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Supermanifold Cohomology and the Wess-Zumino Term of the Covariant Superstring Action

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Abstract. The cohomology theory of supermanifolds is developed. Its basic properties are established and simple examples given. The Wess-Zumino term in the Green-Schwarz covariant superstring action is interpreted as a nontrivial class in the "supersymmetric cohomology" of flat superspace. A quotient supermanifold with nontrivial topology reflecting this class is constructed. It is shown that there is no topological quantization condition for the coefficient of the Wess-Zumino term. The superstring differs from conventional sigma models in this respect because its action is Grassmann-valued and its group manifold (superspace) is noncompact.

1. Introduction

The covariant action discovered by Green and Schwarz for the superstring [1] is not simply the obvious generalization of the supersymmetric particle action. In addition to a kinetic term which does generalize the particle action, there is an additional term which is necessary in order to obtain a somewhat mysterious local supersymmetry. This supersymmetry is needed in order to gauge away unphysical degrees of freedom and establish the equivalence with the light-cone gauge action. Henneaux and Mezincescu, and independently Martinec, subsequently provided a rationale for the extra term, showing it to be a Wess-Zumino (WZ) term in the sense that it could be obtained by applying to the global supersymmetry (SUSY) group the construction that yields the WZ term for sigma models on ordinary group manifolds [2].

The identification of a WZ term in the superstring action raises further questions. Normally such a term is expected only when the group manifold is topologically nontrivial, specifically when the third homology group (for a two-dimensional field theory) does not vanish. There is then a topological quantization

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condition for the coefficient of this term in the action [3]. Yet the group manifold of superstring theory is flat superspace, which is generally assumed to be contractible. How then can a WZ term exist, and should a quantization condition be expected?

For ordinary sigma models, cohomology theory is the appropriate tool for investigating such questions [4]. To date, however, there has been no systematic development of the cohomology theory of general supermanifolds. This is partly due to the fact that previous investigations have shown that the simple (DeWitt) supermanifolds of most interest in physics have no topological properties beyond those of the spacetime on which they are based [5, 6]. In addition, one can show that any closed differential superform on flat superspace is exact, so that the cohomology is trivial [7]. In general, however, it is not clear how much topological information is contained in the cohomology of superforms.

Rogers' definition of supermanifold [8] allows the most general possible topology. The definition has the great advantage that a Rogers supermanifold is in fact an ordinary manifold, so that one knows what is meant by its topology and one knows how to measure it by the usual homotopy and homology groups. One can then ask how the cohomology of superforms ("G^∞ cohomology") compares with these more standard measures. This is the program pursued in the present work. It is shown that there is a homomorphism mapping the cohomology of superforms into that of ordinary forms. This result had been reported in [9]. Further, the G^∞ cohomology is invariant under continuous deformations (homotopies) which do not alter the supermanifold structure. These two results give the precise sense in which G^∞ cohomology contains topological information. The theorem, proven by Kostant [5] within a very different formalism, that the cohomology of a DeWitt supermanifold is essentially trivial is then rederived in the present framework. Some simple examples are given of the behavior of the G^∞ cohomology when nontrivial topology is present in the Grassmann directions.

With the topological meaning of superform cohomology now clear, the WZ term of the superstring action is reconsidered. The topological content of this term is due to the requirement of global SUSY. That is, although the WZ term is a total divergence, it cannot be expressed as the divergence of a manifestly supersymmetric 2-form. It therefore corresponds to a class in the third *supersymmetric* cohomology group of flat superspace, which is *nontrivial*. This difference between the supersymmetric and ordinary cohomologies is only possible because superspace is noncompact. Similar subtleties in the cohomological interpretation of WZ terms should be expected in noncompact sigma models. The topology associated with the requirement of SUSY invariance can actually be visualized. Taking the quotient of flat superspace by a certain discrete subgroup of the SUSY group yields a supermanifold M having precisely the nontrivial topology in question. This topology resides entirely in the Grassmann coordinate directions of M, which are compactified.

If superstring theory could be reformulated on M rather than flat superspace, then a topological quantization condition for the WZ coefficient might be expected by the standard arguments [4]. However, this is not possible. The reason is quite simple. A WZ term is expressed as an integral over some 3-manifold whose boundary is the string world sheet. The arbitrariness in the choice of this 3-manifold leads to a potential ambiguity in the value of the WZ term. Since the action appears in the path integral in the form e^{iS}, an ambiguity quantized in multiples of 2π can be tolerated. This is the usual quantization argument. Unfortunately, the superstring WZ term is actually Grassmann-valued. The ambiguity is also Grassmann-valued and its exponential cannot be made equal to unity. Therefore the WZ term cannot be consistently defined on M, but only on its covering space which is flat superspace and where no quantization condition is to be expected. There are also severe difficulties in defining the path integral for a string moving in M, even without a WZ term.

Is the absence of a quantization condition for the WZ coefficient a Good or a Bad Thing? The superstring action has local SUSY only for a special value of this coefficient, and a quantization condition might have helped to understand this. Unfortunately, this is not the case. A quantization condition on M would actually have related the WZ coefficient to the dimensionless product Tl^2, with T the string tension and l the scale of compactification of M. Thus, any coefficient would be topologically allowed if T and l were chosen appropriately. If the superstring is indeed consistent only for one value of the WZ coefficient, the reason for this is not to be found in topological considerations.

