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# On the Existence of *n*-Geodesically Complete or Future Complete Solutions of Einstein's Field Equations with Smooth Asymptotic Structure

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**Abstract.** It is demonstrated that initial data sufficiently close to De-Sitter data develop into solutions of Einstein's equations $\mathrm{Ric}[g]=\Lambda g$ with positive cosmological constant $\Lambda$, which are asymptotically simple in the past as well as in the future, whence null geodesically complete. Furthermore it is shown that hyperboloidal initial data (describing hypersurfaces which intersect future null infinity in a space-like two-sphere), which are sufficiently close to Minkowskian hyperboloidal data, develop into future asymptotically simple whence null geodesically future complete solutions of Einstein's equations $\mathrm{Ric}[g]=0$, for which future null infinity forms a regular cone with vertex $i^+$ that represents future time-like infinity.

## 1. Introduction

In this paper previous investigations [6, 9] of the existence of asymptotically simple solutions of Einstein's equations $\mathrm{Ric}[\tilde{g}]=\Lambda\tilde{g}$ with cosmological constant $\Lambda \geqslant 0$ [sign $=(-,+,+,+)$] will be extended. We will first discuss the case of positive cosmological constant, since there the results are of a certain completeness now.

In [9] the constraint equations implied by the "regular conformal field equations" [4] on past conformal infinity $\mathscr{I}^-$ have been analysed, which in the case $\Lambda > 0$ is space-like. It turned out that there is no need to consider an analogue of the Lichnerowicz equation if one wants to provide solutions of the constraint equations. Let $S$ be an arbitrary orientable compact 3-dimensional manifold endowed with a Riemannian metric $h_{\alpha\beta}$, $\Lambda$ a positive number, and $d_{\alpha\beta}$ a symmetric trace-free tensor field on $S$ satisfying the equation $D^\alpha d_{\alpha\beta}=0$, where $D$ denotes the covariant Levi-Civita derivative operator for $h_{\alpha\beta}$. Then from these fields a complete "asymptotic initial data set" for the regular conformal field equations can be derived by differentiation and algebra such that $S$ together with these data describes the geometry of a past conformal infinity $\mathscr{I}^-$. Moreover, all these (sufficiently smooth) initial data sets determine unique past asymptotically simple solutions of the equations $\mathrm{Ric}[\tilde{g}]=\Lambda\tilde{g}$, for which $S$ together with the data represents indeed the fields on past conformal infinity. These solutions are "semiglobal" in the sense that they are null geodesically complete in the past. It may be mentioned here that the freedom to specify initial data is essentially the same as in the standard Cauchy problem: if $\Theta$ is a positive function on $S$, the fields obtained by the rescalings $h_{\alpha\beta}\to\Theta^2 h_{\alpha\beta}$, $d_{\alpha\beta}\to\Theta^{-1}d_{\alpha\beta}$ determine the same physical situation as the original fields. Thus one arrives at a rather complete characterization of all past asymptotically simple solutions with compact Cauchy surface. Though the compactness condition is not necessary for the construction of solutions, it appears to be a natural requirement.

Of course, if data of such a generality are admitted, it will be difficult to tell whether the behaviour of the solutions in the far future will be similar as in the case of the conformally flat, geodesically complete De-Sitter space-time, which is asymptotically simple in the future as well. However, it will be shown in Theorem (3.3) that asymptotic initial data sets on the 3-dimensional sphere $S^3$ sufficiently close to an asymptotic De-Sitter data set develop into solutions to $\mathrm{Ric}[\tilde{g}]=\Lambda\tilde{g}$, which are "global" in the sense that they are null geodesically complete, and which moreover are asymptotically simple in the future as well as in the past. Instead of starting from asymptotic initial data one could also consider standard Cauchy data on $S^3$ which are in a suitable sense sufficiently close to De-Sitter data and obtain asymptotically simple solutions. Furthermore the techniques used to derive these results may also be used to show the existence of (weakly) asymptotically simple solutions in suitable "neighbourhoods" of other given (weakly) asymptotically simple solutions of $\mathrm{Ric}[\tilde{g}]=\Lambda\tilde{g}$, $\Lambda\succ 0$.

It has become increasingly clear that the requirement of (weak) asymptotic simplicity is reasonable for solutions of Einstein's equations, if it is to be satisfied either in the past or in the future. In [5, 9] the conditions have been met by taking them into account right in the formulation of the initial value problems from which the solution space-times have been constructed. Here it is seen now that due to the particular propagative properties of Einstein's equations initial data, which are completely arbitrary up to "smallness" and smoothness requirements, *evolve* into space-times with the expected asymptotic behaviour. Furthermore it follows that asymptotic simplicity is a rather stable property of solutions of Einstein's equations with positive cosmological constant.

Of prime interest in the investigations indicated above were the specific properties of the field equations which allow us to derive the global existence result. For a discussion of the early time behaviour of solutions of Einstein's equations with positive cosmological constant, oriented to particular physical applications, the reader might consult for example [14].

The analysis of the case of Einstein's vacuum field equations $\mathrm{Ric}[\tilde{g}]=0$ is considerably complicated by the fact that in general there will not exist conformal extensions of asymptotically simple solutions of these equations in which to "spatial infinity" can be assigned a structure to which the non-physical fields would extend in a smooth way. However, the techniques used in the case considered above, equally apply to the "hyperboloidal initial value problem," for which the existence of "local" solutions under general assumptions on the data has been discussed in [6]. Here "hyperboloidal initial data" for Einstein's equations $\mathrm{Ric}[\tilde{g}]=0$ are given on a 3-dimensional manifold, which represent the geometry of a hypersurface which intersects future null infinity in a space-like 2-dimensional sphere, the boundary of the initial surface. It will be shown in Theorem (3.5) that hyperboloidal initial data, which are sufficiently close to Minkowskian hyperboloidal data, develop into a solution of Einstein's vacuum field equations, which is null geodesically complete in the future and has in fact a smooth asymptotic structure. The conformal extension of this solution can be chosen in such a way that the null generators of the surface $\mathscr{I}^+$, future null infinity, converge to form the regular past null cone of a point $i^+$, which represents future time-like infinity of the solution.

This result appears to reduce the problem of showing the existence of nontrivial "purely radiative solutions" [8] of Einstein's vacuum field equations to the problem of showing the existence of solutions of the standard Cauchy problem, which possess a "piece" of future and past null infinity "near" spatial infinity.

The statements on the existence of solutions as indicated above are obtained by a straightforward application of PDE techniques available in the literature to the initial value problems formulated in Chap. 2. In [11, 12] Kato introduced techniques to deal with linear evolution equations, which he used in [13] to derive existence results for solutions of Cauchy problems for quasi-linear symmetric hyperbolic systems. His methods apply with only a few modifications to Cauchy problems for the symmetric hyperbolic system of reduced conformal field equations considered here.

The regular conformal field equations, from which the reduced equations are extracted, are equivalent to Einstein's equations where the conformal factor is positive. However, they are also regular and meaningful where the conformal factor vanishes or becomes negative. It will be shown in Chap. 3 that solutions to the reduced equations will extend, under suitable "smallness" conditions on the data, into regions, where the conformal factor becomes negative. Moreover, it will be seen that the regular conformal field equations take care that the set of points where the conformal factor vanishes, which will represent conformal infinity for the corresponding "physical" field, has the desired smooth structure. Thus the (semi-) global existence result and the statements about the asymptotic structure are obtained at the same stroke. Decisive for this is the possibility, discussed in [7], to control the evolution of the gauge dependent quantities, the conformal factor, the frame field, and the coordinate system, by a suitable specification of the gauge source fields.

The following exposition is meant to illustrate the essential argument, not to provide the most general results possible. Therefore use will be made freely of some specific properties of the Einstein cosmos, which are not necessarily available in more general situations and whose absence would require a higher technical expenditure. No attempt is made to describe any such generalization, but it may be pointed out that the methods used here extend to the coupled Einstein-Yang-Mills equations.

## 2. The Initial Value Problems

The desired space-times will be obtained from solutions

$$u=(e^{\mu}{}_{k}, \gamma_{i}{}^{j}{}_{k}, \Omega, \Sigma_{i}, s, s_{ij}, d_{ijkl}) \tag{2.1}$$

of the regular conformal field equations. The unknown $u$ comprises the following fields:

– a frame field $e_k = e_k{}^\mu \frac{\partial}{\partial x^\mu}$. It determines by

$$g^{\mu\nu} = \eta^{ik} e_i{}^\mu e_k{}^\nu \quad \text{with} \quad \eta^{ik} = \eta_{ik} = \mathrm{diag}(1, -1, -1, -1) \tag{2.2}$$

the contravariant version of the "non-physical" metric $g_{\mu\nu}$. Here and in the following the frame indices $i, j, k, l, m$ take values $0, 1, 2, 3$ and the summation convention is understood. The same holds for the coordinate indices $\mu, \nu, \lambda, \ldots$. All tensor fields will be thought of as being given with respect to the frame $e_k$;

– connection coefficients $\gamma_i{}^j{}_k$ with respect to $e_k$ of the Levi-Civita connection $\nabla$ derived from $g$, which satisfy

$$\gamma_i{}^j{}_k \eta_{jl} + \gamma_i{}^j{}_l \eta_{jk} = 0, \tag{2.3}$$

$$\nabla_i e_k \equiv \nabla_{e_i} e_k = \gamma_i{}^j{}_k e_j; \tag{2.4}$$

– real valued functions $\Omega$, $s$, where $\Omega$ is the "conformal factor" and [by Eq. (2.8)] $s = \frac{1}{4} \nabla_\mu \nabla^\mu \Omega$;

– a field $\Sigma_i$, which represents [by Eq. (2.7)] the differential of $\Omega$;

– a field $s_{ij}$, representing the trace-free part of the Ricci-tensor $R_{ij}$ of $g$

$$s_{ij} = \tfrac{1}{2}(R_{ij} - \tfrac{1}{4} R \eta_{ij});$$

– the rescaled Weyl-tensor $d_{ijkl} = \Omega^{-1} C_{ijkl}$, obtained from the Weyl-tensor $C_{ijkl}$ of $g$.

