Commun. Math. Phys. 108, 177–192 (1987)

Communications in
Mathematical
Physics

Local Behavior of Solutions of Some Elliptic Equations

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Abstract. We study the local behavior of solutions of some nonlinear elliptic equations. These equations are of interest in differential geometry and mathematical physics.

1. Introduction

Here we shall describe the local behavior of singular positive solutions of certain elliptic equations. Theorem A generalizes in an important manner one of our main results and indeed answers an open problem posed in [A]. There, corresponding upper and lower bounds for the singularities of the solution were given. To obtain Theorem A of this article considerably more arguments are needed.

We point out that when $n=3$, Eq. (1.1) below seems to be relevant in Yang–Mills–Higgs theory. See L. Sibner and R. Sibner [S–S]. We also remark that equations of type (1.1) seem to be relevant to astrophysics, a fact pointed out to the author by J. Serrin, (see [C, F, H]).

Our result reads as follows. Let $\mathrm{B}=\{x\in\mathbb{R}^n:|x|<1, n\geqq 3\}$. Then

Theorem A. *Let $u\in C^2(B\backslash\{0\})$ be a non-negative solution of*

$$\Delta u+|x|^\sigma u^{(n+\sigma)/(n-2)}=0 \quad in \quad B\backslash\{0\}, \tag{1.1}$$

where $-2<\sigma<2$. Then u has either a removable singularity at $\{0\}$ or

$$\lim_{|x|\to 0}|x|^{n-2}(-ln|x|)^{(n-2)/(\sigma+2)}u(x)=l, \tag{1.2}$$

exists and

$$l=\left(\frac{(n-2)}{(\sigma+2)}1/(\sigma+2)\right)^{(n-2)}.$$

(The existence of singular solutions was shown in [A]).

It is interesting to observe that a similar result holds for solutions of equations of the "opposite" sign of (1.1), that is

$$\Delta\psi-|\psi|^{2/(n-2)}\psi=0,$$

in $\{x\in\mathbb{R}^n: |x|>1\}$. Indeed Veron [V] proved

$$\lim_{|x|\to\infty} |x|^{n-2}(\ln|x|)^{(n-2)/2}\psi(x)=l,$$

exists and l is either $\pm((n-2)/\sqrt{2})^{n-2}$ or 0; $l=((n-2)/\sqrt{2})^{n-2}$ if $u>0$.

Next, we would like to say a few words about the proof of Theorem A. In Gidas–Spruck [G–S] the strategy of the proof of the corresponding statement of singular solution of

$$\Delta u+u^q=0,\quad \text{in}\quad B\backslash\{0\}, \frac{n}{n-2}<q<\frac{n+2}{n-2}, \tag{1.3}$$

was based on the important fact that there is only one non-trivial solution of

$$\Delta u+u^q=0\quad \text{in}\quad \mathbb{R}^n\backslash\{0\}.$$

However, there are no non-trivial solutions of (1.1) in $\mathbb{R}^n\backslash\{0\}$. This makes the situation very different. The strategy of our proof is to compute the Laplacian of $v(x)=|x|^{n-2}(-\ln|x|)^{(n-2)/2}u$. Then by means of the change of variable $t=-\ln|x|$ we transform that equation in a time dependent equation. The transformation into a time dependent equation was also done in [V]. But contrary to [V], we then use energy methods to prove that

$$\lim_{|x|\to 0}|x|^{n-2}(-\ln|x|)^{(n-2)/2}u(x)=l,$$

exists and

$$l=\left(\frac{n-2}{\sqrt{2}}\right)^{n-2}.$$

It should be noticed that the method of using time dependent equations in the spirit used here has also been used by L. Simon [S]. For parabolic singularities some related ideas have been applied by Y. Giga and R. Kohn [G–K].

The proof of (1.2) is the main point of our theorem. Roughly it can be described as follows. We introduce polar coordinates $(|x|,\omega)$ in $\mathbb{R}^n\backslash\{0\}$ and

$$v(t,\omega)=|x|^{n-2}(-\ln|x|)^{(n-2)/2}u(x),\quad t=-\ln|x|. \tag{1.4}$$

Because of (1.8) $v\leqq c$, and furthermore satisfies

$$v_{tt}+(n-2)\left(1-\frac{1}{t}\right)v_t+\Delta_\omega v=\frac{v}{t}\left(\frac{(n-2)^2}{2}-v^{2/(n-2)}\right)-\frac{(n-2)n}{4t^2}v. \tag{1.5}$$

The point is to prove that

$$\int_{t_0}^{\infty}\int_{S^{n-1}} v_t^2 d\omega dt<\infty;\quad \int_{t_0}^{\infty}\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega dt<\infty,$$

and $\lim_{t\to+\infty}\int_{S^{n-1}} v_t^2 d\omega=0$, $t_0>0$ a constant. The multiplication of (1.5) by v and the integration over (t_0, ∞) yields

$$\left| \int_{t_0}^{\infty} \int_{S^{n-1}} \frac{v^2}{t} \left(\frac{(n-2)^2}{2} - v^{2/(n-2)} \right) d\omega dt \right| < \infty. \tag{1.6}$$

So if v admits positive a limit it should be $((n-2)/\sqrt{2})^{n-2}$.

The best strategy to prove the existence of limit is by showing that

$$t \to \| v(t, \cdot) - \bar{v}(t) \|_{L^2(S^{n-1})},$$

$(\bar{v}(t) = 1/\omega_{n-1} \int_{S^{n-1}} v(t, \omega) d\omega)$ decays sufficiently fast and by then showing that $\bar{v}(t)$ has a limit as $t \to \infty$.

