A Lipatov Bound for Φ_4^4 Euclidean Field Theory

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Abstract. We bound the large order behavior of these pieces of the renormalized perturbation expansion for Φ_4^4 which do not contain "renormalons" [1]. The bound we obtain has the form of the leading asymptotic behavior computed by the Lipatov method, with the exact value of the "Lipatov constant." Therefore, this paper is a step towards the rigorous study of the large order behavior of Φ_4^4 and towards a proof of existence of the first "renormalon" singularity which should prevent the theory from being Borel summable. Using the results of this paper and the techniques of [15], one can for instance improve [17] the estimate [18] on the finiteness of the radius of convergence of the Borel transform of renormalized Φ_4^4 and obtain that this radius is at least the exact value conjectured in [1].

I. Introduction

In this paper we prove an upper bound of the "Lipatov" type which applies in particular to the convergent graphs of Φ_4^4. We hope that this partial result will be relevant for a more complete future study of the large order behavior of Φ_4^4. This large order behavior is expected to be governed by the presence of a renormalon singularity [1] on the positive real axis of the Borel transform. It happens that this singularity is indeed closer to the origin than the "instanton" singularity on the negative real axis which is responsible for the "Lipatov" behavior of the Φ^4 theory at large order in lower dimensions [2, 3] (see Fig. 1). In the lower dimensions (1, 2, and 3) where the theory is superrenormalizable, the rigorous analysis of the leading behavior of the perturbative expansion has been completed [4–8]. Therefore, we think that the next important objective in this domain should be to find this leading behavior for the renormalized perturbation expansion of Φ_4^4, and more
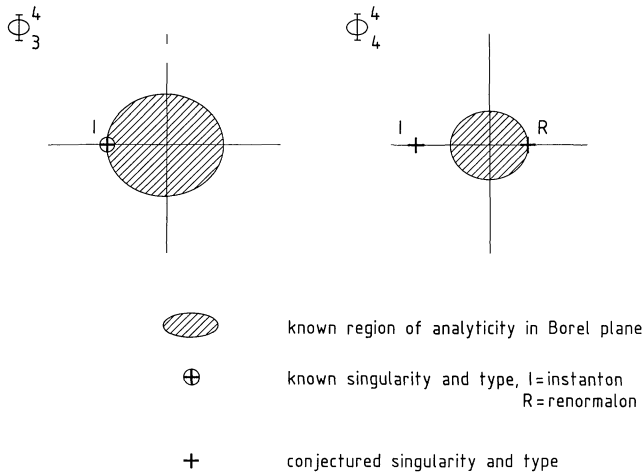
Fig. 1. The Borel plane of φ_3^4 and φ_4^4

precisely to prove the existence of the first renormalon singularity at the right place on the positive real axis. In our opinion one cannot consider that the rigorous study of Φ_4^4 even in the perturbative regime is satisfactory until such a proof is obtained. Indeed, although there is wide agreement that in this regime the theory should be trivial [9] and impressive rigorous results have been obtained [10, 11], we do not know of any direct proof that the renormalized perturbation series of Φ_4^4 actually do not converge, something that certainly nobody believes!

Let us sketch why the results of this paper might be useful for such a proof of existence of the first renormalon of Φ_4^4. Since the renormalon should dominate over the instanton, one needs an upper bound on the pieces of the perturbation expansion responsible for the instanton behavior, and a lower bound on the pieces of the expansion responsible for the renormalon behavior. This paper corresponds to the first part of this program (since upper bounds are usually easier to prove than lower bounds, one might argue that this is the easy part...).

Without entering too much into the technical details, let us remark that renormalons are due to the so-called "useless" pieces of the counterterms which are absorbed into the definition of effective constants in the modern "renormalization-group" improved versions of perturbation theory [12, 13], or in the "partly renormalized phase space expansion" of constructive field theory [14, 15]. Here we prove that the piece of perturbation theory which does not contain these "useless" counterterms is controlled by an upper bound of the "Lipatov" type, with the right value of the constant (Theorem II below). In particular, the sum of all completely convergent graphs at a given order, a piece of the expansion which is easy to define and contains already many graphs is controlled by such a bound (Theorem I below).

These bounds are obtained by combining the Sobolev inequality [16] with a cluster expansion in phase space. This cluster expansion works only for these pieces of the perturbation expansion which have "exponential decay" both in space

and in the separation between the "momentum slices" of the phase space expansion [14, 15] (these slices follow a geometric progression). But such pieces are exactly the ones which do not contain "useless" counterterms [15]. This explains why the "Lipatov bounds" apply only to the corresponding part of the perturbation expansion, hence explains Theorems I and II below.

Combining the bounds of this paper with the "partly renormalized phase space expansion" leads also to the following result [17]:

Theorem. *The radius of convergence of the Borel transform of the renormalized Φ_4^4 series (which was proved to be finite in [18, 13]), is at least $2/\beta_2$, where β_2 is the first non-vanishing coefficient of the β function (with the usual convention of using an interaction $\varphi^4/4!$, $\beta_2 = 3/16\pi^2$ and $2/\beta_2 = 32\pi^2/3$, but in this paper which uses an interaction φ^4, $\beta_2 = 9/2\pi^2$, and $2/\beta_2 = 4\pi^2/9 = (2/3)a^{-1}$, where $a = 3/2\pi^2$ is the constant appearing in the large order Lipatov analysis of this paper).*

At $2/\beta_2$ in the Borel plane the analysis of Parisi [1] can be transcribed in the language of [15], and ultraviolet divergences appear corresponding to six-point subgraphs and similar objects. To complete our program, hence to obtain a proof of existence of the renormalon singularity, one should find a way of ruling out the very unlikely possibility that all these ultra-violet divergences cancel each other completely. This is what we have not yet succeeded to do for one-component φ_4^4 (it seems to involve a subtle problem of linear independence of various singularities over the ring of the analytic functions). For N-component φ_4^4 and N large (but finite), the $1/N$ expansion introduces a hierarchy between these singularities, hence a proof of existence of the first renormalon in this case is possible [17].

II. The Results

Let us introduce some notations to state our results precisely. The perturbative expansion for a connected Schwinger function S_N in the theory with interaction $-g\varphi^4$ is a formal power series in g defined by

$$S_N \approx \sum_n (-g)^n a_n^R, \tag{2.1}$$

where a_n^R is the sum of all renormalized Feynman amplitudes for the connected graphs with n vertices. This requires the choice of a renormalization scheme. In this paper we will consider only a massive φ^4 theory for which one can use the B.P.H.Z. scheme of subtractions at 0 external momenta used in [18]. We will also restrict ourselves to the study of the series (2.1) for the 6–point function at 0 external momenta, which is a typical case. (The pressure, which was studied in [6–8] is no more typical since with our scheme it is 0 at every order in perturbation theory.) Of course, our bounds extend to any Schwinger function at any set of *fixed* external momenta, and even to the case of a massless theory at non-vanishing external momenta (see [19] for the corresponding subtraction scheme), which was the original case studied by Lipatov. It is in fact interesting to remark that in the "critical" dimension 4, the Lipatov constant a (see below) is a pure number which is independent of the mass scale of the theory (in contrast with what happens in lower dimensions).

The Lipatov analysis tells us that at large order one should expect:

$$a_n^R \approx n! \, a^n [1 + \varepsilon(n)]^n, \tag{2.2}$$

where a, the "Lipatov constant" is given by

$$a = e^{\{2 - \inf_{\varphi} (\frac{1}{2} \int [\nabla \varphi(x)]^2 d^4x - \text{Log} \int \varphi^4(x) d^4x)\}}. \tag{2.3}$$

To compute the value of a, let us remark that in 4 dimensions we have the Sobolev inequality:

$$\int \varphi^4(x) d^4x < (K \int [\nabla \varphi(x)]^2 d^4x)^2 \tag{2.4}$$

for all φ in the appropriate Sobolev space. The best constant K for which (2.4) holds is $K = \sqrt{\dfrac{(3/2)}{4\pi}}$ [16]. Then we have:

Lemma 1.
$$a = (4K)^2 = \frac{3}{2\pi^2}. \tag{2.5}$$

Proof. Let us take φ of the form αf, where $\int [\nabla f(x)]^2 d^4x = 1$. We optimize in (2.3) to get:

$$\inf_{\alpha, f} \left(\frac{\alpha^2}{2} - \text{Log}\, \alpha^4 \int \varphi^4(x) d^4x \right) = \inf_f (2 - \text{Log}\, 4^2 \int f^4(x) d^4x) = 2 - \text{Log}(4K)^2.$$

In fact, renormalization should disturb the Lipatov analysis in 4 dimensions [1] so that one should expect, by the "renormalon analysis" instead of (2.2):

$$a_n^R \approx (-1)^n n! \left(\frac{3a}{2} \right)^n [1 + \varepsilon(n)]^n. \tag{2.6}$$

We will show, however, that a "Lipatov bound" holds for large pieces of the perturbation expansion. Let us define first a_n^C as the sum of all Feynman amplitudes with n vertices associated to completely convergent graphs, i.e. graphs which have no divergent subgraphs, hence no subgraphs with less than 6 external legs. We have:

Theorem I. *There exists a function $\varepsilon(n)$ which tends to 0 as n tends to ∞ such that:*

$$a_n^C \leqq n! \, a^n (1 + \varepsilon(n))^n. \tag{2.7}$$

Remark that with our convention the amplitudes for convergent graphs are all positive since the factor $(-1)^n$ has been taken out in (2.1).

To state our second result, we need to introduce the phase space analysis of [14]. We may use a volume cutoff Λ which is a large compact box in $\mathbb{R}^4$, and an ultraviolet cutoff of order M^ϱ, where $M > 1$. The corresponding Gaussian measure $d\mu_{\Lambda, \varrho}$ has covariance:

$$C_{\Lambda, \varrho} = \int_{M^{-2(\varrho+1)}}^{\infty} e^{-\alpha(-\Delta_\Lambda + 1)} d\alpha, \tag{2.8}$$

where Δ_Λ is the Laplacian with 0 Dirichlet boundary conditions on Λ. We decompose the covariance and the corresponding fields into momentum slices [14]:

$$C_{\Lambda,\varrho} = \sum_{i=0}^{\varrho} C_i; \qquad C_i = \int_{M^{-2(i+1)}}^{M^{-2i}} e^{-\alpha(-\Delta_\Lambda+1)}d\alpha, \qquad i \geqq 1,$$

$$C_0 = \int_{M^{-2}}^{\infty} e^{-\alpha(-\Delta_\Lambda+1)}d\alpha; \tag{2.9}$$

$$d\mu = \prod_{i=0}^{\varrho} d\mu_i; \quad \varphi = \sum_{i=0}^{\varrho} \varphi_i.$$

$d\mu_i$ is the measure of covariance C_i and φ_i is the corresponding field.

Remark that one has the estimate [14] (c being a generic name for constants throughout the paper):

$$C_i(x,y) \leqq c \cdot M^{2i} e^{-(1/2)M^i|x-y|}. \tag{2.10}$$

For each graph G, we decompose the propagators of the lines of G as a sum over "momentum assignments" $\mu = \{i_\ell\}$, $\ell \in G$, as in [7]. For a given graph and momentum assignment μ, we call F an AL (almost local) subgraph of G if the indices of the internal lines of G are all higher than the indices of any of its external lines in the assignment μ. The AL subgraphs of G form a forest. The "partly renormalized" amplitude for G is then defined by introducing only, for a given momentum assignment, the counterterms corresponding to the divergent AL subgraphs of G in this assignment (the so-called "useful" counterterms) (see [12–15]). The corresponding "partly renormalized" n-th order of perturbation theory is called a_n^{PR}. Then we have:

Theorem II. *There exists a function $\varepsilon(n)$ which tends to 0 as n tends to ∞ such that:*

$$|a_n^{PR}| \leqq n! \, a^n (1 + \varepsilon(n))^n. \tag{2.11}$$

In the next section we will prove in detail Theorem I. Theorem II is then an easy generalization, since the basic ingredient of Theorem I, namely the "exponential decrease" of the graphs both in space and in the separation of momentum slices also holds for partly renormalized graphs [15]. Since the notations for renormalization are somewhat heavy, we will leave the detailed proof of Theorem II to the reader.

III. The Proof

1. Outline of the Proof

Roughly speaking a_n should be related to $\int d\mu(\varphi)(\int d^4x \varphi^4(x))^n$.

The first remark is that the Lipatov method is a Laplace expansion around critical fields which minimize the functional in (2.3). But these fields saturate the Sobolev inequality (see Lemma 1). On the other hand, the Sobolev inequality is a rigorous upper bound. Hence if one uses the Sobolev inequality to replace $\int \varphi^4$ by $(K \int (\nabla \varphi)^2)^2$, one gets an upper bound without losing anything from

the point of view of the Lipatov analysis. (A lower bound, however, like the one which has been obtained for φ_3^4 in [7, 8], would require a more subtle analysis of the speed at which, at large orders, the typical fields are peaked around the critical fields. Luckily we are not interested in such a bound for the program explained in the introduction.)

The second remark is that the Sobolev inequality changes one vertex into two disconnected vertices, so there is a possible loss of connectivity for the whole graph. But using the cluster expansion in phase space over lattices of cubes adapted to the scale of momenta, we have "exponential decrease" in every direction. Then either the graphs are spread over a large number p of cubes $(p > n(\text{Log} n)^{-\delta}, 0 < \delta < 1)$, many of them "sparsely populated" by vertices, and the corresponding contributions, thanks to the exponential decay, are negligible with respect to $n! K^n$ for *any* $K > 0$ as $n \to \infty$, or there are few cubes "densely populated" by vertices and it is only in this case that we apply the Sobolev inequality, because the loss of connectivity is then harmless since the total volume is small. It is from this second case that the dominant contributions giving rise to the results stated in Theorems I and II come.

To have a better intuition of how we want to apply the Sobolev inequality we prove first a very simple lemma:

Lemma 2. *Let* $\varphi(x)$ *be a field with a fixed cutoff* K, X_0 *a region of* R^4 *of size* $|X_0| \leq \dfrac{n}{(\text{Log} n)^\delta}$, $0 < \delta < 1$. *Then:*

$$\frac{1}{n!} \int d\mu(\varphi) \left(\int_{X_0} d^4 x \varphi^4(x) \right)^n \leq (1 + \varepsilon(n))^n n! \, (4K)^{2n}, \tag{3.1}$$

where $\varepsilon(n)$ *depends also on the cutoff* K. *Remember that by* (2.5) $a = (4K)^2$, *hence the right-hand side of* (3.1) *is the same as the one of* (2.7) *or* (2.11).

