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The Singleton Dipole

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Abstract. The space of solutions of the dipole equation

$$(\square - \tfrac{5}{4}\varrho)^2 \phi(x) = 0,$$

in a $3+2$ de Sitter space with curvature constant ϱ, contains a complete Gupta-Bleuler triplet, consisting of pure gauge modes, physical modes, and "scalar" or "auxiliary" modes. Indefinite metric quantization is carried out precisely as in more conventional gauge theories. The associated Lagrangian and Hamiltonian field theory formulations reveal an interesting interplay between fields on the de Sitter manifold and their boundary values at spatial infinity.

1. Introduction

The physical role of singletons [1], as fundamental constituents of massless particles [2], has recently been extended to include hadrons [3]. The unusual properties of singleton fields [4], that makes them unobservable at least at the classical level, opens up the possibility of unusual statistics, intermediary between classical and Bose-Einstein (or Fermi-Dirac) [3]. In its simplest form, a singleton field theory in de Sitter space provides a composite model that is exactly equivalent to QED, at least in the limit of vanishing curvature. In a second stage massive particles appear and singletons seem ideally suited to assume the role usually assigned to quarks.

Singleton field theory cannot be formulated directly in flat space – not, that is, as a relativistic operator quantum field theory. To achieve an autonomous flat space formulation it is necessary to give up the idea of local quantum field operators as far as the singletons are concerned. But it is possible to construct a field theory in terms of Green's functions, or Feynman rules, in the manner that was attempted about 15 years ago in the context of conformal invariance and operator product expansions. The S-matrix is expressed in terms of n-point functions that are defined on Minkowski space. Two- and three-point functions (and perhaps four-point functions) must be specified a priori, although originally obtained as flat space limits of vacuum expectation values of products of field operators defined on de Sitter space. The constant curvature of de Sitter space thus appears as the parameter of a deformation that makes an operator formulation possible. This is another aspect of the efficient infrared regularization that is introduced through curvature.

The quantization of singleton fields is characterized by an indefinite metric and a local gauge group. Some time ago [4] we carried out quantization associated with the rac (= bosonic singleton) wave equation

$$(\Box + m^2)\phi(x) = 0\,, \qquad m^2 = -\tfrac{5}{4}\varrho\,.$$

The associated propagator is logarithmic and the interpretation becomes less clear than one could hope for (Sect. 2). The main purpose of this paper is to present a better alternative. Logarithms will be avoided by replacing the Klein-Gordon equation by an equation of higher order. The ghost that is introduced thereby is just the degree of freedom that is needed to provide the gauge modes with canonically conjugate field variables. It should be pointed out that the interpretation of the electromagnetic potential as a singleton-composite field is also facilitated by this new formulation of the free singleton theory. This is especially true of the flat space limit, as will be shown elsewhere.

The Klein-Gordon dipole equation

$$(\Box + m^2)^2\phi(x) = 0$$

has been examined many times, mostly in flat space [5, 6]. It demands an indefinite metric Hilbert space, even in flat space, but attempts to devise a physical interpretation have nevertheless not been completely abandoned [6]. The problem, reduced to its most elemental aspect, is the following. There are two kinds of solutions (in flat space):

$$\phi = e^{-ip\cdot x}\,, \qquad \phi' = te^{-ip\cdot x}\,, \qquad p\cdot x = Et - \mathbf{p}\cdot\mathbf{x}\,, \qquad E^2 - \mathbf{p}^2 = m^2\,.$$

The first type satisfies the Klein-Gordon equation and spans an invariant subspace. In canonical quantization, the two types are mutually conjugate and the invariant Hilbert space metric is zero when restricted to the subspace of solutions of the first type. The theory remains unitary when interactions are introduced only if these zero-norm modes remain uncoupled; in other words, interactions must be gauge invariant. The analogy with electrodynamics is striking, with solutions of the second type playing the role of the "scalar" photons. But the physical content of QED rests on the existence of transverse modes, and here the parallel breaks down, for the flat space dipole has nothing that corresponds to transverse modes. In group theoretical terms it is only an indecomposable doublet, Scalar→Gauge, while the Gupta-Bleuler triplet of QED, Scalar→Transverse→Gauge, is the essence of its structure and its physical interpretation.

Curvature brings nothing essentially new as far as this triplet structure of electrodynamics is concerned [7], but the effect on the dipole is dramatic. If ϱ is the de Sitter curvature constant and $m^2 = -\tfrac{5}{4}\varrho$, then the singleton appears as an additional set of solutions of the dipole equation. Group theoretically it takes its place in the middle of an indecomposable representation, Scalar→Singletons

→Gauge of the de Sitter group. The doublet turns into a triplet and becomes a complete gauge theory. One way to study this theory is to examine the dipole propagator. This is not logarithmic, and the interpretation of this new singleton field theory is much more transparent than the one proposed previously [4] (Sect. 3).