2. Supermanifolds and Their Cohomology

A supermanifold M is an ordinary real manifold equipped with an analytic structure best described in terms of Grassmann coordinates [8]. In each coordinate chart on a supermanifold there will be m even coordinates x^μ and n odd coordinates θ^α, taking values in a fixed Grassmann algebra B_L having L anticommuting generators $v_1, v_2, \dots, v_L$. The Grassmann coordinates are related to the ordinary real coordinates by expanding them in terms of the generators,

$$\begin{aligned} x^\mu &= x_0^\mu + \sqrt{-1}\,x_{ij}^\mu v_i v_j + x_{ijkl}^\mu v_i v_j v_k v_l + \dots \equiv x_\Gamma^\mu v_\Gamma\,, \\ \theta^\alpha &= \theta_i^\alpha v_i + \sqrt{-1}\,\theta_{ijk}^\alpha v_i v_j v_k + \dots \equiv \theta_\Gamma^\alpha v_\Gamma\,. \end{aligned} \tag{2.1}$$

Here the coefficients x_Γ^μ and θ_Γ^α may be used as real coordinates. Γ denotes an increasing sequence of distinct integers between 1 and L inclusive, or the single integer 0. The supermanifold M therefore has real dimension $2^{L-1}(m+n)$. The factors $\sqrt{-1}$ appear in (2.1) where necessary to ensure that each Grassmann coordinate $z^A = z_\Gamma^A v_\Gamma$ is real in the sense of being equal to its complex conjugate, where complex conjugation by definition reverses the order of v_i factors in a product. The symbol v_Γ denotes the product of all the v_i whose subscripts appear in the sequence Γ, including the $\sqrt{-1}$ where necessary, with $v_0 \equiv 1$. It is easily checked that v_Γ contains $\sqrt{-1}$ precisely when the length of the sequence Γ has the form $4k+2$ or $4k+3$.

The defining property of a supermanifold is that the transition functions relating the coordinates in overlapping charts should be G^∞ functions (otherwise known as superfields) in the Grassmann coordinates. Such functions are polynomials in the θ coordinates,

$$F(x,\theta) = f_0(x) + f_\alpha(x)\theta^\alpha + f_{\alpha\beta}(x)\theta^\alpha\theta^\beta + \dots\,. \tag{2.2}$$

Furthermore, the coefficient functions in this expansion are required to admit Taylor expansions about the *body* x_0 of x in powers of the *soul* $s(x) \equiv x - x_0$:

$$f(x) = f(x_0) + s(x^\mu)\partial_\mu f(x_0) + \tfrac{1}{2}s(x^\mu)s(x^\nu)\partial_\mu\partial_\nu f(x_0) + \dots \quad (2.3)$$

This series terminates if L is finite. The G^∞ conditions are very restrictive, since a priori the transition functions could have had arbitrary dependence on the real coordinates z_I^A. The effects of these conditions on the topology of M have been explored in [10, 11]. A G^∞ function can be differentiated with respect to either the real or Grassmann coordinates. By the chain rule these derivatives are connected by

$$\partial F/\partial z_I^A = v_I \partial F/\partial z^A . \quad (2.4)$$

Not all supermanifolds have proven useful in physics. One must require in addition to the properties above that the body coordinates x_0^μ alone parametrize an m-dimensional manifold called the body of M which can serve as the physical spacetime in a theory based on M. M is normally assumed to be a vector bundle over its body. Such an M will be called a DeWitt supermanifold [12]. For example, the flat superspace with m even and n odd coordinates, $B_L^{m,n}$, is a trivial vector bundle over its body R^m. Furthermore, in physics the Grassmann algebra is assumed to contain arbitrarily many independent elements, not merely L such elements. In field theory, unitarity is violated for finite L because the Green's functions with more than L fermionic fields vanish identically. Therefore one either sets $L = \infty$ directly or works with a sequence of supermanifolds constructed over the algebras B_L for $L \to \infty$. Setting $L = \infty$ is awkward because expressions like (2.3) and (2.5) below become infinite series for which convergence must be defined and proven. To avoid this, the limiting procedure $L \to \infty$ will be adopted here. A careful discussion of this limit has been given by Rogers [13] and will now be sketched.

The key observation is that the G^∞ functions over B_L form a subalgebra of those over B_{L+1}. By Eqs. (2.2) and (2.3), a G^∞ function is determined by its coefficient functions $f_{\alpha\dots\sigma}(x_0)$, so each G^∞ function over B_L can be identified with the G^∞ function over B_{L+1} having the same coefficients. Furthermore, the flat superspace $B_L^{m,n}$ itself is a subspace of $B_{L+1}^{m,n}$. (Rogers showed that curved superspaces for different values of L are also nested, but this fact will not be needed here.) This means that whenever a physical quantity threatens to vanish because of cancellations in the algebra B_L, one can enlarge both the superspace and the function algebra to avert it. Physical Green's functions do not depend on the value of L once this value is large enough to avoid accidental vanishing. This is evident from the fact that standard path integral calculations depend only on the formal algebraic properties of the fermion fields and never refer to their values. Mathematically, this procedure is called a direct limit: the superspace and function algebra for $L = \infty$ are defined as the infinite union of the nested superspaces and algebras for all finite L. Each Green's function is computed using a sufficiently large but finite L, and convergence questions do not arise.