Proof. Applying the Sobolev inequality the left-hand side of (3.1) satisfies the inequality:

$$[(\text{l.h.s. } (3.1))] \leq \frac{1}{n!} K^{2n} \int d\mu(\varphi) \left(\int_{X_0} d^4 x (\nabla \varphi(x))^2 \right)^{2n}. \tag{3.2}$$

As the vertices are of $(\nabla \varphi)^2$ type, all the possible Feynman graphs that one obtains by integrating over $d\mu(\varphi)$ are closed loops, each propagator being ΔC, where C is the covariance of φ. Therefore,

$$\int d\mu(\varphi) \left(\int_{X_0} (\nabla \varphi(x))^2 \right)^{2n} = \sum_{p=1}^{2n} \frac{1}{p!} \sum_{\substack{t_1, \ldots, t_p \\ \sum_{j=1}^{p} t_j = 2n}} \sum_{\mathbb{P}} \prod_{j=1}^{p} [\text{closed loop } \mathscr{L}_j](2t_j - 1)!! \tag{3.3}$$

and

$$[\text{closed loop } \mathscr{L}_j] = \int_{X_0^{t_j}} dx_1 \ldots dx_{t_j} (-\Delta) C(x_1, x_2)(-\Delta) \ldots C(x_{t_j}, x_1),$$

where p is the number of closed loops, $t_1, \ldots, t_p$ are the numbers of vertices present in the $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_p$ loops, respectively. $\sum_{\mathbb{P}}$ is the sum over all the partitions of the

$x_1, \dots, x_{2n}$ vertices into groups of $t_1, t_2, \dots, t_p$ elements. Using the exponential decay of the covariance C it is easy to prove that

$$[\text{closed loop } \mathcal{L}_j] \leq C|X_0|, \tag{3.4}$$

where C depends on the cutoff. Therefore,

$$\int d\mu(\varphi) \left(\int_{X_0} (\nabla\varphi(x))^2\right)^{2n} \leq \sum_{p=1}^{2n} \frac{1}{p!} (C|X_0|)^p \sum_{\substack{t_1, \dots, t_p \\ \sum_{j=1}^{p} t_j = 2n}} \frac{(2n)! \, (2t_1-1)!! \dots (2t_p-1)!!}{t_1! \dots t_p!}$$

$$\leq (2n)! \, 2^{2n} \sum_{p=1}^{2n} \frac{(C|X_0|)^p}{p!} \binom{2n}{p}. \tag{3.5}$$

It is easy to prove that if $C|X_0| < \dfrac{n}{(\mathrm{Log}\,n)^\delta}$, $0 < \delta < 1$,

$$\sum_{p=1}^{2n} \frac{(C|X_0|)^p}{p!} \binom{2n}{p} \leq e^{\frac{Cn}{(\mathrm{Log}\,n)^\varepsilon}} \quad \text{with} \quad \varepsilon > 0$$

$$\leq (1 + \varepsilon(n))^n. \tag{3.6}$$

Therefore, remembering that $(2n)! \leq 2^{2n}(n!)^2$, the lemma is proven.

Remark. This simple lemma suggests which are the dominant contributions to a_n^C for $n \to \infty$. The main part of the work consists in making this argument rigorous. This means that we must be able to distinguish different situations according to whether $|X_0|$ is large or small (because we need a cluster expansion) and prove that the contributions to a_n^C when $|X_0|$ is large are negligible. Moreover, as a_n^C has to be computed in the limit of the u.v. cutoff going to ∞, we must also be able to get rid of the cutoff dependence in Lemma 1. This requires that we take care of the convergent factors which appear when we consider the different scales of momenta.

2. The General Definitions, the Cluster Expansion

We start considering b_n, the sum of all the n-th order graphs (connected or not) with UV and volume cutoffs.

$$b_n = \frac{1}{n!} \int \varphi(y_1) \dots \varphi(y_N) \left(\int_\Lambda \varphi^4(x) d^4x\right)^n d\mu_\Lambda(\varphi), \tag{3.7}$$

where $\varphi = \sum_{i=0}^{\varrho} \varphi_i$ [see (2.9)] and

$$C_{\Lambda,\varrho} = \sum_{i=0}^{\varrho} C_i \quad [\text{see (2.8)}].$$

$\varphi(y_1), \dots, \varphi(y_N)$ are the external fields (we will take $N = 6$ and 0 external momenta, hence the y_j's are integrated over Λ). For each scale i, $i = 0, \dots, \varrho$, we introduce a lattice $\mathbb{D}_i$ covering the volume Λ by cubes of side M^{-i}, the cubes of $\mathbb{D}_{i-1}$ being the union of M^4 cubes of $\mathbb{D}_i$ (M is a fixed integer). We define $\mathbb{D} = \bigcup \mathbb{D}_i$. We start

decomposing b_n as a sum of terms such that in each of them all the $(4n+N)$ fields in (3.7) have a momentum assignment μ, i.e. we write:

$$\varphi^4(x) = \sum_{i_1,i_2,i_3,i_4} \varphi_{i_1}(x)\varphi_{i_2}(x)\varphi_{i_3}(x)\varphi_{i_4}(x)$$

$$= \sum_{\mu(v)} \varphi_{i_1(v)}\varphi_{i_2(v)}\varphi_{i_3(v)}\varphi_{i_4(v)}(x_v), \tag{3.8}$$

where $\mu(v) = (i_1(v), i_2(v), i_3(v), i_4(v))$, and we can also assume (multiplying by a combinatorial factor that we do not write explicitly) that $i_1(v) \geqq i_2(v) \geqq i_3(v) \geqq i_4(v)$. Hence

$$\prod_{v=1}^{n} \varphi^4(x_v) = \sum_{\mu} \prod_{v=1}^{n} (\varphi_{i_1(v)}\varphi_{i_2(v)}\varphi_{i_3(v)}\varphi_{i_4(v)})(x_v). \tag{3.9}$$

The same, of course, has to be done for the external fields $\varphi(y_1), ..., \varphi(y_N)$ which, anyway as one can easily see, do not play any relevant role in the game except that of providing at the end enough external legs not to require any final subtraction.

In each term of our sum we want that the vertex v with position x_v lies in a well defined cube of scale $i_1(V)$, which will be called the "localization cube" of V. In general, a field $\varphi(x_v)$ of index $i_2(v)$, $i_3(v)$, or $i_4(v)$ has also to be thought of as being associated to the cube of $\mathbb{D}_{i_2}$, $\mathbb{D}_{i_3}$, or $\mathbb{D}_{i_4}$, respectively, which contains the vertex position x_v. We call

$$\tilde{X}_0 = \{\Delta_{i_1(v)}\}, \qquad v=1, ..., n, \tag{3.10}$$

the set of all localization cubes.

Similarly, we define

$$\tilde{X}_0^{(2)((3),(4))} = \{\Delta_{i_2(v)(i_3(v), i_4(v))}\}, \qquad v=1, ..., n, \tag{3.11}$$

and the set

$$X \equiv \tilde{X}_0 \bigcup_{\ell=2}^{4} \tilde{X}_0^{(\ell)} \tag{3.12}$$

that we will use later on. X is uniquely defined by $\tilde{X}_0$ and μ: $X = X(\tilde{X}_0, \mu)$. Therefore, we decompose b_n in the following way:

$$b_n = \sum_{\mu} \sum_{q=1}^{n} \sum_{\substack{n_1,...,n_q \\ \sum_{\ell=1}^{q} n_\ell = n}} \sum_{|X_0|=q} \frac{1}{n!} \sum_{\mathbb{P}} \int_{\Delta_1^{n_1} \times ... \times \Delta_q^{n_q}} dx_{i_1} ... dx_{i_n} \int d\mu(\varphi)R(\varphi),$$

where $$\tag{3.12}$$

$$R(\varphi) \equiv R(\varphi; \mu) \equiv \prod_{v=1}^{n} (\varphi_{i_1(v)}\varphi_{i_2(v)}\varphi_{i_3(v)}\varphi_{i_4(v)})(x_v) \prod_{J=1}^{6} \varphi_{i(J)}(x_J),$$

and where now we define $X_0 = \{\Delta_1, ..., \Delta_q\}$. $(n_1, ..., n_q)$ are the numbers of vertices in the cubes $\Delta_1, ..., \Delta_q$, respectively. $\sum_{\mathbb{P}}$ is the sum over the ways in which the n vertices can be regrouped in groups of $n_1, ..., n_q$ elements, and $\mathbb{P} \equiv (i_1, ..., i_n)$. Remark that X_0 is now an ordinary set of different cubes, in contrast with $\tilde{X}_0$, which was a sequence of not necessarily distinct cubes. Observe also that the lowest

value of q is μ-dependent. For simplicity we avoid hereafter to write explicitly the product $\varphi(y_1) \ldots \varphi(y_{N=6})$ in $R(\varphi)$, and we just remember at the end that it is present. We apply the cluster expansion to each term of the sum (3.12). We call B_i the set of the 3-dimensional sides of the cubes of $\mathbb{D}_i$ and put $B = \bigcup B_i$. For each $\gamma \subset B_i$ we define

$$C_{i,\gamma} = \int_{M^{-2(i+1)}}^{M^{-2i}} e^{-\alpha(-\Delta_{\Lambda,\gamma} + 1)} d\alpha, \tag{3.13}$$

where $\Delta_{\Lambda,\gamma}$ is the Laplacian with 0 Dirichlet boundary conditions on $\partial\Lambda \cup \gamma$. To each $b \in B_i$ we associate a variable s_b, $0 \leq s_b \leq 1$, and we define

$$C_i(\{s\}) = \sum_{\gamma \subset B_i} \prod_{b \in \gamma} (1 - s_b) \prod_{b \notin \gamma} s_b C_{i,\gamma}. \tag{3.14}$$

We order B_i in an arbitrary way and apply a cluster expansion to each term of the sum (3.12). Namely, we define I_b and P_b by

$$I_b\left[\left(\prod_j \frac{d}{ds_{b_J}} C_i(J)\right)\left(\prod_K C_i(K)\right)\right] = \left(\prod_j \frac{d}{ds_{b_J}} C_i(J)\right)\Bigg|_{s_b=1} \left(\prod_K C_i(K)\right)\Bigg|_{s_b=0}, \tag{3.15_1}$$

$$P_b\left[\left(\prod_j \frac{d}{ds_{b_J}} C_i(J)\right)\left(\prod_K C_i(K)\right)\right] = \left(\prod_j \frac{d}{ds_{b_J}} C_i(J)\right)\Bigg|_{s_b=1} \int_0^1 \frac{d}{ds_b}\left(\prod_K C_i(K)\right) ds_b,$$

and

$$\int d\mu(\varphi) R(\varphi) = \int d\mu_{\{s\}}(\varphi) R(\varphi)|_{\{s\}=\{1\}} = \prod_{i=0}^{\varrho} \prod_{b \in B_i} (I_b + P_b) \int d\mu_{\{s\}}(\varphi) R(\varphi), \tag{3.15_2}$$

where we omit the index Λ in the measure (with Dirichlet b.c. on $\partial\Lambda$) to simplify the notations. Using the path representation of the propagator as in [20, 26], it is easy to check that:

$$\left|\frac{d}{ds_b} C_i(x, y)\right| \leq (\text{const}) M^{2i} e^{-\frac{1}{2}M^i d(x, b, y)}, \tag{3.16}$$

where $d(x, b, y) = \text{dist}(x, b) + \text{dist}(b, y)$.

Remark. The definition (3.15) is such that each covariance will be derived at most once. Therefore, developing the product $\prod_b (I_b + P_b)$ we cannot arbitrarily interchange the I_b and the P_b's, but we must keep them fixed in the order they appear.

Developing the multiple products in (3.16), we have

$$\int d\mu(\varphi) R(\varphi) = \prod_{i=0}^{\varrho} \sum_{\Gamma_i}\left[\sum_{\{x,x',b\}} \prod_{b \in \Gamma_i} \frac{d}{ds_b} C_i(x, x')\right]\int d\mu_\Gamma(\varphi) \bar{R}(\varphi), \tag{3.17}$$

where we always omit to write explicitly the integrals with respect to the s_b variables, where Γ_i denotes a generic set of $b = B_i$ and $\sum_{\{x,x',b\}}$, hereafter written as $\sum_{\mathcal{I}}$, is the sum over all possible ways in which couples of vertices can be associated to the b's $\in \Gamma_i$.

$d\mu_\Gamma = \prod_{i=0}^{\varrho} d\mu_{i,\Gamma_i}$, where $d\mu_{i,\Gamma_i}$ is the Gaussian measure over the field φ_i whose covariance has Dirichlet b.c. on $\partial\Lambda \cup \Gamma_i^c$.

As usual at each scale i, Γ_i can produce different connected regions $Y_i^{(1)}, \ldots, Y_i^{(s)}$ such that $d\mu_i$ has Dirichlet b.c. on $\partial Y_i^{(1)}, \ldots, \partial Y_i^{(s)}$. Nevertheless some derived covariances due to the previous remark can connect some of these regions. It is useful to think this connectedness property associated to the cluster expansion in terms of connection between cubes belonging to $X \cap \mathbb{D}_i \equiv X_i$. In fact [see (3.11)] to each field $\varphi_{i_\ell(v)}(x_v)$ we can associate a well defined cube $\Delta_{i_\ell(v)} \in X$ such that $x_v \in \Delta_{i_\ell(v)}$ [for any term of (3.12)]. We will say that two cubes in X_i are elementary connected if they belong to the same region $Y_i^{(k)}$ with Dirichlet b.c. or if there is a derivated covariance C_i which connects two vertices belonging to these two cubes. Such a connection is called *elementary horizontal connection*.

In each slice i a generic term of the cluster expansion will decompose X_i in the union of disconnected regions $X_i^{(1)}, \ldots, X_i^{(s_i)}$. A region $X_i^{(J)}$ is connected if any pair of cubes in it can be linked through a chain of cubes of $X_i^{(J)}$ which are connected through elementary connections.

It is also natural to introduce between the cubes of X the notion of *elementary vertical connection*: two cubes are elementary vertically connected if they are associated to the same vertex and have different indices: $\{\Delta_{i_1(v)}, \Delta_{i_2(v)}, \Delta_{i_3(v)}, \Delta_{i_4(v)}\}$ are elementary vertically connected and, of course, $\Delta_{i_1(v)} \subseteqq \Delta_{i_2(v)} \subseteqq \Delta_{i_3(v)} \subseteqq \Delta_{i_4(v)}$.

These rules can be pictured as in Fig. 2 or [14, 15]; we represent propagators by horizontal lines joining cubes with the same index, and vertices by vertical dashed lines joining the cubes which contain the position of the vertex and have indices corresponding to the 4 half-legs (or fields) hooked to the vertex.