Singleton field theory is a gauge theory with a difference: the gauge structure is non-local and is closely associated with boundary conditions [8]. In Sect. 4, we study fourth-order wave equations in general, with special attention to surface terms that arise in the variation of the Lagrangian. The dynamics associated with the boundary is richer than in second order theories, for the fourth-order system on the manifold may couple to a second order system on the boundary. In fact, however, this possibility is realized in one case only: the fourth order equation must be a dipole and the "mass" must be that of the singleton (to ensure the required asymptotic behavior). The singleton dipole is thus a unique system in which the interplay between the space time manifold and its boundary is most intimate.

In Sect. 5, we study the Hamiltonian formulation. An interesting aspect is that, on the "physical subspace" (defined by the "Lorentz condition") the Hamiltonian reduces to a surface integral, which reminds us of the situation of the ADM energy in general relativity.

2. Second Order Wave Equation

The physics of the Klein-Gordon equation is affected by curvature in many ways. The analysis is in some respects simpler with curvature than it is without it. It is enriched by an interesting fine structure in the region of very low mass – squared masses of the order of magnitude of the curvature constant [9].

The Klein-Gordon equation is

$$(\Box + q\varrho)\phi = 0, \qquad \Box \equiv \partial_\mu g^{\mu\nu}\partial_\nu .$$

The mass term is expressed as a multiple q of the curvature constant ϱ. The simplest expressions for the metric components are

$$g^{00} = (1+\varrho r^2)^{-1}, \qquad -g^{ij} = \delta^{ij} + \varrho x^i x^j, \qquad g^{0i} = 0 .$$

The (intrinsic) coordinates $t, \mathbf{x}$ are global and $r = |\mathbf{x}|$.

Though the wave equation can easily be solved directly, one gains more insight, and more quickly, by evaluating the two-point function. Let $\{\phi^i\}, i = 1, 2, \ldots$ be any complete set of solutions, then the associated two-point function is

$$D(x, x') = \sum_i \pm \phi^i(x)\bar{\phi}^i(x') \qquad (x = \mathbf{x}, t),$$

in which $\bar{\phi}$ denotes the complex conjugate of ϕ. If ϕ^i is a stationary solution with well defined angular momentum, then the sum will be referred to as the Fourier expansion of the function D. Canonical quantization introduces the hermitian quantum field operator

$$\hat{\phi}(x) = \sum_i (\phi^i(x) a_i + \bar{\phi}^i(x) a_i^*),$$

and the vacuum state $|0\rangle$, with $a_i|0\rangle=0$. The physical interpretation requires that the solutions ϕ^i, and consequently the one-particle states $a_i^*|0\rangle$, have positive energy. The two-point function is the vacuum expectation value

$$D(x,x')=\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle .$$

All important physical characteristics of the chosen set of solutions of the wave equation (the "modes"), including positivity, causality, unitarity, and relativistic invariance, are easily obtained by inspection of this function.

The requirement of relativistic (i.e. de Sitter) invariance goes far towards fixing the two-point function. To impose it, it is very convenient to interpret space time as (a covering of) a 4-dimensional hyperboloid

$$y^2\equiv y_0^2+y_5^2-\mathbf{y}^2=\varrho^{-1}$$

in $3+2$ dimensional pseudo-Euclidean space. The intrinsic coordinates $\mathbf{x},t$ are related to (y_α), $\alpha=0,1,2,3,5$ by

$$y_5+iy_0=Ye^{i\tau},\qquad \mathbf{y}=\mathbf{x},\qquad Y\equiv(\varrho^{-1}+r^2)^{1/2},\qquad \tau\equiv t\varrho^{1/2}.$$

In terms of y_α, the wave operator takes the form

$$\Box=\partial^2-\varrho\hat{N}(\hat{N}+3),\qquad \hat{N}\equiv y^\alpha\partial_\alpha .$$

The two-point function is a relativistic invariant if and only if it can be expressed as a function of the invariant pseudo-Euclidean scalar product $z\equiv\varrho y\cdot y'$. The two-point function satisfies the wave equation, and if it is relativistically invariant, it may be regarded as a (generalized) function of one variable,

$$D(z)=\sum_i \pm\phi^i(y)\bar{\phi}^i(y') .$$

The wave equation then reduces to

$$[(1-z^2)\partial_z^2-4z\partial_z+q]D(z)=0 .$$

We now study this hypergeometric differential equation.

Consider the expansion

$$D(z)=\sum_{n=0}^{\infty} a_n z^{-\lambda-2n} .$$

The indicial equation gives

$$\lambda=\lambda_\pm=\tfrac{3}{2}\pm(q+\tfrac{9}{4})^{1/2},$$

and the recursion relation for the coefficients is

$$n(n+1)a_n=(n+\tfrac{1}{4})(n+\tfrac{3}{4})a_{n-1},\qquad n\geqq 1 .$$

Now we can investigate the unitarity of the theory.