The regular conformal field equations are given by

$$e^\mu{}_{k,\nu} e^\nu{}_j - e^\mu{}_{j,\nu} e^\nu{}_k = (\gamma_j{}^i{}_k - \gamma_k{}^i{}_j) e^\mu{}_i, \tag{2.5}$$

$$\begin{aligned} &\gamma_l{}^i{}_{j,\mu} e^\mu{}_k - \gamma_k{}^i{}_{j,\mu} e^\mu{}_l + \gamma_k{}^i{}_m \gamma_l{}^m{}_j - \gamma_l{}^i{}_m \gamma_k{}^m{}_j - \gamma_m{}^i{}_j (\gamma_k{}^m{}_l - \gamma_l{}^m{}_k) \\ &= \Omega d^i{}_{jkl} + 2(\eta^i{}_{[k} s_{l]j} - \eta_{j[k} s_{l]}{}^i) + \tfrac{1}{6} R \eta^i{}_{[k} \eta_{l]j}, \end{aligned} \tag{2.6}$$

$$\nabla_i \Omega = \Sigma_i, \tag{2.7}$$

$$\nabla_i \Sigma_k = -\Omega s_{ik} + s \eta_{ik}, \tag{2.8}$$

$$\nabla_i s = -\Sigma^j s_{ji} - \tfrac{1}{12} R \Sigma_i - \tfrac{1}{24} \Omega \nabla_i R, \tag{2.9}$$

$$\nabla_k s_{lj} - \nabla_l s_{kj} = \Sigma_i d^i{}_{jkl} - \tfrac{1}{12} \eta_{j[l} \nabla_{k]} R, \tag{2.10}$$

$$\nabla_i d^i{}_{jkl} = 0. \tag{2.11}$$

Assuming that $\Omega$ is positive, all equations can be derived from the "conformal field equations,"

$$\mathrm{Ric}[\Omega^{-2} g] = \Lambda(\Omega^{-2} g), \tag{2.12}$$

which may be understood as a condition on the metric $g$ and the conformal factor $\Omega$. In terms of the "physical metric,"

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}, \tag{2.13}$$

Eq. (2.12) is just Einstein's field equation with cosmological constant $\Lambda$. This has been discussed in [4, 7, 9] and here will be mentioned only a few properties of the regular conformal field equations, which are important for the following.

Obviously Eq. (2.12) is invariant under rescalings of the form

$$(g, \Omega) \rightarrow (\Theta^2 g, \Theta\Omega)$$

with positive functions $\Theta$. Consequently the same is true for Eqs. (2.5)–(2.11) if the fields given by $u$ and the Ricci-scalar $R$ are transformed appropriately. As shown in [7] the real-valued function $R$ may be given arbitrarily near some initial surface. Its choice determines the propagation of the conformal factor $\Omega$ uniquely. Thus $R$ may be considered as the "gauge source function" for the conformal factor.

The trace-free part of (2.12) is just Eq. (2.8). It can be shown that for a solution of the system (2.5)–(2.11) the quantity

$$\Lambda' = 6\Omega s - 3\Sigma_j \Sigma^j + \tfrac{1}{4}\Omega^2 R \tag{2.14}$$

is a constant which may be identified with the cosmological constant $\Lambda$. Thus $\Lambda$ is fixed already by the choice of initial data on some initial surface. Equation (2.14) is then just the trace of Eq. (2.12), and it is seen that Eqs. (2.5)–(2.11) are essentially equivalent to (2.12), where the conformal factor is positive. Notice, however, that the system (2.5)–(2.11) is regular and that the structure of its principal part is preserved even if the conformal factor vanishes or becomes negative.

To solve a Cauchy problem for the regular conformal field equations, one has to split the system into constraint equations and propagation equations. The constraint equations have been given explicitly in [6, 9] and will not be reproduced here. The analysis of the propagation equations is done most easily in the spin frame formalism and has been discussed in all possible generality in [7]. The unknowns are represented then by the following fields:

– a pseudo-orthonormal frame $e_{aa'} = e^{\mu}{}_{aa'} \dfrac{\partial}{\partial x^{\mu}}$ satisfying

$$g(e_{aa'}, e_{bb'}) = \varepsilon_{ab}\varepsilon_{a'b'},$$

which may be related to an orthonormal frame by

$$e_{aa'} = \sigma^{k}{}_{aa'} e_k,$$

where the constant van der Waerden symbols $\sigma^{k}{}_{aa'}$ are given more explicitly by

$$\begin{aligned} e_{00'} &= \frac{1}{\sqrt{2}}(e_0 + e_1), \qquad & e_{01'} &= \frac{1}{\sqrt{2}}(e_1 + ie_2), \\ e_{10'} &= \frac{1}{\sqrt{2}}(e_1 - ie_2), \qquad & e_{11'} &= \frac{1}{\sqrt{2}}(e_0 - e_3), \end{aligned} \tag{2.15}$$

– the spin connection coefficients $\Gamma_{aa'bc} = \Gamma_{aa'(bc)}$,

– $s$, $\Omega$ and the differential $\Sigma_{aa'}$ of $\Omega$,

– the Ricci-spinor $\phi_{aba'b'} = \phi_{(ab)(a'b')} = \bar{\phi}_{aba'b'}$,

– the rescaled Weyl-spinor $\varphi_{abcd} = \varphi_{(abcd)} = \Omega^{-1}\Psi_{abcd}$, where $\Psi_{abcd}$ is the Weyl-spinor.

Here and in the following the conventions of [7] are used. In particular, the spinor indices $a, b, \ldots, h, a', b', \ldots, h'$ take values 0, 1 and the summation convention holds. Use will be made freely of this notation, where the unknown is given by

$$u = (e_{aa'}{}^{\mu}, \Gamma_{aa'bc}, \Omega, \Sigma_{aa'}, s, \phi_{aba'b'}, \varphi_{abcd}), \tag{2.16}$$

and the notation (2.1). It is assumed that the transition between the frame fields is always achieved by (2.15).

To obtain the symmetric hyperbolic reduced conformal field equations one has to choose beside the Ricci-scalar a "coordinate gauge source function"

$$F^{\mu} = \tfrac{1}{2} \nabla^{ff'} \nabla_{ff'} x^{\mu}, \tag{2.17}$$

and a "frame gauge source function"

$$F_{ab} = \tfrac{1}{2} \nabla^{ff'} (g(\nabla_{ff'} e_{ah'}, e_b{}^{h'})). \tag{2.18}$$

As shown in [7] the choice of these functions, which is completely arbitrary, determines the propagation of the coordinate system and the frame field off a suitable initial surface uniquely.

To find gauge source functions which will be useful for our purpose, we study first two particular solutions of (2.5)–(2.11) which are related to the De-Sitter and the Minkowski space-time, respectively.

In the following the 3-dimensional sphere $S^3$ will always be thought of as being given as the submanifold

$$S^3 = \left\{ x^A \in \mathbb{R}^4 \Big/ \sum_{A=1}^{4} (x^A)^2 = 1 \right\}$$

of $\mathbb{R}^4$. The restrictions of the functions $x^A$, $A = 1, 2, 3, 4$, on $\mathbb{R}^4$ to $S^3$ will again be denoted by $x^A$. The vector fields

$$\left.\begin{aligned} c_1 &= x^1 \frac{\partial}{\partial x^4} - x^4 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2} \\ c_2 &= x^1 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^4} \\ c_3 &= x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^4} - x^4 \frac{\partial}{\partial x^3} \end{aligned}\right\} \tag{2.19}$$

on $\mathbb{R}^4$ are tangent to $S^3$ and will always be considered as vector fields on $S^3$. Let $d\omega^2$ denote the line element obtained as pull-back of the line element $\sum_{A=1}^{4} (dx^A)^2$ on $\mathbb{R}^4$ to $S^3$. Then the fields $c_r$ (indices $r, s, t$ will always be assumed to take values 1, 2, 3 and the summation convention is assumed) constitute a globally defined frame on $S^3$ orthonormal with respect to $d\omega^2$. The diffeomorphism

$$S^3 \ni (x^1, x^2, x^3, x^4) \to \begin{pmatrix} x^1 + ix^2 & -x^3 + ix^4 \\ x^3 + ix^4 & x^1 - ix^2 \end{pmatrix} \in \mathrm{SU}(2)$$

of the sphere $S^3$ onto the Lie-group SU(2) maps the vector fields (2.19) onto left invariant vector fields. This is the reason for picking the particular frame (2.19) on

$S^3$. It finds application in the relation (2.35), which is basic for the construction of a certain class of maps. Another reason will become clear in the discussion of the proof of Theorem (3.1).

The Einstein-cosmos is given by the manifold $M^E = \mathbb{R} \times S^3$ together with the Lorentz-metric $g^E = dt^2 - d\omega^2$ with $t \in \mathbb{R}$. The vector fields $c_0 = \frac{\partial}{\partial t}$ and $c_r$ as given by (2.19) constitute a frame, which is defined globally on $M^E$ and satisfies $g^E(c_k, c_j) = \eta_{kj}$. The connection coefficients of the Levi-Civita connection determined from $g^E$ with respect to the frame $c_k$ are given by $\gamma^E{}_j{}^l{}_k = \varepsilon_{0jik}\eta^{il}$, where $\varepsilon_{ijkl}$ is the totally skew tensor with $\varepsilon_{0123} = 1$.

The two "conformal factors"

$$\Omega^D = \cos t\,, \quad \text{respectively} \quad \Omega^M = \cos t + x^1 \tag{2.20}$$

on $M^E$ allow to determine two solutions $u^D$, respectively $u^M$ of the regular conformal field equations (2.5)–(2.11), which both comprise the frame field $c_k$ [thus implying by (2.2) the metric $g^E$] and the connection coefficients $\gamma^E{}_i{}^j{}_k$, and are defined on the manifold $M^E$. Equation (2.5) is satisfied by $c^\mu{}_k = c_k(x^\mu)$ and $\gamma^E{}_i{}^j{}_k$ for any coordinate system $x^\mu$ on $M^E$. From (2.6) follows

$$\left.\begin{aligned} &d^D{}_{ijkl} = d^M{}_{ijkl} = 0\\ &R^D = R^M = -6\\ &s^D{}_{ij} = s^M{}_{ij} = \delta^0{}_i\delta^0{}_k - \tfrac{1}{4}\eta_{ik}\,. \end{aligned}\right\} \tag{2.21}$$

Equation (2.7) is solved by setting

$$\Sigma_k{}^D = c_k(\Omega^D)\,, \qquad \Sigma_k{}^M = c_k(\Omega^M)\,, \tag{2.22}$$

and Eq. (2.8) is solved by setting

$$s^D = -\tfrac{1}{4}\cos t\,, \qquad s^M = -\tfrac{1}{4}(\cos t - 3x^1)\,. \tag{2.23}$$

The remaining equations will then be satisfied as well.

Only parts of the solutions $u^D$, respectively $u^E$ are of physical interest. Denote by

$$M^D = \{(t,x) \in \mathbb{R} \times S^3 / |t| \prec \tfrac{\pi}{2}\}$$

a maximal connected domain in $M^E$, where $\Omega^D$ is positive. The manifold $M^D$ together with the metric $g^D = (\Omega^D)^{-2} g^E$ is just De-Sitter space-time. Defining a coordinate $\tau$, $-\infty \prec \tau \prec \infty$, on $M^D$ by setting $\mathrm{tg}\, t = \sinh \tau$, the metric $g^D$ writes

$$g^D = d\tau^2 - \cosh^2 \tau\, d\omega^2\,.$$

It satisfies the equation

$$\mathrm{Ric}[g^D] = \Lambda_0 g^D$$

with cosmological constant $\Lambda_0 = -3$ [which is negative because of the sign convention (2.2)]. The two space-like hypersurfaces

$$\mathscr{J}^{D\pm} = \{t = \pm \tfrac{\pi}{2}\}$$

of $M^E$, where $\Omega^D$ vanishes but $\Sigma_0{}^D \neq 0$, represent future, respectively past conformal infinity of $(M^D, g^D)$. They are Cauchy surfaces for the conformally extended space-time. Thus $u^D$ may be thought of as being obtained either by analytic continuation of the fields $\Omega^D$, $(\Omega^D)^2 g^D$ or as an extension of a solution of the regular conformal field equations (2.5)–(2.11) for data given on $\mathscr{I}^{D\pm}$ or on some Cauchy surface of $(M^D, g^D)$. The last point of view will be of importance in the following.