We say that a region $Q \subset X$ is connected if all the cubes in Q are connected one to each other through a chain of horizontal or vertical connections. For any Q we
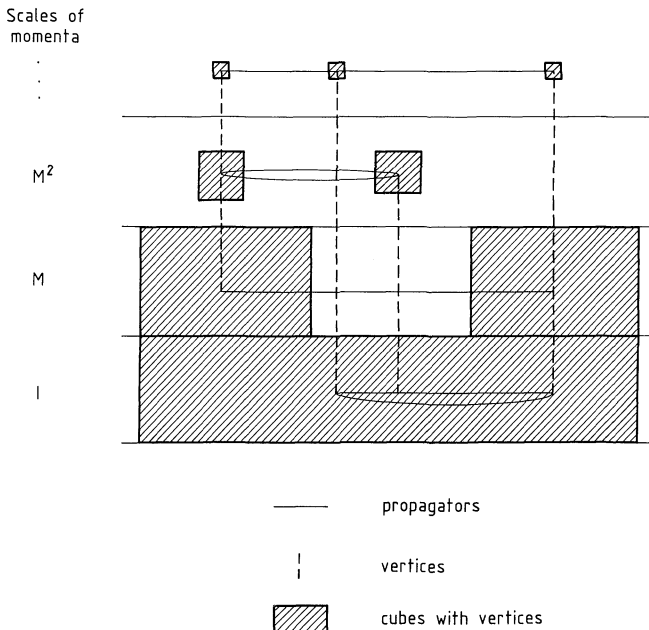

Fig. 2. The phase space expansion

call $Q_{\geq i}$ the set $Q \cap \left(\bigcup_{j \geq i} \mathbb{D}_j \right)$. If $Q_{\geq i}$ is connected through elementary connections *of slices* $j \geq i$, and maximal with these properties, we call it an "almost local" subgraph or in short an "AL subgraph" of X.

Hence the generic term of the cluster expansion made at every scale produces at the level i a decomposition of X in AL subgraphs $X_{\geq i}^{(\ell)}$ with ℓ running on a finite set. As what only appears in the integration region are the cubes of X_0 and not of X, we define also

$$X_{0,\geq i}^{(\ell)} = X_{\geq i}^{(\ell)} \cap X_0. \tag{3.18}$$

It turns out useful to decompose the $\sum_r \equiv \prod_{i=0}^{\varrho} \sum_{\Gamma_i}$ in a product of smaller sums so that for each sum we know at each scale i how many disconnected regions $X_{\geq i}^{(\ell)}$ are present and also at which scale two regions previously disconnected become connected. This can be easily summarized by introducing the notion of tree (connected to the notion introduced in [13] although different). A tree θ (see Fig. 3) is geometrically defined as in [13]; at each bifurcation b a frequency h_b is associated, at each scale i the number of branches that the slice of frequency i crosses tells us the number of disconnected regions. The frequency at a bifurcation
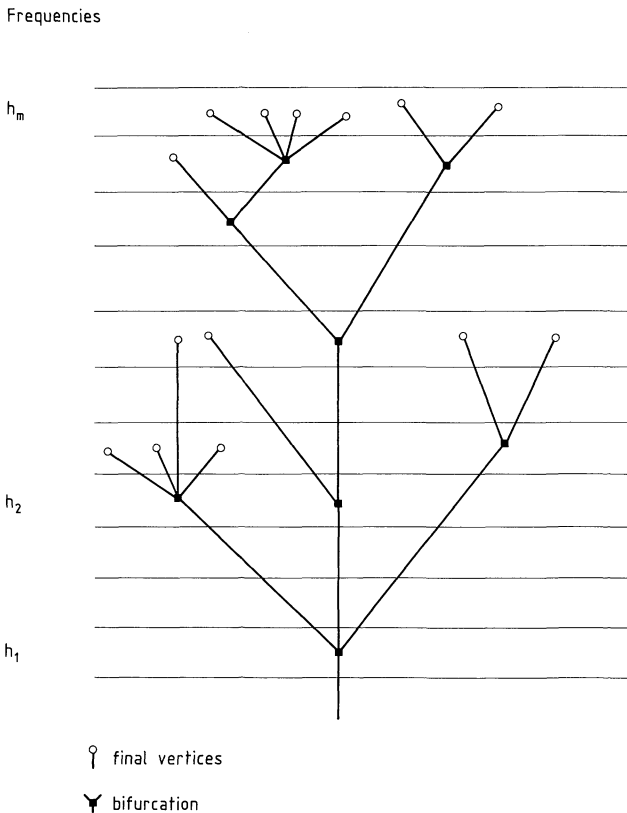

Fig. 3. The tree θ

b defines at which scale some of the disconnected regions at higher scales become connected. To consider all the possibilities we have to sum over all the frequencies $\{h_b\}$ and over all the possible trees. Of course, the frequencies h_b have to satisfy $h_b > h_{b'}$ if b is a bifurcation higher than b' in the tree and they can run only over the values provided by μ (μ is a sequence of $4n$ numbers). The final vertices (the top of the final lines), whose number we call $v(\theta)$, correspond to connected subsets of X_0 which at that frequency are disconnected from all the other subsets of X. The sum over these trees is of the following type:

$$\sum_{\kappa} \sum_{\{h_b\}} \sum_{\theta,\, v(\theta) = \kappa} \frac{1}{n(\theta)} \cdots , \tag{3.19}$$

where $n(\theta) = \prod_b s_b!$, s_b is the number of branches merging in the bifurcation b. The factor $1/n(\theta)$ arises as we sum independently over all the possible ways of characterizing the final vertices saying which cubes of X_0 belong to each of them. Remark that it arises also because of our convention that two trees with two subtrees, merging in a given bifurcation, when permutated are thought of as different (if these two subtrees have different shapes).

We can therefore write a general decomposition of b_n in the following way

$$b_n = \sum_{q=1}^{n} \sum_{\kappa=1}^{q} \sum_{n_1,\ldots,n_q} \sum_{\theta,\, v(\theta)=\kappa} \sum_{\mu} \sum_{\{h_b\}} \sum_{|X_0|=q} {\sum_{\Gamma}}' \sum_{\mathscr{I}} \left[\prod_{i=0}^{\varrho} \left(\prod_{b \in \Gamma_i} \frac{d}{ds_b} C_i(x,x') \right) \right]$$

$$\times \sum_{\mathbb{P}} \left[\frac{1}{n(\theta)} \frac{1}{n!} \int_{\Lambda_{\Gamma}^{n_1} \times \ldots \times \Lambda_{q}^{n_q}} dx_{i_1} \ldots dx_{i_n} \int d\mu_{\Gamma} \bar{R}(\varphi) \right], \tag{3.20}$$

where $\sum_{\mu}$ has now some constraints due to the parameters $q, \kappa, \{n_1, \ldots, n_q\}$ and θ fixed by the previous sums and $\sum_{\Gamma}'$ is the collection of sums over Γ_i, subject to the constraints that $i \in \mu$, and that the connected regions produced are compatible with the tree structure θ. We need to refine further our decomposition asking that in each term of b_n one also fixes how many cubes are in each of the κ disconnected pieces; in this case the initial sum becomes:

$$\sum_{q=1}^{n} \sum_{\kappa=1}^{q} \sum_{\substack{(q_1,\ldots,q_\kappa)(n_1,\ldots,n_q) \\ \sum_{j=1}^{\kappa} q_j = q}} \sum .$$

In the decomposition (3.20) of b_n we still have to impose some constraints as in fact our goal is not to bound b_n but a_n^C, the sum of all Feynman amplitudes with n vertices associated to completely convergent graphs. Therefore, we require that the terms of (3.20) satisfy some conditions to exclude unwanted contributions.

i) The first condition is that after all the cluster expansions have been performed in each term of (3.20) the set $X = X_{\geq 0}$ be connected. This is imposed by the obvious fact that the trees θ we are considering are trees and not forests, i.e. they end on the bottom with a single line (the trunk). The excluded contributions correspond, after the Gaussian integration has been performed, to some (not all!) disconnected graphs which are not present in a_n.

ii) A second condition on (3.20) has to be imposed to exclude ultraviolet divergences and therefore also the need of counterterms. In (3.20) fixed

$(q, \kappa, \{q\}, \{n\}, \vartheta, \mu, \{h_b\})$ one has still to sum over $(X_0, \Gamma, \mathscr{J}, \mathbb{P})$. Each set $(X_0, \mathbb{P})$ defines completely X [see (3.12)] as we know how many cubes are in X_0, where they are located and how many and which vertices v are contained in each cube of X_0 and therefore of X. As θ has been previously fixed we know also completely at each level the structure of "AL subgraphs" $\{X_{\le i}^{(\ell)}\}$, that is the family of the maximal connected sets of cubes of $\bigcup_{j \ge i} \mathbb{D}_j$. The requirement to impose is that each "AL subgraph" in each term of the huge sum (3.20) has at least 5 (in fact 6) external legs, where an external leg to $X_{\le i}^{(\ell)}$ is a field $\varphi_h(x_v)$ with $x_v \in X_{\le i}^{(\ell)}$ and $h < i$, or one of the N (here $N = 6$) true external legs. Therefore, we put equal to 0 all the terms of the sum in which this condition is not satisfied. This constraint which will be denoted by $\chi(\text{c.s.c.})$ (convergent subgraph condition) is essentially a constraint over $(X_0, \mathscr{J}, \mathbb{P})$. Imposing conditions i) and ii) we get rid of the divergent AL subgraphs but still we are neither restricted to connected nor to completely convergent Feynman graphs. Nevertheless, as all the amplitudes are positive this just produces an overestimate for our upper bound (which will turn out to be irrelevant).

In a more compact notation, denoting b_n with conditions i) and ii) imposed by $S_n(\Lambda, \varrho)$, we have:

$$S_n(\Lambda, \varrho) = \sum_{(q, \kappa, \{q\}, \{n\}, \theta)} \sum_{\mu, \{h_b\}} \sum_{(X_0, \Gamma, \mathscr{J}, \mathbb{P})} \chi(\text{c.s.c.}) \cdot \left[\prod_{b \in \Gamma} \frac{d}{ds_b} C_{i(b)}(x, x') \right]$$

$$\times \left[\frac{1}{n(\theta)n!} \int_{\Lambda_1^{n_1} \times \ldots \times \Lambda_q^{n_q}} dx_{i_1} \ldots dx_{i_n} \int d\mu_\Gamma(\varphi) \bar{R}(\varphi) \right]. \tag{3.22}$$

This decomposition given, we start studying the case where q or $\gamma \equiv |\Gamma|$ is large (which here means $\ge n(\text{Log}\,n)^{-\delta}$ for some $\delta \in]0, 1[$).

This case corresponds to the situation envisaged in Sect. III.1 when p is large. We prove that the total contribution $S_{n, \text{large}}(\Lambda, \varrho)$ to $S_n(\Lambda, \varrho)$ coming from those terms in (3.22) satisfying one or both of these conditions is negligible with respect to $(4K)^{2n}n!$ when $n \to \infty$.

The other case when both q and γ are "small" ($\le n(\text{Log}\,n)^{-\delta}$) will produce essentially the estimate for a_n^C suggested by Lemma 2, but with $\varepsilon(n)$ cutoff-independent.

3. The Case q or γ Large

The goal of this subsection is to prove the following lemma.

Lemma 3. *If q or γ is larger than $n(\text{Log}\,n)^{-\delta}$, there exists $\tilde{\varepsilon}$ depending on δ, such that the following bound holds, uniform in ϱ and Λ:*

$$|S_{n, \text{large}}(\Lambda, \varrho)| \le C^n \frac{n!}{e^{n(\text{Log}\,n)^{\tilde{\varepsilon}}}}. \tag{3.23}$$

Proof. The proof of this result is straightforward but very long. We start discussing the general strategy. We have to distinguish two subcases:

 I) $|X_0| \ge n(\text{Log}\,n)^{-\delta'}$, $0 < \delta' < 1$,

 II) $|X_0| < n(\text{Log}\,n)^{-\delta'}$ and $\gamma > n(\text{Log}\,n)^{-\delta}$, $0 < \delta < \delta'$.

δ and δ' will be fixed during the proof ($\delta' = 9/10$ and $\delta = 1/10$ are o.k.).

Case I. In Appendix A we prove the following bound for the cluster expansion:

$$\sum_{\Gamma}^{*} \sum_{\mathscr{J}} \left[\prod_{b \in \Gamma} \frac{d}{ds_b} C_{i(b)}(x, x') \right] \leq (\text{const})^n \prod_{i=0}^{\varrho} M^{2i\gamma_i} e^{-\alpha_{\ell}^{s_i} d(X_i^{(\ell)})} \qquad (3.24)$$

for some α, $0 < \alpha < 1$, where γ_i is the number of bonds of Γ of scale i $(\gamma_i \equiv |\Gamma_i|)$; $d(X_i^{(\ell)})$ is the scaled tree distance of the set of cubes $X_i^{(\ell)} \equiv X_{\geq i}^{(\ell)} \cap \mathbb{D}_i$ and s_i is the number of disconnected $X_{\geq i}^{(\ell)}$ regions, which is determined by θ and $\{h_b\}$. We bound $S_{n,1}(\Lambda, \varrho)$ (1 means that we are in the case I for $S_{n,\text{large}}$) in the following way:

$$|S_{n,1}(\Lambda, \varrho)| \leq (\text{const})^n \sum_{(q, \kappa, \{q\}, \{n\})} \sum_{\mu} \sum_{\{h_b\}} \sum_{\theta} \sum_{X_0, |X_0| = q} \prod_{i=0}^{\varrho} M^{2i\gamma_i} e^{-\alpha_{\ell}^{s_i} d(X_i^{(l)})}$$

$$\times \left\{ \frac{1}{n(\theta) \prod_{\Delta \subset X_0} n(\Delta)!} \sup_{(\Gamma, \mathbb{P}, \mathscr{J})} \left| \chi(\text{c.s.c.}) \int_{\Delta_1^{n_1} \times \ldots \times \Delta_q^{n_q}} dx_{i_1}, \ldots, dx_{i_n} \int d\mu_{\Gamma}(\varphi) \bar{R}(\varphi) \right| \right\} ,$$

$$\qquad (3.25)$$

where we used that

$$\frac{1}{n!} \sum_{\mathbb{P}} 1 = \frac{1}{\prod_{\Delta \subset X_0} n(\Delta)!} , \qquad (3.26)$$

$n(\Delta)$ being the number of fields $\varphi_{i_1(v)}(x_v)$ in $R(\varphi)$ with $x_v \in \Delta$. It is immediate to recognize that $\sum_{(q, \kappa, \{q\}, \{n\})} 1 \leq (\text{const})^n$; therefore, we look for an estimate of the brackets in (3.25) uniform in $(q, \kappa, \{q\}, \{n\})$ such that we can perform the remaining sums and get the estimate of the lemma.