Wave propagation is unitary if the two-point function is the integral kernel of a positive operator. We shall show that this implies that λ must satisfy the bound $\lambda\geqq\frac{1}{2}$. In fact, the requirement that the one-particle states $a_i^*|0\rangle$ have positive energy means that the so far ill defined generalized function $z^{-\lambda-n}$ must be interpreted as a distribution with positive frequencies, namely

$$z^{-\nu}=(\varrho YY'/2)^{-\nu}\sum_{k=0}^{\infty} e^{-i\tau(\nu+k)}C_k^\nu(\mathbf{y}\cdot\mathbf{y}'/YY').$$

The functions C_k^ν are Gegenbauer polynomials, and they have the following expansion in terms of Legendre functions:

$$C_k^\nu(x)=\frac{1}{(2\nu-2)!}\sum_l \frac{(k+2\nu-l-3)!!\,(k+2\nu+l-2)!!}{(k-l)!!\,(k+l+1)!!}(2l+1)P_l(x).$$

The summation is over $l=k, k-2, \ldots, 1$ or 0. The distribution $z^{-\nu}$ is positive if and only if all the coefficients in this series are positive or zero, which implies that $\nu\geqq 1/2$. This must hold for $\nu=\lambda+n$, $n\geqq 0$, so we must have $\lambda\geqq 1/2$.

The case of interest is just the limiting case:

$$q=-\tfrac{5}{4},\quad \lambda_-=\tfrac{1}{2},\quad \lambda_+=\tfrac{5}{2}.$$

It may be seen that the lowest energy that occurs in the Fourier series is λ. The singleton has minimum energy $E_0=1/2$, so we would expect a two-point function of the type $\sum a_n z^{-(1/2)-2n}$. However, $\lambda_+-\lambda_-$ is an integer. The general theory of differential equations predicts that only the larger of the two solutions of the indicial equation may give rise to a power series that solves the differential equation. In fact, if $\lambda=\lambda_+=5/2$, we get the two-point function

$$D_+(z)=z^{-5/2}\,{}_2F_1(\tfrac{5}{4},\tfrac{7}{4};2;z^{-2}),$$

but if $\lambda=\lambda_-=\frac{1}{2}$ the recursion relation fails already for $n=0$. The other solution,

$$D_-(z)=\sum_{n=0}^{\infty} b_n z^{-(1/2)-2n}+\ln z D_+(z),$$

is unavoidably logarithmic.

The propagator $D_+(z)$ contains modes with lowest energy 5/2, and therefore does not contain the singleton. The logarithm in $D_-(z)$ complicates the picture, for the expansion

$$D_-(z)=\sum_i \pm\phi^i(y)\bar{\phi}^i(y')$$

includes mode functions of the form te^{iEt}, and the space of one particle states does not have a basis consisting of energy eigenstates. It is nevertheless possible to quantize the theory, as was shown elsewhere [4], but here we shall find an alternative treatment that avoids logarithms and is more interesting in other ways as well.

3. Fourth Order Wave Equation

If Δ is an ordinary, second order differential operator, for which the origin is a regular singular point, consider the equations $\Delta f(x)=0$ and $\Delta^2 g(x)=0$, in a neighborhood of $x=0$. Let λ_+ and λ_- be the solutions of the indicial equation. If the difference $\lambda_+-\lambda_-$ is non-integral, then the first equation will have two solutions by power series:

$$f_+(x)=x^{\lambda_+}\sum_{n=0}^{\infty} a_n x^n\,,\qquad f_-(x)=x^{\lambda_-}\sum_{n=0}^{\infty} b_n x^n\,.$$

For the other, fourth order equation, λ_+ and λ_- will be double solutions of the indicial equation, and the two additional solutions will normally be logarithmic.

Now suppose that $s=\lambda_+-\lambda_-$ is a non-negative integer. The power series solution $f_+(x)$ will still exist, but $f_-(x)$ will be replaced by a logarithmic function. On the other hand, one will gain an additional power series solution of the fourth order equation. In fact, set

$$g(x)=x^{\lambda_-}\sum_{n=0}^{\infty} g_n x^n=x^{\lambda_+}\sum_{n=-s}^{\infty} g_{n+s}x^n\,,$$

then it is always possible to choose the coefficients so that

$$\Delta g(x)=f_+(x)\,,\quad \therefore\ \Delta^2 g(x)=0\,.$$

Consequently, we can avoid logarithms in our two-point function if we replace our wave operator by its square.