A maximal connected domain in $M^E$, where $\Omega^M$ is positive is given by

$$M^M = \{0 \leqq \chi < \pi,\ \chi - \pi < t < \pi - \chi\},$$

where the function $\chi$ on $S^3$, respectively $M^E$ is defined by setting $x^1 = \cos\chi$. In terms of the coordinates $\tau, r$ on $M^M$, $-\infty < \tau < \infty$, $0 \leqq r < \infty$, given by

$$\tau + r = \mathrm{tg}\frac{t+\chi}{2}, \qquad \tau - r = \mathrm{tg}\frac{t-\chi}{2},$$

the metric $g^M = (\Omega^M)^{-2} g^E$ writes $g^M = d\tau^2 - dr^2 - r^2 d\sigma^2$, where $d\sigma^2$ denotes the standard line element on the 2-dimensional sphere $S^2$. Thus $(M^M, g^M)$ is just Minkowski space-time given here in standard polar coordinates. The surfaces future and past conformal infinity for this space-time are represented by the null surfaces

$$\mathscr{I}^{M\pm} = \{0 < \chi < \pi,\ t = \pm(\pi - \chi)\},$$

where $\Omega$ vanishes but $d\Omega$ does not. They are neither Cauchy surfaces for $M^E$ nor for the conformal completion of $M^M$ because of the "holes" at the three points

$$i^{\pm} = \{\chi = 0,\ t = \pm\pi\}, \qquad i^0 = \{\chi = \pi,\ t = 0\}$$

which represent future, respectively past time-like and space-like infinity for Minkowski space. At these points not only $\Omega^M$ but also its differential vanishes, but its Hessian is non-degenerate. The intersection of the Cauchy surface $\{t = t_0\}$, where $-\pi < t_0 < \pi$, of the Einstein-cosmos with $M^M$ is a Cauchy surface for Minkowski-space only if $t_0 = 0$. Fix a number $t_0$, $0 < t_0 < \pi$ and consider the surface with boundary

$$\bar{S} = \{0 \leqq \chi \leqq \pi - t_0,\ t = t_0\}. \tag{2.24}$$

Such a surface will be called "hyperboloidal," since in the case $t_0 = \frac{\pi}{2}$ the intersection of $\bar{S}$ with $M^M$ is a space-like unit hyperboloid. The future domain of dependence (for this and related causal notions see [10])

$$D^+(\bar{S}) = \{0 \leqq \chi \leqq \pi - t,\ t_0 \leqq t \leqq \pi\} \tag{2.25}$$

of $\bar{S}$ in the Einstein-cosmos comprises the part of Minkowski space in the future of $\bar{S}$ as well as $\bar{S}$ and

$$H^+(\bar{S}) = \{\chi = \pi - t,\ t_0 \leqq t \leqq \pi\}, \tag{2.26}$$

the Cauchy horizon of $\bar{S}$, which is generated by segments of the past directed null geodesics emanating from the point $i^+$ in the Einstein-cosmos. In the hyperboloidal initial value problem dara are described on a surface, which similar to $\bar{S}$

intersects future null infinity of the prospective solution space-time in a space-like 2-dimensional sphere. It will be seen in Theorem (3.5) that for suitable hyperboloidal initial data the domain of the dependence of the solution will have a future boundary which forms a cone similar to $H^+(\bar{S})$ given by (2.26).

As in these two examples, the solution of Einstein's equation we are looking for will be obtained by restricting suitably certain solutions of the "reduced conformal field equations" which represent the propagational part of the regular conformal field equations. These solutions will provide a metric $g$ on a manifold which is diffeomorphic to $M^E$. Equations (2.5) contain as unknowns the coefficients $e^\mu{}_k$ of the frame within a coordinate basis. Since no global coordinate systems exist on $S^3$ one would have to patch together local solutions. To avoid the lengthy technical discussion arising from this difficulty, one may try, as suggested by the examples above, to take advantage of the fact that there exist globally defined frames on $M^E$, and to express the field equations in terms of such a frame. This will result in a representation (2.42)–(2.46) of the equations, which allows us to dispense with any further coordinate considerations. How this can be achieved will be discussed now.

Let $(M, g)$ be a Lorentz-space such that the manifold $M$ is diffeomorphic to $M^E$. A map $\phi$, defined on an open subset $U$ of $M$, will be called a "cylinder map," if it maps $U$ diffeomorphically onto a set $I \times S^3$, $I$ an open interval in $\mathbb{R}$, such that the sets $\phi^{-1}(\{t\} \times S^3)$, $t \in I$, are space-like Cauchy surfaces and the curves $I \ni t \to \phi^{-1}((t, x))$, $x \in S^3$, are time-like with respect to the metric $g$. The cylinder map $\phi$ may be used to pull back to $U$ the function $x^0 = t$, which takes values in $I$, and the functions $x^A$, $A = 1, \ldots, 4$ on $S^3$, whose pull-back to $U$ will again be denoted by the same symbol. Since the system of functions so obtained has rank four on $U$, one may take $x^0$ together with a suitable choice of three of the functions $x^A$, to obtain a coordinate system in a certain neighbourhood of a given point of $U$. Coordinates constructed in this way will be called "cylinder coordinates" and denoted by $x^\mu$, $\mu = 0, 1, 2, 3$ (which involves possibly a renumeration of the functions $x^A$). In the following will be assumed that the manifold $M$ can be covered by domains of cylinder maps.

Given a cylinder map $\phi$ and thus the functions $x^0$, $x^A$ on its domain $U$, one can define a frame field $c_k$ on $U$ with $c_0 = \dfrac{\partial}{\partial x^0}$ and $c_1, c_2, c_3$ as in (2.19). The frame coefficients $e^\mu{}_k$, respectively $e^\mu{}_{aa'}$ with respect to a coordinate basis may now be replaced in the unknown $u$ given by (2.1), respectively (2.16) by the coefficients $e^j{}_k$, respectively $e^j{}_{aa'}$ defined by

$$e_k = e^j{}_k c_j, \quad \text{respectively} \quad e_{aa'} = e^j{}_{aa'} c_j \quad \text{on } U. \tag{2.27}$$

(Here indices belonging to the same "family" are used in two different ways. However, the upper index of $e^j{}_k$ will, with the exception of formulae (2.29), only be contracted with the index of the fields $c_j$, while indices attached to any tensor field will always refer to the lower index of $e^j{}_k$, which counts the member of the frame $\{e_k\}_{k=0,1,2,3}$). In the two examples studied above the coefficients $e^j{}_{aa'}$ coincide then with the van der Waerden symbols $\sigma^j{}_{aa'}$.

It is obvious how Eqs. (2.6)–(2.11), if written in the spin-frame formalism, have to be reexpressed in terms of the coefficients $e^j{}_{aa'}$ and derivatives in the direction of $c_j$. However, to obtain a suitable form for the reduced equations, Eq. (2.5), which is the condition that the torsion tensor vanishes, needs further discussion. The torsion tensor $t_{aa'}{}^{bb'}{}_{cc'}$ may be thought of as being given by the quantities $t^{cc'}_{ab}=\frac{1}{2}t_{ah'}{}^{cc'}{}_{b}{}^{h'}$, $t^{cc'}_{a'b'}=\frac{1}{2}t_{ha'}{}^{cc'h}{}_{b'}$, which satisfy by definition the equations

$$t^{cc'}_{ab}e_{cc'}(x^{\mu})=\nabla_{(a}{}^{f'}\nabla_{b)f'}x^{\mu}\,,\qquad t^{cc'}_{a'b'}e_{cc'}(x^{\mu})=\nabla^{f}{}_{(a'}\nabla_{b')f}x^{\mu} \tag{2.28}$$

for any coordinate system $x^{\mu}$ on $M$, in particular for cylinder coordinates. Assume that for a given cylinder map $\phi$ the field $c_j$ are defined as above and set [using (2.4)]

$$\begin{aligned} s^{j}_{ab}&=e^{k}{}_{a}{}^{f'}c_k(e^{j}{}_{bf'})-\Gamma_{a}{}^{f'cc'}{}_{bf'}e^{j}{}_{cc'}+e^{k}{}_{a}{}^{f'}e^{l}{}_{bf'}\varepsilon_{0kil}\eta^{ij}\,,\\ s^{j}_{a'b'}&=e^{kf}{}_{a'}c_k(e^{j}{}_{fb'})-\Gamma^{f}{}_{a'}{}^{cc'}{}_{fb'}e^{j}{}_{cc'}+e^{kf}{}_{a'}e^{l}{}_{fb'}\varepsilon_{0kil}\eta^{ij}\,. \end{aligned} \tag{2.29}$$

Then Eq. (2.28) may be written in the form

$$\begin{aligned} t^{cc'}_{ab}e_{cc'}(x^{\mu})&=s^{j}_{ab}c_j(x^{\mu})+\varepsilon_{ab}G^{\mu}\,,\\ t^{cc'}_{a'b'}e_{cc'}(x^{\mu})&=s^{j}_{a'b'}c_j(x^{\mu})+\varepsilon_{a'b'}G^{\mu}\,, \end{aligned} \tag{2.30}$$

with

$$G^{\mu}=\tfrac{1}{2}(\nabla_{aa'}\nabla^{aa'}x^{\mu}-e^{jff'}e^{k}{}_{ff'}c_j(c_k(x^{\mu})))\,, \tag{2.31}$$

which holds for any coordinate system, in particular for the cylinder coordinates associated with the map $\phi$.

The first term on the right of Eq. (2.31) is just the coordinate gauge source function. Since this may be given arbitrarily in an initial value problem for Eq. (2.5)–(2.11), one may expect that it can be chosen such that $G^{\mu}$ vanishes. In the following it will be shown that this can be done and that the requirement $G^{\mu}\equiv 0$ amounts to the construction of a particular type of cylinder map.