Several different cohomology theories can be constructed for a supermanifold M [9]. First, since M is an ordinary manifold, there is the usual de Rham cohomology of real-valued differential forms. Such forms locally appear as polynomials in the (anticommuting) coordinate differentials dz_I^A with smooth real-valued coefficient functions. The differential operator is

$$d = dz_\Gamma^A \frac{\partial}{\partial z_\Gamma^A}. \tag{2.5}$$

The cohomology, denoted $H_{DR}^k(M)$, is the group of closed ($d\omega = 0$) mod exact ($\omega = d\psi$) k-forms. The wedge product of forms defines a multiplication of cohomology classes which makes the union of the $H_{DR}^k(M)$ into a ring. Next, there is the de Rham cohomology of Grassmann-valued forms. This is defined exactly as above except that the differential forms have smooth B_L-valued coefficient functions. It will be denoted $H_{DR}^k(M; B_L)$. Both of these cohomology theories measure the ordinary topology of M as a real manifold. Indeed, a B_L-valued form ω can be decomposed into real-valued forms ω_Γ via $\omega = v_\Gamma \omega_\Gamma$. ω is closed or exact iff all the ω_Γ are, so these cohomology theories contain exactly the same information.

A cohomology theory which takes account of the supermanifold structure of M is the cohomology $H_G^k(M)$ of G^∞ differential forms. These forms locally appear as polynomials in the Grassmann coordinate differentials dz^A with G^∞ functions for coefficients. Note that the differentials $d\theta^\alpha$ commute with each other but anticommute with dx^μ and with θ^β. The differential operator here is

$$d = dz^A \frac{\partial}{\partial z^A}, \tag{2.6}$$

which is different from the previous cases. A priori it is not clear that G^∞ cohomology has any topological significance at all. The two basic properties of G^∞ cohomology to be established here are its relation to de Rham cohomology and its invariance under G^∞ homotopies (homotopies which preserve the supermanifold structure). These results give the precise sense in which G^∞ cohomology contains topological information. As a corollary, Kostant's theorem [5] (which he expressed in a quite different formalism) that the cohomology of a DeWitt supermanifold is just that of its body will be rederived in the present language.

The relation between the G^∞ and de Rham cohomologies of a supermanifold is expressed by the existence of a map $h: H_G^k(M) \to H_{DR}^k(M; B_L)$ which is a homomorphism of the cohomology groups (and of the ring structure as well). h is first defined at the level of forms. From any G^∞ form ω one can produce a corresponding B_L-valued form $h\omega$ by replacing each dz^A by $v_\Gamma dz_\Gamma^A$. This map does not depend on the choice of coordinates. If $w = w(z)$ are G^∞ functions of the old coordinates, then

$$dz^A = dw^B \frac{\partial z^A}{\partial w^B},$$

while

$$v_\Gamma dz_\Gamma^A = v_\Gamma dw_\Sigma^B \frac{\partial z_\Gamma^A}{\partial w_\Sigma^B} = dw_\Sigma^B \frac{\partial z^A}{\partial w_\Sigma^B} = dw_\Sigma^B v_\Sigma \frac{\partial z^A}{\partial w^B},$$

and these expressions are indeed related by $dw^B \to v_\Sigma dw_\Sigma^B$. Furthermore, this map commutes with $d: hd = dh$ on G^∞ forms. For example, for a 1-form $\omega = f_B(z) dz^B$,

$$hd\omega = hdz^A \frac{\partial f_B}{\partial z_A} dz^B = v_\Gamma dz_\Gamma^A \frac{\partial f_B}{\partial z^A} v_\Sigma dz_\Sigma^B, \tag{2.7}$$

while

$$dh\omega = df_B v_\Sigma dz_\Sigma^B = dz_\Gamma^A \frac{\partial f_B}{\partial z_\Gamma^A} v_\Sigma dz_\Sigma^B = dz_\Gamma^A v_\Gamma \frac{\partial f_B}{\partial z^A} v_\Sigma dz_\Sigma^B . \tag{2.8}$$

Equation (2.4) is the key to all these manipulations. Since h commutes with d, it sends closed (exact) forms to closed (exact) forms and hence gives a map of cohomology classes as claimed. Since h also preserves sums and products of forms, it is a homomorphism of cohomology rings. It is not an isomorphism, however. As shown in [9], h is an isomorphism for the zeroth cohomology groups, a one-to-one map for the first cohomology groups, and need not be one-to-one or onto for the higher groups.

The map h is used implicitly whenever one has a map $f: W \to M$ from an ordinary manifold (such as the world sheet of a string) into a supermanifold and desires to pull back a G^∞ form on M to a B_L-valued form which might be integrated on W. One first applies h to get a B_L-valued form on M and then pulls this back.

The basic property which gives topological significance to the de Rham cohomology is its homotopy invariance: homotopy-equivalent manifolds have the same cohomology. The G^∞ cohomology of a supermanifold is similarly invariant under homotopies which preserve the supermanifold structure. This qualification is important, because a single real manifold often admits many supermanifold structures. The G^∞ cohomology is not an invariant of the underlying manifold, but of its G^∞ structure only. The proof of this property is almost identical to the classical proof for de Rham cohomology [14].