Taking $\Delta=\Box-\frac{5}{4}\varrho$, we get the following recursion relation for g_n $(x=z^{-2})$

$$n(n+1)g_{n+1}-(n+\tfrac{1}{4})(n+\tfrac{3}{4})g_n=-\tfrac{1}{4}a_n\,,\qquad n\geqq 0\,.$$

The recursion relation for a_n, with $\lambda=5/2$, is

$$n(n+1)a_n=(n+\tfrac{1}{4})(n+\tfrac{3}{4})a_{n-1}\,,$$

and this allows us to rewrite our first equation as

$$c_n-c_{n+1}=1/n(n+1)\,,\qquad c_n\equiv g_n/a_{n-1}\,.$$

A solution is

$$c_n=-\sum_{l=1}^{n-1}\frac{1}{l(l+1)}=-\frac{n-1}{n}\,,$$

which leads to the following two-point function,

$$D(z)=z^{-1/2}\,{}_2F_1(\tfrac{1}{4},\tfrac{3}{4};1;z^{-2})\,, \tag{3.1}$$

for the wave equation

$$(\Box-\tfrac{5}{4}\varrho)^2\phi(x)=0\,.$$

This is the wave equation that will be used from now on. The non-logarithmic two-point function is unique except for the possibility of adding a constant multiple of $D_+(z)$. As we shall see, this corresponds to a change of gauge.

The first two terms in the expansion of $D(z)$ are

$$D(z)=z^{-1/2}+\tfrac{3}{16}z^{-5/2}+\ldots\,.$$

The lowest energies are contained in

$$z^{-1/2}=(\varrho YY'/2)^{-1/2}\{e^{-i(\tau-\tau')/2}+P_1e^{-3i(\tau-\tau')/2}+2P_2e^{-5i(\tau-\tau')/2}+\ldots\}\,,$$
$$z^{-5/2}=(\varrho YY'/2)^{-5/2}\{e^{-5i(\tau-\tau')/2}+\ldots\}\,.$$

The argument of the Legendre functions is

$$\mathbf{y}\cdot\mathbf{y}'/YY'=\frac{\mathbf{x}\cdot\mathbf{x}'}{rr'}\left(1+\frac{1}{\varrho r^2}\right)^{-1/2}\left(1+\frac{1}{\varrho r'^2}\right)^{-1/2};$$

consequently, the P_2-term conceals a contribution with angular momentum zero. The contributions with energy 5/2 and angular momentum zero are of two kinds,

$$(YY')^{-5/2}e^{-5i(\tau-\tau')/2} \quad \text{and} \quad (r^2+r'^2)(YY')^{-5/2}e^{-5i(\tau-\tau')/2}.$$

This bespeaks the presence of two types of modes, namely

$$\phi_1=(y_5+iy_0)^{-5/2}, \qquad \phi_2=r^2(y_5+iy_0)^{-5/2}.$$

The first satisfies the second order wave equation and appears as the contribution of lowest energy in $D_+(z)$; the other satisfies

$$(\Box-\tfrac{5}{4}\varrho)\phi_2\propto\phi_1,$$

and the squared wave operator thus annihilates it.

We are now in a position to describe the space $\mathscr{V}'$ of one-particle states associated with the fourth order wave equation. It is precisely the space of modes that occur in the Fourier expansion of $D(z)$.

Let us begin with the subspace $\mathscr{V}_g$ of $\mathscr{V}'$ that consists of those modes that satisfy the second order wave equation and appear in the expansion of $D_+(z)$. The lowest energy in this space is 5/2, and this space carries the unitarizable, irreducible representation $D(5/2,0)$ of so(3, 2). We observe that all these modes satisfy

$$\lim_{r\to\infty} r^{1/2}\phi(\mathbf{x},t)=0 \quad \text{(Gauge modes)}.$$

Next, consider the subspace $\mathscr{V}$ of $\mathscr{V}'$ that consists of those modes that satisfy the second order wave equation, whether they contribute to $D_+(z)$ or not. The lowest energy mode in this space has energy 1/2, and consequently $\mathscr{V}$ contains modes in addition to those that appear in $\mathscr{V}_g$. However, these additional modes do not form an invariant subspace complementing $\mathscr{V}_g$. The structure of $\mathscr{V}$ is precisely analogous to the space of modes of the electromagnetic potential, after the Lorentz condition has been imposed. The invariant subspace $\mathscr{V}_g$ corresponds to longitudinal modes and the complement is analogous to the space of transverse modes. The representation of so(3, 2) realized in $\mathscr{V}$ is the nondecomposable $D(1/2,0)\to D(5/2,0)$. If we define $\tilde{\phi}$, a function of t and the angles, by

$$\tilde{\phi}(t,\Omega)=\lim_{r\to\infty} r^{1/2}\phi(\mathbf{x},t),$$

then the contribution from the gauge modes is removed, and we obtain the irreducible singleton representation $D(1/2,0)$, realized in terms of functions on the cone at infinity.

The representations $D(E_0,s)$ considered so far are algebraic representations of so(3, 2). The corresponding unitary representations of SO(3, 2) will be described below.

We have seen that $D_+(z)$ is the only invariant two-point function that satisfies the second order wave equation, and that it does not contain the singleton modes.