Let $S$ be a smooth space-like Cauchy surface of $M$, which is mapped by a diffeomorphism $\psi$ onto the sphere $S^3$ and let, assuming a time orientation, $n$ be the future directed unit normal vector field on $S$. Maps of the form

$$M\supset U\ni p\to(x^0(p),x^A(p))\in\mathbb{R}\times\mathbb{R}^4 \tag{2.32}$$

will now be constructed, which will be seen to define cylinder maps of the desired type. The functions $x^0, x^A$ are obtained as solutions of the system of semi-linear wave equations

$$\nabla_k\nabla^k x^0=0\,,\qquad \nabla_k\nabla^k x^A=-x^A\left(\sum_{B=1}^{4}\nabla_k\left(\frac{x^B}{|x|}\right)\nabla^k\left(\frac{x^B}{|x|}\right)\right), \tag{2.33}$$

with $|x|=\left(\sum_{A=1}^{4}(x^A)^2\right)^{1/2}$. Data for these equations are prescribed on $S$ such that:

$$\left.\begin{aligned} &S\ni q\to x^A(q)\in S^3\quad\text{describes the diffeomorphism }\psi\,,\\ &x^0|_S=t_0\quad\text{for some}\quad t_0\in\mathbb{R}\,,\\ &\sum_{A=1}^{4}x^A n(x^A)=0\,,\qquad n(x^0)\succ 0\quad\text{on }S\,. \end{aligned}\right\} \tag{2.34}$$

**Lemma (2.1).** *A solution of Eqs.* (2.33) *satisfying the initial conditions* (2.34) *defines by* (2.32) *a cylinder map $\phi$, that maps a certain neighbourhood $U$ of $S$ diffeomorphically onto a neighbourhood of $\{t_0\}\times S^3$ in $\mathbb{R}\times S^3$.*

*If $G^\mu$, given by* (2.31), *is defined with respect to this cylinder map and associated cylinder coordinates $x^\mu$, then $G^\mu \equiv 0$ on $U$.*

Where it is positive the function $|x|$ calculated from a solution $x^A$ of (2.33) satisfies the equation, $\nabla_k \nabla^k(|x|)=0$, as may be verified by direct calculation, using the second of Eqs. (2.33). Since by (2.34) one has $|x|\equiv 1$, $n(|x|)\equiv 0$ on $S$, it follows that $|x|\equiv 1$ in the neighbourhood $U$ of $S$, where the solution of the initial value problem (2.33), (2.34) exists. By (2.34) the map $\phi: u \to \mathbb{R}\times S^3$ so obtained has rank 4 on $S$ and thus, possibly after restricting $U$ suitably, defines a diffeomorphism onto the set $\phi(U)$ containing $\{t_0\}\times S^3$ such that the surfaces $\{x^0=\text{const}\}$ are space-like Cauchy surfaces and the curves $\{x^A=\text{const}\}$ are time-like.

Using the functions $x^0, x^A$ on $U$ to define vector fields $c_0=\frac{\partial}{\partial x^0}$, and $c_1, c_2, c_3$ as in (2.19) one finds by direct calculation:

$$\left.\begin{aligned} &c_{(j}(c_{k)}(x^A)) = -(\delta^0_j\delta^0_k-\eta_{jk})x^A = -x^A\left(\sum_{B=1}^4 c_j(x^B)c_k(x^B)\right),\\ &c_j(c_k(x^0))=0, \end{aligned}\right\} \tag{2.35}$$

whence, observing $|x|\equiv 1$,

$$\left.\begin{aligned} e^{jff'}e^k{}_{ff'}c_j(c_k(x^A)) &= -x^A\left(\sum_{B=1}^4 e^{jff'}c_j(x^B)e^k{}_{ff'}c_k(x^B)\right)\\ &= -x^A\sum_{B=1}^4 \nabla_k\left(\frac{x^B}{|x|}\right)\nabla^k\left(\frac{x^B}{|x|}\right),\\ e^{jff'}e^k{}_{ff'}c_j(c_k(x^0)) &= 0. \end{aligned}\right\} \tag{2.36}$$

Thus by (2.33) one has $G^\mu \equiv 0$ on $U$ if $G^\mu$ is defined with respect to the cylinder map $\phi$ and related cylinder coordinates.

*Remark (2.2).* i) If $x^{\mu'}$ is an arbitrary coordinate system on $U$, it may locally be given in the form $x^{\mu'}=x^{\mu'}(x^\mu)$, where $x^\mu$ is one of the cylinder coordinate systems considered above. Assuming $c_j$ as above, one finds again

$$\nabla_k\nabla^k x^{\mu'} = e^{jff'}e^k{}_{ff'}c_j(c_k(x^{\mu'})),$$

but the vector fields $c_j$ are now not necessarily related to the coordinates $x^{\mu'}$ as they are related to the coordinates $x^\mu$. Therefore, the condition $G^\mu=0$ is in fact a requirement on the cylinder map which serves to define the fields $c_j$.

ii) Using (2.35), (2.36) one can show that the gauge condition $G^\mu=0$ implies that the cylinder map used to construct the fields $c_j$ in the definition (2.31) of $G^\mu$ must always be obtained as a solution of an initial value problem for Eqs. (2.35). For the following it is sufficient to see that the condition $G^\mu=0$ can always be satisfied and leaves the usual freedom to specify initial conditions [notice that in (2.34) one could also have $t_0=t_0(q)$ a function on $S$].

For the function $G^\mu$, which is used to single out a class of cylinder coordinates, thus fixing the coordinate function $F^\mu$, for the frame gauge source function $F_{ab}$ and for the function $R$, which determines the evolution of the conformal factor, we will now take in the general situation we want to consider the same functions as are obtained in the case of the two examples discussed above. For space-times which "deviate not too much" from those of these examples it may then be expected that the cylinder map, the frame field, and a conformal factor satisfying the appropriate positivity conditions will exist globally on the prospective solution space-time.

Thus we require

$$G^{\mu}=0\,, \qquad F_{ab}=0\,, \qquad R=-6\,. \tag{2.37}$$

As shown in [7], respectively pointed out in Remark (2.2) this leaves the freedom to specify the conformal factor, the frame field and the coordinates as well as the normal derivatives of these fields on a suitable hypersurfaces. By (2.30) the first of conditions (2.37) implies that the vanishing of the torsion tensor then is equivalent to

$$s^{j}{}_{ab}=0\,, \qquad s^{j}_{a'b'}=0\,. \tag{2.38}$$

On the other hand, we have:

**Lemma (2.3).** *If in* (2.30) *holds*

$$\left.\begin{aligned} -s^{j}_{01}-s^{j}_{0'1'}&=0\,, & -s^{j}_{00}+s^{j}_{1'1'}&=0\,,\\ s^{j}_{11}-s^{j}_{0'0'}&=0\,, & s^{j}_{10}+s^{j}_{1'0'}&=0\,, \end{aligned}\right\} \tag{2.39}$$

*then* $G^{\mu}=0$ *and*

$$\left.\begin{aligned} -t^{aa'}_{01}-t^{aa'}_{0'1'}&=0\,, & -t^{aa'}_{00}+t^{aa'}_{1'1'}&=0\,,\\ t^{aa'}_{11}-t^{aa'}_{0'0'}&=0\,, & t^{aa'}_{10}+t^{aa'}_{1'0'}&=0\,. \end{aligned}\right\} \tag{2.40}$$

Using the definitions (2.29), (2.31) one finds the identity

$$G^{\mu}=\tfrac{1}{4}(s^{j}_{10}+s^{j}_{1'0'}-s^{j}_{01}-s^{j}_{0'1'})c_{j}(x^{\mu})$$

which holds for any coordinate system $x^{\mu}$. From this follows the first assertion and the second is then an immediate consequence of the relations (2.30).

*Remark (2.4).* The subsystem (2.42) of the following system of reduced conformal field equations are just Eqs. (2.39) in explicit form. Thus the system (2.42)–(2.46) implies the system (2.40), (2.43)–(2.46) and allows us to determine the gauge source function $F^{\mu}$ from $G^{\mu}=0$ and (2.31). The discussion of the system (2.40), (2.43)–(2.46) in [7] therefore shows that a solution of the system (2.42)–(2.46), which satisfies the constraint equations implied by the regular conformal field equations on some initial surface $S$, will in fact be a solution of the complete system of regular conformal field equations in the domain of dependence of the surface $S$ with respect to the metric supplied by that solution.

With the definition

$$\begin{gathered} \gamma^{ab}_{cd}=\tfrac{1}{2}\Gamma_{(c}{}^{e'f(a}\Gamma_{d)e'}{}^{b)}{}_{f}\,, \qquad \gamma^{ab}_{c'd'}=\tfrac{1}{2}\Gamma^{e}{}_{(e'}{}^{f(a}\Gamma_{d')e}{}^{b)}{}_{f}\,,\\ \Gamma_{aa'}{}^{bb'}{}_{cc'}=\Gamma_{aa'}{}^{b}{}_{c}\varepsilon_{c'}{}^{b'}+\bar{\Gamma}_{aa'}{}^{b'}{}_{c'}\varepsilon_{c}{}^{b}\,, \end{gathered} \tag{2.41}$$

and assuming $\nabla_{aa'}\Gamma_{bb'cd}$ to be given by the same expression which one would obtain if the $\Gamma_{bb'cd}$ were the components of a spinor field, the reduced conformal field equations in the gauge (2.37) may be written:

$$\left.\begin{aligned}
&2e^j{}_{00'}c_j(e^i{}_{11'})-e^j{}_{01'}c_j(e^i{}_{10'})-e^j{}_{10'}c_j(e^i{}_{01'})\\
&\quad=-(\Gamma_0{}^{f'cc'}{}_{1f'}+\Gamma^f{}_{0'}{}^{cc'}{}_{f1'})e^i{}_{cc'}+(e^j{}_0{}^{f'}e^k{}_{1f'}+e^{jf}{}_{0'}e^k{}_{f1'})\varepsilon_{0j}{}^i{}_k,\\
&(e^j{}_{00'}+e^j{}_{11'})c_j(e^i{}_{01'})-e^j{}_{10'}c_j(e^i{}_{11'})-e^j{}_{10'}c_j(e^i{}_{00'})\\
&\quad=(\Gamma_1{}^{f'cc'}{}_{1f'}-\Gamma^f{}_{0'}{}^{cc'}{}_{f0'})e^i{}_{cc'}-(e^j{}_1{}^{f'}e^k{}_{1f'}-e^{jf}{}_{0'}e^k{}_{f0'})\varepsilon_{0j}{}^i{}_k,\\
&(e^j{}_{00'}+e^j{}_{11'})c_j(e^i{}_{01'})-e^j{}_{01'}c_j(e^i{}_{00'})-e^j{}_{01'}c_j(e^i{}_{11'})\\
&\quad=-(\Gamma_0{}^{f'cc'}{}_{0f'}-\Gamma^f{}_{1'}{}^{cc'}{}_{f1'})e^i{}_{cc'}+(e^j{}_0{}^{f'}e^k{}_{0f'}-e^{jf}{}_{1'}e^k{}_{f1'})\varepsilon_{0j}{}^i{}_k,\\
&2e^j{}_{11'}c_j(e^i{}_{00'})-e^j{}_{10'}c_j(e^i{}_{01'})-e^j{}_{01'}c_j(e^i{}_{10'})\\
&\quad=(\Gamma_1{}^{f'cc'}{}_{0f'}+\Gamma^f{}_{1'}{}^{cc'}{}_{f0'})e^i{}_{cc'}-(e^j{}_1{}^{f'}e^k{}_{0f'}+e^{jf}{}_{1'}e^k{}_{f0'})\varepsilon_{0j}{}^i{}_k,
\end{aligned}\right\}\tag{2.42}$$