In Sect. 3 we shall give an extension of Theorem A for non-positive solutions. In Sect. 4 we shall give a new simple proof of the Harnack inequality for non-negative solutions of (1.1). This was previously proved in the paper of Gidas and Spruck [G–S]. But their proof is not easy to follow.

We also mention that radial solutions of $\Delta u + u^q = 0$ have been studied by Fowler [F] and Hopf [H]. Ni and Serrin [N–S] have informed us of work in preparation in which they study singular radial solutions for some general classes of equations.

Finally, we shall explicitly recall the known results about (1.1) and (1.3). In [G–S], Theorem 3.3, and in [A], Theorem 1, it was proved that if $u \geqq 0$ is a solution of (1.3) or (1.1), then either u has a removable singularity at the origin or in case u satisfies (1.3)

$$u(x) \leq C|x|^{-2/(q-1)}, \tag{1.7}$$

and

$$\limsup_{|x| \to 0} |x|^{2/(q-1)} u(x) \geqq C^{-1}, \tag*{(1.7)^1}$$

or in case u satisfies (1.1)

$$u(x) \leqq C|x|^{2-n}(-\ln|x|)^{(2-n)/2} \tag{1.8}$$

and

$$\limsup_{|x| \to 0} |x|^{n-2}(-\ln|x|)^{(n-2)/2} u(x) \geqq C^{-1}, \tag*{(1.8)^1}$$

$C > 0$ a constant. As a matter of fact in [G–S] as well as in [A] the reverse inequality of (1.7) and (1.8) was claimed. However, in both cases there is an assertion which seems to require further explanation. The precise statement is given in Sect. 5. This statement is either an immediate consequence of Theorem C given in Sect. 5 or in case of solution of (1.1) follows at once from the much more delicate result in Theorem A of this paper.

2. Proof of Theorem A

We begin by considering the average

$$\bar{u}(r)=\frac{1}{\omega_{n-1}}\int_{S^{n-1}} u(r,\omega)d\omega, \quad 0<r<1,$$

where ω_{n-1} is the volume of the sphere S^{n-1}. From now on we shall assume that in Eq. (1.1), $\sigma=0$. The cases $\sigma\neq 0$, $-2<\sigma<2$ are treated in exactly the same manner. Taking the average in (1.1) we obtain

$$\bar{u}''+\frac{n-1}{r}\bar{u}'+\bar{u}^{n/(n-2)}\leqq 0, 0<r\leqq 1.$$

We show the following

Lemma 1. *Any non-negative solution of (1.1) satisfies*

$$(-\ln r)^{(n-2)/2} r^{n-2}\bar{u}(r)\leqq\left(\frac{n-2}{\sqrt{2}}\right)^{n-2}, \quad 0<r<r_0, \tag{2.1}$$

for some $r_0>0$.

Proof. Define $A(s)=\bar{u}(s^{-1/(n-2)})$. Then

$$\ddot{A}(s)+\frac{1}{(n-2)^2}s^{-2(n-1)/(n-2)}A(s)^{n/(n-2)}\leqq 0.$$

Let $B(r)=rA(r^{-1})$ with r closes to zero. Then B satisfies

$$\ddot{B}(r)+\frac{1}{(n-2)^2}\frac{1}{r^2}B(r)^{n/(n-2)}\leqq 0.$$

It follows from [A] p. 778 that B is non-decreasing and $B(0)=0$. These facts imply that

$$\dot{B}(\rho)\geqq\frac{1}{(n-2)^2}\frac{B(\rho)^{n/(n-2)}}{\rho}, \quad \rho \ \text{near} \ 0.$$

(See proof of Lemma 1 of [A].) By considering $C(\rho)=B(\lambda\rho)$, $0<\lambda<1$, we may suppose that the above relation holds for $0<\rho\leqq 1$. Integrating from ρ to 1 we obtain

$$-\frac{(n-2)}{2}[-B(\rho)^{-2/(n-2)}+B(1)^{-2/(n-2)}]\geqq-\frac{1}{(n-2)^2}\ln\rho.$$

Hence

$$\left(\frac{n-2}{2}\right)B(\rho)^{-2/(n-2)}\geqq\frac{(n-2)}{2}B(1)^{-2/(n-2)}+\frac{1}{(n-2)^2}(-\ln\rho).$$

Since $B(1)\geqq 0$ we get

$$\frac{2}{(n-2)}B(\rho)^{2/(n-2)}\leqq(n-2)^2(-\ln\rho)^{-1}.$$

So

$$\left(\frac{1}{(n-2)}\right)^{(n-2)/2} B(\rho) \leqq \frac{(n-2)^{(n-2)}}{2^{(n-2)/2}}(-\ln \rho)^{-(n-2)/2}.$$

The definition of $B(\rho)$ yields

$$\left(\frac{1}{(n-2)}\right)^{(n-2)/2} (-\ln \rho)^{(n-2)/2} \rho \bar{u}(\rho^{1/(n-2)}) \leqq \left(\frac{n-2}{2^{1/2}}\right)^{n-2}.$$

Setting $\rho = r^{n-2}$ we obtain Lemma 1.

Lemma 2. *For any* $\gamma \in (0, 1]$, *there exists* $C_\gamma > 0$ *such that*

$$\| v(t, \cdot) - \bar{v}(t) \|_{L^2(S^{n-1})} \leqq C_\gamma t^{-\gamma} \tag{2.2}$$

for $t \geqq t_0$.