Table 1. Comparison between the indefinite metric structures of electrodynamics (in flat space) and the singleton dipole (in de Sitter space). The representation $D(0,\lambda)$ of the Poincaré group has mass zero and helicity λ

		QED (Flat space)	Singleton dipole (de Sitter space)	Equations hold in
Wave equation		$\Box A_\mu = 0$	$(\Box - \frac{5}{4}\varrho)^2\phi = 0$	$\mathscr{V}'$
Lorentz cond.		$\partial \cdot A = 0$	$(\Box - \frac{5}{4}\varrho)\phi = 0$	$\mathscr{V}$
Gauge modes		$F_{\mu\nu} = 0$	$\lim r^{1/2}\phi(x,t) = 0$	$\mathscr{V}_g$
Group		Poincaré	SO(3, 2)	Representation space
Re-	Scalar	$D(0,0)$	$D(5/2,0)$	
pre-		$\downarrow$	$\downarrow$	$\mathscr{V}'$
sen-	Physical	$D(0,1)\oplus D(0,-1)$	$D(1/2,0)$	$\mathscr{V}$
ta-		$\downarrow$	$\downarrow$	
tion	Gauge	$D(0,0)$	$D(5/2,0)$	$\mathscr{V}_g$

The situation is precisely as in electrodynamics with the Lorentz condition imposed. The two-point function $D(z)$ contains the singleton modes and, besides, additional modes that do not satisfy the "Lorentz condition;" that is, the second order wave equation. These modes of $\mathscr{V}'$ complement $\mathscr{V}$, but do not form an invariant subspace. The total representation of so(3, 2) in $\mathscr{V}'$ is $D(5/2,0) \to D(1/2,0) \to D(5/2,0)$. See Table 1 for a detailed comparison with electrodynamics.

The "function" $D(z)$ is actually the integral kernel of an operator; it is not positive. The mode expansion has the form

$$D(z) = \sum_i [\phi_g{}^i(y)\bar{\phi}_s{}^i(y') + \phi_s{}^i(y)\bar{\phi}_g{}^i(y') + \phi_g{}^i(y)\bar{\phi}_g{}^i(y')] + \sum_i \phi_p{}^i(y)\bar{\phi}_p{}^i(y'), \quad (3.2)$$

where the subscripts refer to *g*auge, *s*calar, and *p*hysical modes. Unitarity is saved in the usual way: the "Lorentz condition" (the second order wave equation) must be imposed on the physical asymptotic states, and the gauge modes must be decoupled by gauge invariance. Thus everything decouples except the singleton modes.

The flat space dipole equation $(\Box + m^2)^2\phi = 0$ can be handled in analogous fashion, but in that case the modes that remain after imposing the "Lorentz condition" $(\Box + m^2)\phi = 0$ are all lost to gauge invariance. The total representation is $D(0,0) \to D(0,0)$; there is nothing that corresponds to the singleton middle member in the triplet $D(5/2,0) \to D(1/2,0) \to D(5/2,0)$, and no physics.

4. Lagrangian Field Theory

Singleton field theory is a gauge theory with a difference: the separation of gauge modes is non-local. A pure gauge field is characterized by the fact that it falls off faster than $r^{-1/2}$ when r tends to infinity [4]. The pure, on-shell gauge field can also be characterized by the inequality $E \geq l + \frac{5}{2}$; the physical singleton modes have energy $E=l+\frac{1}{2}$ [1]. (Energy is measured in units of $\varrho^{1/2}$.) But this is also essentially non-local, for any cut-off in volume (or in time) will mask the singleton component. A most striking characteristic of singletons is their extremely poor localizability. The boundary surface at spatial infinity must be taken more seriously than in other field theories. This is also indicated by the extremely slow rate at which the physical modes decrease towards infinity. Finally, curvature is essentially a volume cut-off, and physics on the boundary is the infrared problem [10].

Since the singleton wave operator is a fourth order differential operator, we consider briefly the problem of setting up a canonical formalism for an equation of the type

$$(A\partial_t^4+B\partial_t^2+C)\phi=0\,, \tag{4.1}$$

in which A is a real function, B and C are formally self-adjoint differential operators, and all three are independent of the time. As Lagrangian density we try the general expression ($\dot\phi=\partial\phi/\partial t$)

$$\mathscr{L}=\dot\phi a\dot\psi+\psi b\psi+\phi c\psi+\dot\phi d\dot\phi+\phi e\phi\,.$$

Here $a,b,\ldots,e$ are either real functions or formally self-adjoint differential operators, independent of the time. The dot denotes the time derivative, and the field ψ is introduced to cover for $\ddot\phi$.

The Euler-Lagrange equations are

$$-a\ddot\psi+c\psi-2d\ddot\phi+2e\phi=0\,,\qquad -a\ddot\phi+2b\psi+c\phi=0\,.$$

In order that ψ can be eliminated in favor of $\ddot\phi$, we need that $a,b\neq 0$, and we take $a=2b=A$ with no essential loss of generality. (Only the scale of ψ is fixed.) Thus $A\psi=A\ddot\phi-c\phi$. Eliminating ψ we are left with the Euler-Lagrange equation for ϕ, and for this to take the form (4.1) we must fix

$$d=c+\tfrac{1}{2}B\,,\qquad e=\tfrac{1}{2}(cA^{-1}c-C)\,,$$

while c remains arbitrary. The resulting Lagrange density is, up to surface terms (that alone depend on the choice of c)

$$\dot\phi A\dot\phi+\tfrac{1}{2}\ddot\phi A\ddot\phi+\tfrac{1}{2}\dot\phi B\dot\phi-\tfrac{1}{2}\phi C\phi\,.$$

It remains to discuss the surface terms.