$$\left.\begin{aligned}
&2\nabla_{00'}\Gamma_{11'}{}^{ab}-\nabla_{01'}\Gamma_{10'}{}^{ab}-\nabla_{10'}\Gamma_{01'}{}^{ab}-\gamma^{ab}_{10}-\gamma^{ab}_{1'0'}\\
&\quad=\Omega\varphi^{ab}{}_{01}+\tfrac{1}{2}\varepsilon_0{}^{(a}\varepsilon_1{}^{b)}+\phi^{ab}{}_{0'1'},\\
&(\nabla_{00'}+\nabla_{11'})\Gamma_{10'}{}^{ab}-\nabla_{10'}\Gamma_{11'}{}^{ab}-\nabla_{10'}\Gamma_{00'}{}^{ab}+\gamma^{ab}_{11}-\gamma^{ab}_{0'0'}\\
&\quad=-\Omega\varphi^{ab}{}_{11}-\tfrac{1}{2}\varepsilon_1{}^a\varepsilon_1{}^b+\phi^{ab}{}_{0'0'},\\
&(\nabla_{00'}+\nabla_{11'})\Gamma_{01'}{}^{ab}-\nabla_{01'}\Gamma_{00'}{}^{ab}-\nabla_{01'}\Gamma_{11'}{}^{ab}-\gamma^{ab}_{00}+\gamma_{1'1'}{}^{ab}\\
&\quad=\Omega\varphi^{ab}{}_{00}+\tfrac{1}{2}\varepsilon_0{}^a\varepsilon_0{}^b-\phi^{ab}{}_{1'1'},\\
&2\nabla_{11'}\Gamma_{00'}{}^{ab}-\nabla_{10'}\Gamma_{01'}{}^{ab}-\nabla_{01'}\Gamma_{10'}{}^{ab}+\gamma^{ab}_{01}+\gamma_{0'1'}{}^{ab}\\
&\quad=-\Omega\varphi^{ab}{}_{10}-\tfrac{1}{2}\varepsilon_1{}^{(a}\varepsilon_0{}^{b)}-\phi^{ab}{}_{1'0'},
\end{aligned}\right\}\tag{2.43}$$

$$\left.\begin{aligned}
&-\nabla^f{}_{0'}\varphi_{111f}=0,\\
&-\nabla^f{}_{0'}\varphi_{ab0f}+\nabla^f{}_{1'}\varphi_{ab1f}=0 \quad \text{in the order} \quad ab=11,10,00,\\
&\nabla^f{}_{1'}\varphi_{000f}=0,
\end{aligned}\right\}\tag{2.44}$$

$$\left.\begin{aligned}
&-\nabla_0{}^{f'}\phi_{bc1'f'}=-\varphi_{0bcf}\Sigma^f{}_{1'},\\
&-\nabla_0{}^{f'}\phi_{bc0'f}+\nabla_1{}^{f'}\phi_{bc1'f}=-\varphi_{0bcf}\Sigma^f{}_{0'}+\varphi_{1bcf}\Sigma^f{}_{1'},\\
&\nabla_1{}^{f'}\phi_{bc0'f'}=\varphi_{1bcf}\Sigma^f{}_{0'},
\end{aligned}\right\}\tag{2.45}$$

$$\left.\begin{aligned}
&(\nabla_{00'}+\nabla_{11'})\Omega=\Sigma_{00'}+\Sigma_{11'},\\
&(\nabla_{00'}+\nabla_{11'})\Sigma_{bb'}=-\Omega(\phi_{0b0'b'}+\phi_{1b1'b'})+s(\varepsilon_{0b}\varepsilon_{0'b'}+\varepsilon_{1b}\varepsilon_{1'b'}),\\
&(\nabla_{00'}+\nabla_{11'})s=-(\phi_{0b0'b'}+\phi_{1b1'b'})\Sigma^{bb'}+\tfrac{1}{2}(\Sigma_{00'}+\Sigma_{11'}).
\end{aligned}\right\}\tag{2.46}$$

In the following it will always be assumed that in these equations the frame $e_{aa'}=e^j_{aa'}c_j$ is written in the form $\frac{1}{2}(e^j_{aa'}+\bar{e}^j_{aa'})c_j$ to make the system "symmetric" [see (3.1)]. Given a solution of the system written this way, one may, following the discussion in [7], subtract the complex conjugates of Eqs. (2.42) (not assuming now that $e^j{}_{aa'}=\bar{e}^j{}_{aa'}$ necessarily holds) suitably from Eqs. (2.42) to obtain a linear homogeneous symmetric hyperbolic system of equations for the quantities $e^j{}_{aa'}-\bar{e}^j{}_{aa'}$. Since initial data for (2.42)–(2.46) will be given such that $e^j{}_{aa'}=\bar{e}^j{}_{aa'}$ holds initially, this equality will hold everywhere by the uniqueness property of symmetric hyperbolic system. Thus Eqs. (2.42)–(2.46) will be satisfied with the frame written in the original way.

Before initial data sets will be specified for Eqs. (2.42)–(2.46) it is convenient to collect a few facts, which allow us to reduce the remaining gauge freedom on the initial surface. Let $(M, g)$, $S$, $\psi$ be the Lorentz-space, the Cauchy surface and the diffeomorphism of $S$ onto $S^3$, which have been considered in Lemma (2.1). The requirements listed in the following lemma restrict, for a given choice of the conformal factor near $S$, the freedom to select a cylinder map and a frame near $S$ to the choice of the diffeomorphism $\psi$ and the "origin of time" $t_0$.

**Lemma (2.5).** *There exists a cylinder map of the type considered in Lemma (2.1) with related cylinder coordinates $x^\mu$ and frame field $c_j$ and a g-orthonormal frame field $e_k$, $k=0,1,2,3$; both defined in some neighbourhood of $S$ in $M$, such that*

i) *The fields $e_r$, $r=1,2,3$, are tangent to $S$, satisfy $h(e_a, e_b) = -\delta_{ab}$, where $h$ is the interior metric implied on $S$ by $g$, and have a unique expansion*

$$e_r = e^s_r c_s \quad \text{with} \quad e^s{}_r = \begin{cases} > 0 & \text{if} \quad s = r, \\ 0 & \text{if} \quad s > r. \end{cases}$$

ii) *$e_0$, which coincides on $S$ with the future directed unit normal of $S$, is given by $e_0 = \frac{\partial}{\partial x^0} = c_0$ on $S$. Furthermore $x^0 = t_0$ on $S$, $t_0 \in \mathbb{R}$.*

iii) *The connection coefficients $\gamma_i{}^j{}_k$ with respect to $e_k$ satisfy on $S$:*

> *$\gamma_r{}^t{}_s e_t = D_r e_s$, where $D_r$ denotes the covariant derivative in the direction of $e_r$ with respect to the Levi-Civita connection defined on $S$ by $h$, $\chi_{rs} = -\gamma^0_{rs} = -\gamma_r{}^s{}_0$ is the second fundamental form on $S$, given in the frame $e_r$, $\gamma_0{}^i{}_k = 0$ on $S$.*

iv) *The frame gauge source function of $e_k$ vanishes near $S$.*

Since the fields $c_r$, $r=1,2,3$, given together with $\psi$ are tangent to $S$, condition i) is just an orthonormalization prescription to fix in a unique way an orthonormal basis of tangent vectors at each point of $S$. Denote by $\tilde{e}_k$ the $g$-orthonormal frame on $S$, where $\tilde{e}_0$ is the future directed unit normal vector and the $\tilde{e}_r$ satisfy i). Assume that the frame $\tilde{e}_k$ is extended to a neighbourhood of $S$ by parallel transport in the direction of $\tilde{e}_0$. Then the connection coefficients $\tilde{\gamma}_i{}^j{}_k$ with respect to $\tilde{e}_k$ have the properties listed in iii). Set $e_k = L^j{}_k \tilde{e}_j$, where $L^j{}_k$ denotes a function, which maps some neighbourhood of $S$ into the Lorentz-group. As shown in [7] the requirement that the frame gauge source function for $e_k$ vanishes near $S$ implies a system of semi-linear wave equations for the functions $L^j{}_k$, $j, k = 0,1,2,3$. Solving these equations with the initial conditions $L^j{}_k = \delta^j{}_k$, $\tilde{e}_0(L^j{}_k) = 0$ on $S$, one obtains a unique frame $e_k$ near $S$ which satisfies i), iii), iv). If in (2.34) the initial conditions are specified by setting $n(x^A) = 0$, $n(x^0) = 1$ on $S$, the cylinder map obtained by solving (2.33) near $S$ will be such that iii) holds.

The reduced conformal field equations (2.42)–(2.46) will now be considered as equations on the manifold $\mathbb{R} \times S^3$, on which the fields $c_j$ are known, and data will be specified on a surface $S_{t_0} = \{t_0\} \times S^3$, where the gauge conditions i)–iii) will be assumed. The further structure of the initial data sets will depend on the particular problem to be considered.

In the "De-Sitter-type initial value problem" are given "asymptotic initial data" on $S_{t_0}$, which will correspond to the (non-physical) fields implied on past conformal infinity of a past asymptotically simple solution of Einstein's equations $\mathrm{Ric}[\tilde{g}] = \Lambda_0 \tilde{g}$, $\Lambda_0 < 0$. For easy comparison with the standard solution $u^D$ of that situation we take as the initial data surface $S = S_{-\frac{\pi}{2}}$ and as the value for the cosmological constant $\Lambda_0 = -3$. The asymptotic initial data may be specified by giving two tensor fields $h$ and $d$, where

- $h$ is a negative definite metric on $S$,
- $d$ is a trace-free symmetric on $S$ with vanishing divergence with respect to $h$:

$$D^r d_{rs} = 0 \quad \text{on } S. \tag{2.47}$$

Here it is assumed, as in the following, that $e_r$, $e^s{}_r$, $\gamma_r{}^s{}_t$, $D_r$ denote the quantities associated to $h$ as described in i), iii) of Lemma (2.5) (with $\psi$ the identity), and tensor fields are given by their components in the frame $e_r$. Denote by ${}^3R_{rs}$, ${}^3R$, ${}^3B_{rst}$ the Ricci-tensor, the Ricci-scalar, and the Bach tensor of $h$, respectively.

With the choice $\Lambda_0 = -3$, $R = -6$, an *asymptotic initial data set for the De-Sitter-type initial value problem*, denoted in the following collectively by $u_0{}^{DT}$, is given by:

$$\left.\begin{aligned}
&e^r{}_s, \gamma_r{}^s{}_t \text{ as above},\\
&e^j{}_0 = \delta^j{}_0, \quad e^0{}_r = 0, \quad \gamma_0{}^i{}_k = \gamma_i{}^0{}_k = \gamma_i{}^k{}_0 = 0,\\
&\Omega = 0, \quad \Sigma_k = \delta^0{}_k, \quad s = 0,\\
&s_{00} = -\tfrac{1}{4}({}^3R + 3), \quad s_{0r} = 0,\\
&s_{rs} = {}^3R_{rs} + \tfrac{1}{4}\delta_{rs}({}^3R - 1),\\
&d_{r0st} = \tfrac{1}{2}{}^3B_{rst}, \quad d_{r0s0} = d_{rs}.
\end{aligned}\right\} \tag{2.48}$$

*Remark (2.6).* (i) It has been shown in [9] that by $h$, $d$ and (2.48) all possible asymptotic initial data satisfying the constraint equations implied on past conformal infinity by the regular conformal field equations (2.5)–(2.11) with $R = -6$ and (2.14) with $\Lambda' = -3$ are given. "Conformally equivalent" data obtained by rescalings $h \to \Theta^2 h$, $d \to \Theta^{-1} d$ with a positive function $\Theta$ on $S$ and related transformations of the quantities given by (2.48), determine the same physical solutions. Because of the condition $s = 0$ these rescalings represent the remaining conformal freedom.