Lemma. *For any supermanifold M, $H_G^k(M) = H_G^k(M \times B_L^{1,0})$.*

Proof. Let x denote a point of M and t the even coordinate of $B_L^{1,0}$. Then there are natural G^∞ maps

$$\pi: M \times B_L^{1,0} \to M, (x,t) \to x , \qquad s: M \to M \times B_L^{1,0}, x \to (x,0) . \tag{2.9}$$

The corresponding maps π^* and s^* on forms can be used to pull back G^∞ forms from M to $M \times B_L^{1,0}$ or vice versa. (The choice of the slice $t=0$ in the definition of s is arbitrary. Any constant t slice could be used.) Now $\pi s = 1$, so $s^*\pi^* = 1$, but $s\pi \neq 1$ and $\pi^* s^* \neq 1$. The plan is to find a G^∞ operator K acting on forms on $M \times B_L^{1,0}$ with the property

$$1 - \pi^* s^* = \pm (dK \pm Kd) . \tag{2.10}$$

This shows that $1 - \pi^* s^*$ sends closed forms to exact forms, hence sends every cohomology class to the trivial one. At the level of cohomology, then, $\pi^* s^* = s^* \pi^* = 1$, so that π^* and s^* are inverse isomorphisms between the cohomologies of M and $M \times B_L^{1,0}$.

A G^∞ form on $M \times B_L^{1,0}$ can be split into pieces which contain dt and pieces which do not. Precisely, such a form is uniquely a sum of two types of forms, namely $(\pi^*\phi)F(x,t)$ and $(\pi^*\phi)F(x,t)dt$, where ϕ is a G^∞ form on M and F is a G^∞ function. The operator K will be integration over t:

$$K(\pi^*\phi)F(x,t) = 0 , \qquad K(\pi^*\phi)F(x,t)dt = (\pi^*\phi) \int_0^t F(x,u)du . \tag{2.11}$$

The integral here is a contour integral within $B_L^{1,0}$, depending parametrically on x. It is independent of the path joining the endpoints [12]. Further, it is a G^∞ function of the upper endpoint t, so is completely determined once known for soulless values of t. For such values the contour may be taken within the body R^1 of $B_L^{1,0}$. At this point the classical calculation [14] applies to show that $1-\pi^* s^* = (-1)^{k-1}(dK - Kd)$ when acting on k-forms, which completes the proof.

Definition. Let M and N be supermanifolds. Two G^∞ maps $f, g: M \to N$ are G^∞ homotopic iff there exists a G^∞ map $F: M \times B_L^{1,0} \to N$ such that $F(x,t) = f(x)$ for $t_0 \geqq 1$ and $F(x,t) = g(x)$ for $t_0 \leqq 0$.

Because $F(x,t)$ is G^∞ it is completely determined once $F(x,t_0)$ is known. Therefore a G^∞ homotopy is really just an ordinary homotopy by a sequence of maps which are all G^∞.

Theorem. *Two maps $f, g: M \to N$ which are G^∞ homotopic induce identical maps $f^*, g^*: H_G(N) \to H_G(M)$ in G^∞ cohomology.*

Proof. Let π be the projection of $M \times B_L^{1,0}$ onto M as in the lemma. Let s_0 and s_1 map M to the slices $t=0$ and $t=1$ of $M \times B_L^{1,0}$ respectively (Fig. 1). By the lemma, s_0^* and s_1^* are both inverses to π^* in cohomology, so they are equal. Hence the induced maps $f^* = s_1^* F^*$ and $g^* = s_0^* F^*$ are equal.

One says that two supermanifolds M and N have the same G^∞ homotopy type, or are G^∞ homotopic, if there are G^∞ maps $f: M \to N$ and $g: N \to M$ such that fg and gf are each G^∞ homotopic to the identity map. It follows from the theorem that two such supermanifolds have the same G^∞ cohomology. Thus the G^∞ cohomology is invariant under homotopies preserving the G^∞ structure.

Corollary (Kostant [5]). *Let M be a DeWitt supermanifold, M_0 its body, and $\bar{M}$ the even Grassmann extension of the body (the canonical DeWitt supermanifold having body M_0 and even coordinates only). Then*

$$H_G(M) = H_G(\bar{M}) = H_{DR}(M_0; B_L). \tag{2.12}$$

Proof. The first equality is established by showing that M and $\bar{M}$ are G^∞ homotopic. This depends on Rogers' result that any DeWitt supermanifold has a special atlas with especially simple transition functions [15]. If (x,θ) and (y,ϕ) are the coordinates in overlapping charts in this atlas, then y depends only on x, not on θ, and ϕ depends on θ only linearly. Given such an atlas for M, simply send $\theta \to f(t)\theta$ in each chart, where $f(t)$ is G^∞ and $f(t_0)$ is 0 for $t_0 \leqq 0$, 1 for $t_0 \geqq 1$, and smooth in between. This is consistent across overlaps of charts and gives a G^∞ homotopy of M into $\bar{M}$.

The second equality is true because G^∞ forms on $\bar{M}$ are in one-to-one correspondence with B_L-valued forms on its body. Given a G^∞ form on $\bar{M}$, apply h

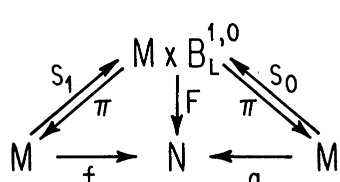

Fig. 1

to get a B_L-valued form and then restrict it to M_0. Conversely, given a B_L-valued form on M_0, a G^∞ form on $\bar{M}$ is obtained by replacing each dx_0^μ by dx^μ and each coefficient function by its unique G^∞ extension [8]. These maps on forms commute with d and are easily seen to give the desired isomorphisms in cohomology.