Consider the case of the invariant wave equation

$$(\Box^2+2u\Box+v)\phi=0\,,\qquad \Box\equiv\partial_\mu g^{\mu\nu}\partial_\nu\,, \tag{4.2}$$

in which u,v are constants. Invariance limits the choice of surface terms but still leaves some ambiguity. We consider

$$\begin{aligned}
&\mathscr{L}=\alpha\mathscr{L}_4+\beta\mathscr{L}_3,\\
&\mathscr{L}_4=(-g)^{1/2}\{g^{\mu\nu}\phi_\mu\psi_\nu-u\phi\psi+\tfrac{1}{2}\psi\psi+\tfrac{1}{2}w\phi\phi\}\,,\\
&\mathscr{L}_3=\tfrac{1}{3}\tilde\Box\{\tfrac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu+\tfrac{1}{2}\gamma_1\phi\phi+\gamma_2\phi\psi\}\,,\\
&\phi_\mu\equiv\partial_\mu\phi\,,\qquad w\equiv u^2-v\,,\qquad \tilde\Box\equiv\partial_\mu(-g)^{1/2}g^{\mu\nu}\partial_\nu\,,
\end{aligned}\tag{4.3}$$

where α, β, γ are real parameters. The variational principle is

$$\delta L = 0\,, \qquad L = \alpha L_4 + \beta L_3\,, \qquad L_{4,3} = \int d^4x \mathscr{L}_{4,3}\,.$$

For the time being, we look at ψ as shorthand for $(\Box + u)\phi$, and evaluate

$$\begin{aligned}\delta L_4 = &\int d^4x(-g)^{1/2}\{\delta\psi(\psi - \Box\phi - u\phi) - \delta\phi(\Box\psi + u\psi - w\phi)\}\\ &+ \int d^4x\partial_\mu(-g)^{1/2}g^{\mu\nu}\{\delta\psi\phi_\nu + \psi_\nu\delta\phi\}\,.\end{aligned} \tag{4.4}$$

The second term is a surface term. If this term could be ignored, then we could vary ϕ and ψ independently, since independent variation of ψ would give the constraint $\psi = (\Box + u)\phi$. The surface term is

$$\int d^4x\partial_\mu(-g)^{1/2}g^{\mu\nu}F_\nu\,, \tag{4.5}$$

$$F_\nu = \phi_\nu(\Box + u)\delta\phi + \delta\phi_\nu(\Box + u)\phi\,. \tag{4.6}$$

We are concerned with the behavior of F_r as $r \to \infty$. To investigate this we specialize the metric.

In de Sitter space

$$\begin{aligned}&\Box = (\varrho^{-1} + r^2)^{-1}\partial_\tau^2 + r^{-2}[\mathbf{L}^2 - \hat{N}(\hat{N}+1)] - \varrho\hat{N}(\hat{N}+3)\,,\\ &\hat{N} \equiv y^\alpha\partial_\alpha = r\partial_r\,, \qquad \tau = \varrho^{1/2}t\,,\end{aligned} \tag{4.7}$$

in which $\mathbf{L}^2$ is the operator of total angular momentum. The integral (4.5) reduces to

$$\int d^4x\partial_\mu(-g)^{1/2}g^{\mu\nu}F_\nu = \int dt d\Omega \lim_{r\to\infty} r^2 g^{rr}F_r\,,$$

$$\lim_{r\to\infty} r^2 g^{rr}F_r = \lim_{r\to\infty}(-\varrho r^4 F_r)\,.$$

Suppose that there is real N such that

$$\lim r^{-N}\phi(x) = \tilde{\phi}(t, \Omega)\,;$$

then the leading contribution to r^4F_r is

$$2Nr^3[u - \varrho N(N+3)]\delta\phi\phi\,,$$

and this must be finite. One way to ensure this is to have ϕ fall off faster than $r^{-3/2}$ as $r \to \infty$. The more interesting possibility is to choose

$$u = \varrho N(N+3)\,. \tag{4.8}$$

The next leading contribution to r^4F_r is now finite, provided

$$N = -\tfrac{1}{2}\,, \qquad u = -\tfrac{5}{4}\varrho\,. \tag{4.9}$$

In this case $\phi \sim r^{-1/2}$, $\psi = (\Box + u)\phi \sim r^{-5/2}$, and

$$\lim r^4 F_r = -\tfrac{1}{2}(\tilde{\phi}V\delta\tilde{\phi} + \delta\tilde{\phi}V\tilde{\phi})\,, \tag{4.10}$$

$$V = \lim r^2(\Box + u) = \partial_\tau^2 + \mathbf{L}^2 + \tfrac{1}{4}\,. \tag{4.11}$$

We adopt (4.9) from now on.