Proof. We follow Veron [V]. Set

$$t = -\ln |x|, \quad \phi(t, \theta) = |x|^{n-2} u(x),$$

then

$$\phi(t, \theta) \leqq C t^{(2-n)/2}, \tag{2.3}$$

and ϕ satisfies

$$\phi_{tt} + (n-2)\phi_t + \Delta_\omega \phi + \phi^{n/(n-2)} = 0. \tag{2.4}$$

We consider the average of ϕ, $\bar{\phi}(t)$. We recall the Poincaré inequality

$$\int_{S^{n-1}} (\phi - \bar{\phi}) \Delta_\omega (\phi - \bar{\phi}) d\omega \leqq (1-n) \int_{S^{n-1}} |\phi - \bar{\phi}|^2 d\omega. \tag{2.5}$$

We now subtract from (2.4) the corresponding equation for $\bar{\phi}$, then we multiply that equation by $(\phi - \bar{\phi})$ and we use Hölder inequality and the inequality

$$(ab)^{1/2} \leqq \frac{(n-1)}{2} a + \frac{1}{2(n-1)} b, \quad a, b > 0,$$

to get from (2.3) and (2.5) that

$$X_{tt} + (n-2) X_t - (n-1) X \geqq -\frac{C_1}{t^n}, \quad t \geqq t_0, \tag{2.6}$$

where $X(t) = \| \phi(t, \cdot) - \bar{\phi}(t) \|^2_{L^2(S^{n-1})}$ and $C_1 > 0$ is a constant. The homogeneous associated equation to (2.6)

$$Y_{tt} + (n-2) Y_t - (n-1) Y = 0, \tag{2.7}$$

admits the two linearly independent solutions

$$\begin{cases} Y_1(t) = \exp((1-n)t) \\ Y_2(t) = \exp(t) \end{cases}. \tag{2.8}$$

A particular solution of

$$Y_{tt}+(n-2)Y_t-(n-1)Y=-\frac{C_1}{t^n} \tag{2.9}$$

is given by

$$Y_p(t)=C_1 e^{-(n-1)t}\int_{t_0}^{t} e^{nl}\left(\int_l^{\infty}\frac{e^{-s}}{s^n}ds\right)dl.$$

As $X(t)\leqq Ct^{2-n}$, basic comparison principles implies $X(t)\leqq C_2Y_1(t)+Y_p(t)$ for t large, where $C_2>0$ is a constant. This is because $Y_p(t)\leqq C_3t^{-n}$ for t large, $C_3>0$ a constant. Hence (2.2) follows.

Lemma 3. *v, v_t, v_{tt} and $|\nabla_\omega v|$ are uniformly bounded.*

Proof. From Lemma 1 and the Harnack inequality cf. [G–S] Theorem 3.1 (or see Appendix I of this paper) we get

$$u(x)\leqq C\frac{(-\ln|x|)^{-(n-2)/2}}{|x|^{n-2}},$$

from which it follows that v is bounded. It is now standard from well known elliptic estimates to have that v_t, v_{tt} and $|\nabla_\omega v|$ are uniformly bounded.

Lemma 4. (i) $\int_{t_0}^{\infty}\int_{S^{n-1}} v_t^2\, d\omega dt<\infty$, (ii) $\int_{t_0}^{\infty}\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega dt<\infty$ *and* (iii) $\int_{S^{n-1}} v_t^2\, d\omega = 0$.

Proof. Equation (1.5) implies after a multiplication by v_t:

$$\frac{1}{2}\int_{S^{n-1}}(v_t^2)_t d\omega+(n-2)\int_{S^{n-1}}\left(1-\frac{1}{t}\right)v_t^2 d\omega-\frac{1}{2}\int_{S^{n-1}}|\nabla_\omega v|_t^2 d\omega$$
$$=\frac{1}{t}\int_{S^{n-1}}\left(\frac{(n-2)^2}{4}(v^2)_t-\frac{(n-2)}{2(n-1)}(v^{2(n-1)/(n-2)})_t\right)d\omega-\frac{(n-2)n}{8}t^{-2}\int_{S^{n-1}}(v^2)_t d\omega. \tag{2.10}$$

Lemma 3 and integration by parts on (t_0,∞) yields (i).

We prove (ii). We multiply (1.5) by $v-\bar v$ to get

$$\int_{t_0}^{T}\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega dt=\int_{t_0}^{T}\int_{S^{n-1}} v_{tt}(v-\bar v)d\omega dt+(n-2)\int_{t_0}^{T}\int_{S^{n-1}}(1-t^{-1})v_t(v-\bar v)d\omega dt$$
$$+\int_{t_0}^{T}\int_{S^{n-1}} t^{-1}v^{n/(n-2)}(v-\bar v)d\omega dt$$
$$-\tfrac{1}{2}(n-2)^2\int_{t_0}^{T}\int_{S^{n-1}} t^{-1}v(v-\bar v)d\omega dt$$
$$+(n-2)\frac{n}{4}\int_{t_0}^{T}\int_{S^{n-1}} t^{-2}v(v-\bar v)d\omega dt.$$

It follows from (i) and Lemma 2 that

$$\int_{t_0}^{T}\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega dt\leq\int_{t_0}^{T}\int_{S^{n-1}} v_{tt}(v-\bar v)d\omega dt+C$$

with $C > 0$ a constant independent of T. We next observe that

$$\int_{t_0}^{T} v_{tt}(v-\bar{v})dt = v_t(v-\bar{v})|_{t_0}^{T} - \int_{t_0}^{T} v_t^2\,dt + \frac{1}{\omega_{n-1}}\int_{t_0}^{T} v_t\left(\int_{S^{n-1}} v_t d\omega\right)dt.$$

Hence by Fubini's theorem and Hölder inequality we obtain

$$\left|\int_{t_0}^{T}\int_{S^{n-1}} v_{tt}(v-\bar{v})d\omega dt\right| \leqq C,$$

with $C > 0$ a constant independent of T. Therefore we obtain (ii).