Finally, a simple example will be given of the behavior of G^∞ cohomology when there is nontrivial topology in the soul directions. Consider the cylinder C_Γ formed from $B_L^{1,0}$ by compactifying a single coordinate direction x_Γ into a circle. As ordinary manifolds, the C_Γ are all diffeomorphic to each other and are homotopy-equivalent to a circle. Therefore $H_{DR}^k(C_\Gamma)=R$ for $k=0,1$ and is trivial for $k>1$. Similarly $H_{DR}^k(C_\Gamma; B_L)=B_L$ for $k=0,1$ and is trivial otherwise. The closed but nonexact form generating H_{DR}^1 is dx_Γ. This is because the function x_Γ on $B_L^{1,0}$ is not periodic, so there is no corresponding function on the quotient space C_Γ.

The various C_Γ have different G^∞ structures and are not G^∞ homotopic. The G^∞ cohomology may therefore distinguish between them. There is no global function $x=v_\Sigma x_\Sigma$ on any of the manifolds because there is no function x_Γ. Therefore dx is not exact. However, cdx *is* exact if the constant c is proportional to v_i for any index i in the sequence Γ. The function cx does exist because multiplication by c annihilates the troublesome term $v_\Gamma x_\Gamma$ (no sum) in x. Hence $H_G^0(C_\Gamma)=B_L$, but $H_G^1(C_\Gamma)$ is generated by a single element dx subject to the relations $v_i dx=0$ for each i in the sequence Γ. The map h in this example sends the generator dx in H_G^1 to $v_\Gamma dx_\Gamma$ (no sum) in H_{DR}^1. It is one-to-one but not onto, since for example nothing maps to the generator dx_Γ of H_{DR}^1 in general. The fact that H_G^1 is not freely generated is the algebraic reflection of the fact that a particular real coordinate direction has been compactified rather than an entire Grassmann coordinate direction. The reader may wish to study the G^∞ cohomology of various cylinders and tori involving compactification in several directions.

3. The Green-Schwarz Wess-Zumino Term

Superstring theory may be viewed as a dynamical theory of maps from the two-dimensional world sheet of the string into the flat superspace $B_L^{m,n}$. Customarily $m=10$ and $n=32$. The spinor coordinates θ^α are assumed to be Weyl as well as Majorana. This can be viewed as a restriction on the allowed maps. For simplicity only one set of θ^α's will be used, corresponding to the $N=1$ heterotic string rather than the $N=2$ superstring. This is purely for convenience: none of the conclusions below depend on specific properties of any particular superstring theory. The image W of the world sheet in $B_L^{m,n}$ is an immersed 2-manifold representing the path swept out by the moving string. It is natural to construct an action functional by integrating some SUSY-invariant G^∞ 2-form over W. Here the SUSY group acts on $B_L^{m,n}$ by

$$Q(\eta^\alpha)(x^\mu, \theta^\alpha)=(x^\mu+i\bar{\eta}\gamma^\mu\theta, \theta^\alpha+\eta^\alpha), \qquad T(a^\mu)(x^\mu, \theta^\alpha)=(x^\mu+a^\mu, \theta^\alpha). \quad (3.1)$$

The group parameters η^α and a^μ are respectively odd and even bodyless elements of B_L, and the Dirac matrices are in the Majorana representation. Of course one wants the action to be invariant under the full Poincaré group also, but the above transformations already close and will be of most interest. Some relations obeyed by the group elements are

$$Q(\eta)Q(\lambda)=Q(\eta+\lambda)T(i\bar{\eta}\gamma\lambda), \qquad Q(-\eta)Q(-\lambda)Q(\eta)Q(\lambda)=T(2i\bar{\eta}\gamma\lambda). \quad (3.2)$$

The second equation is the analog for finite group elements of the fundamental anticommutator of the SUSY algebra.

The SUSY-invariant G^∞ forms are not difficult to classify. The only SUSY-invariant G^∞ functions other than constants are functions of the form $v_{12\ldots\hat{k}\ldots L}f(x)$ (at most one subscript missing), which are uninteresting and will be ignored both because they have no body and because they do not survive the limit $L\to\infty$. Up to normalization, the only SUSY-invariant G^∞ 1-forms are $\omega^\alpha\equiv d\theta^\alpha$ and $\omega^\mu\equiv dx^\mu-i\bar{\theta}\gamma^\mu d\theta$. It is easy to see that any SUSY-invariant G^∞ form of higher degree must be a polynomial in the ω^A with constant coefficients: certainly any G^∞ form can be written as a polynomial in the ω^A, and if the form is invariant so are the coefficients of this polynomial.

The kinetic or metric term in the covariant superstring action is [16]

$$S_1=-\tfrac{1}{2}\int_W \omega^\mu\, {}^*\omega_\mu. \quad (3.3)$$

Here the invariant 1-forms ω^μ are restricted to W, and * is the Hodge dual on W which contains the dependence on the metric.

If X is a 3-manifold in $B_L^{m,n}$ whose boundary is the world sheet W then the WZ term in the action can be expressed in terms of a SUSY-invariant 3-form Ω_3 restricted to X:

$$S_2=\int_X \Omega_3, \qquad \Omega_3=-i(C\gamma_\mu)_{\alpha\beta}\omega^\mu\omega^\alpha\omega^\beta=-idx^\mu d\bar{\theta}\gamma_\mu d\theta, \quad (3.4)$$

with C the charge conjugation matrix. Here boundary conditions are implicitly chosen such that the world sheet is a closed manifold. One can question whether such boundary conditions are appropriate for a string theory, but this issue will be ignored here.