We write

$$\bar{R}(\varphi) = \left(\prod_{J=1}^{n} \varphi_{i_1(J)}^{\alpha_1(J)}(x_J) \right) \left(\prod_{J=1}^{n} \varphi_{i_2(J)}^{\alpha_2(J)}(x_J) \right) \left(\prod_{J=1}^{n} \varphi_{i_3(J)}^{\alpha_3(J)}(x_J) \right) \left(\prod_{J=1}^{n} \varphi_{i_4(J)}^{\alpha_4(J)}(x_J) \right)$$

$$\equiv A(\varphi) B(\varphi) C(\varphi) D(\varphi), \qquad (3.27)$$

where $\alpha_{\ell}(J) = +1$ or 0 depending on $(\vartheta, \mu, X_0, \Gamma)$. By Hölder inequality

$$\left| \int d\mu_{\Gamma}(\varphi) \bar{R}(\varphi) \right| \leq \left(\int A^4(\varphi) d\mu(\varphi) \right)^{1/4} \ldots \left(\int D^4(\varphi) d\mu(\varphi) \right)^{1/4}. \qquad (3.28)$$

To estimate the right-hand side of (3.28), we denote, for $\Delta \in \mathbb{D}_i$, $\ell_1(\Delta)$ [respectively $\ell_j(\Delta)$] the number of fields $\varphi_{i_1(v)}(x_v)$ [respectively $\varphi_{i_j(v)}(x_v)$] with $i_1(v) = i$ [respectively $i_j(v) = i$] and $x_v \in \Delta$. The number of fields with frequency i in $A^4(\varphi)$ is $4 \sum_{\Delta \subset X_0 \cap \mathbb{D}_i} \ell_1(\Delta)$. We order the cubes in $X_0 \cap \mathbb{D}_i$, so that

$$\ell_1(\Delta_1) \geq \ell_1(\Delta_2) \geq \ldots \geq \ell_1(\Delta_{\# X_0 \cap \mathbb{D}_i}), \qquad (3.29)$$

and perform the integration with respect to $d\mu_i(\varphi)$, that is we sum over all possible contraction schemes, contracting the variables in the order given by the ordering (3.29) of the cubes to which they belong. Taking into account the exponential decay of the covariance C_i [see (2.10)] we get

$$\int A_i^4(\varphi) d\mu_i(\varphi) \leq (\text{const})^{\# (A_i)} \prod_{\Delta \subset X_0 \cap \mathbb{D}_i} (2\ell_1(\Delta)!) M^{i \# (A_i)}$$

$$\leq (\text{const})^{\# (A_i)} M^{i \# (A_i)} \prod_{\Delta \subset X_0 \cap \mathbb{D}_i} (\ell_1(\Delta)!), \qquad (3.30)$$

where $\#(A_i) \equiv \sum\limits_{\Delta \subset X_0 \cap \mathbb{D}_i} \ell_1(\Delta)$. Finally, we get

$$\left(\int A^4(\varphi)d\mu(\varphi)\right)^{1/4} \leqq (\mathrm{const})^n \prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^{1/2} M^{\sum_i i\#(A_i)}, \tag{3.31}$$

and similar results when A is substituted by B, C, D with the obvious replacements

$$\prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^{1/2} \to \prod_{\Delta \subset X_0^{(J)}} (\ell_J(\Delta)!)^{1/2},$$

$$M^{\sum_i i\#(A_i)} \to M^{\sum_i i\#(B_i, C_i, D_i)}.$$

As $\ell_1(\Delta) \leqq n(\Delta)$, $\forall \Delta \in X_0$ [in $R(\varphi)$ this inequality would be equality] we have remembering the volume factor $\prod\limits_v M^{-4i_1(v)}$ due to the integration over the Δ's:

$$\prod_{i=0}^{\varrho} M^{i2\gamma_i}\{\text{bracketed term in (3.25)}\}$$

$$\leqq \prod_v M^{-(i_1(v)-i_2(v))} M^{-(i_1(v)-i_3(v))} M^{-(i_1(v)-i_4(v))}$$

$$\times \frac{1}{n(\theta) \prod\limits_{\Delta \subset X_0} (n(\theta)!)^{1/2}} \prod_{\Delta \subset X_0^{(2)}} (\ell_2(\Delta)!)^{1/2} \prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{1/2} \prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2}, \tag{3.33}$$

and

$$|S_{n,1}(\Lambda, \varrho)| \leqq (\mathrm{const})^n \sum_\mu \sum_{\{h_b\}} \sum_\theta \left\{ \sum_{X_0, |X_0|=q} e^{-\alpha \sum_i \sum_{\ell=1}^{s_i} d(X_{0,i}^{(\ell)})} \prod_v M^{-(i_1(v)-i_2(v))} \right.$$

$$\times M^{-(i_1(v)-i_3(v))} M^{-(i_1(v)-i_4(v))} \frac{1}{n(\theta) \prod\limits_{\Delta \subset X_0} (n(\Delta)!)^{1/2}}$$

$$\times \left. \prod_{\Delta \subset X_0^{(2)}} (\ell_2(\Delta)!)^{1/2} \prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{1/2} \prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2} \right\}. \tag{3.34}$$

To estimate the brackets in (3.34) we proceed in the following way, we first bound the $\ell_j(\Delta)$, $j=1, \ldots, 4$ with the corresponding (larger) values we obtain substituting $R(\varphi)$ to $\bar{R}(\varphi)$, and by a slight abuse of notations we still denote them in the same way period. Then the following relations hold

$$\sum_{\Delta \subset X_0^{(J)}} \ell_J(\Delta) = n, \quad J = 1, 2, 3, 4. \tag{3.35}$$

The following bounds are then proved in Appendix B:

$$\prod_{\Delta \subset X_0^{(J)}} \ell_J(\Delta)! \leqq \prod_v M^{(4+\varepsilon)(i_1(v)-i_J(v))} \prod_{\Delta \subset X_0} \ell_1(\Delta)!, \tag{3.36}$$

where ε has to satisfy $2M^\varepsilon < 1$.

We plug the following inequality, with $\eta > 0$ and small

$$\prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{1/2} \prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2}$$

$$\leqq \prod_v M^{\eta(4+\varepsilon)(i_1(v)-i_3(v))} \prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^\eta \prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{\frac{1}{2}-\eta} \prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2}$$

$$\tag{3.37}$$

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into the right-hand side of (3.34), and we get

$$
|S_{n,1}(\Lambda,\varrho)| \leqq \left(\prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^\eta \prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{\frac{1}{2}-\eta} \prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2} \right) \cdot (\text{const})^n
$$

$$
\times \sum_\mu \sum_{\{h_b\}} \sum_\theta \left\{ \sum_{X_0, |X_0|=q} e^{-\alpha \sum_i \sum_{\ell=1}^{s_i} d(X_{0,i}^{(\ell)})} \right.
$$

$$
\times \prod_v M^{-(i_1(v)-i_2(v))-(i_1(v)-i_3(v))-(i_1(v)-i_4(v))}
$$

$$
\times \frac{1}{n(\theta) \prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^{1/2}} \prod_{\Delta \subset X_0^{(2)}} (\ell_2(\Delta)!)^{1/2} \right\} . \tag{3.38}
$$

Using (3.35)

$$
\prod_{\Delta \subset X_0} \ell_1(\Delta)! \leqq \prod_{\Delta \subset X_0} e^{\ell_1(\Delta)} (\ell_1(\Delta)-1)! \leqq e^n (n-q)! \leqq e^n \frac{n!}{q!} \leqq e^n \frac{n!}{\left(\frac{n}{(\text{Log}\, n)^{\delta'}} \right)!} ,
$$

and moreover,

$$
\prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{\frac{1}{2}-\eta} \prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2} \leqq (n!)^{1-\eta} c^n , \tag{3.39}
$$

so that for some (δ and η dependent) $\tilde{\varepsilon}$:

$$
\left(\prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^{1/2} \right)^\eta \left(\prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{\frac{1}{2}-\eta} \right) \left(\prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2} \right)
$$

$$
\leqq (\text{const})^n \frac{n!}{\left(\frac{n}{(\text{Log}\, n)^{\delta'}}! \right)^\eta} \leqq (\text{const})^n \frac{n!}{e^{n(\text{Log}\, n)^{\tilde{\varepsilon}}}} , \tag{3.40}
$$

we reduce the proof of Case I of Lemma 3 to the following bound:

$$
\sum_\mu \sum_{\{h_b\}} \sum_\theta \left\{ \sum_{|X_0|=q} e^{-\alpha \sum_i \sum_{\ell=1}^{s_i} d(X_{0,i}^{(\ell)})} \prod_v M^{-(i_1(v)-i_2(v))} M^{(1-\varepsilon_1)(i_1(v)-i_3(v))} M^{-(i_1(v)-i_4(v))} \right.
$$

$$
\times \frac{1}{n(\theta) \prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^{1/2}} \prod_{\Delta \subset X_0^{(2)}} (\ell_2(\Delta)!)^{1/2} \right\} \leqq (\text{const})^n , \tag{3.41}
$$

where $\varepsilon_1 = \eta(4+\varepsilon)$.

This estimate which is based essentially on (3.36), on the presence of the scaled tree decay factor and on the constraint that all the AL subgraphs are convergent is proven in Appendix B. This completes the proof of the Case I.

Case II. The proof in this case goes essentially as in the previous one. The only difference is that, as now q can be small, we cannot get the term $e^{-n(\text{Log}\, n)^{\tilde{\varepsilon}}}$ as in (3.40). In this case we simply obtain

$$
\prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^\eta \prod_{\Delta \subset X_0^{(3)}} (\ell_3(\Delta)!)^{\frac{1}{2}-\eta} \prod_{\Delta \subset X_0^{(4)}} (\ell_4(\Delta)!)^{1/2} \leqq C^n n! .
$$

The needed factor comes from replacing (3.24), the bound on the factor produced by the cluster expansion, by a better estimate which uses the fact that γ/q

$> (\mathrm{Log}\, n)^{\delta' - \delta}$. Indeed, in a lattice $\mathbb{D}_i$ there can be at most $\mathrm{const}\, r^4$ faces of B_i at distance less or equal than rM^{-i} of a given cube of $\mathbb{D}_i$. Therefore, at least half of the γ faces of Γ have to be far from X_0 (in the relevant scale). More precisely, choosing c small enough, we have necessarily $\gamma' \equiv |\Gamma'| \geq \gamma/2$, where $\Gamma' \equiv \bigcup_i \Gamma_i'$ and

$$\Gamma_i' = \left\{ b \in B_i \cap \Gamma \mid \mathrm{dist}\,(b, X_0^i) \geq cM^{-i}(\mathrm{Log}\, n)^{\frac{\delta' - \delta}{4}} \right\}. \tag{3.42}$$

Collecting a piece of the decay (3.16) of the cluster propagators before applying (3.24) we have the estimate

$$\sum_\Gamma^* \sum_{\mathscr{I}} \left(\prod_{b \subset \Gamma} \frac{d}{ds_b} C_{i(b)}(x, x') \right)$$

$$\leqq O(1)^n \prod_{i=0}^{\varrho} M^{2i\gamma_i} e^{-\alpha \sum_{\ell=1}^{s_i} d(X_i^{(\ell)})} \prod_{b \in \Gamma'} e^{-O(1)M^i d(b, X_0)}$$

$$\leqq O(1)^n \left(\prod_i M^{2i\gamma_i} e^{-\alpha \sum_{l=1}^{s_i} d(X^{\{l\}})} \right) e^{-O(1)n \left[(\log n)^{\frac{\delta' - \delta}{4} - \delta} \right]}, \tag{3.43}$$

which completes the proof of (3.23) in this second case, provided δ is chosen close to 0 and δ' close to 1.

4. The Case q and γ Small

It is from this case, that $S_n(\Lambda, \varrho)$ gets its larger contribution, called $S_{n,3}(\Lambda, \varrho)$, and it is only now that we use the Sobolev inequality. We will prove in fact the following lemma:

Lemma 4. If q and γ are smaller than $n(\mathrm{Log}\, n)^{-\delta}$, then $S_{n,3}(\Lambda, \varrho)$ satisfies the following bound, uniform in Λ, ϱ:

$$|S_{n,3}(\Lambda, \varrho)| \leqq [1 + \varepsilon(n)]^n ((4K)^2)^n n!. \tag{3.43}$$

From Lemmas 3 and 4, Theorem 1 obviously follows.

Proof. We start again from expression (3.22)

$$S_{n,3}(\Lambda, \varrho) = \sum_{(q, \kappa, \{q\}, \{n\}, \theta)} \sum_\mu \sum_{\{h_b\}} \sum_{\substack{(X_0, \Gamma, \mathscr{I}, \mathbb{R}) \\ |X_0| = q}} \chi(\mathrm{c.s.c.}) \,[\text{cluster exp factors}]$$

$$\times \left[\frac{1}{n(\theta)n!} \int_{\Lambda_1^{n_1} \times \, \ldots \, \times \Lambda_q^{n_q}} dx_{i_1} \ldots dx_{i_n} \int \bar{R}(\varphi) d\mu_\Gamma(\varphi) \right]. \tag{3.44}$$

We remark that fixed $(\mu, X_0, \mathscr{I}, \mathbb{P})$ we know which of the original fields in $R(\varphi)$ [see (3.12)] have been transformed in half derived covariances and which are external lines of the AL subgraphs. We have to make a consistent choice, for each AL subgraph, of at least 5 particular external legs. (This is somehow the vertical analogue of the cluster expansion.) A definite prescription for such a choice can be the following one. For any AL subgraph $Y_{\geq 0}$ of X [recall $X = X(\mu, X_0)$] of the first scale we choose a set of at most 5 vertices localized in Y to which at least five external legs of Y are hooked such that vertices with external legs of lowest possible

indice are taken in priority. Then we go on, fixing for any AL subgraph of scale i at most 5 vertices localized in it with the same property as before and with the rule that vertices already chosen in lower slices are taken in priority if possible. Now we reserve the name of "external legs" to the external legs chosen in such a way. Given this prescription (of course, not the only one possible) and fixed $(\mu, X_0, \mathscr{J}, \mathbb{P})$, that is X being completely determined, we can regroup the set V of the n vertices into two disjoint subsets:

$$V = V_1 \cup V_2, \qquad V_1 \cap V_2 = \emptyset, \qquad (3.45)$$

$V_1 \equiv \{$vertices to which derived covariances or external legs are hooked$\}$, $V_2 \equiv V - V_1$.