Variations that vanish on the boundary give us

$$[(\Box + u)^2 - w]\phi(x) = 0, \qquad r < \infty,$$

valid in the interior of de Sitter space. In order that this equation admit solutions that fall off as $r^{-1/2}$ when $r \to \infty$, it is neccessary that $w = u^2 - v$ is zero. We suppose so from now on and thus end up with the singleton dipole equation

$$(\Box - \tfrac{5}{4}\varrho)^2 \phi(x) = 0, \qquad r < \infty,$$

in the interior.

Returning to (4.4), we shall now regard ϕ and ψ as two independent fields. Having fixed the asymptotic behavior of ϕ and ψ, we can write (4.4) as follows:

$$\delta L_4 = \int d^4x(-g)^{1/2}\{\delta\psi(\psi - \Box\phi - u\phi) - \delta\phi(\Box + u)\psi\} + \tfrac{1}{2}\int d^4x(-g)^{1/2}\delta_\infty(x)\{\delta\psi(x)\phi(x) + 5\psi(x)\delta\phi(x)\}. \tag{4.12}$$

Here δ_∞ is the distribution, defined for functions F that satisfy

$$\lim_{r\to\infty} r^3 F(x) = \tilde{F}(t, \Omega),$$

by

$$\int d^4x(-g)^{1/2}\delta_\infty(x)F(x) = \tfrac{1}{3}\int d^4x \tilde{\Box} F(x) = \varrho \int dt d\Omega \tilde{F}(t, \Omega). \tag{4.13}$$

The coefficient of $\delta\psi$ in δL_4 gives us

$$\psi = (\Box + u)\phi - \tfrac{1}{2}\delta_\infty(x)\phi(x) \qquad (\beta = 0). \tag{4.14}$$

This is problematical.

It does not seem possible to live with the last term in (4.14), for if we try to use this equation to eliminate ψ, then we encounter $\delta_\infty(x)\delta_\infty(x)$. We avoid this problem by including the surface term L_3 in L. First, we shall show that $\gamma_1 = \varrho/4$. In fact,

$$g^{\mu\nu}\phi_\mu\phi_\nu = (\varrho^{-1} + r^2)^{-1}\dot{\phi}\dot{\phi} - (\delta^{ij} + \varrho r^i r^j)\partial_i\phi\partial_j\phi$$
$$\simeq r^{-2}\{\dot{\phi}\dot{\phi} - (r^2\delta^{ij} - r^i r^j)\partial_i\phi\partial_j\phi - \tfrac{1}{4}\phi\phi\} - (\varrho/4)\phi\phi.$$

The last term decreases too slowly and must be compensated by $\gamma_1\phi\phi$, so $\gamma_1 = \varrho/4$. This gives

$$\delta L_3 = \int d^4x(-g)^{1/2}\delta_\infty(x)\{-\delta\phi(\Box + u)\phi + \gamma_2(\phi\delta\psi + \psi\delta\phi)\}.$$

The coefficients of $\delta\psi$ and $\delta\phi$ in $\delta L = 0$ are

$$\alpha[\psi - \Box\phi - u\phi + \tfrac{1}{2}\delta_\infty\phi] + \beta\gamma_2\delta_\infty\phi = 0,$$
$$\alpha[-(\Box + u)\psi + \tfrac{5}{2}\delta_\infty\psi] + \beta\delta_\infty[\gamma_2\psi - (\Box + u)\phi] = 0.$$

Thus $\beta\gamma_2$ must be equal to $-\alpha/2$ to avoid a surface term in the expression for ψ, and β must be 2α in order that ϕ satisfy the dipole equation: $\beta = 2\alpha$, $\gamma_2 = -\tfrac{1}{4}$, in which case the variational equations become

$$\psi = (\Box + u)\phi, \qquad (\Box + u)^2\phi = 0,$$

with $u = -\tfrac{5}{4}\varrho$.

The Lagrangian is thus finally fixed up to overall scale,

$$L=\alpha\int d^4x(-g)^{1/2}\{g^{\mu\nu}\phi_\mu\psi_\nu-u\phi\psi+\tfrac{1}{2}\psi\psi\}+\frac{\alpha}{3}\int d^4x\tilde{\Box}\left\{g^{\mu\nu}\phi_\mu\phi_\nu+\frac{\varrho}{4}\phi\phi-\frac{1}{2}\phi\psi\right\}.$$

In this form it is ready for passage to the Hamiltonian formalism in the next section. Another form is

$$L=-\frac{\alpha}{2}\int d^4x(-g)^{1/2}\{(\Box+u)\phi\}^2+\frac{\alpha}{3}\int d^4x\tilde{\Box}\left\{g^{\mu\nu}\phi_\mu\phi_\nu+\frac{\varrho}{4}\phi\phi\right\}.$$

5. The Hamiltonian

The singleton dipole is a model of a system for which the boundary is of exceptional dynamical importance. It is of some interest to cast it in Hamiltonian form. We note that the boundary also plays a certain role in massless field theories, and that boundary terms in the Hamiltonian have been investigated in that context [8].