One normally expects the occurrence of a WZ term to be associated with nontrivial topology – specifically a nontrivial H^3 – in the space of values of the fields. Here, however, this space is $B_L^{m,n}$, whose G^∞ cohomology is trivial by Kostant's theorem. Indeed, Ω_3 can be obtained as d of various 2-forms. However, these 2-forms cannot be chosen to be SUSY-invariant. This is the formal version of the familiar statement that when the Wess-Zumino term is expressed as an integral over W it is SUSY-invariant but not manifestly so. Defining the supersymmetric cohomology $H_{\mathrm{SUSY}}(B_L^{m,n})$ to be the set of closed SUSY-invariant G^∞ forms modulo those which are differentials of SUSY-invariant G^∞ forms, Ω_3 represents a nontrivial class in $H^3_{\mathrm{SUSY}}(B_L^{m,n})$.

The cohomology of invariant differential forms on a Lie group is a familiar object in mathematics. For example, it is a classical theorem that the cohomology of invariant forms on a compact connected Lie group is identical to the ordinary de Rham cohomology of the group [17]. This theorem is relevant to the classification of Wess-Zumino terms in ordinary sigma models with compact groups. There too it is the invariant cohomology which is of interest for constructing invariant Lagrangians, but by the theorem this need not be distinguished from the de Rham cohomology. Therefore the Wess-Zumino terms can be classified using the de Rham cohomology of the group manifold. The critical difference for superstrings is that the SUSY group is noncompact, so the theorem does not apply. Because H_G and H_{SUSY} can be different, a Wess-Zumino term is possible despite the trivial topology of $B_L^{m,n}$.

Is H_{SUSY} only a formal algebraic construction or does it have a topological interpretation? Suppose a supermanifold M_L could be found such that the SUSY-invariant forms on $B_L^{m,n}$ were in one-to-one correspondence with *all* G^∞ forms on M_L, this correspondence preserving the action of d. Then $H_{\mathrm{SUSY}}(B_L^{m,n}) = H_G(M_L)$, and the nontriviality of $H^3_{\mathrm{SUSY}}(B_L^{m,n})$ could be visualized in terms of the topology of M_L. The natural way to construct M_L is to take the quotient of $B_L^{m,n}$ by the SUSY group. Unfortunately, any two points in $B_L^{m,n}$ having the same body coordinates can be connected by an element of the group, so that this quotient space is just the body R^m of $B_L^{m,n}$, which is not a supermanifold at all.

Fortunately, the SUSY group has a discrete subgroup DSUSY which is generated by the $Q(\eta)$ and $T(a)$ for which the components η_T^α and a_T^μ are integers [18]. Remarkably, for G^∞ forms invariance under SUSY and DSUSY are equivalent. This is true because ω^α and ω^μ are the only DSUSY-invariant 1-forms, and any invariant G^∞ form must be a polynomial in these with constant coefficients. Intuitively, the point is that supersymmetries are translations along soul directions, and superfields are polynomial functions of soul coordinates. A polynomial which is invariant under integer translations automatically has continuous translation invariance as well. Now, defining $M_L = B_L^{m,n}/\mathrm{DSUSY}$ gives a supermanifold with the non-trivial topology reflected in the WZ term. M_L is a fiber bundle over its body R^m, but the fibers are topologically nontrivial [10]. This is the simplest type of supermanifold which is less trivial than the DeWitt type. Each fiber is in fact a torus bundle over a torus. A picture of the fibers in the lowest-dimensional case $m=n=1$, $L=2$ is given in [10]. It is easy to see that M_L embeds in M_{L+1}, so that the direct limit construction described in Sect. 2 defines a topologically nontrivial limiting manifold $\bigcup_L M_L$. The subscript L will be suppressed from now on.

The ordinary translation group in Euclidean space has a one-parameter family of discrete subgroups: the subgroups of translations by integer multiples of some fixed length. One can see that these subgroups are all isomorphic by choosing units in which the fixed length is equal to unity. Similarly, the DSUSY group contains a scale parameter which was implicitly set equal to unity above. More generally one restricts the components a_T^μ to be integer multiples of a length l, while the components η_T^α are multiples of $l^{1/2}$. Then l is the scale of compactification of the soul coordinates of M. For the present, l will be set equal to unity, but it will reappear later.

Since the WZ term reflects the nontrivial topology of M, it is tempting to try to define a theory of superstrings moving in M rather than in $B_L^{m,n}$. Supersymmetry would be automatic in such a theory because every G^∞ form on M is SUSY-invariant. The classical theory of closed strings on M splits into two sectors: contractible strings, and strings which are not contractible because they wind around the compactified soul directions of M. The contractible strings correspond to ordinary closed superstrings in the covering space $B_L^{m,n}$. Strings in M with nonzero winding numbers correspond to open strings in $B_L^{m,n}$ with an unusual boundary condition: their endpoints must be related by a DSUSY transformation. One might expect a topological quantization condition for the coefficient of the WZ term in such a theory, as usual in nonlinear sigma models, whereas there was no reason for such a condition in the theory on $B_L^{m,n}$. It will be shown that there is no consistent way to define the WZ term in the theory on M, hence no topological quantization condition.