We interchange some of the sums of (3.44) in the following way:

$$\sum_{\mu} \sum_{(X_0, \Gamma, \mathscr{J}, \mathbb{P})} \chi(\text{c.s.c.}) \sum_{|V_1|} \sum_{\mu_{V_1}} \sum_{(X_0, \Gamma, \mathscr{J}, \mathbb{P})} \chi(\text{c.s.c.}) \sum_{\mu_{V_2}}^{*},$$

where μ_{V_1} is the sequence of the first $|V_1|$ terms of μ. In this case $(X_0, \Gamma, \mathscr{J}, \mathbb{P})$ must satisfy some constraints imposed by μ_{V_1}, namely the cubes of X_0 which are localization cubes for the vertices $v \in V_1$ must have a well defined scale fixed by μ_{V_1}. The remaining localization cubes associated to the vertices of V_2 must be such that the "c.s.c." condition is satisfied; this imposes some constraints on $i_1(v)$ when $v \in V_2$, therefore to μ_{V_2}. This is the meaning of the $*$ in $\sum\limits_{\mu_{V_2}}^{*}$. In fact, $\sum\limits_{\mu_{V_2}}^{*}$ must be such that

a) X_0 does not vary when μ_{V_2} varies,

b) the "c.s.c." is satisfied by all $\mu = (\mu_{V_1}, \mu_{V_2})$ when μ_{V_2} varies at fixed μ_{V_1}. We can rewrite (3.44) in the following way:

$$S_{n,3}(\Lambda, \varrho) = \sum_{(q, \kappa, \{q\}, \{n\}, \theta)} \sum_{|V_1|} \sum_{\mu_{V_1}} \sum_{\{h_b\}} \sum_{(X_0, \Gamma, \mathscr{J}, \mathbb{P})} \chi(\text{c.s.c.}) \,[\text{cluster exp factors}]$$

$$\times \left\{ \frac{1}{n(\theta)n!} \int_{\Delta_1^{n_1} \times \ldots \times \Delta_q^{n_q}} dx_{i_1} \ldots dx_{i_{|V_2|}} \int d\mu_\Gamma \bar{\bar{R}}(\varphi) \right.$$

$$\left. \times \left(\sum_{\mu_{V_2}}^{*} \prod_{v \subset V_2} (\varphi_{i_1(v)} \varphi_{i_2(v)} \varphi_{i_3(v)} \varphi_{i_4(v)})(x) \right) \right\} \qquad (3.46)$$

and

$$\sum_{\mu_V}^{*} (\varphi_{i_1(v)} \varphi_{i_2(v)} \varphi_{i_3(v)} \varphi_{i_4(v)})(x) \leq \sum_{J=0}^{\varrho} (\varphi^{[\leq J]}(x))^4 \delta(J = \max \{\ell \mid x \in \Delta \in \mathbb{D}_\ell \cap X_0\})$$

$$\equiv (\varphi_{X_0}(x))^4, \qquad (3.47)$$

$$\varphi^{[\leq J]}(x) \equiv \sum_{\ell=0}^{J} \varphi_\ell(x), \qquad (3.48)$$

$$\bar{\bar{R}}(\varphi) = \prod_{i=0}^{\varrho} \prod_{s=1}^{\kappa_i} \left(\prod_{v \in V_1 \cap X_{\geq i}^{(s)}} \varphi_i^{\alpha_i(v)}(x_v) \right), \qquad (3.49)$$

where $\alpha_i(v)$ can be 1 or 0 depending on the terms of (3.22). k_i is the number of connected components of $X_{\geq i}^{(s)}$, and it is fixed by θ. For each couple (i, s) we bound the corresponding products of fields in $\bar{\bar{R}}(\varphi)$ by the rule (this product has to be

even, otherwise, the contribution after Gaussian integration is 0)

$$(\varphi_i(x_1) \ldots \varphi_i(x_{2m})) \leqq \frac{1}{2^{2m}} \prod_{\kappa=0}^{m-1} (\varphi_i^2(x_{2\kappa+1}) + \varphi_i^2(x_{2\kappa+2})), \qquad (3.50)$$

obtaining

$$\bar{R}(\varphi) \leqq c \sum_{\mathscr{L}} \hat{R}_{\mathscr{L}}(\varphi), \qquad (3.51)$$

where $c \sum_{\mathscr{L}} 1 = 1$ and $\hat{R}_{\mathscr{L}}(\varphi) \geqq 0$, $\forall \mathscr{L}$.

We bound (3.46) using (3.51) and we omit, for simplicity, to write the sum over $\mathscr{L}$, calling $\hat{R}$ just the generic $\hat{R}_{\mathscr{L}}$ (of course, our estimates will be uniform in $\mathscr{L}$). Then we partially resum over $\{n\} = \{n_1, \ldots, n_q\}$ leaving specified, in each term of the sum defining $S_{n,3}$, only how the vertices of V_1 are distributed in the q cubes of X_0. We call their numbers $\{\tilde{n}\} \equiv \{\tilde{n}_1, \ldots, \tilde{n}_q\}$, where by definition $\sum_{\ell=1}^{q} \tilde{n}_\ell = n - v''$ $\equiv w = |V_1|$ and $v'' \equiv |V_2|$. All this together produces the following inequality:

$$|S_{n,3}(\Lambda, \varrho)| \leqq \sum_{(q, \kappa, \{q\}, \{\tilde{n}\}, \theta)} \sum_{|V_1|} \sum_{\mu_{V_1}} \sum_{\{h_b\}} \sum_{(X_0, \Gamma, \mathscr{J})} \chi(\text{c.s.c.}) \, [\text{cluster exp factors}]$$

$$\times \left\{ \sum_{\mathbb{P}} \frac{1}{n(\theta) w!} \int d\mu_\Gamma \int_{\Delta_1^{\tilde{n}_1} \times \ldots \times \Delta_q^{\tilde{n}_q}} dx_{i_1} \ldots dx_{i_w} \hat{R}(\varphi) \right.$$

$$\left. \times \left[\frac{1}{v''!} \left(\int_{X_0} dx (\varphi_{X_0}(x))^4 \right)^{v''} \right] \right\}, \qquad (3.52)$$

where X_0 is now also understood as an integration region of Λ (the union of its cubes) and $\sum_{\mathbb{P}}$ is the sum over the ways in which the $x_1, \ldots, x_w$ vertices of V_1 can be organized in groups of $\tilde{n}_1, \ldots, \tilde{n}_q$ elements.

We can apply now the Sobolev inequality to $\int_{X_0} dx [\varphi_{X_0}(x)]^4$, as a consequence of our use of Dirichlet boundary conditions. Indeed, inequality (2.4) holds for functions of $H_0^1(\Lambda)$, the Sobolev space of functions φ with square integrable gradient which vanish on the boundary of Λ. Then the measure $d\mu_\Gamma$ has a support included in H_0^1 due to the ultraviolet cutoff, hence the sample fields are very regular and the Dirichlet boundary conditions enforce that they vanish on $\partial\Lambda$ (see [24, 25] for the definition of Gaussian measures and properties of their support). We obtain for the bracketed term of (3.52):

$$\{(3.52)\} \leqq \sum_{\mathbb{P}} \int d\mu_\Gamma \frac{1}{n(\theta)} \int_{\Delta_1^{\tilde{n}_1} \times \ldots \times \Delta_q^{\tilde{n}_q}} dx_{i_1} \ldots dx_{i_w} \hat{R}(\varphi)$$

$$\times K^{2v''} \frac{1}{v''!} \left(\int_{X_0} dx (\nabla \varphi_{X_0}(x))^2 \right)^{2v''}. \qquad (3.53)$$

Let u be the number of ordinary φ fields in $\hat{R}(\varphi)$; such a field φ must be hooked to a vertex of V_1. Therefore,

$$u \leqq 80q + 6\gamma \leqq 86 \frac{n}{(\text{Log} \, n)^\delta}, \qquad (3.54)$$

because remembering the way we decided to pick the external legs the number of the vertices chosen is certainly bounded by $5|X| \leqq 20|X_0|$ and the number of fields

hooked to derived covariances is bounded by 6γ (3 at most per end of derived covariance, and a covariance has two ends). We perform now the Gaussian integration in (3.52) and we look for an estimate, uniform in $\tilde{\mathbb{P}}$ and Γ, of

$$\frac{1}{n(\theta) \prod\limits_{\ell=1}^{q} \tilde{n}_\ell!} \int d\mu_\Gamma \int_{\Delta_1^{\tilde{n}_1} \times \ldots \times \Delta_q^{\tilde{n}_q}} dx_1 \ldots dx_w \hat{R}(\varphi) \frac{K^{2v''}}{v''!} \left(\int_{X_0} dx (\nabla \varphi_{X_0}(x))^2 \right)^{2v''}, \quad (3.55)$$

where
$$w \leq \alpha \frac{n}{(\text{Log} n)^\delta} \qquad (\text{obviously } \alpha = 22 \text{ is O.K.}),$$

$$\tag{3.56}$$

$$v'' \geq n - \alpha \frac{n}{(\text{Log} n)^\delta} \quad \text{for some} \quad \alpha > 0.$$

The integration in (3.55) gives rise to a sum over all the possible contraction schemes between the u fields φ and the $4v''$ $\nabla \varphi$ fields. Each contraction scheme consists of products of terms of the following kind:
  a) derived covariances of the cluster expansion;
  b) ordinary covariances between ordinary fields;
  c) "propagators": $C(-\Delta)C \ldots (-\Delta)C$ hence chains with $k > 0$ insertions of $(\nabla \varphi)^2$ vertices joining vertices of V_1;
  d) closed loops made of covariances with $k > 0$ insertions of $(-\Delta)$.

We omit the s-dependence when unnecessary and we do not write the subscript X indicating that C and the Laplacian Δ are computed with Dirichlet boundary conditions on ∂X. We want to regroup the different contraction schemes one gets performing the Gaussian integration in (3.55) in the following way: In each group of contraction schemes we assume that between the u ordinary fields φ, $2r$ of them, forming the set $\mathcal{G}_{2r}$, contract with $2r$ $\nabla \varphi$ fields forming the set $\mathcal{F}_{2r}$, to become ends of chains, and the remaining contract together, giving ordinary propagators. To write more explicitly this decomposition we label with an index v the u fields in $\hat{R}(\varphi)$, $v \in \{1, \ldots, u\}$ and with an index μ the $4v''$ fields $\nabla \varphi$, $\mu \in \{1, \ldots, 4v''\}$. (There should not be any possible confusion with the assignment μ.) We write

$$[(3.55)] = \frac{1}{n(\theta) \prod\limits_{\ell=1}^{q} \tilde{n}_\ell!} \int_{X_0^{2v''}} dx_1 \ldots dx_{2v''} \frac{K^{2v''}}{v''!} \left[\int d\mu_\Gamma \hat{R}(\varphi) \prod_{J=1}^{2v''} (\nabla \varphi_{X_0}(x_J))^2 \right]$$

$$= \sum_{r=0} \sum_{\mathcal{G}_{2r}} \sum_{\mathcal{F}_{2r}} \int_{\Delta_1^{\tilde{n}_1} \times \ldots \times \Delta_q^{\tilde{n}_q}} dx_1 \ldots dx_w$$

$\qquad \times$ [contractions of the fields φ with indices v not in $\mathcal{G}_{2r}$]

$\qquad \times$ [contributions of the r chains and of the closed loops]. (3.57)

First of all we remark that we look for estimates of these factors which are uniform in $\mathcal{G}_{2r}$ and $\mathcal{F}_{2r}$, therefore their estimates will be multiplied by

$$\sum_{\mathcal{G}_{2r}} \sum_{\mathcal{F}_{2r}} 1 = \binom{u}{2r} \binom{4v''}{2r} \leq [1 + \varepsilon(n)]^n, \quad (3.58)$$

remembering (3.56) and the fact that $2r \leq u$. Therefore, once we have chosen the $2r$ fields φ which are hooked to $\nabla \varphi$'s to form the end of the r chains, the contribution

in (3.57): [chains + loops] has the following form:

$$[\text{chains} + \text{loops}] = \frac{K^{2v''}}{v''!} \int_{X_0^{2v''}} dx_1 \dots dx_{2v''} \int d\mu_\Gamma$$

$$\times \prod_{\ell=1}^{r} \varphi_{i_\ell}(\tilde{x}_\ell) \prod_{\ell'=1}^{r} \varphi_{i_{\ell'}}(\tilde{y}_{\ell'}) \prod_{s=1}^{2v''} (\nabla\varphi_{X_0})^2(x_s), \qquad (3.59)$$

where $\varphi_{i_\ell}(\tilde{x}_\ell)$, $\varphi_{i_{\ell'}}(\tilde{y}_{\ell'})$ are the $2r$ fields φ which produce the ends of the r chains (some $\tilde{x}$ and $\tilde{y}$ can be equal). The remaining $\nabla\varphi$ fields not in $\mathscr{F}_{2r}$ contract between themselves, forming the closed loops and the interior of the r chains. We prove the following lemma:

Lemma 5.

$$|[\text{chains} + \text{loops}]| \leq \sum_{m=0}^{2v''} \sum_{p=0}^{2(v''-m/2)} \frac{1}{p!} (\bar{C}|X_0|)^p$$

$$\times \sum_{\substack{t_1 \dots t_p \\ \sum_{j=1}^{p} t_j = 2(v''-m/2)}} \frac{2(v''-r)!\,(2t_1-1)!! \dots (2t_p-1)!!}{t_1! \dots t_p!}$$

$$\times \sum_{\substack{t_1' \dots t_r' \\ \sum_{j=1}^{r} t_j' = (m-r)}} \frac{(2t_1'-1)!! \dots (2t_r'-1)!!}{t_1'! \dots t_r'!} \frac{K^{2v''}}{v''!}$$

$$\times \sum_{(i_1 \dots i_r)} \sum_{(j_1 \dots j_r)} \int d\underline{z} d\underline{z}'$$

$$\times \{C(\tilde{x}_{i_1}, z_1) \dots C(\tilde{x}_{i_r}, z_r)\mathbb{P}(z, z')C(z_r', \tilde{y}_{j_1}) \dots C(z_r', \tilde{y}_{j_r})\},$$

$$(3.60)$$

where the operator $\mathbb{P}(\underline{z}, \underline{z}')$ is the product of r operators:

$$\mathbb{P} = \prod_{k=1}^{r} P_k(z_k, z_k'; t_k'); \qquad P_k(z_k, z_k'; t_k') \equiv [(-\Delta)C(-\Delta) \dots C(-\Delta)](z_k, z_k'),$$

$$(3.61)$$

the chain in (3.61) having t_k' insertions. The different regions of integration should be clear from the previous formulas (see (3.57)).

Proof. This estimate is the generalization of the estimate of Lemma 2. In fact, the right-hand side of (3.59) when $r = 0$ is just the right-hand side of (3.2) with the only two differences that the measure is now s-dependent and that $\varphi_{X_0}(x)$ [defined in (3.47)] is different from $\varphi(x)$.

To prove (3.60) one follows (with some generalizations) the proof of Lemma 2. First of all we regroup the contraction schemes depending on how many vertices $(\nabla\varphi)^2$ will produce the interiors of the r chains. Their number is m and this is the origin of the first sum in (3.60), then we further divide the contraction schemes in subgroups where the number of closed loops p, the number of the vertices present in each of the p closed loops, $t_1, \dots, t_p$ and the number of the vertices present in each of the r chains $t_1', \dots, t_r'$ are fixed.