The momenta conjugate to ϕ and ψ are

$$\pi=\frac{\delta}{\delta\dot{\phi}}L=\alpha(-g)^{1/2}\{g^{0\mu}\psi_\mu+2\delta_\infty g^{0\mu}\phi_\mu\},\qquad \pi'=\frac{\delta}{\delta\dot{\psi}}L=\alpha(-g)^{1/2}g^{0\mu}\phi_\mu.$$

The Hamiltonian is $H=\int d^3x(-g)^{1/2}\mathscr{H}$, with

$$\mathscr{H}=\dot{\phi}\pi+\dot{\psi}\pi'-\mathscr{L}=\alpha\{g^{00}\dot{\phi}\dot{\psi}-g^{ij}\phi_i\psi_j+u\phi\psi-\tfrac{1}{2}\psi\psi\}$$
$$+\alpha\delta_\infty\left\{g^{00}\dot{\phi}\dot{\phi}-g^{ij}\phi_i\phi_j-\frac{\varrho}{4}\phi\phi+\frac{1}{2}\phi\psi\right\}.$$

We verify that

$$\frac{d}{d\tau}\mathscr{H}=\alpha\{\dot{\psi}(\Box\phi+u\phi-\psi)+\dot{\phi}(\Box+u)\psi\}+2\alpha\delta_\infty\dot{\phi}(\Box\phi+u\phi-\psi)=0$$

by virtue of the field equations.

Originally, in the context of the homogeneous, fourth order wave equation, the Lorentz condition had the effect of eliminating the "scalar" modes. One should expect that the only contribution to the Hamiltonian that remains, when the space is cut down by the Lorentz condition, is that of physical singleton modes (since the gauge modes have zero norm). But the gauge modes uncouple only at the boundary, so the implication is that the Hamiltonian must reduce to a surface integral. Indeed, the Lorentz condition is simply $\psi=0$, and then

$$H|_{\text{Lorentz}}=\alpha\int d^3x(-g)^{1/2}\delta_\infty\left\{g^{00}\dot{\phi}\dot{\phi}-g^{ij}\phi_i\phi_j-\frac{\varrho}{4}\phi\phi\right\}$$
$$=\alpha\varrho\int d\Omega\{(\partial_\tau\tilde{\phi})^2+\tilde{\phi}(\mathbf{L}^2+\tfrac{1}{4})\tilde{\phi}\}.$$

It remains to determine the scale.

The free wave equation for $\tilde{\phi}$ is

$$\nabla\tilde{\phi}(t,\Omega)=\{\partial_\tau^2+\mathbf{L}^2+\tfrac{1}{4}\}\tilde{\phi}(t,\Omega)=0.$$

The solutions

$$\tilde{\phi}_{lm} = [2\varrho^{1/2}(l+\tfrac{1}{2})]^{-1/2} e^{-i\tau(l+1/2)} Y_{lm}(\Omega)$$

are normalized to

$$\int d\Omega \tilde{\phi}^*_{lm} i\overleftrightarrow{\partial}_t \tilde{\phi}_{l'm'} = \delta_{ll'}\delta_{mm'}\,.$$

The expectation value of H on one of these states is $\alpha\varrho^{1/2}(l+1/2)$ so if H generates $i\partial_t$ then $\alpha=1$. The reproducing kernel is

$$\sum_{lm} \tilde{\phi}_{lm}(t,\Omega)\tilde{\phi}^*_{lm}(t',\Omega') = \varrho^{-1/2}\sum_{lm} e^{-i\tau(l+1/2)}\frac{1}{2l+1} Y_{lm}(\Omega)\overline{Y_{lm}(\Omega')}$$

$$= \frac{1}{8\pi}\lim_{r\to\infty}(rr'/\varrho y\cdot y')^{1/2} = \frac{1}{8\pi}\lim_{r\to\infty}(rr')^{1/2}D(z)\,,$$

where $D(z)$ is the two-point function (3.1).

We end with the following two remarks:

(a) Among all possible free fourth-order equations (in flat or de Sitter spaces) for scalar fields, it is the Rac dipole equation – and only this one – that gives rise to a unitary theory.

(b) The formula for the energy of the Rac field as a surface term at infinity has a strong analogy with the ADM [11] formula of the energy of the gravitational field. One can deduce a similar formula for the Fermi singleton of the de Sitter group – the Di. This suggests very strongly that the mysterious spinor field that appears in the analog of Witten's demonstration [12] of the positivity of the energy in general relativity, for the case of de Sitter boundary conditions, is exactly the Di singleton.

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