We show (iii). Let

$$g(t) = \int_{S^{n-1}} v_t^2\,d\omega.$$

Since $v_t v_{tt}$ is uniformly bounded we get that

$$\dot{g}(t) = 2\int_{S^{n-1}} v_t v_{tt} d\omega,$$

is uniformly bounded. If $g(t) \not\to 0$ as $t \to \infty$, then given $\varepsilon > 0$ there exists a sequence $t_i \to \infty$ so that $g(t_i) > 2\varepsilon$. Let M be chosen so that $|\dot{g}(t)| \leqq M$. Therefore if $|t - t_j| < \varepsilon/M$, then

$$g(t) > g(t_j) - \left|\int_{t}^{t_j} \dot{g}(s)ds\right| > \varepsilon.$$

Let now $\{t'_j\}$ be a subsequence of $\{t_j\}$ satisfying $t'_{j+1} > t'_j + \varepsilon/M$, $t'_0 > t_0$. Since

$$\int_{t'_{j-1}}^{t'_j} g(t)dt \geqq \int_{t'_{j-1}}^{t'_{j-1}+\varepsilon/M} g(t)dt \geqq \frac{\varepsilon^2}{M},$$

we obtain

$$\sum_{j=1}^{N}\int_{t'_{j-1}}^{t'_j} g(t)dt > \frac{\varepsilon^2}{M}N \to \infty \quad \text{as} \quad N \to \infty,$$

contradicting (i).

Lemma 5. *$\bar{v}(t)$ admits a limit as $t \to \infty$.*

Proof. From Lemma 3 and Arzela–Ascoli's theorem for any sequence $\{t_n\} \to \infty$ there exists a sequence $\{t_{n_k}\}$ and $l(\omega)$ such that $v(t_{n_k}, \omega) \to l(\omega)$ uniformly on S^{n-1}. Hence $\bar{v}(t_{n_k}) \to l$ and $l(\omega) = l$ from Lemma 2. Assume now

$$l = \lim_{n_k \to \infty} v(t_{n_k}, \omega), \quad l' = \lim_{n_k \to \infty} v(s_{n_k}, \omega).$$

Since Lemma 1 implies

$$\frac{(n-2)^2}{2} - \bar{v}^{2/(n-2)} \geqq 0, \tag{2.11}$$

we have

$$\bar{v}_{tt} + (n-2)\left(1 - \frac{1}{t}\right)\bar{v}_t = \frac{\bar{v}}{t}\left(\frac{(n-2)^2}{2} - \bar{v}^{2/(n-2)}\right) - \frac{n(n-2)}{4t^2}\bar{v} + \frac{1}{t}\left(\bar{v}^{n/(n-2)} - \overline{v^{n(n-2)}}\right).$$

Hence from (2.11) we have

$$\bar{v}_{tt} + (n-2)\left(1 - \frac{1}{t}\right)\bar{v}_t \geqq \frac{1}{t}(\bar{v}^{n/(n-2)} - \overline{v^{n/(n-2)}}) - \frac{(n-2)n}{4t^2}\bar{v}. \tag{2.12}$$

But

$$\bar{v}^{n/(n-2)} - \overline{v^{n/(n-2)}} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} (\bar{v}^{n/(n-2)} - v^{n/(n-2)})d\omega.$$

So from Lemma 2 we have

$$|\bar{v}^{n/(n-2)} - \overline{v^{n/(n-2)}}| \leqq \| v - \bar{v} \|_{L^2(S^{n-1})} \leqq \frac{C'}{t}.$$

To this end we observe that by taking subsequences we may suppose that $t_{n_k} > s_{n_k}$. Since from (iii) in Lemma 4 we have $\bar{v}_t \to 0$ as $t \to \infty$, and since the right-hand side of (2.12) is integrable in (t_0, ∞), we conclude by integrating the above relation from s_{n_k} to t_{n_k} and letting $n_k \to \infty$ that

$$(n-2)(l - l') \geqq 0.$$

In the same way $l \leqq l'$ and $v(t, \cdot) \underset{t \to +\infty}{\to} l$ uniformly on S^{n-1}.

Proof of Theorem A. It is consequence of (1.6) in the introduction and Lemma 2 of [A]. See also Lemma 3.3 of this article.

Remark. It is clear that the proof above can be easily modified to get the results stated in the introduction for

$$\Delta u + |x|^\sigma u^{(n+\sigma)/(n-2)} = 0, \quad -2 < \sigma < 2, \quad \sigma \neq 0.$$

3. A Suitable Theorem A for Nonpositive Solutions

In this section we shall consider non-positive solutions of (1.1), more precisely of

$$\Delta u + u|u|^{2/(n-2)} = 0 \quad \text{in} \quad B = \{x: 0 < |x| < 1\}. \tag{3.1}$$

For such solution no estimate of the form

$$|u(x)| \leqq C|x|^{(2-n)}(-\ln|x|)^{(2-n)/2}, \tag{3.2}$$

is known, but we shall assume (3.2) is satisfied for some $C > 0$. Then as in [V] (for the equation of the opposite sign) we obtain.