The usual quantization condition for a WZ coefficient is obtained by demanding that the path integral be well-defined. This means that it must be independent of the choice of the 3-manifold X with $\partial X = W$. The path integral for the superstring has the form

$$Z = \int Dx^\mu(\sigma,\tau) D\theta^\alpha(\sigma,\tau) \exp iT\left(-\tfrac{1}{2}\int_W \omega^\mu {}^*\omega_\mu + \beta \int_X \Omega_3 + \dots\right). \tag{3.5}$$

For the heterotic string there are additional terms in the action which describe the left-moving bosonic variables [19]. The integral over all metrics on W has also been suppressed. To understand whether the topological quantization argument applies in this case one must carefully define the meaning of this path integral. In particular, one might worry that the presence of the integrals over $\theta^\alpha(\sigma,\tau)$, which are Berezin integrals, could invalidate the argument. This is because a Berezin integral depends more on the algebraic structure of the integrand than on its value, so it may not be sufficient to make the exponential of the WZ term single-valued by adjusting its coefficient.

A provisional definition of the path integral (3.5) is obtained by restricting σ and τ to discrete values and integrating over finitely many variables $x^\mu(\sigma,\tau)$, $\theta^\alpha(\sigma,\tau)$. The derivatives in the action are replaced by some discrete approximation. This amounts to replacing the world sheet by a polyhedral approximation or lattice, the integration variables being the locations of the lattice points. Eventually the number of lattice points is taken to infinity. When the action is evaluated for such a polyhedral world sheet, it will be a function of the lattice point locations. The integration over the θ^α is multiple Berezin integration, and the integration over the x^μ is contour integration in $B_L^{m,n}$ which is essentially equivalent to ordinary integration over the bodies x_0^μ [12]. The WZ term makes sense if X is a 3-manifold whose boundary is the polyhedral world sheet. Note that X need not be discretized.

Further discussion is in order concerning this definition of the path integral. One is integrating over the space of all maps from the discrete set of lattice points (σ,τ) into M. This space is itself a supermanifold, namely $M \times M \times \dots \times M = M^k$ with k the number of lattice points. $x^\mu(\sigma,\tau)$ and $\theta^\alpha(\sigma,\tau)$ are the coordinates on this supermanifold. The topological nontriviality of M should not complicate the definition of the integrations, at least if W is contractible in M (a slightly stronger assumption than W being a boundary, which was already assumed). For then a single coordinate chart in M can be found containing W, and one need only integrate over this topologically trivial chart. Consideration of this case is sufficient for showing that a WZ term cannot be defined on M. If one wanted to go further and try to define the path integral for arbitrary W, not necessarily contractible, serious problems would arise in defining integration over the topologically nontrivial supermanifold M^k. It was shown in [15] that one can integrate over a general supermanifold only when it has a covering by charts with especially simple transition functions: the even coordinates must transform independently of the odd ones. Intuitively this is because one uses Berezin integration for the odd coordinates and ordinary integration for the even ones. To define integration globally one needs some global distinction between even and odd coordinates. This is not possible for M^k. Typical transition functions relating overlapping charts in M^k have the form of DSUSY group transformations:

$$\theta'^{\alpha}(\sigma,\tau)=\theta^{\alpha}(\sigma,\tau)+\eta^{\alpha}(\sigma,\tau), \qquad x'^{\mu}(\sigma,\tau)=x^{\mu}(\sigma,\tau)+i\bar{\eta}(\sigma,\tau)\gamma^{\mu}\theta(\sigma,\tau). \tag{3.6}$$

These do not have the required form. In principle the transition functions might take the necessary form in terms of some other choice of coordinates, but in fact this bizarre possibility can be ruled out. One argument is that since M is the quotient of $B_L^{m,n}$ by the DSUSY group, the parameters η^{α} of this group must appear in the transition functions. But these parameters carry a spinor index, so they can only appear in the transformation law of the vector x^{μ} in conjunction with another spinor. Thus x^{μ} cannot transform independently of θ^{α}. Evidently there are many reasons why a consistent superstring theory cannot be constructed on M.

Now consider the effect of choosing two different 3-manifolds X and Y which each have boundary W and which together form a closed 3-manifold C. The action changes by a multiplicative factor

$$\exp iT\beta \int_C \Omega_3 \tag{3.7}$$

which is actually independent of the path integration variables which parametrize the discretization of W. C may be any closed 3-manifold in M, and is not discretized. Therefore the presence of Berezin integrals in (3.5) does not alter the usual consistency condition for a WZ term: one must guarantee that the phase factor (3.7) is unity for any C.

The mere fact that Ω_3 represents a nontrivial class in $H_G^3(M)$ does not imply that the integral in (3.7) can be nonzero. If Ω_3 is in the kernel of the map h, then it represents the trivial class in $H_{DR}^3(M;B_L)$ and the integral over any C will be zero. In this case the WZ term is well-defined on M with any coefficient, so there is no quantization condition. If, on the other hand, Ω_3 is *not* in the kernel of h, it represents a nontrivial class in $H_{DR}^3(M;B_L)$. By the usual duality between cohomology and homology, a 3-manifold C can be found which gives a nonzero integral in (3.7). This integral will be a pure soul Grassmann number, because of the two factors $d\theta$ in Ω_3 which each contribute at least one factor v_i. No adjustment of the coefficient of Ω_3 can make the phase equal to unity, since for a pure soul Grassmann number q, $e^{iq}=1+iq-\frac{q^2}{2}+\dots$ cannot be equal to unity: the term in iq containing the fewest v_i factors cannot be cancelled by any other term in the series. Therefore there is no way to define the WZ term on M: its coefficient is topologically quantized to zero. The best one can do is to return to the covering space $B_L^{m,n}$, where the WZ term makes sense but has no reason to be quantized. Thus, whether or not Ω_3 is in the kernel of h, there is no quantization condition for the WZ coefficient on $B_L^{m,n}$.