(ii) The set of solutions of (2.47) has been described in [2].

(iii) The asymptotic initial data set $u_0{}^D = u^D|_{S-\frac{\pi}{2}}$ implied on $S$ by the solution $u^D$ may be obtained by choosing $h$ to be such that the fields $c_r$ are orthonormal, i.e. $e^r{}_s = \delta^r{}_s$, and that the field $d_{rs}$ vanishes. Although the "nearness" of an abstractly given initial data set on $S$ to an asymptotic De-Sitter data set should be formulated more precisely in terms of the possible diffeomorphisms $\psi$ and the remaining conformal freedom, it will be said in the following and will be adequate for our purpose that the data $u_0{}^{DT}$ are "close" to asymptotic De-Sitter data, if the fields $e^r{}_s - \delta^r{}_s$, $d_{rs}$ and consequently the (componentwise) difference $u_0{}^{DT} - u_0{}^D$ are "sufficiently" small with respect to a suitable norm on an appropriate function space. A similar remark holds for hyperboloidal and Minkowskian hyperboloidal initial data.

The hyperboloidal initial value problem has been described at length in [6]. Here the conditions on the data will be given, which are important for the discussion in the next chapter. Let $\bar{S}$ be now the surface given by (2.24) and let $Z$ denote its boundary

$$Z=\{\chi=\pi-t_0,\ t=t_0\}.$$

The collection of fields

$$u_0{}^H=(e^j{}_k\gamma_i{}^j{}_k,\Omega,\Sigma_i,s,s_{ik},d_{ijkl}) \tag{2.49}$$

given on $\bar{S}$ will be called a *hyperboloidal initial data set*, if

$$\left.\begin{array}{l}
\text{on } Z: \quad \Omega=0, \quad \Sigma_i\Sigma^i=0, \quad \Sigma_0<0,\\
\text{on } \bar{S}\setminus Z: \quad \Omega>0,\\
\text{on } \bar{S}: \quad e^j{}_0=\delta^j{}_0, \quad e^0{}_r=0,\\
e^r{}_s, \text{ respectively } \gamma_i{}^j{}_k \text{ satisfy the conditions stated in i), iii) of Lemma (2.2).}\\
\text{The fields given by } u_0{}^H \text{ satisfy the conformal constraint equations implied on space-like surfaces by (2.5)–(2.11) with}\\
R=-6 \text{ and (2.14) with } \Lambda'=0 \text{ on } S.
\end{array}\right\} \tag{2.50}$$

From $u_0{}^H$ will be constructed in the next chapter "extended hyperboloidal initial data" $\mathring{u}_0{}^H$, which will be defined on the whole of $S_{t_0}\supset\bar{S}$ and will serve as initial data set for Eqs. (2.42)–(2.46).

## 3. The Structure of the Solutions

Before discussing the existence of solutions to the initial value problems formulated in Sect. 2, the reduced conformal field equations will be written in a slightly different way. Set $v=(\mathrm{Re}\,u,\mathrm{Im}\,u)$ and let $v'$ denote either $v^D\equiv(\mathrm{Re}\,u^D,\mathrm{Im}\,u^D)$ or $v^M\equiv(\mathrm{Re}\,u^M,\mathrm{Im}\,u^M)$. Setting now $v=v'+w$, Eqs. (2.42)–(2.46), written as a system of real equations for $w$, take the form:

$$A^0(w)\frac{\partial}{\partial t}w+\sum_{r=1}^{3}A^r(w)c_r(w)+B(t,x,w)\cdot w=0 \tag{3.1}$$

for an unknown $w$ which takes values in $\mathbb{R}^N$ for some $N\in\mathbb{N}$. The properties of the system (3.1), which are important for the following, are:

i) The entries of the matrix-valued function $B=B(t,x,z)$, defined for $(t,x,z)\in\mathbb{R}\times S^3\times\mathbb{R}^N$, are polynomials in $z$ of degree at most one with coefficients, which are real-analytic functions on $\mathbb{R}\times S^3$, periodic in $t$.

ii) The entries of the matrix-valued function $A^j(z)$, $j=0,1,2,3$, are polynomials in $z\in\mathbb{R}^N$ of degree at most one with constant coefficients. The matrices are symmetric ${}^tA^j(z)=A^j(z)$, $z\in\mathbb{R}^N$, and the matrix $A^0(0)$ is diagonal with diagonal elements not smaller than $\frac{1}{\sqrt{2}}$.

On the space $C^\infty(S^3, \mathbb{R}^N)$ of smooth $\mathbb{R}^N$-valued functions on $S^3$ define for $m \in \mathbb{N}$ the norm:

$$\|w\|_m = \left\{ \sum_{k=0}^{m} \int |D^k w|^2 d\mu \right\}^{1/2}, \tag{3.2}$$

where the integration is over $S^3$ and the following notation is used:

$$|D^0 w|^2 = |w|^2, \quad |D^k w|^2 = \sum_{r_1, \ldots, r_k = 1}^{3} |D_{r_1} \ldots D_{r_k} w|^2,$$

with $|\ |$ being the standard euclidean norm on $\mathbb{R}^N$, $D$, $d\mu$ denote respectively the covariant Levi-Civita derivative operator and the volume element associated with the metric $d\omega^2$ on $S^3$, and $D_r$ denotes the covariant derivative in the direction of the field $c_r$. For $m \in \mathbb{N}$ let now $H^m(S^3, \mathbb{R}^N)$ be the Hilbert-space which is obtained as the completion of the space $C^\infty(S^3, \mathbb{R}^N)$ in the norm (3.2). In most cases it will be convenient now to consider the unknown $w = w(t, x)$ as a function of $t$ which takes values in the function space $H^m(S^3, \mathbb{R}^N)$ with "space variable" $x \in S^3$ for some $m \in \mathbb{N}$.

For $\delta \in \mathbb{R}$, $m \in \mathbb{N}$ with $0 < \delta < \dfrac{1}{\sqrt{2}}$, $m \geqq 2$, set

$$D^m_\delta = \{ w \in H^m(S^3, \mathbb{R}^N) / (z, A^0(w) z) > \delta(z, z)\ \forall z \in \mathbb{R}^N \},$$

where $(\ ,\ )$ is the standard scalar product on $\mathbb{R}^N$. This set is well defined because $H^m(S^3, \mathbb{R}^N)$ is embedded for $m \geqq 2$ continuously into $C^0(C^3, \mathbb{R}^N)$, and it contains a neighbourhood of the origin in $H^m(S^3, \mathbb{R}^N)$ by property (ii) of Eqs. (3.1).

**Theorem (3.1).** *Suppose $m \geqq 4$, $D$ is a bounded open subset of $H^s(S^3, \mathbb{R}^N)$ with $D \subset D^m{}_\delta$. If $w_0 \in D$ is given as initial condition for Eq.* (3.1), *then:*

i) *There exists a $T > 0$ and a unique solution $w(t)$ of* (3.1), *defined on* $[0, T]$, *with $w(0) = w_0$ and*

$$w \in C(0, T;\ D) \cap C^1(0, T;\ H^{m-1}(S^3, \mathbb{R}^N)).$$

ii) *There is an $r > 0$ such that the corresponding $T$ can be chosen common to all initial conditions in the open ball $B_r(w_0)$ with center $w_0$ and radius $r$ (such that $B_r(w_0) \subset D$).*

iii) *If $r$ and $T$ are chosen as in* ii) *and $w_0{}^n \in B_r(w_0)$ with $\|w_0{}^n - w_0\|_m \to 0$ as $n \to \infty$, then for the solutions $w^n(t)$ with $w^n(0) = w_0{}^n$ holds $\|w^n(t) - w(t)\|_m \to 0$ uniformly in $t$, $t \in [0, T]$.*

iv) *If the solution $w(t)$ in* i) *exists on $[0, T_0]$ for some $T_0 > 0$, then the solutions to all initial conditions in $B_r(w_0)$ exist on $[0, T_0]$ if $r > 0$ is sufficiently small.*

*Remark (3.2).* By i) local existence of solutions is asserted for any $w_0 \in D^m{}_\delta$. Of particular interest in the following, however, is the fact that $w \equiv 0$ is a solution of (3.1), which is defined on $[0, T]$ for any $T > 0$, whence by iv) for any $T_0 > 0$ there exists an $r > 0$ such that for $w_0 \in B_r(0)$ the solution $w(t)$ exists on $[0, T_0]$. Because of the particular structure of (3.1) solutions to (3.1) exist in fact on intervals of the form $[-T_1, T_2]$, $T_1, T_2 > 0$. Furthermore, they are of class $H^m(]-T_1, T_2[ \times S^3)$ (with the obvious definition of $H_m$) and in particular of class $C^{m-2}([-T_1, T_2] \times S^3)$. By iii) one has $w^n(t, x) \to w(t, x)$ uniformly on $[-T_1, T_2] \times S^3$.

The results above are analogues of results obtained by Kato in [13] for solutions of symmetric hyperbolic systems for Hilbert-space-valued functions on $\mathbb{R}^n$. The first part of his proof consists in showing certain properties, among them the $m$-accretiveness, of operators of a specific type, which are naturally associated with the equations, and in demonstrating that conditions (i′), (ii‴), and (iii) in Theorem I of [12] are satisfied. Considering, as indicated in the beginning, $S^3$ as underlying manifold of SU(2), $d\mu$ as Haar measure, the fields $c_r$ as left invariant vector fields and using further structures on the group such as the convolution product to replace corresponding structures on $\mathbb{R}^n$, one can mimick, with only a few modifications due to the non-commutativity of the group SU(2), each step in the first part of Kato's proof in the present setting (the complete proof of the $m$-accretiveness of the operator associated with (3.1) may be given on SU(2) in close analogy to the exposition in [3]). The second part of Kato's proof, which relies on general properties of Banach spaces, and of function spaces similar to $H^m(S^3, \mathbb{R}^N)$, and in particular on the results derived in [11, 12] for linear evolution equations in Banach spaces, applies then in much the same way to the present situation. To avoid what would amount essentially to a lengthy repetition of known arguments, we leave out the details.