Theorem B. *If $u \in C^2(B \backslash \{0\})$ satisfies (3.1) and (3.2) then*

$$\lim_{x \to 0} |x|^{(n-2)}(-\ln|x|)^{(n-2)/2}u(x) = l,$$

exists and $l = \pm((n-2)/\sqrt{2})^{(n-2)}$ or 0. If $l = 0$ then the singularity is removable.

Because of the inequality

$$|v|v|^{q-1} - \bar{v}|\bar{v}|^{q-1}| \leqq C\|v\|_{L^\infty}^{q-1}|v - \bar{v}|,$$

we get

$$|v|v|^{q-1}-\frac{1}{\omega_{n-1}}\int_{S^{n-1}} v|v|^{q-1}d\omega| \leqq C\|v\|_{L^\infty}^{q-1}\left(|v-\bar{v}|+\frac{1}{\omega_{n-1}}\|v-\bar{v}\|_{L^1(S^{n-1})}\right),$$

(see [V] p. 35). Hence using the notations of Lemma 2 we get with the help of (3.2) the differential inequality

$$X_{tt}(t)+(n-2)X_t(t)-(n-1-\varepsilon)X(t)\geqq 0,$$

for t large enough, $\varepsilon>0$ arbitrarily small. As in Lemma 2 this implies: For any $\gamma\in(0,n-1)$ there exists $C_\gamma>0$ such that

$$\|v(t,\cdot)-\bar{v}(t)\|_{L^2(S^{n-1})}\leqq C_\gamma\exp(-\gamma t), \tag{3.3}$$

for $t\geqq t_0$.

Next we multiply (1.5) by $v-\bar{v}$ to obtain

$$\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega=\int_{S^{n-1}}B(t,\omega)(v-\bar{v})d\omega dt, \tag{3.4}$$

where $B(t,\omega)$ is bounded. Hence it follows from (3.3) that $\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega$ is exponentially decaying.

Since Lemma 1 does not remain valid we replace it as follows.

Lemma 3.1. *The energy associated to Eq. (1.5)*

$$E(t)=\frac{1}{2}t\int_{S^{n-1}}v_t^2 d\omega-\frac{1}{2}t\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega+\frac{(n-2)}{2(n-1)}\int_{S^{n-1}}|v|^{2(n-1)/(n-2)}d\omega$$
$$-\frac{(n-2)^2}{4}\int_{S^{n-1}}v^2 d\omega+(n-2)\frac{n}{8}t^{-1}\int_{S^{n-1}}v^2 d\omega,$$

has the following properties

(a) $$\frac{dE(t)}{dt}<0 \quad \textit{for} \quad t>t_0,$$

(b) $$\lim_{t\to\infty}E(t)=L>-\infty \quad \textit{exists.}$$

Proof. To see the first part we multiply (1.5) by tv_t and we integrate over S^{n-1} to obtain

$$\frac{dE}{dt}=\frac{1}{2}\int_{S^{n-1}}v_t^2 d\omega-(n-2)\int_{S^{n-1}}(t-1)v_t^2 d\omega-\frac{1}{2}\int_{S^{n-1}}|\nabla_\omega v|^2 d\omega-(n-2)\frac{n}{8}\int_{S^{n-1}}t^{-2}v^2 d\omega.$$

Hence $(dE/dt)<0$ and therefore $\lim_{t\to\infty}E(t)$ exists. (b) follows from (3.4).

On the other hand from (3.2) we get that Lemma 3 remains valid and therefore the limit set of $\{v(t,\cdot)\}$ is not empty and all its elements are constant. The main point is now to prove

Lemma 3.2.

$$\lim_{t\to\infty}t\int_{S^{n-1}}v_t^2 d\omega=0.$$

Such a relation implies that

$$\lim_{t\to\infty} E(t) = L = \left(\frac{(n-2)}{2(n-1)}|l|^{2(n-1)/(n-2)} - \frac{(n-2)}{4}l^2\right)\omega_{n-1}.$$

But the set of $l \in \mathbb{R}$ such that

$$\frac{(n-2)}{2(n-1)}|l|^{2(n-1)/(n-2)} - \frac{(n-2)^2}{4}l^2 = L/\omega_{n-1},$$

is discrete (0.2.3, 4 elements). It follows from (1.6), that Theorem B holds.

Proof of Lemma 3.2. From Lemma 3.1 we get that $\int_{t_0}^{\infty} t \int_{S^{n-1}} v_t^2 d\omega dt < \infty$. We now differentiate (1.5) to get

$$\begin{aligned} &w_{tt} + (n-2)(1-t^{-1})w_t + (n-2)t^{-2}w + \Delta_\omega w \\ &= t^{-2}v|v|^{2/(n-2)} - \frac{n}{(n-2)}t^{-1}|v|^{2/(n-2)}w + \tfrac{1}{2}(n-2)^2 t^{-1}w \\ &\quad - \tfrac{1}{2}(n-2)^2 t^{-2}v + (n-2)\frac{n}{2}t^{-3}v - (n-2)\frac{n}{4}t^{-2}w, \end{aligned} \tag{3.5}$$

with $w = v_t$. Proceeding as above (see (3.4)) we get that $\int_{S^{n-1}} |\nabla_\omega v_t|^2 d\omega$ is exponentially decaying. Next, we multiply the Eq. (3.5) by $tv_{tt} = tw_t$ and we integrate over S^{n-1} to get the relation