This explains the sum:

$$
\sum_{m=0}^{2v''} \sum_{p=0}^{2(v''-m/2)} \frac{1}{p!} \sum_{\substack{t_1 \ldots t_p \\ \sum\limits_{j=1}^{\varrho} t_j = 2(v''-m/2)}} \sum_{\substack{t_1' \ldots t_{r'}' \\ \sum\limits_{j=1}^{r} t_j' = m-r}} , \tag{3.62}
$$

where $1/p!$ avoids overcountings which otherwise will arise as we assign the vertices to the p loops in all possible ways, and we sum over $(t_1, \ldots, t_p)$ in all possible ways. Then we look for an estimate of the generic term of this sum uniform in the assignments of the vertices to a well defined chain or loop; this produces a factor:

$$
\frac{(2v''-r)!}{t_1! \ldots t_p! t_1'! \ldots t_r'!}. \tag{3.63}
$$

At this stage we have to remember that we still have to count the various contraction schemes for the $V\varphi$ fields inside each chain and closed loops. They will produce the following factors:

$$
\prod_{j=1}^{p} (2t_j-1)!! \, [\text{closed loop } \mathscr{L}_j] \cdot \prod_{k=1}^{r} (2t_k'-1)!! \, [\text{chain } C_k], \tag{3.64}
$$

where $(2t_j-1)!!$ and $(2t_k'-1)!!$ are just the numbers of the contraction schemes respectively inside the j-th loop and the k-th chain, and finally,

$$
\begin{aligned}
[\text{closed loop } \mathscr{L}] &= \int_{X_0^t} dx_1 \ldots dx_t \\
&\times \sum_{i_1=1}^{i(x_1) \wedge i(x_2)} \ldots \sum_{i_t=1}^{i(x_t) \wedge i(x_1)} ((-\varDelta)C_{i_1}(x_1, x_2) \ldots (-\varDelta)C_{i_t}(x_t, x_1)) \\
&= \int_{X_0^t} dx_1 \ldots dx_t ((-\varDelta)C^{[\le i(x_1) \wedge i(x_2)]}(x_1, x_2) \\
&\times \ldots (-\varDelta)C^{[\le i(x_t) \wedge i(x_1)]}(x_t, x_1)), \tag{3.65}
\end{aligned}
$$

where $i \wedge j \equiv \min\{i, j\}$, and the highest frequency of the covariances C depend on X since they are the propagators of the $V\varphi_X$ fields which are constrained by definition (3.47),

$$
\begin{aligned}
&\prod_{k=1}^{r} [\text{chain } C_k] \\
&= \int \sum_{(i_1, \ldots, i_r)} \sum_{(j_1, \ldots, j_r)} \{C(\tilde{x}_{i_1}, z_1) \ldots C(\tilde{x}_{i_r}, z_r) \mathbb{P}(\underline{z}, \underline{z}') C(z_1', \tilde{y}_{j_1}) \ldots C(z_r', \tilde{y}_{j_r})\} d\underline{z} d\underline{z}',
\end{aligned}
$$

$$\tag{3.66}$$

where the operator $\mathbb{P}(\underline{z}, \underline{z}')$ is defined in (3.61). Again the covariances $C(\tilde{x}, z)$ and $C(z', \tilde{y})$ have to be thought of more precisely as $C_{\tilde{i}}$ and $C_{\tilde{j}}$, where $\tilde{i}$ (respectively $\tilde{j}$) is the momentum assignment of the corresponding field $\varphi(\tilde{x})$ [respectively $\varphi(\tilde{y})$], subject to the constraint that $\tilde{i} \le i(z)$ [respectively $\tilde{j} \le i(z')$], where $i(z)$ and $i(z')$ are the upper limit over frequencies coming from definition (3.47) of $V\varphi_{X_0}(z)$ or $V\varphi_{X_0}(z')$. The proof of the lemma will then be achieved by the proof of the following inequality:

$$
[\text{closed loop } \mathscr{L}] \le \bar{c}|X_0| \tag{3.67}
$$

with $\bar{c}$ cutoff independent. Inequality (3.67) is proven using repeatedly the fact that as a quadratic form $\sum_i C_i(s) \leqq C \equiv (-\varDelta+1)^{-1} \leqq (-\varDelta)^{-1}$ to bound all the integrations except the last one (due to translation invariance). More precisely, $C^{[\leqq i(x_1) \wedge i(x_2)]}(x_1, x_2)$ is the kernel of a positive quadratic form C' [since $i(x_1) \wedge i(x_2)$ is symmetric] which we can write as $B \times B$, where B is the kernel of the square root of this quadratic form. Since x_1 and x_2 are integration variables with identical ranges, the closed loop then can be written as an integral over z of $B(z, \cdot) \times (-\varDelta)C(-\varDelta) \dots C(-\varDelta) \times B(\cdot, z)$. Hence we have "symmetrized" in x_1 and x_2. Applying repeatedly $C \leqq (-\varDelta)^{-1}$ we get rid of the chain, and have only to bound the sum over z of $B(z, \cdot) \times (-\varDelta)^{-1} \times B(\cdot, z)$. This reconstructs just $\mathrm{Tr}\, C' \times (-\varDelta)^{-1}$. Then we rewrite C' as a sum over frequencies of convex combinations of propagators $C_{j,Y}$ with some specified set Y of Dirichlet b.c. For any fixed frequency j we use the path integral representation of C just as in Appendix C (since the argument is exactly the same except that it is somewhat shorter and easier, we do not repeat it). This gives a bound on the integrand corresponding to $C_{j,Y} \times (-\varDelta)$ which is a constant independent of j and Y. Hence after the last integration corresponding to the trace Tr, we get for any closed loop the bound:

$$[\text{closed loop}] \leqq \bar{c} \sum_j \mathrm{Vol}(X_0 \cap \mathbb{D}_j) \leqq \bar{c}|X_0|.$$

This completes the proof of Lemma 5.

Now to estimate the chains we define

$$F_{\{x\}}(z_1, \dots, z_r) = \sum_\sigma \prod_{k=1}^r C(x_{\sigma(k)}, z_k), \qquad G_{\{x\}}(z_1', \dots, z_r') = \sum_{\sigma'} \prod_{k=1}^r C(z_k', x_{\sigma'(k)}),$$

(3.68)

where the sums are over all permutations σ and σ' of $\{1, \dots, r\}$. Then the sum over all ways of pairing the chains with their corresponding variables x and y can be simply written as $\langle F_{\{x\}}, \mathbb{P}G_{\{y\}} \rangle$. Since $\mathbb{P}$ is a positive form we have

$$\langle F_{\{x\}}, \mathbb{P}G_{\{y\}} \rangle \leqq \langle F_{\{x\}}, \mathbb{P}F_{\{x\}} \rangle^{1/2} \langle G_{\{y\}}, \mathbb{P}G_{\{y\}} \rangle^{1/2},$$

(3.69)

and now we have again a symmetric form. Proceeding as in the proof of (3.67), hence using again that as a quadratic form $C(s) \leqq (-\varDelta)^{-1}$, we can bound each operator P_k (as a quadratic form) by $(-\varDelta)$, hence $\mathbb{P}$ by $\otimes^r(-\varDelta)$, and conclude that:

$$\langle F_{\{x\}}, \mathbb{P}F_{\{x\}} \rangle \leqq \sum_\sigma \sum_{\sigma'} \prod_{k=1}^r C_{i_k}(x_{\sigma(k)}, z_k)(-\varDelta)(z_k, \bar{z}_k)C_{j_k}(\bar{z}_k, x_{\sigma'(k)}),$$

(3.70)

where i_k and j_k are the momentum assignments of the fields φ at the beginning and at the end of the k-th chain. The net result is that now defining $\{\bar{x}\}$ and $\{\bar{y}\}$ as copies of the variables $\{x\}$ and $\{y\}$, we are reducing a chain to the product of the square root of two simplified chains of the type $C_i(x, \cdot)(-\varDelta)C_j(\cdot, \bar{x})$. We call such an object a "modified propagator" $\Gamma_{i,j}(x, \bar{x})$. We have the bound [to compare with (2.10)] expressed by the following lemma.

Lemma 6. *Suppose $i \leqq j$. Then*

$$|\Gamma_{i,j}(x, y)| = |C_i(s)(x, \cdot)(-\varDelta)C_j(s)(\cdot, y)| \leqq (\text{const})M^{2i}e^{-O(1)M^i|x-y|}. \quad (3.71)$$

The proof of this lemma is given in Appendix C, using the path representation of the propagator as in [26].

The bound (3.71) on $\Gamma_{i,j}$ allows us to do one of the sums over the permutations σ, σ' in the development of $\langle F, \mathbb{P}F \rangle$. In fact, writing M^{2i} in (3.71) as $M^i M^j M^{-(j-i)}$, the factor $M^{-(j-i)}$ allows us, knowing j, to sum over the index i, paying a constant. The exponential decrease in (3.71) allows us to choose the cube of D_i. Then if the number of fields $\varphi_i(x)$ with x in this cube is bounded by a constant, the sum over one of the permutations gives just a constant factor. If the number of fields $\varphi_i(x)$ is large and not uniformly bounded, then one has to remember that since these fields are associated to vertices of V_1, there must be many derived covariances starting from these vertices and as their b's must be all different (see Sect. III.2), by a standard argument we can use a fraction of their exponential decay [see (3.16)] to bound the sum over all different choices of fields in the cube by a constant independent on r. The second sum over the permutations σ in the development of $\langle F, \mathbb{P}F \rangle$ gives a factor $r!$. The same argument applies to $\langle G, \mathbb{P}G \rangle$, and taking into account the square root in (3.69) we conclude that the contribution from all the chains can be bounded by

$$\prod_{k=1}^{r} [\text{chain } C_k] \leqq (\text{const})^r r! \left(\prod_{v \in \mathscr{G}_{2r}} M^{i(v)} \right), \tag{3.72}$$

where $\{i(v)\}$ are the momentum assignments of the fields φ which form the ends of the r chains.

Collecting together (3.72) and Lemma 5, we get

$$|[\text{chains} + \text{loops}]| \leqq (\text{const})^r \frac{(2v'' - r)! \, r!}{v''!} K^{2v''} \cdot 2^{2(v'' - m/2)} 2^{(m-r)}$$

$$\times \sum_{m=0}^{2v''} \sum_{p=0}^{2(v'' - m/2)} \frac{1}{p!} (\bar{C}|X_0|)^p$$

$$\times \left(\sum_{\substack{t_1, \ldots, t_p \\ \sum_j t_j = 2(v'' - m/2)}} 1 \right) \left(\sum_{\substack{t'_1, \ldots, t'_r \\ \sum_\ell t'_\ell = (m-r)}} \right) \left(\prod_{v \in \mathscr{G}_{2r}} M^{i(v)} \right)$$

$$\leqq (\text{const})^r 2^{4v''} K^{2v''} v''! \left(\sum_{m=0}^{2v''} \binom{m-r}{r} \right)$$

$$\times \sum_{p=0}^{2v''} \frac{1}{p!} (\bar{C}|X_0|)^p \binom{2v''}{p} \left(\prod_{v \in \mathscr{G}_{2r}} M^{i(v)} \right), \tag{3.73}$$

remembering (3.6) of Lemma 2 and since $\binom{m}{r} \leq [(1 + \varepsilon(n)]^n$ we get:

$$[\text{r.h.s. of (3.73)}] \leqq (1 + \varepsilon(n))^n (4K)^{2v''} v''! \left(\prod_{v \in \mathscr{G}_{2r}} M^{i(v)} \right). \tag{3.74}$$

Collecting together (3.74), (3.57), and (3.52), we get

$$|S_{n,3}(\Lambda, \varrho)| \leqq ((1 + \varepsilon(n))^n (4K)^{2v''} v''!)$$

$$\times \left\{ \sum_{(q, \kappa, \{q\}, \{\tilde{n}\}, \theta)} \sum_{|v_1|} \sum_{\mu_{V_1}} \sum_{\{h_b\}} \sum_{(X_0, \Gamma, \mathscr{J})} \chi(\text{c.s.c.})[\text{cluster exp factors}] \right.$$

$$\times \left. \sum_{\mathbb{P}} n(\theta) \int_{\Delta_1^{\tilde{n}_1} \times \cdots \times \Delta_q^{\tilde{n}_q}} d\tilde{x}_{i_1} \ldots d\tilde{x}_{i_n} \left| \int \hat{R}(\varphi) d\mu_\Gamma \left(\prod_{v \in \mathscr{G}_{2r}} M^{i(v)} \right) \right| \right\}, \tag{3.75}$$

where we have reexpressed the first factor of (3.57) as a Gaussian integration over the product of the $(u-2r)$ fields of $\hat{R}$. It is clear that this term can be estimated as in Lemma 3, observing that now neither $|X_0|$ nor γ are large, so we just do not get any corresponding factor $\exp[-n(\text{Log}\,n)^{\tilde{\epsilon}}]$, and $4n$ is replaced here by $u \leq 4(n-v'')$, hence the corresponding bound is:

$$|[\text{bracketed term in the r.h.s. of (3.75)}]| \leq (1+\varepsilon(n))^n (n-v'')!. \qquad (3.76)$$

The only subtlety in this proof of (3.76) along the lines of Lemma 3 is the following. We have to sum over X_0, which is the set of the localization cubes of *all* the vertices (not only of those of V_1). However, some cubes of X_0, forming the set Z, can contain only vertices of V_2. This creates a slight difficulty because the connections to sum over these cubes seem to disappear after the contributions of chains and loops have been factorized and estimated. However, due to the "c.s.c." condition and our choice of V_1, these cubes are not free to be anywhere in Λ. The only possibility is that they must contain at least one face b of some derived covariance (hence, in the language of Appendix A they are joined to cubes of $Z' \equiv X_0 - Z$ by s.e. connections). This implies that we can organize at the end our sum over X_0 in the following way: fixing Z' we sum first over the cubes of Z using the exponential decay (3.16) of the propagators which attach at least one of their faces to a cube of Z'. This results in a constant per cube of Z, hence a factor $(1+\varepsilon(n))^n$. We sum then over the cubes of Z' (forming the set Z'') which contain vertices of V_1, but are not localization cubes of vertices of V_1. Using (3.47) and our rule of choice of vertices of V_1, this sum gives a factor F, which is just the number of cubes which contain a cube of $Z'-Z''$. Then the sum over the cubes of $Z'-Z''$ is exactly as before, since it corresponds to the sum over localization cubes of V_1. The only thing to check is that the additional factor F is controlled by a fraction of the exponential vertical decrease (B.25) (corresponding to the positive power counting), which is obvious.

When we plug (3.76) in (3.75) we obtain:

$$|S_{n,3}(\Lambda, \varrho)| \leq (1+\varepsilon(n))^n (4K)^{2n} n!, \qquad (3.77)$$

which completes the proof of Lemma 4.