By the choice of the frame field coefficients in the asymptotic initial data set $u_0{}^{DT}$, respectively $v_0{}^{DT} = (\mathrm{Re}\,u_0{}^{DT}, \mathrm{Im}\,u_0{}^{DT})$ given by (2.48) one will always have $w_0{}^{DT} = v_0{}^{DT} - v^D|_S \in D_\delta{}^m$ if $v_0{}^{DT} \in H^m(S^3, \mathbb{R}^N)$, $m \geqq 2$, and if $v' = v^D$ is chosen to derive the system (3.1). Thus Theorem (3.1) implies the existence of local solutions of the regular conformal field equations for all sufficiently smooth initial data sets as given by (2.48). This has been discussed in all generality in [9]. Moreover, one has

**Theorem (3.3).** *Suppose $m \geqq 4$. There is a number $r > 0$ such that initial conditions $v_0{}^{DT}$, obtained as above from* (2.48), *with $\|v_0{}^{DT} - v^D|_S\|_m < r$ determine a unique solution $u^{DT}(t)$ of the regular conformal field equations* (2.5)–(2.11) *with $R = -6$ and* (2.14) *with $\Lambda' = -3$, which exists on $[-\frac{\pi}{2}, T_0]$ with $T_0 > \frac{\pi}{2}$ and such that:*

i) *The fields supplied by $u^{DT}$, in particular the conformal factor $\Omega^{DT}$ and the frame coefficients, are of class $C^{m-2}([-\frac{\pi}{2}, T_0] \times S^3)$ and the frame coefficients define by* (2.2) *a Lorentz-metric of class $C^{m-2}$.*

ii) *There exists a function $S^3 \ni x \to t(x) \in \mathbb{R}$ with $-\frac{\pi}{2} < t(x) < T_0$ and*

$$\Omega^{DT} > 0 \quad on \quad M^{DT} = \{(t, x) \in \mathbb{R} \times S^3 / -\tfrac{\pi}{2} < t < t(x)\},$$

$$\Omega^{DT} = 0 \quad on \quad \mathscr{I}^{DT+} = \{(t(x), x) / x \in S^3\},$$

*$\mathscr{I}^{DT+}$ is a hypersurface of $[-\frac{\pi}{2}, T_0] \times S^3$ of class $C^{m-2}$ which is space-like with respect to $g^{DT}$.*

*In particular one has: The metric $\tilde{g}^{DT} = (\Omega^{DT})^{-2} g^{DT}$ on $M^{DT}$ is a solution of class $C^{m-2}$ (with curvature tensor of class $C^{m-2}$) of Einstein's equations* $\mathrm{Ric}[\tilde{g}] = \Lambda_0 \tilde{g}$, *$\Lambda_0 = -3$, which is asymptotically simple, thus null geodesically complete, and for which the surfaces $\mathscr{I}^{DT-} = \{(-\frac{\pi}{2}, x) / x \in S^3\}$ and $\mathscr{I}^{DT+}$ represent past and future conformal infinity.*

If $v' = v^D$ is used to obtain Eq. (3.1), Theorem (3.1) and Remark (3.2) show that for sufficiently small $r > 0$ there exists a unique solution $u^{DT}(t)$ of the reduced conformal field equations (2.42)–(2.46) satisfying the initial conditions, defined for $t\in[-\frac{\pi}{2}, T_0]$ with $T_0=\pi$, and such that on $M_0=[-\frac{\pi}{2}, T_0]\times S^3$ holds:

$$|\Omega^{DT}-\Omega^{D}|\leqq\tfrac{1}{2}, \quad |(e^{DT})^i{}_k-\delta^i{}_k|<\varepsilon. \tag{3.3}$$

Here $\varepsilon>0$ is assumed small enough [see e.g. (3.6)] to ensure that $\det((e^{DT})^i{}_k)\neq 0$ [whence the metric $g^{DT}$ defined by (2.2) from $(e^{DT})^i{}_k$ is regular on $M_0$], that $\frac{\partial}{\partial t}$ is time-like, and that for $-\frac{\pi}{2}\leqq t\leqq T_0$ the surface $\{t\}\times S^3$ is space-like in $M_0$ with respect to the metric $g^{DT}$. The smoothness properties stated in i) will hold by Remark (3.2).

From $\Omega^D(T_0,x)=-1$ ensues by (3.3) that $\Omega^{DT}(T_0,x)<0$ for $x\in S^3$. The initial conditions $\Omega^{DT}(-\frac{\pi}{2},x)=0$, $\Sigma_0^{DT}(-\frac{\pi}{2},x)=1$, $x\in S^3$, imply therefore that the function $t(x)$, and hence the sets $M^{DT}$, $\mathscr{I}^{DT+}$ are well defined. Since $u_0{}^{DT}$ given by (2.48) satisfies the constraints on $\mathscr{I}^{DT-}$ and $u^{DT}$ satisfies the reduced conformal field equations, $u^{DT}$ satisfies by Remark (2.4) the regular conformal field equations on the domain of dependence of $\mathscr{I}^{DT-}$ with respect to $g^{DT}$, which is $M_0$. From Eq. (2.14), which holds with $\Lambda'=-3$, and Eq. (2.7) ensues that $(\Sigma^{DT})^i$, the gradient of $\Omega^{DT}$, is time-like where $\Omega^{DT}$ vanishes, whence $\mathscr{I}^{DT+}$ is a hypersurface of class $C^{m-2}$, which is space-like and thus transverse to the $t$-coordinate lines. This shows the first statement and ii). The final remark follows from the properties of the regular conformal field equations discussed in Sect. 2, from the behaviour of the Weyl tensor under conformal rescalings and from the fact that $\tilde{g}^{DT}$ satisfies the field equations on $M^{DT}$.

*Remark (3.4).* (i) Instead of using asymptotic initial data, one could also start from standard Cauchy data for $\mathrm{Ric}[\tilde{g}]=\Lambda_0\tilde{g}$, which describe the geometry of a Cauchy surface in the prospective solution space-time and construct from those a conformal initial data set. If this is close enough in the sense used above to a conformal De-Sitter initial data set, it follows by the arguments above that it will also determine an asymptotically simple solution.

(ii) In a similar way as the existence of asymptotically simple solutions close to De-Sitter space-time has been shown here, one can demonstrate the existence of (weakly) asymptotically simple solutions close to other given (weakly) asymptotically simple solutions. This may, however, require more lengthy and more technical arguments, since in general an analogue of the cylinder map will not be available.

(iii) Values of the cosmological constant other than $\Lambda_0=-3$ may be dealt with by conformal rescalings with constant conformal factors.

(iv) If required, the differentiability statements for the field $\tilde{g}^{DT}$ and its curvature tensor may be strengthened somewhat and formulated in terms of Sobolev-spaces.

Let now $\bar{S}$ be the surface given by (2.24). Set $S=\bar{S}\backslash Z$, where $Z$ is the boundary of $\bar{S}$. Writing $S^3$ for $S_{t_0}$, we may consider $S$, $\bar{S}$ as subsets of $S^3$. Denote by $F$ the set of $\mathbb{R}^N$-valued functions on $S$, which are obtained as restrictions to $S$ of functions in $C^\infty(S^3,\mathbb{R}^N)$, and let $\|w\|_{m,S}$ for $w\in F$ denote the norm defined by (3.2), where the integration is now over $S$. By $H^m(S,\mathbb{R}^N)$ we will denote the Sobolev-space obtained as completion of $F$ in the norm $\|\cdot\|_{m,S}$.

Set $v_0{}^H = (\mathrm{Re}\, u_0{}^H, \mathrm{Im}\, u_0{}^H)$, where $u_0{}^H$ is a "vector" as given by (2.49). Since for $m \geqq 4$ the function space $H^m(S, \mathbb{R}^N)$ may be embedded continuously into the space $C^{m-2}(\bar{S}, \mathbb{R}^N)$ of functions $w$ on $S$, which together with their derivatives $D_{r_1} \dots D_{r_k} w$, $k \leqq m-2$, are bounded and uniformly continuous on $S$, it makes sense to say that the functions given by $u_0{}^H$ satisfy (2.50), if $v_0{}^H \in H^m(S, \mathbb{R}^N)$. We call $v_0{}^H$ then a hyperboloidal initial data set on $S$.

To proceed as in the case of the De-Sitter type initial value problem, the hyperboloidal initial data sets on $S$ will be extended to $S^3$. For any $m \geqq 0$ there exists a linear "extension operator" $E$, which maps $H^m(S, \mathbb{R}^N)$ into $H^m(S^3, \mathbb{R}^N)$, such that for any $w \in H^m(S, \mathbb{R}^N)$ holds

$$(Ew)(x) = w(x) \quad \text{a.e. in } S, \quad \|Ew\|_m \leqq K \|w\|_{m,S},$$

with a constant $K = K(m) > 0$ [1]. Assume $m \geqq 4$ and make a fixed choice of such an extension operator. For an hyperboloidal initial data set $v_0{}^H \in H^m(S, \mathbb{R}^N)$, we will call $\mathring{v}_0^H \in H^m(S^3, \mathbb{R}^N)$, defined by

$$\mathring{v}_0^H = E(v_0{}^H - v_0{}^M|_S) + v_0{}^M,$$

with $v_0{}^M = v^M|_{S_{t_0}}$, the "extended hyperboloidal initial data set."

It then holds

$$\|\mathring{v}_0^H - v_0{}^M\|_m \leqq K \|v_0{}^H - v_0{}^M|_S\|_{m,S}. \tag{3.4}$$

By the structure of $v_0{}^M$ and by (2.50) we may furthermore assume that the extension $\mathring{v}_0^H$ is such that:

$$\mathring{e}^k{}_0 = \delta^k{}_0, \quad \mathring{e}^0{}_j = \delta^0{}_j, \quad \mathring{\gamma}_0{}^i{}_k = 0 \quad \text{on } S_{t_0}. \tag{3.5}$$

It will be seen later that the fact that the extension is not unique and that $\mathring{v}_0^H$ does not satisfy necessarily the constraint equations on $S_{t_0} \setminus \bar{S}$, creates no difficulties.