$$\frac{dJ}{dt} = (-(n-2)t + (n-3/2)) \int_{S^{n-1}} w_t^2 d\omega - \frac{1}{2}\int_{S^{n-1}} |\nabla_\omega w|^2 d\omega, \tag{3.6}$$

where $J(t)$ is the natural energy associated to Eq. (3.5):

$$\begin{aligned} J(t) &= \tfrac{1}{2}t \int_{S^{n-1}} w_t^2 d\omega - \tfrac{1}{2}t \int_{S^{n-1}} |\nabla_\omega w|^2 d\omega + \int_{S^{n-1}} \int_t^\infty t^{-1}v|v|^{2/(n-2)} w_t dt d\omega \\ &\quad - \frac{n}{(n-2)} \int_{S^{n-1}} \int_t^\infty |v|^{2/(n-2)} w w_t dt d\omega - \frac{(n-2)^2}{4} \int_{S^{n-1}} w^2 d\omega - \tfrac{1}{2}(n-2)^2 \\ &\quad \cdot \int_{S^{n-1}} \int_t^\infty t^{-1} v w_t dt d\omega + (n-2)\frac{n}{2} \int_{S^{n-1}} \int_t^\infty t^{-2} v w_t dt d\omega \\ &\quad - (n-2)\frac{(n+4)}{4} \int_{S^{n-1}} \int_t^\infty t^{-1} w w_t dt d\omega. \end{aligned}$$

Now since $(dJ/dt) \leqq 0$ and $\int_{S^{n-1}} |\nabla_\omega v_t|^2 d\omega$ is exponentially decaying it follows

$$\lim_{t\to\infty} J(t) = \tfrac{1}{2}\lim_{t\to\infty} t \int_{S^{n-1}} w_t^2 d\omega = C < \infty \text{ exists.}$$

So integrating (3.6) over (t_0, ∞) we obtain

$$\int_{t_0}^{\infty} \int_{S^{n-1}} t v_{tt}^2 d\omega dt < \infty. \tag{3.7}$$

We conclude the proof of the lemma.

$$\frac{d}{dt} t \int_{S^{n-1}} v_t^2 d\omega = \int_{S^{n-1}} v_t^2 d\omega + 2t \int_{S^{n-1}} v_t v_{tt} d\omega.$$

Let $\{t_k\}$, $\{s_k\}$ be two sequences such that $t_k, s_k \to \infty$. Integrating from t_k so s_k. We obtain

$$\left| t_k \int_{S^{n-1}} v_t^2 d\omega - s_k \int_{S^{n-1}} v_t^2 d\omega \right| \leqq \int_{l_k}^{\infty} \int_{S^{n-1}} v_t^2 d\omega dt + \int_{l_k}^{\infty} \int_{S^{n-1}} t v_t^2 d\omega dt + \int_{l_k}^{\infty} \int_{S^{n-1}} t v_{tt}^2 d\omega dt,$$

where $l_k = \min(t_k, s_k)$. It follows that $\lim_{t\to\infty} t \int_{S^{n-1}} v_t^2 d\omega$ exists. But since tv_t^2 is integrable we obtain the claim.

We finish the proof by proving

Lemma 3.3. *If $l = 0$, then the singularity is removable.*

Proof. We follow [A] Lemma 2 pp. 778–779. Let

$$f(x) = (-\ln|x|^2)^{-s} u(x), \quad s > 0.$$

Then f satisfies

$$\Delta f + \sum_{i=1}^{n} b_i(x) f_{x_i} = f(x)\left[4s(s-1)\frac{(-\ln|x|^2)^{-2}}{|x|^2} + 2s(n-2)\frac{(-\ln|x|^2)^{-1}}{|x|^2} - |u|^{2/(n-2)} \right]$$

with $b_i(x) = -4s(-\ln|x|^2)^{-1}(x_i/|x|^2)$. So if in Theorem B $l = 0$, then $\Delta|f| + \sum_{i=1}^{n} b_i(x)|f|_{x_i} \geqq 0$, near the origin, and

$$|f| = 0((-\ln|x|)^{-(n-2)/2-s}|x|^{-(n-2)}).$$

We then consider the comparison function

$$\psi(r) = \int_r^{1/2} \frac{(-\ln t)^{-2s}}{t^{n-1}} dt, \quad 0 < r < \tfrac{1}{2}.$$

Hence since $\Delta\psi + \sum_{i=1}^{n} b_i(x)\psi_{x_i} = 0$, $\psi(x) \geqq C\frac{(-\ln|x|)^{2s}}{|x|^{n-2}}$ $C > 0$ a constant we conclude that if $M = \max_{|x|=r_0} |f(x)|$, r_0 small, then for every $\varepsilon > 0$ there exists $r(\varepsilon) < r < r_0(r(\varepsilon) \to 0$ as $\varepsilon \to 0)$ such that $f(x) \leqq \varepsilon\psi(|x|) + M$ when $r(\varepsilon) \leqq |x| \leqq r_0$. Therefore, $|f(x)|$ is bounded and as in the proof of Lemma 2 in [1] the singularity must be removable.

4. Appendix I

In this appendix we shall give a simple proof of the Harnack inequality for positive solution of (1.1).