Appendix A

Proof of Inequality (3.24)

We show first how to obtain the tree decay factor in (3.24). Let us consider a derived propagator $d/ds_b C_i$ of scale i as a s.e. connection (special elementary connection) connecting together the cubes at its ends *and* the two cubes of $\mathbb{D}_i$ containing the face b. (Hence it can connect at most 4 cubes, and does not coincide with our former definition of elementary horizontal connections.) As before we have the notion of s-connected region, which is just a maximal set of cubes joined by at least a tree of s.e. connections. Remark that any connected region with the former definition is included in a s-connected region; in fact, s-connected regions

are just the former connected regions plus a corridor of neighbouring cubes (which might glue some of the previously disconnected regions into a single s-connected piece). In a straightforward way we can obtain tree decay in the s-connected regions. Simply pick a piece, say one half, of the exponential decay (3.16) of each derived cluster propagator. Using at each scale the tree T which defines the connectedness of all the cubes of a given s-connected region Y, we can bound this piece of exponential decay by $O(1)^{|Y|} \exp[-O(1)M^i d_T(Y)]$, where $d_T(Y)$ is the sum over the lines of the tree T of the distance between the corresponding cubes of Y. This is better than $O(1)^{|Y|} \exp[-O(1)d(Y)]$, where $d(Y)$ is the ordinary (scaled) tree distance, computed with the shortest tree. Now we use the trick that $M^i \geqq O(1) \sum\limits_{j=1,\dots,i} M^i$ to replicate this decay over all lower scales. But by the remark above, tree decay in the s-connected regions immediately imply tree decay in the ordinary regions of (3.24) [up to $O(1)$ per cube]. Now the products of all the replicated tree decays over all scales is easily transformed into a product over all scales of a scaled tree decay for each region $X_i^{(1)}$, since these regions are defined as being connected through elementary connections *of scales* $\geqq i$. Remarking that the factor $O(1)$ per cube can be absorbed into the $(\text{const})^n$ of (3.24), it remains only to control the sums over Γ and $\mathscr{J}$ using the remaining half of the decay (3.16).

This is easily done as follows. Paying a factor 8^n we can choose which of the $8n$ fields are used to form derived covariances. Then we can use the remaining decay (3.16) to sum, for each such field, over all the faces b, hence choose the derived covariance to which it hooks. This sum gives a constant per such field, hence a total factor $O(1)^n$. In this way, both sums over Γ and $\mathscr{J}$ are done at the same time, and the proof of (3.24) is therefore completed.

Appendix B

Proof of the Inequality (3.36)

To simplify notations, we suppose that $J = 2$ in inequality (3.36); the other cases are absolutely identical. We have:

$$\prod_{\Delta \subset X_0^{(2)}} \ell_2(\Delta)! = \prod_{j=0}^{\varrho} \prod_{\Delta_j \subset X_0^{(2)} \cap \mathbb{D}_j} \ell_2(\Delta_j)!. \tag{B.1}$$

We define

$$\ell_1(\Delta_i; J) \equiv \#\{\varphi_{i_1}(x) \mid i_1 = i, \ x \in \Delta_i \subset \mathbb{D}_i, \ i_2(x) = J\}, \tag{B.2}$$

$$\begin{aligned}
\ell_2(\Delta_J) &= \sum_i \sum_{\Delta_i \subset \Delta_J} \ell_1(\Delta_i; J) \\
&= \sum_{\Delta_{J+1} \subset \Delta_J} \ell_1(\Delta_{J+1}; J) + \sum_{\Delta_{J+1} \subset \Delta_J} \sum_{i=J+2}^{\varrho} \sum_{\Delta_i \subset \Delta_{J+1}} \ell_1(\Delta_i; J) \\
&\equiv \sum_{\Delta_{J+1} \subset \Delta_J} (\ell_1(\Delta_{J+1}; J) + \bar{\ell}_1(\Delta_{J+1}; J)),
\end{aligned} \tag{B.3}$$

$$\ell_2(\Delta_J)! = \frac{\ell_2(\Delta_J)!}{\prod_{\Delta_{J+1} \subset \Delta_J} (\ell_1(\Delta_{J+1}; J) + \bar{\ell}_1(\Delta_{J+1}; J))!}$$

$$\times \prod_{\Delta_{J+1} \subset \Delta_J} (\ell_2(\Delta_{J+1}; J) + \bar{\ell}_1(\Delta_{J+1}; J))!$$

$$\leqq (M^4)^{\ell_2(\Delta_J)} \prod_{\Delta_{J+1} \subset \Delta_J} 2^{\ell_1(\Delta_{J+1}; J) + \bar{\ell}_1(\Delta_{J+1}; J)}$$

$$\times \prod_{\Delta_{J+1} \subset \Delta_J} \ell_1(\Delta_{J+1}; J)! \prod_{\Delta_{J+1} \subset \Delta_J} \bar{\ell}_1(\Delta_{J+1}; J)!$$

$$= (2M^4)^{\ell_2(\Delta_J)} \prod_{\Delta_{J+1} \subset \Delta_J} \ell_1(\Delta_{J+1}; J)! \prod_{\Delta_{J+1} \subset \Delta_J} \bar{\ell}_1(\Delta_{J+1}; J)!, \quad (B.4)$$

$$\bar{\ell}_1(\Delta_{J+1}; J) = \sum_{\Delta_{J+2} \subset \Delta_{J+1}} \left(\ell_1(\Delta_{J+2}; J) + \sum_{i=J+3}^{\varrho} \sum_{\Delta_i \subset \Delta_{J+2}} \ell_1(\Delta_i; J) \right)$$

$$= \sum_{\Delta_{J+2} \subset \Delta_{J+1}} (\ell_1(\Delta_{J+2}; J) + \bar{\ell}_1(\Delta_{J+2}; J)), \quad (B.5)$$

$$\bar{\ell}_1(\Delta_{J+1}; J)! = \frac{\bar{\ell}_1(\Delta_{J+1}; J)!}{\prod_{\Delta_{J+2} \subset \Delta_{J+1}} (\ell_1(\Delta_{J+2}; J) + \bar{\ell}_1(\Delta_{J+2}; J))!}$$

$$\times 2^{\bar{\ell}_1(\Delta_{J+1}; J)} \prod_{\Delta_{J+2} \subset \Delta_{J+1}} \ell_1(\Delta_{J+2}; J)! \prod_{\Delta_{J+2} \subset \Delta_{J+1}} \bar{\ell}_1(\Delta_{J+2}; J)!$$

$$\leqq (2M^4)^{\bar{\ell}_1(\Delta_{J+1}; J)} \prod_{\Delta_{J+2} \subset \Delta_{J+1}} \ell_1(\Delta_{J+2}; J)! \prod_{\Delta_{J+2} \subset \Delta_{J+1}} \bar{\ell}_1(\Delta_{J+2}; J)!.$$

$$(B.6)$$

Therefore,

$$\prod_{\Delta_{J+1} \subset \Delta_J} \bar{\ell}_1(\Delta_{J+1}; J)! \leqq (2M^4)^{\sum_{\Delta_{J+1} \subset \Delta_J} \bar{\ell}_1(\Delta_{J+1}; J)} \prod_{\Delta_{J+2} \subset \Delta_J} \ell_1(\Delta_{J+2}; J)!$$

$$\times \prod_{\Delta_{J+1} \subset \Delta_J} \prod_{\Delta_{J+2} \subset \Delta_{J+1}} \bar{\ell}_1(\Delta_{J+2}; J)! \quad (B.7)$$

and iterating

$$\ell_2(\Delta_J)!$$

$$\leqq (2M^4)^{\ell_2(\Delta_J) + \sum_{\Delta_{J+1} \subset \Delta_J} \sum_{i=J+2}^{\varrho} \sum_{\Delta_i \subset \Delta_{J+2}} \ell_1(\Delta_i; J) + \sum_{\Delta_{J+1} \subset \Delta_J} \sum_{\Delta_{J+2} \subset \Delta_{J+1}} \sum_{i=J+3}^{\varrho} \sum_{\Delta_i \subset \Delta_{J+2}} \ell_1(\Delta_i; J) \ldots}$$

$$\times \prod_{i=J+1}^{\varrho} \prod_{\Delta_i \subset \Delta_J} \ell_1(\Delta_i; J)!$$

$$= (2M^4)^{\sum_{\Delta_{J+1} \subset \Delta_J} \ell_1(\Delta_{J+1}; J) + 2 \sum_{\Delta_{J+2} \subset \Delta_J} \ell_1(\Delta_{J+2}; J) + 3 \sum_{\Delta_{J+3} \subset \Delta_J} \ell_1(\Delta_{J+3}; J) + \ldots}$$

$$\times \prod_{i=J+1}^{\varrho} \prod_{\Delta_i \subset \Delta_J} \ell_1(\Delta_i; J)!$$

$$= \prod_{\substack{v\{i_2(v) = J\} \\ \{i_1(v) = J+1\}}} 2M^4 \prod_{\substack{v\{i_1(v) = J+2\} \\ \{i_2(v) = J\}}} (2M^4)^2 = \prod_{v; i_2(v) = J} (2M^4)^{(i_1(v) - J)} \prod_{\Delta \subset \Delta_J} \ell_1(\Delta; J)!.$$

$$(B.8)$$

Finally,

$$\prod_J \prod_{\Delta_J \subset X_0^{(2)} \cap \mathbb{D}_j} \ell_2(\Delta_J)! = \prod_{\Delta \subset X_0^{(2)}} \ell_2(\Delta)!$$

$$\leq \prod_v (2M^4)^{(i_1(v) - i_2(v))} \prod_J \prod_{\Delta_J \subset X_0^{(2)} \cap \mathbb{D}_j} \prod_{\Delta \subset \Delta_J} \ell_1(\Delta; J)!$$

$$\leq \prod_v M^{(4+\varepsilon)(i_1(v) - i_2(v))} \prod_{\Delta \subset X_0} \ell_1(\Delta)!, \quad \text{with} \quad 2M^{-\varepsilon} \leq 1.$$
(B.9)

Proof of Inequality (3.41). Using the previous definitions we have

$$\prod_v M^{-(i_1(v) - i_2(v))} M^{-(i_1(v) - i_3(v))} M^{-(i_1(v) - i_4(v))}$$

$$= \prod_{i_1, i_2} M^{-(i_1 - i_2) \sum_{\Delta_{i_1}} \ell_1(\Delta_{i_1}; i_2)}$$

$$\times \prod_{i_1, i_3} M^{-(i_1 - i_3) \sum_{\Delta_{i_1}} \ell_1(\Delta_{i_1}; i_3)} \prod_{i_1, i_4} M^{-(i_1 - i_4) \sum_{\Delta_{i_1}} \ell_1(\Delta_{i_1}; i_4)}, \quad \text{(B.10)}$$

where $\ell_1(\Delta_{i_i}; i_3) = \#$ fields $\varphi_{i_1}(x)$, $x \in \Delta_i$, $i_3(x) = i_3$ ($i_4(x) = i_4$) and $i_1 \geq i_2 \geq i_3 \geq i_4$ run over the values prescribed by μ (which is fixed) and Δ_i run over the cubes of X_0 (also fixed). We can also write two more expressions for the (right-hand side) of (B.10), namely

$$\prod_v M^{-(i_1(v) - i_2(v))} M^{-(i_1(v) - i_3(v))} M^{-(i_1(v) - i_4(v))}$$

$$= \prod_{i_1, i_2} M^{-(i_1 - i_2) 3 \sum_{\Delta_{i_1}} \ell_1(\Delta_{i_1}; i_2)}$$

$$= \prod_{i_1, i_3} M^{-(i_1 - i_3) 2 \sum_{\Delta_{i_1}} \ell_1(\Delta_{i_1}; i_3)} = \prod_{i_1, i_4} M^{-(i_1 - i_4) \sum_{\Delta_{i_1}} \ell_1(\Delta_{i_1}; i_4)}, \quad \text{(B.11)}$$

where

$$\ell_2(\Delta_{i_2}; i_3) = \# \text{ fields } \varphi_{i_2}(x) \quad \text{with} \quad x \in \Delta_{i_2} \quad \text{and} \quad i_3(x) = i_3,$$

$$\ell_3(\Delta_{i_3}; i_4) = \# \text{ fields } \varphi_{i_3}(x) \quad \text{with} \quad x \in \Delta_{i_3} \quad \text{and} \quad i_4(x) = i_4,$$

and

$$\prod_v M^{-(i_1(v) - i_2(v))} M^{-(i_1(v) - i_3(v))} M^{-(i_1(v) - i_4(v))}$$

$$= \prod_\ell M^{-(h_\ell - h_{\ell+1})[3 \#(\text{ext} i_2 \text{ to } X_{\geq h_\ell}) + 2 \#(\text{ext} i_3 \text{ to } X_{\geq h_\ell}) + \#(\text{ext} i_4 \text{ to } X_{\geq h_\ell})]}, \quad \text{(B.12)}$$

where $\{h_\ell\}$ is the set of momentum assignments given by μ, ordered in a monotone decreasing way ($h_\ell > h_{\ell+1}$) and

$$\#(\text{ext} i_2 \text{ to } X_{\geq h_\ell}) = \#\{\varphi_{i_1}(x) \text{ with } i_1 \geq h_\ell, \ x \in X_{\geq h_\ell}, \ i_2(x) < h_\ell\},$$

$$\#(\text{ext} i_3 \text{ to } X_{\geq h_\ell}) = \#\{\varphi_{i_2}(x) \text{ with } i_2 \geq h_\ell, \ x \in X_{\geq h_\ell}, \ i_3(x) < h_\ell\},$$

$$\#(\text{ext} i_4 \text{ to } X_{\geq h_\ell}) = \#\{\varphi_{i_3}(x) \text{ with } i_3 \geq h_\ell, \ x \in X_{\geq h_\ell}, \ i_4(x) < h_\ell\}.$$

From (3.36)

$$\prod_{\Delta \subset X_0^{(2)}} (\ell_2(\Delta)!)^{1/2} \leqq \prod_v M^{(2+\varepsilon/2)(i_1(v)-i_2(v))} \left(\prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^{1/2} \right)$$

$$= \prod_\ell M^{(2+\varepsilon/2)(h_\ell - h_{\ell+1}) \, \# \, (\text{ext} i_2 \text{ to } X_{\geq h_\ell})} \left(\prod_{\Delta \subset X_0} (\ell_1(\Delta)!)^{1/2} \right).$$

(B.13)

Substituting these estimates in the brackets of (3.41) we get

$$\{ \ \} \leqq \sum_{X_0, |X_0| = q} e^{-\sum\limits_i \sum\limits_{\ell=i}^{K_i} d(X_{0,i}^{(\ell)})} \frac{1}{n(\theta)}$$

$$\times \left[\prod_\ell M^{-(h_\ell - h_{\ell+1})[(1-\varepsilon_1) \, \# \, (\text{ext} i_2 \text{ to } X_{\geq h_\ell}) + (2-\varepsilon_0) \, \# \, (\text{ext} i_3 \text{ to } X_{\geq h_\ell}) + 1 \, \# \, (\text{ext} i_4 \text{ to } X_{\geq h_\ell})]} \right],$$

