due to the Green–Schwarz mechanism.

Next we have to examine gravitational anomalies.

Then let $\psi \in \mathrm{Diff}\, M$ and let $O_g M$ and $O_{\psi^* g} M$ be the orthonormal frame bundles corresponding to the ψ-related metrics g and $\psi^* g$. In order to have the cancellation of gauge and gravitational anomalies, we have to have not only the imbedding of the bundle $O_g M$ in the gauge bundle P, but we have to have also an "action" of $\mathrm{Diff}\, M$ on the image $i(O_g M) \subset P$. To be more specific, we need for all $\psi \in \mathrm{Diff}\, M$ an imbedding $i_\psi: O_{\psi^{-1*} g} M \to P$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & i(O_gM) \arrow[rr, "\hat{\psi}"] \arrow[dd] & & i_\psi(O_{\psi^{-1*}g}M) \arrow[dd] \\
O_gM \arrow[ur, "i"] \arrow[rr, "l(\psi)"] \arrow[dd] & & O_{\psi^{-1*}g}M \arrow[ur, "i_\psi"] \arrow[dd] & \\
 & M \arrow[rr, "\psi"] & & M \\
M \arrow[ur, "id"] \arrow[rr, "\psi"] & & M \arrow[ur, "id"] &
\end{tikzcd}
\tag{8.8}
$$

Here $l(\psi): O_gM \to O_{\psi^{-1*}g}M$ is the bundle isomorphism obtained by restricting to O_gM the lift of ψ (i.e. $l(\psi) \in \operatorname{Aut} LM$), while $\hat{\psi}$ maps $i(O_gM)$ isomorphically onto $i_\psi(O_{\psi^{*-1}g}M)$. We have now the following theorem.

Theorem (8.9). *Assume that there exists an imbedding $i: O_gM \to P$. One can then construct diagram (8.8) if and only if there is a lift of* Diff M *in* Aut P.

Proof. If such a lift exists and is denoted by $\hat{l}$, then we can define $\hat{\psi} = \hat{l}(\psi)|_{O_gM}$ and $i_\psi = \hat{\psi} \circ i \circ l(\psi)^{-1}$. Vice versa, if we have the commutative diagram (8.8), then we can extend $\hat{\psi}$ by equivariance to P and hence obtain an element $\tilde{\psi} \in \operatorname{Aut} P$.

It is easy to verify that the map $\psi \mapsto \tilde{\psi}$ is a lift. □

From what has been said up to now, we can deduce that, if there is an imbedding of the orthonormal frame bundle into the gauge bundle and if the condition of Theorem (8.9) is satisfied, then gravitational and gauge anomalies of the gauge bundle and of the reduced bundle can compensate for each other. In fact, we can write down the following equation:

$$i_Z((TK(A) - TK(A_\omega)) \wedge q) = i_Z(d\tilde{B} \wedge q), \tag{8.10}$$

which is valid for any vector field Z in aut P. So the anomalies are cancelled since we can write down for the coboundary operator in Aut P:

$$\delta(\tilde{B} \wedge q) = i_{(\cdot)}(d\tilde{B} \wedge q) + di_{(\cdot)}(\tilde{B} \wedge q). \tag{8.11}$$

The price we have to pay is that, in requiring the validity of Theorem (8.9), we put serious limitations on the gauge bundle P; the existence of the lift of Diff M would likely imply that P is trivial (see Sect. 1). That is, it would likely imply that there are no higher dimensional analogs of instantons[27].

Notice that if O_gM is imbedded in a trivial bundle P, then given a flat connection A_0 on P, we can obviously write $\forall X \in \operatorname{diff} M$:

$$\operatorname{aut} P \ni \hat{l}(X) = \hat{l}(X)^v + \hat{l}(X)^h \quad \text{for all } X \in \operatorname{diff} M,$$

[27] It would be very interesting to know precisely whether it is possible or not to construct a non-trivial principal bundle, with structure group $E_8 \times E_8$ or $SO(32)$, which admits a lift for Diff M. If such a bundle exists, it could provide a different framework than the one described here and below. As we mentioned in Sect. 1, according to ref. [11], there are no known examples of such non-trivial bundles

where $\hat{l}$ is defined as in the proof of Theorem (8.9) and $\hat{l}(X)^v$ and $\hat{l}(X)^h$ are, respectively, the A_0-vertical and the A_0-horizontal component of $\hat{l}(X)$. The gravitational anomaly will not depend on $\hat{l}(X)^h$, so, in this situation, we can recover a "gauge interpretation of gravitational anomalies," analogous to the one discussed in Sect. 4[28].

Also notice that if P is a trivial bundle and if $O_g M$ is again imbedded in P, then the Pontrjagin classes of M, which correspond to ad-invariant polynomials on both Lie G and Lie $SO(n)$ are trivial.

In particular the first Pontrjagin class of M must be zero.

Moreover, if $G = SO(32)$ then all the Pontrjagin classes of M are trivial[29].

We know, on the contrary, that E_8 or $E_8 \times E_8$ has fewer irreducible ad-invariant polynomials, namely the first three irreducible polynomials of E_8 are the ones with 2, 8 or 16 entries. So we could have, in principal, a ten dimensional manifold M with second Pontrjagin class different from zero.

Nevertheless, if M is assumed to be the product of a six-dimensional manifold times a four-dimensional one, then the vanishing of the first Pontrjagin class of both manifolds, would imply the vanishing of the second Pontrjagin class of the product manifold.

Finally let again $P(M, G)$ be a trivial bundle and let M^4 be a compact four-dimensional submanifold of M. Then any reduced bundle of $P(M, G)$, restricted to M^4, will have the second Chern class equal to zero, irrespectively of the fact that G is $SO(32)$ or $E_8 \times E_8$.

That is, if P is a trivial bundle, then we do not have "instantons" on any four-dimensional compact submanifold of M.

Finally a word of caution: the imbedding scheme outlined above seems to be a rather extreme condition in which a theory can be defined. So all the above conclusions should be considered as a first attempt to interpret a rather unusual situation in field theory.

A different cancellation mechanism will be presented in [1], but then we will be in the framework of sigma models.

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28 Moreover if $\hat{l}$ is exactly the horizontal lift with respect to A_0, then $\hat{l}(X)^v = 0$, so the gravitational anomaly completely disappears

29 An important set of manifolds which have all Pontrjagin classes trivial, is the set of stably-parallelizable manifolds, which includes all the spheres (and all n-dimensional manifolds imbedded in $\mathbb{R}^{n+1}$)

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