To obtain the upper estimate for singular solutions of (1.1), that is, estimate (1.8), we used the Harnack inequality (see Lemma 1 of [A]). This inequality is a

consequence of Theorem 3.1 in [G–S]. However, their proof is rather complicated.
We begin by recalling the well known fact

Lemma 4.1. *There are no non-negative C^∞ solutions of (1.1) in $\mathbb{R}^n \backslash K$, where $K \subset \mathbb{R}^n$ is a compact set of $\mathbb{R}^n$.*

Proof. We assume $0 \in K$ and $K \subseteq \{x \in \mathbb{R}^n : |x| < 1\}$. We then consider the average of u, $\bar{u}$, center at the origin. By averaging (1.1) we get (assuming $\sigma = 0$)

$$\bar{u}_{rr} + (n-1)\frac{\bar{u}_r}{r} + \bar{u}^{n/(n-2)} \leqq 0, r > 1.$$

Next, we make the following changes of variables

$$v(r) = \bar{u}(r^{-1/(n-2)}), \quad r \leqq 1, \quad f(r) = rv(r^{-1}), \quad r \geqq 1.$$

We obtain the differential inequality

$$f_{rr} + \frac{1}{(n-2)^2}\frac{f^{n/(n-2)}}{r^2} < 0, \quad r > 1.$$

Therefore $f_{rr} \leqq 0$ and because $f \geqq 0$ we obtain that $f_r \geqq 0$. Hence

$$f_r(r) = f_r(r_0) + \int_{r_0}^{r} f_{rr}(s)ds \leqq f_r(r_0) - \frac{1}{(n-2)^2}\frac{f^{n/(n-2)}(r_0)}{r_0} + \frac{1}{(n-2)^2}\frac{f^{n/(n-2)}(r_0)}{r}. \tag{4.1}$$

If there is $r_0 \geqq 1$ so that

$$f_r(r_0) - \frac{1}{(n-2)^2}\frac{f^{n/(n-2)}(r_0)}{r_0} \leq 0. \tag{4.2}$$

Then by letting $r \to \infty$ in (4.1) we conclude that $f_r(r) < 0$, for r sufficiently large. This is a contradiction. So, since (4.2) never holds we have

$$f_r(r) \geqq \frac{1}{(n-2)^2}\frac{f^{n/(n-2)}(r)}{r}, \quad r \geqq 1.$$

Hence, by integrating from 1 to r we obtain

$$-\frac{(n-2)}{2} f^{-2/(n-2)}(r) + \frac{(n-2)}{2} f^{-2/(n-2)}(1) \geqq \frac{1}{(n-2)^2} \ln r.$$

Since as $r \to \infty$, $f(r) \to C \leqq \infty$, we get a contradiction.

Lemma 4.2. *If $u \geqq 0 \in C^\infty(\Omega)$ is a solution of (1.1) in Ω. Then*

$$\sup_{x \in \tilde{\Omega}} u(x) \leqq C(\tilde{\Omega}, n),$$

where $\tilde{\Omega} \subset\subset \Omega$ and $C(\tilde{\Omega}, n) > 0$ is a constant depending only on $\tilde{\Omega}$ and n but independent of u.

Proof. This lemma follows at once from Lemma 4.1, (see [G–S, II], pp. 887–890). Indeed suppose there is a sequence of C^∞ solutions, say $u_i \geqq 0$, and a sequence of points $p_i \to p, p_i, p \in \text{closure}(\tilde{\Omega})$ such that $M_i = u_i(p_i) \to \infty$ as $i \to \infty$. We consider

$$v_i(x) = \lambda_i^{n-2} u_i(\lambda_i x + p_i), \quad |x| \leqq \lambda_i^{-1},$$

where we have assumed that $B_1(p_i) \subset \Omega$ and where $\lambda_i \to 0$ is defined by $\lambda_i^{n-2} u_i(p_i) = 1$. If $p_i \to p \in \tilde{\Omega}$ and $B_{1/2}(p) \subset \tilde{\Omega}$, then standard elliptic estimates and a diagonalization procedure imply we can find v and a subsequence $i \to \infty$ such that

$$v_i \to v \quad \text{in} \quad C^2(\mathbb{R}^n),$$

$$\Delta v + v^{n/(n-2)} = 0 \quad \text{in} \quad \mathbb{R}^n, v(0) = 1.$$

But this contradicts Lemma 4.1. If $p \in \partial\tilde{\Omega}$ then we argue as in [G–S, II] p. 892. This completes the proof of this lemma.

Lemma 4.3. *Let $u \geqq 0$, $\in C^\infty(B \backslash \{0\})$ be a solution of (1.1) in $B \backslash \{0\}$, where $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Then*

$$u(x) \leqq |x|^{2-n}, \quad |x| \leqq 1/2, \tag{4.3}$$

$$\sup_{x \in B(x,|x|/2)} u(x) \leqq C \inf_{x \in B(x,|x|/2)} u(x), \tag{4.4}$$

where $|x| < 1/2$ and $C > 0$ is a constant independent of u and x;

$$\sup_{\varepsilon_0 \leqq |x| \leqq (1+\theta)\varepsilon_0} u(x) \leqq C \inf_{\varepsilon_0 \leqq |x| \leqq (1+\theta)\varepsilon_0} u(x), \tag{4.5}$$

where $C > 0$, $0 < \theta < \frac{1}{2}, \varepsilon_0 > 0$ and small, are constants independent of u.

Proof. We prove (4.3). Let $x_0 \neq 0$, $|x_0| \leqq \frac{1}{2}$. We consider

$$w(x) = |x_0|^{n-2} u(|x_0| x + x_0), \quad |x| < 1.$$

By Lemma 4.2 $w(x) \leqq C$ *if* $|x| \leqq \frac{1}{2}$. In particular $|x_0|^{n-2} u(x_0) = w(0) \leqq C$. This proves (4.3).