If $v_0{}^H \subset H^m(S, \mathbb{R}^N)$, $m \geqq 4$, is an hyperboloidal initial data set on $S$, then by (3.5) holds $\mathring{v}_0{}^H - v_0{}^M \in D_\delta{}^m$. Thus Theorem (3.1) implies a local existence theorem for the hyperboloidal initial value problem. This has been discussed in detail in [6]. Moreover, we have

**Theorem (3.5).** *Assume $m \geqq 4$. There is a number $r > 0$ such that for any hyperboloidal initial data set $v_0{}^H$ on S with $\|v_0{}^H - v_0{}^M|_S\|_{m,s} < r$, there exists on some interval $[t_0, T_0]$, with $T_0 > \pi$, a unique solution $\mathring{v}^H(t)$ of the reduced conformal field equations (2.42)–(2.46) satisfying $\mathring{v}^H(t_0) = \mathring{v}_0^H$ and such that*

(i) *The fields supplied by $\mathring{v}^H$, in particular $\mathring{\Omega}^H$ and the frame coefficients, may be obtained as restrictions to $M_0 = [t_0, T_0] \times S^3$ of functions of class $C^{m-2}$ defined on a neighbourhood of $M_0$ in $\mathbb{R} \times S^3$. The frame coefficients define by (2.2) a Lorentz metric $\mathring{g}^H$ with the same smoothness properties.*

(ii) *There exists a point $i^+ \in ]t_0, T_0[ \times S^3$ such that the causal past $J^-(i^+)$ of $i^+$ and the future domain of dependence $D^{H+}(\bar{S})$ of $\bar{S}$ in the Lorentz-space $(M_0, \mathring{g}^H)$ coincide.*

(iii) *The restriction $v^H$ (respectively $g^H$, $\Omega^H$) of $\mathring{v}^H$ (respectively $\mathring{g}^H$, $\mathring{\Omega}^H$) to $D^{H+}(\bar{S})$ is uniquely determined by the initial data set $v_0{}^H$ (i.e. independent of the chosen extension $\mathring{v}_0^H$) and $v^H$ provides on $D^{H+}(\bar{S})$ a solution of the regular conformal field equations (2.5)–(2.11) and of (2.14) with $\Lambda' = 0$.*

(iv) *The function $\Omega^H$ is positive on $M^H = D^{H+}(\bar{S}) \backslash H^{H+}(\bar{S})$, where $H^{H+}(\bar{S})$ denotes the Cauchy horizon of $\bar{S}$. Furthermore, $\Omega^H$ vanishes on $H^{H+}(\bar{S})$, $d\Omega^H \neq 0$ on $\mathscr{J}^{H+} = H^{H+}(\bar{S}) \backslash \{i^+\}$, and $d\Omega^H = 0$ but the Hessian of $\Omega^H$ is non-degenerate at $i^+$.*

*In particular one has: The metric $\tilde{g}^H = (\Omega^H)^{-2} g^H$ on $M^H$ is a solution of class $C^{m-2}$ (with curvature tensor of class $C^{m-2}$) of Einstein's vacuum field equations* $\mathrm{Ric}[\tilde{g}] = 0$, *which is future asymptotically simple, thus future null geodesically complete, for which $\mathscr{J}^{H+}$ represents future null infinity and $i^+$ future time-like infinity.*

Assume that $v' = v^M$ has been chosen to derive Eq. (3.1). Then Theorem (3.1), Remark (3.2), and relation (3.4) imply that $r > 0$ can be chosen so small that for $v_0{}^H$ satisfying the estimate above there exists a unique solution $\overset{*}{v}{}^H(t)$ of the reduced field equations (2.42)–(2.46), which is defined on $[t_0 - \varepsilon, T_0 + \varepsilon]$ with $T_0 = \frac{3}{2}\pi$ and some small $\varepsilon > 0$, satisfies $\overset{*}{v}{}^H(t_0) = \overset{*}{v}_0{}^H$, and which is such that on $M_0$ holds

$$|(\overset{*}{e}{}^H)^i{}_k - \delta^i{}_k| \leqq \tfrac{1}{16}, \tag{3.6}$$

$$|\overset{*}{\Sigma}_0{}^H - \Sigma_0{}^M| \leqq \tfrac{1}{2} \sin t_0 . \tag{3.7}$$

The functions supplied by $\overset{*}{v}{}^H$ will then satisfy the smoothness conditions stated in (i). The same is true for the metric $\overset{*}{g}{}^H$ defined by (2.2) from $(\overset{*}{e}{}^H)^i{}_k$, since the latter are invertible by (3.6). Moreover, (3.6) implies that $|((\overset{*}{e}{}^H)^{-1})^i{}_k - \delta^i{}_k| \leqq \frac{1}{12}$, from which ensues

$$|\overset{*}{g}{}^H(c_i, c_k) - \eta_{ik}| = |((\overset{*}{e}{}^H)^{-1})^l{}_i ((\overset{*}{e}{}^H)^{-1})^j{}_k \eta_{lj} - \eta_{ik}| < \tfrac{1}{4} .$$

This in turn implies that with respect to $\overset{*}{g}{}^H$

$$\{t\} \times S^3 \text{ is space-like for } t_0 \leqq t \leqq T_0 , \quad \frac{\partial}{\partial t} \text{ is time-like on } M_0 . \tag{3.8}$$

We take $\dfrac{\partial}{\partial t}$ as future directed.

Since $\overset{*}{v}{}^H$ satisfies on $M_0$ the reduced conformal field equation (2.42)–(2.46) and its restriction to $S$ satisfy by assumption the constraint equations, it follows from Remark (2.4) that (iii) is true.

The Cauchy horizon $H^{H+}(\bar{S})$ is generated by null geodesic segments which have past endpoint on $Z$ [10]. Since $Z$ is smooth, it follows that $H^{H+}(\bar{S})$ represents near $Z$ a $\overset{*}{g}{}^H$-null hypersurface of class $C^{m-2}$. By (2.50) the null direction tangent to $H^{H+}(\bar{S})$ is given on $Z$ by $\overset{*}{\Sigma}{}^H{}_k$. On $Z$ $\overset{*}{g}{}^H$-null vectors $n$, $l$ are defined uniquely if $n$ is orthogonal to $Z$, $n^k l_k = 1$, and $l_k = \overset{*}{\Sigma}{}^H{}_k$. Here and in the following components are given with respect to the frame $\overset{*}{e}{}^H{}_k$. On suitable open sets $U$ of $Z$ the fields $n$, $l$ may be complemented by complex vector fields $m$, $\bar{m}$ "tangent" to $Z$ with $m_k \bar{m}^k = 1$. The pseudo-orthonormal frame so obtained on $U$ may be propagated along the null generators of $H^{H+}(\bar{S})$ which end on $U$ by parallel transport in the direction of $l$.

On $H^{H+}(\bar{S})$ the regular conformal field equations (2.5)–(2.11) are satisfied. Transvecting, where the Cauchy-horizon is a hypersurface of class $C^{m-2}$, Eqs. (2.7), (2.8) suitably with $l^k$, $m^k$, one obtains (dropping the $H$'s and *'s from now on)

$$\left.\begin{aligned} &l^k \nabla_k \Omega = l^j \Sigma_j, \\ &l^k \nabla_k (l^j \Sigma_j) = -\Omega (s_{ik} l^i l^k), \\ &l^k \nabla_k (m^j \Sigma_j) = -\Omega (s_{ik} l^i m^k). \end{aligned}\right\} \tag{3.9}$$

By (2.50) the functions $\Omega$, $l^k \Sigma_k$, $m^k \Sigma_k$ vanish on $U$. Thus following from $Z$ a null generator on $H^+(\bar{S})$ one will have $\Omega = 0$ until one possibly arrives at a caustic point. On that part of the Cauchy horizon where $\Omega$ vanishes, Eqs. (2.8), (2.9) give, observing $R = \text{const}$,

$$\left.\begin{aligned} &l^k \nabla_k (m^j \Sigma_j) = s, \\ &l^k \nabla_k s = -(n^j \Sigma_j)(s_{ik} l^i l^k). \end{aligned}\right\} \tag{3.10}$$

Finally, one obtains from (2.8) by transvecting with $m^i$, $\bar{m}^k$, where the Cauchy-horizon is of class $C^{m-2}$ and $\Omega = 0$,

$$s = -\varrho (n^k \Sigma_k), \tag{3.11}$$

where $\varrho$ is a measure for the convergence of the null generators. From (3.10) follows that $n^j \Sigma_j$ and $s$ cannot vanish at the same point on a null generator, since $n^j \Sigma_j = 1$ on $Z$. Equation (3.11) shows then that $n^k \Sigma_k$ or equivalently $d\Omega$ vanishes at a point $p \in H^+(\bar{S})$ if and only if $p$ is a caustic point.

Set $Z_t = (\{t\} \times S^3) \cap H^+(\bar{S})$ for $t_0 \leqq t \leqq T_0$, such that $Z_{t_0} = Z$. Assume that $d\Omega \neq 0$ on $Z_t$ for $t_0 \leqq t < t_1$, $t_1 < T_0$. By the preceding discussion we then have $\Omega = 0$, $\Sigma_k \Sigma^k = 0$ on $Z_t$ for $t_0 \leqq t \leqq t_1$. If $d\Omega \neq 0$ on $Z_t$, this implies that not all the functions $c_r(\Omega)$, $r = 1, 2, 3$, can vanish at the same point of $Z_{t_1}$. Thus $Z_{t_1}$ is of class $C^{m-2}$ and $H^+(\bar{S})$ extends through $Z_{t_1}$ with this smoothness property. It is, however, impossible to have $d\Omega \neq 0$ on $Z_t$ for $t_0 \leqq t \leqq T_0$. Since $\Sigma_0{}^M(t_0, x) = -\sin t_0$, $\Sigma_0{}^M(\frac{3}{2}\pi, x) = 1$, the estimate (3.7) shows that $\Sigma_0{}^H(t_0, x) < 0$, $\Sigma_0(\frac{3}{2}\pi, x) > 0$ for $x \in S^3$, Therefore, on any null geodesic tangent to $H^+(\bar{S})$ at $Z$, which must end on $\{T_0\} \times S^3$ by (3.8), will be a point where $\Sigma_0{}^H$ vanishes. Let $t_1 < T_0$ be as above, such that $\Sigma_0{}^H \neq 0$ on $Z_t$ for $t < t_1$ and that there is a point $i^+ \in Z_{t_1}$, where $\Sigma_0{}^H = 0$. Then by $\Sigma_k \Sigma^k = 0$ on $Z_{t_1}$ we have

$$d\Omega = 0 \quad \text{at } i^+ .$$

Equation (2.8), which holds by continuity at $i^+$, since it holds on $H^+(\bar{S})$ for $t < t_1$, and (3.9), (3.11) give, since $\Omega = 0$ on $Z_{t_1}$

$$\nabla_i \Sigma_k = s \eta_{ik}, \qquad s \neq 0 \quad \text{at } i^+ .$$

Hence $i^+$ is a non-degenerate and therefore isolated critical point of the function $\Omega$.

It follows that all null geodesics on the surface $H^+(\bar{S})$ must converge at the point $i^+$ such that $H^+(S)$ is the past light cone of the point $i^+$. This establishes (ii) and (iii). The final remark of Theorem (3.5) follows by the same argument as the final remark of Theorem (3.3).

*Remark (3.6).* The analysis of the structure of $H^+(\bar{S})$ is based on a discussion of the field equations, which does not only involve the reduced equations but the complete set of Eqs. (2.8), which are the trace free part of (2.12) with $\Lambda = 0$, of Eqs. (2.9), which "replace" Eq. (2.14), the trace of (2.12), as well as of (2.7). These equations, however, seem not to be sufficient to deduce the occurrence of a critical point of $\Omega$, which has been forced here by requiring the estimate (3.7). One might think to use the equation (in N.P. notation)

$$D\varrho = \varrho^2 + \sigma\bar{\sigma} + \phi_{00}$$

contained in (2.6) to show the divergence of $\varrho$. However, since $\phi_{00}$ is obtained with $s_{ik}$ as a solution of (2.10), it appears difficult to derive any information on the behaviour of $\phi_{00}$.

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