In fact, Ω_3 is not in the kernel of h, so the WZ term does not make sense on M. To see this one must show that $h\Omega_3$ cannot be expressed as d of any SUSY-invariant 2-form, where this 2-form is no longer required to be G^∞. Explicitly,

$$h\Omega_3 = -i(dx_0^\mu + iv_i v_j dx_{ij}^\mu + \ldots)(v_k d\bar{\theta}^k + \ldots)\gamma_\mu(v_l d\theta_l + \ldots). \tag{3.8}$$

The terms involving dx_0^μ are exact and can be dropped: there is a global function x_0^μ on M since the body of M has not been compactified. Then

$$\begin{aligned} h\Omega_3 &= v_{ijkl} dx_{ij}^\mu d\bar{\theta}_k \gamma_\mu d\theta_l + O(v^6) \\ &= 2v_{1234}(dx_{12}^\mu d\bar{\theta}_3\gamma_\mu d\theta_4 + dx_{34}^\mu d\bar{\theta}_1\gamma_\mu d\theta_2 - dx_{13}^\mu d\bar{\theta}_2\gamma_\mu d\theta_4 \\ &\quad - dx_{24}^\mu d\bar{\theta}_1\gamma_\mu d\theta_3 + dx_{14}^\mu d\bar{\theta}_2\gamma_\mu d\theta_3 + dx_{23}^\mu d\bar{\theta}_1\gamma_\mu d\theta_4) + \ldots. \end{aligned} \tag{3.9}$$

The invariant B_L-valued forms are built from the 1-forms ω_Γ^A, where $\omega^A = v_\Gamma \omega_\Gamma^A$. For example, $\omega_\Gamma^\alpha = d\theta_\Gamma^\alpha$ and

$$\omega_{ij}^\mu = dx_{ij}^\mu + \bar{\theta}_j \gamma^\mu d\theta_i - \bar{\theta}_i \gamma^\mu d\theta_j.$$

These are complete in the sense that any invariant form is a polynomial in these with invariant functions as coefficients. However, now that the G^∞ restriction has been dropped, there are many invariant functions other than constants.

The only invariant 2-form whose differential could possibly yield the terms displayed in Eq. (3.9) is

$$v_{1234}(\omega_{12}^\mu \omega_{\mu 34} - \omega_{13}^\mu \omega_{\mu 24} + \omega_{14}^\mu \omega_{\mu 23}). \tag{3.10}$$

However, the differential of (3.10) does not reproduce (3.9). For example, the terms involving dx_{12}^μ and dx_{34}^μ appear with the same sign in (3.9) but with opposite signs in (3.10). Therefore $h\Omega_3$ is not exact, and in fact there exists a 3-manifold C over which its integral is proportional to v_{1234}.

Now that the absence of any topological quantization condition for the WZ coefficient has been shown, what would the consequences of such a condition have been? Could it have helped to explain the unit coefficient in the superstring action? For dimensional reasons, the phase in (3.7) is proportional to $\beta T l^2$, with β the WZ coefficient, T the string tension, and l the compactification scale appearing in the DSUSY group. This is the quantity which would have been quantized. Clearly, any value of β is possible if T and l are chosen appropriately. Alternatively, given that $\beta = 1$ is correct, l becomes quantized in terms of T. This would have had consequences for the spectrum of the string theory on M if such a theory were sensible.

4. Conclusions

In this paper supermanifold techniques were used to understand the topology associated with the Wess-Zumino term in the covariant superstring action. In order to do this it was necessary to develop the cohomology theory of supermanifolds. The precise sense in which the cohomology of superforms contains topological information was explained. An important theme here was the interplay between the topology as measured by superforms versus ordinary forms. Although flat superspace is contractible and has trivial G^∞ cohomology, it is the supersymmetry-invariant cohomology which is relevant to the superstring, and this is non-trivial. The quotient supermanifold M which was constructed allows one to visualize the topology resulting from the requirement of SUSY-invariance. The usual topological quantization argument for the coefficient of a WZ term leads to radically different conclusions when the action is Grassmann-valued. The topological quantization condition for the WZ term on M actually restricts its coefficient to be zero, while on flat superspace there is no quantization condition at all. Such a quantization condition would not have explained the unit coefficient of the WZ term in any case. At most it would have quantized the compactification scale of the soul directions on M in terms of the string tension.

The cohomological analysis of the WZ term differed from that in ordinary sigma models in two respects. First, the superstring action is Grassmann-valued. This meant that any phase ambiguity in the path integral was also Grassmann-valued and could not be eliminated by adjusting the WZ coefficient. Second, superspace is noncompact, unlike the group manifolds in most sigma models of interest. This made it possible for the invariant cohomology to differ from the usual de Rham cohomology and motivated the construction of the manifold M on which the identity of these cohomology groups was restored. Much of this analysis would apply to the study of WZ terms on noncompact sigma models. There too the WZ terms are classified by the invariant cohomology while the quantization conditions are determined by the de Rham cohomology, these being different in general. If a compactified manifold M can be constructed whose topology reflects the invariant cohomology, there may be interesting relations between the noncompact sigma model and the corresponding model on M. In the present work, the construction of M relied heavily on the polynomial behavior of superfields in soul directions. It is not clear how M might be constructed for general noncompact sigma models.

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