Equations (4.4) and (4.5) follow from (4.3) by using standard arguments. Indeed we write (1.1) as $\Delta u + u^{2/(n-2)} u = 0$. Equation (4.3) implies that we can apply standard results for linear equation to conclude (4.4) and (4.5). We refer to the proof of Theorem 3.1 of Gidas and Spruck [G–S] for further details.

5. Appendix II.

In this appendix we shall give a new proof of a theorem of Gidas and Spruck.

In [G–S], Gidas and Spruck studied positive singular solutions of

$$\Delta u + u^q = 0 \quad \text{in} \quad B \backslash \{0\}, \frac{n}{(n-2)} < q < \frac{(n+2)}{(n-2)}.$$

where B is the unit ball in $\mathbb{R}^n, n \geqq 3$.

In Theorem 3.3 of [G–S] where they claimed the estimate

$$u(x) \geqq C|x|^{-2/(q-1)}, \tag{5.1}$$

$C > 0$ a constant, the statement:

"If $\liminf_{x\to 0} |x|^{2/(q-1)}u(x)=0$, then the Harnack inequality implies that

$$\lim_{x\to 0} |x|^{2/(q-1)}u(x)=0''.$$

seems to require more explanation. The same statement was made later in [A].

However, as we shall see below, one can modify their proof of the theorem stated below in such a way that one only need to use

$$u(x)\leqq C/|x|^{2(q-1)}. \tag{5.2}$$

Using (5.2) we prove

Theorem B. (Gidas–Spruck). Let $u\in C^2(B\backslash\{0\})$, $u\geqq 0$ *be a solution of*

$$\Delta u+|x|^\sigma u^q=0 \quad \text{in} \quad B\backslash\{0\}, \tag{5.3}$$

where

$$1<\frac{(n+\sigma)}{(n-2)}<q<\frac{(n+2)}{(n-2)}, \; -2<\sigma<2$$

and $q\neq(n+2+2\sigma)/(n-2)$. *Then u has either a removable singularity at* $\{0\}$ *or*

$$\lim_{|x|\to 0} |x|^{(2+\sigma)/(q-1)}u(x)=C_0, \tag{5.4}$$

where

$$C_0=\left(\frac{(2+\sigma)(n-2)}{(q-1)^2}\left(q-\frac{(n+\sigma)}{(n-2)}\right)\right)^{1/(q-1)}$$

Clearly (5.4) is a stronger statement than (5.1).

Proof. If u is a solution of (5.3), then it follows from the work of Gidas and Spruck that

$$u(x)\leqq C/|x|^{(2+\sigma)/(q-1)} \tag{5.5}$$

with x closes to the origin and $C>0$ a constant. Then we consider

$$t=-\ln|x|,$$
$$v(t,\omega)=|x|^{(2+\sigma)/(q-1)}u(r,\omega),$$
$$r=|x|, \quad \omega\in S^{n-1}, \quad t\in\mathbb{R}.$$

Because of (5.5) v is bounded and moreover we have

$$v_{tt}+av_t+\Delta_\omega v-C_0^{q-1}v+v^q=0$$

with

$$a=\frac{(n-2)}{(q-1)}\left(\frac{n+2+2\sigma}{n-2}-q\right),$$
$$C_0=\left(\frac{(2+\sigma)(n-2)}{(q-1)^2}\left(q-\frac{(n+\sigma)}{(n-2)}\right)\right)^{1/(q-1)},$$
$$q\neq(n+2+2\sigma)/(n-2).$$

Repeating the proof of their Theorem 1.4 (it should be observed that here one only need the fact that v is bounded) we conclude that for each sequence $\{t_k\}$, $t_k \to \infty$, there exists a subsequence t'_k such that

$$v(t'_k, \omega) \to v(\omega) \quad \text{as} \quad t'_k \to \infty,$$

and where $v(\omega)$ satisfies

$$\Delta_\omega v - C_0{}^{(q-1)} v + v^q = 0 \quad \text{on} \quad S^{n-1}. \tag{5.6}$$

It is shown in Appendix B of Gidas and Spruck [G–S] that the only solutions of (5.6) are $v = 0$ or $v = C_0$. Hence because the limit set of a smooth function is connected

$$v(t, \omega) \to C_0 \quad \text{or} \quad v(t, \omega) \to 0 \quad \text{as} \quad t \to \infty,$$

(observe that when $q = n/(n-2)$, then we do not know a priori that the limit set of v is a discrete set). If the latter occurs, it follows that

$$\lim_{x \to 0} |x|^{(2+\sigma)/(q-1)} u(x) = 0. \tag{5.7}$$

Then we define the auxiliary function

$$v(x) = |x|^s u(x), \quad s > 0.$$

By computing the Laplacian of this function and then by using maximum principle, exactly as in the proof of Theorem 2 of [A] pp. 785–786 we conclude that if (5.7) occurs then the singularity is removable.

Acknowledgements. The author wishes to thank the referee for several relevant comments. This paper is based on the preprint: "Local behavior of solutions of some elliptic equations," IMA Preprint Series #178, which was completed during the visit of the author to the Institute for Mathematics and its Applications during the Spring Semester of 1985. This work was also partially supported by N.S.F. Grant No. 85-12711.

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Communicated by L. Nirenberg

Received September 27, 1985, in revised form May 21, 1986