(B.14)

where $\varepsilon_1 \equiv \varepsilon/2 + \eta(4+\varepsilon)$, $\varepsilon_0 \equiv \eta(4+\varepsilon)$; θ and $\{h_b\}$ fix for a definite $h_{\ell+1}$ how many connected components are present. Let this number be $K_{h_{\ell+1}}$; θ also fixes how many disconnected components at level h_ℓ merge in a single component t of level $h_{\ell+1}$. Let this number be $s_{h_\ell}^{(t)}$; then

$$[(B.14)] = \prod_\ell \prod_{t=1}^{K_{h_{\ell+1}}}$$
$$\times M^{-(h_\ell - h_{\ell+1}) \sum\limits_{r=1}^{s_{h_\ell}^{(t)}} [(1-\varepsilon_1) \, \# \, (\text{ext} i_2 \text{ to } X_{\geq h_\ell}^{(r)}) + (2-\varepsilon_0) \, \# \, (\text{ext} i_3 \text{ to } X_{\geq h_\ell}^{(r)}) + \# \, (\text{ext} i_4 \text{ to } X_{\geq h_\ell}^{(r)})]}.$$

(B.15)

Then

$$\sum_{X_0, |X_0| = q}^* e^{-\sum\limits_i \sum\limits_{\ell=1}^{K_i} d(X_{0,i}^{(\ell)})} \leqq (\text{const})^n \prod_\ell \prod_{t=1}^{K_{h_{\ell+1}}} M^{4(h_\ell - h_{\ell+1}) S_{h_\ell}^{(t)}}.$$

(B.16)

Putting together (B.14), (B.15), and (B.16) we obtain the inequality

$$\{(3.41)\} \leq \frac{1}{n(\theta)} \prod_\ell \prod_{t=1}^{K_{h_{\ell+1}}}$$

$$\times M^{-(h_\ell - h_{\ell+1}) \left\{ \sum\limits_{r=1}^{s_{h_\ell}^{(t)}} [(1-\varepsilon_1) \, \# \, (\text{ext} i_2 \text{ to } X_{\geq h_\ell}^{(r)}) + (2-\varepsilon_0) \, \# \, (\text{ext} i_3 \text{ to } X_{\geq h_\ell}^{(r)}) + \# \, (\text{ext} i_4 \text{ to } X_{\geq h_\ell}^{(r)})] - 4S_{h_\ell}^{(t)} \right\}}.$$

(B.17)

By the "convergent subgraph condition" we have imposed that from each $X_{\geq h_\ell}^{(r)}$ AL subgraph must exit at least 6 external lines. We must verify that this condition is enough to ensure that the bracketed coefficient of $(h_\ell - h_{\ell+1})$ in (B.17), denoted by $\{(B.17)\}$ is strictly positive, uniformly in $(\mu, \{h_b\}, \theta, X_0)$. Simple inspection of (B.17) shows that without the "c.s.c." this is not true in some definite cases, for instance

when

$$\#(\mathrm{ext}\, i_2 \text{ to } X^{(r)}_{\geqq h_\ell}) \leqq 4, \qquad \forall r,$$

and   (B.18)

$$\#(\mathrm{ext}\, i_3 \text{ to } X^{(r)}_{\geqq h_\ell}) = {}^{\#}(\mathrm{ext}\, i_4 \text{ to } X^{(r)}_{\geqq h_\ell}) = 0.$$

These are not the only possible cases but are the worst ones; therefore, if we can treat them using the "c.s.c.," we can also treat all the other dangerous cases. Observe that given $X'_{\geqq h_\ell}$ if $\#(\mathrm{ext}\, i_2 \text{ to } X^{(r)}_{\geqq h_\ell}) = 0$, then the "c.s.c." condition imposes $2\#(\mathrm{ext}\, i_3 \text{ to } X^{(r)}_{\geqq h_\ell}) + \#(\mathrm{ext}\, i_4 \text{ to } X^{(r)}_{\geqq h_\ell}) \geqq 6$ which implies that this term in $\{(\text{B.17})\}$ is positive and only helps. If $\#(\mathrm{ext}\, i_2 \text{ to } X^{(r)}_{\geqq}) = 1$, then $2\#(\mathrm{ext}\, i_3 \text{ to } X^{(r)}_{\geqq h_\ell}) + \#(\mathrm{ext}\, i_4 \text{ to } X^{(r)}_{\geqq h_\ell}) \geqq 3$, and this term produces a negative contribution, dangerous for the uniform positivity of $\{(\text{B.17})\}$. Similar effects happen when $\#(\mathrm{ext}\, i_2) = 2, 3, 4$. We are therefore reduced to study these dangerous cases: $\#(\mathrm{ext}\, i_2 \text{ to } X^{(r)}_{\geqq h_\ell}) \in [1, 4] \cap Z$, and $\#(\mathrm{ext}\, i_{3, 4} \text{ to } X^{(r)}_{\geqq h_\ell})$ as small as possible, but still compatible with the "c.s.c." condition.

The origin of the fact that the factor $\{(\text{B.17})\}$ can be negative is due to the estimate (B.13) for $\displaystyle\prod_{\Delta \subset X^{(r)}_0} (\ell_r(\Delta)!)^{1/2}$, which produces the factors $\displaystyle\prod_v M^{(2+\varepsilon/2)(i_1(v) - i_2(v))}$. It is easy to check that without these factors $\{(\text{B.17})\}$ would be always > 0. We observe now that when $S^{(t)}_{h_\ell} = 1$ the problem is easily solved. In fact, in this case

$$\#(\mathrm{ext}\, i_2 \text{ to } X_{\geqq h_\ell}) \leqq 4,$$ (B.19)

which implies that

$$\sum_{J=0}^{h_{\ell+1}} \sum_{\Delta_J \subset X^{(r)}_0} \sum_{i_1 = h_\ell}^{\varrho} \sum_{\Delta_{i_1} \subset \Delta_J} \ell_1(\Delta_{i_1}; J) \leqq 4.$$ (B.20)

Therefore,

$$\ell_2(\Delta_J) = \sum_{i_1} \sum_{\Delta_{i_1} \subset \Delta_J} \ell_1(\Delta_{i_1}; J) \leqq C \sum_{i_1 = 0}^{h_{\ell+1}} \sum_{\Delta_{i_1} \subset \Delta_J} \ell_1(\Delta_{i_1}; J),$$ (B.21)

and now the inequality (3.36) can be applied to the right-hand side of (B.21); then the factors $M^{(2+\varepsilon/2)(h_\ell - h_{\ell+1})\#(\mathrm{ext}\, i_2 \text{ to } X \geqq h)}$, for these h_ℓ for which (B.19) holds are missing and again $\{(\text{B.17})\}$ is > 0. The only more delicate case which can appear is when $S^{(t)}_{h_\ell}$ is large (this depends on the shape of θ). In this case, in fact, we can have

$$\#(\mathrm{ext}\, i_2 \text{ to } X^{(r)}_{\geqq h_\ell}) \leqq 4, \qquad S^{(t)}_{h_\ell} \gg O(1),$$ (B.22)

therefore, we can have $S^{(t)}_{h_\ell} \leqq \ell_2(\Delta_J) \leqq 4 S^{(t)}_{h_\ell}$, $J < h_\ell$. It is the factor $n(\theta)$ which helps us at this moment; in fact, recall its definition given after (3.19); it contains a factor $\dfrac{1}{(S^{(t)}_{h_\ell})!}$, hence if $\ell_2(\Delta_J) \leqq 2 S^{(t)}_{h_\ell}$, we have $\dfrac{(\ell_2(\Delta_J)!)^{1/2}}{(S^{(t)}_{h_\ell})!} \leqq C^{\ell_2(\Delta_J)}$, and again $\{(\text{B.17})\} > 0$, uniformly. If $\ell_2(\Delta_J) = 4 S^{(t)}_{h_\ell}$, then we estimate

$$\frac{(\ell_2(\Delta_J)!)^{1/2}}{(S^{(t)}_{h_\ell})!} \leqq (\ell_2(\Delta_J)!)^{1/4} C^{\ell_2(\Delta_J)},$$ (B.23)

and for $(\ell_2(\Delta_J)!)^{1/4}$ we apply the inequality (B.13) which now due to the 1/4 instead of 1/2 gives a better factor

$$M^{(1+\varepsilon/4)(h_\ell - h_{\ell+1}) \, \#(\text{ext}\, i_2 \text{ to } X_{\geq h_\ell}^{(r)})},$$ (B.24)

which changes the $(1-\varepsilon_1) \#(\text{ext}\, i_1)$ of {(B.17)} in $(2-\tilde{\varepsilon}_1) \#(\text{ext}\, i_2 \text{ to } X_{\geq h_\ell}^{(r)})$, which is $(2-\tilde{\varepsilon}_1)4$ and makes again {(B.17)} uniformly >0.

To summarize we have the following estimate

$$\{(\text{B.17})\} \leq (\text{const})^n \prod_v M^{-\delta''[(i_1(v)-i_2(v))+(i_1(v)-i_3(v))+(i_1(v)-i_4(v))]}$$ (B.25)

for some $\delta'' > 0$, and it is now an easy task (see [13] or [14]) to prove inequality (3.41), remembering that the presence of the true external fields $\varphi(y_1), ..., \varphi(y_N)$ $(N=6)$ breaks the translation invariance and makes the corresponding volume factor $|\Lambda|$ disappear.

Appendix C

We prove in this appendix the following lemma:

Lemma 6. *Suppose $i \leq j$. Then*

$$|\Gamma_{i,j}(x,y)| = |C_i(s)(x, \cdot)(-\Delta)C_j(s)(\cdot, y)| \leq O(1)M^{2i} e^{-O(1)M^i|x-y|}.$$ (C.1)

Proof. We remark first that $C_j(s)$ is a linear convex combination of $C_{j,Y}$, Y being a set of faces of cubes of $\mathbb{D}_j$ on which Dirichlet boundary conditions have been imposed (of course, Y contains ∂X_j, the boundary of the volume made of the cubes of X of index j). Hence it is enough to prove (C.1) for a fixed $C_{j,Y}$. We write first the Laplacian in (C.1) as $(-\Delta+1)-1$ and we bound the piece with the -1, which is simply the convolution of C_i with C_j, using the usual bound [14]:

$$C_i(\{s\})(x,y) \leq O(1)M^{2i} e^{-(1/2)M^i|x-y|}.$$ (C.2)

This is easy and gives a much better bound than the right-hand side of (C.1). Then we write $C_{j,Y}$ as $C_j - [C_j - C_{j,Y}]$ and evaluate separately both pieces. On the piece C_j, which has no Dirichlet restrictions, the action of $(-\Delta+1)$ can be computed easily, since C_j is just the product (in momentum space) of $(-\Delta+1)^{-1}$ by the cutoff [see (2.9)]. Multiplying by $(-\Delta+1)$ we get the cutoff. Let us introduce the notation:

$$a_i \equiv M^{-2(i+1)}, \quad b_i \equiv M^{-2i}.$$

For this first piece C_j we have the bound:

$$|C_i(s) \cdot [e^{-a_j(-\Delta+1)} - e^{-b_j(-\Delta+1)}]| \leq \int_{a_i+a_j}^{b_i+a_j} + \int_{a_i+b_j}^{b_i+b_j} e^{-\alpha(-\Delta+1)} d\alpha$$

$$\leq 2 \int_{a_i}^{2b_i} e^{-\alpha(-\Delta+1)} d\alpha \leq 2(C_i + C_{i-1})$$

$$\leq O(1)M^{2i} e^{-O(1)M^i|x-y|},$$ (C.3)

where for the first inequality we bound the difference by the sum, for the second and third inequalities we used the fact that $i \leq j$ and $M > 2$, and for the last one

we used (C.2). Inequality (C.3) is similar to (C.1). Hence it remains only to bound $C_i(s)(-\varDelta+1)C_j^Y$, where $C_j^Y = C - C_{j,Y}$ has the path interpretation:

$$C_j^Y(z, y) = \int_{M^{-2(j+1)}}^{M^{-2j}} dt e^{-t} \int P_t^Y(z, y) d\omega, \tag{C.4}$$

where $P_t^Y(x, y)d\omega$ is the conditional Wiener measure on the sets of all paths starting at x at time 0 and ending at y at time t and crossing at least one of the faces of Y. We will bound the action of $(-\varDelta+1)$ on (C.4) using an analysis of Wiener paths. We have (using the notations of [26, Sect. V]):

$$C_j^Y(z, y) = \int_0^{b_j} e^{-t_1} E_y\{\chi(\tau_Y \in dt_1\} \int_{\sup\{0, a_j - t_1\}}^{b_j - t_1} p(t_2, \omega(t_1), z)e^{-t_2} dt_2, \tag{C.5}$$

where E_y is the Wiener expectation for paths starting at y, $\chi(\tau_Y \in dt_1)$ is the characteristic function for all paths whose first hitting time for Y lies in the time interval $[t_1, t_1 + dt_1]$, and $p(t, x, y)$ is the transition density for the free Wiener process. (This formula just says that if a path ω hits Y, it has to hit it a first time, and then to go freely to its end.) Now we can apply the operator $(-\varDelta+1)$ on the kernel $p(t_2, \omega(t_1), z)$, and integrate over dt_2, since there are no Dirichlet conditions on p. This gives just again the difference of the cutoffs on t_2, hence the operator [acting on the $\omega(t_1)$ variable]:

$$e^{-\sup\{0, a_j - t_1\}(-\varDelta+1)} - e^{-(b_j - t_1)(-\varDelta+1)}. \tag{C.6}$$

Again we can bound the corresponding difference of operators by the sum of the absolute values of the corresponding quantities. Now we remark that each operator $e^{-c(-\varDelta+1)}$ is positive pointwise in direct space. This is obviously true also for $C_i(s)(x, z)$ and for E_y. Therefore, we can bound the corresponding convolution integral by increasing $C_i(s)$ to C_i. But the action of $e^{-c(-\varDelta+1)}$ on C_i is just to shift the bounds of integration by c [as was done in (C.3)]. Hence we can bound each of the two terms generated by (C.6) by (using again the kernel p):

$$\int_0^{b_j} e^{-t_1} E_y\{\chi(\tau_Y \in dt_1\} \int_{a_i+c}^{b_i+c} p(t_3, \omega(t_1), x)e^{-t_3} dt_3, \tag{C.7}$$

where c is either $\sup\{0, a_j - t_1\}$ or $b_j - t_1$. But using a formula similar to (C.5) in the reverse way, this is bounded by

$$\int_{a_i+c}^{b_i+c+b_j} dt e^{-t} \int P_t^Y(x, y) d\omega. \tag{C.8}$$

Now we can increase (C.8) by suppressing the condition that the path ω crosses Y and replacing the bounds of integration by a_i and $3b_i$ (since b_j and c are bounded by b_i). This is then easily estimated by the right-hand side of (C.1) (as in [14]). This completes the proof of Lemma 6.

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