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The Gibbs Measures and Partial Differential Equations

I. Ideas and Local Aspects

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Abstract. We investigate the connections of the Gibbs measures, which appear in Euclidean Field Theory, and the corresponding partial differential equations of Classical Euclidean Field Theory.

1. Preliminaries

Let $\mathscr{F}$ denote the family of bounded open sets $\Lambda \subset R^d$ with piecewise $\mathscr{C}^n$, for some $n \geqq 1$, boundary $\partial\Lambda$. Let $\mathscr{F}_0$ be a countable base of $\mathscr{F}$, i.e.

$$\mathscr{F}_0 := \{\Lambda_n\}_{n\in\mathbb{N}} \subset \mathscr{F} \quad \text{such that for every} \quad n\in\mathbb{N}, \Lambda_n \subset \Lambda_{n+1} \qquad (1)$$
$$\text{and } \forall \Lambda \in \mathscr{F}\ \exists n\ \Lambda \subset \Lambda_n .$$

For $\Lambda \subset R^d$ we denote $\Lambda^c := R^d \backslash \Lambda$.

Let (Ω, Σ) be a standard Borel space. We assume that in Σ there is a distinguished family of σ-algebras of local events $\{\Sigma_\Lambda\}_{\Lambda\in\mathscr{F}}$, which generates Σ and is compatible, i.e.

$$\Lambda_1, \Lambda_2 \in \mathscr{F} : \Lambda_1 \subset \Lambda_2 \ \Rightarrow\ \Sigma_{\Lambda_1} \subset \Sigma_{\Lambda_2} . \qquad (2)$$

For any open set $Q \subset R^d$ we define the σ-algebra Σ_Q as the σ-algebra generated by $\{\Sigma_\Lambda : \Lambda \in \mathscr{F}, \Lambda \subset Q\}$. For arbitrary set $Q \subset R^d$ we define

$$\Sigma_Q := \{\bigcap \Sigma_{\tilde{Q}} : \tilde{Q} \text{ open}, \tilde{Q} \supset Q\} . \qquad (3)$$

In particular we have the family of σ-algebras $\{\Sigma_{\Lambda^c}\}_{\Lambda\in\mathscr{F}}$ with the property

$$\Lambda_1, \Lambda_2 \in \mathscr{F} : \Lambda_1 \subset \Lambda_2 \ \Rightarrow\ \Sigma_{\Lambda_2^c} \subset \Sigma_{\Lambda_1^c} . \qquad (4)$$

We define the σ-algebra at infinity by

$$\Sigma_\infty := \bigcap_{\Lambda\in\mathscr{F}} \Sigma_{\Lambda^c} . \qquad (5)$$

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For $\Lambda \subsetneqq R^d$, $\mathfrak{A}_\Lambda$ denotes the set of bounded Σ_Λ measurable real functions on (Ω, Σ). If $\Lambda = R^d$ we will write $\mathfrak{A} \equiv \mathfrak{A}_{R^d}$.

By $\mathscr{M}$ we denote the set of probability measures on (Ω, Σ). For $\mu \in \mathscr{M}$ and a measurable function F, by $\mu(F)$ or simply μF we denote the expectation value of F with a measure μ. The conditional expectation of F with respect to a σ-algebra Σ', associated to a measure μ is denoted by $E_\mu(F|\Sigma')$.

We will write

$$\mu(F, G) := \mu FG - \mu F \mu G . \tag{6}$$

For $\mu \in \mathscr{M}$, if

$$\forall \Lambda \in \mathscr{F} \;\; \forall F \in \mathfrak{A}_\Lambda, \qquad E_\mu(F|\Sigma_{\Lambda^c}) \in \mathfrak{A}_{\partial\Lambda}, \tag{7}$$

we say that μ has the local Markov property, and if

$$\forall Q \subset R^d \;\; \forall F \in \mathfrak{A}_Q, \qquad E_\mu(F|\Sigma_{Q^c}) \in \mathfrak{A}_{\partial Q} \tag{8}$$

we say (eventually restricting to the sets Q with sufficiently smooth boundary) that μ has the global Markov property and we write $\mu \in \mathrm{GMP}$. The global Markov property imply the local one, but the converse is not true in general (e.g. [7, 20]).

A local specification ([5, 12, 16]) is a family $\mathscr{E} := \{E^{\cdot}_{\Lambda^c}\}_{\Lambda \in \mathscr{F}}$, which consists of functions

$$E^{\cdot}_{\Lambda^c} : \Omega \times \Sigma \to [0, 1], \tag{9}$$

such that

i) $\forall \Lambda \in \mathscr{F} \;\; \forall \omega \in \Omega$, $E^{\omega}_{\Lambda^c}(\cdot) \in \mathscr{M}$,

ii) $\forall \Lambda \in \mathscr{F} \;\; \forall F \in \mathfrak{A}$, $E^{\cdot}_{\Lambda^c}(F) \in \mathfrak{A}_{\Lambda^c}$,

iii) the compatibility condition holds i.e.

$$\Lambda_1, \Lambda_2 \in \mathscr{F} : \Lambda_1 \subset \Lambda_2 \;\Rightarrow\; E_{\Lambda_2^c}(F E_{\Lambda_1^c} G) = E_{\Lambda_2^c}(FG) \tag{10}$$

for any $G \in \mathfrak{A}$, $F \in \mathfrak{A}_{\Lambda_1^c}$. □

A local specification $\mathscr{E} = \{E^{\cdot}_{\Lambda^c}\}_{\Lambda \in \mathscr{F}}$ is called Markov if it fulfills the following condition, which can be essentially written as (cf. [14]):

$$\forall \Lambda \in \mathscr{F} \;\; \forall F \in \mathfrak{A}_\Lambda, E^{\cdot}_{\Lambda^c}(F) \in \mathfrak{A}_{\partial\Lambda} . \tag{11}$$

The set of Gibbs measures for $\mathscr{E}$ is defined by

$$\mathscr{G}(\mathscr{E}) := \{\mu \in \mathscr{M} : \forall \Lambda \in \mathscr{F} \; \mu E_{\Lambda^c} = \mu\} . \tag{12}$$

The set of its extremal points [i.e. the set of those Gibbs measures for $\mathscr{E}$ which have no nontrivial convex decompositions in $\mathscr{G}(\mathscr{E})$] is denoted by $\partial\mathscr{G}(\mathscr{E})$.

2. The Ground Specifications and the Ground Gibbs States

Let $\mathscr{T}$ be a Fréchet space of real functions on R^d such, that $\mathscr{D} \subset \mathscr{T} \subset \mathscr{H}$ densely and continuously, where $\mathscr{D}$ is $\mathscr{C}_0^\infty$ with the usual topology [19] and $\mathscr{H}$ is a Hilbert space. Let $\mathscr{T}'$ be the topological dual of $\mathscr{T}$. Let $\mathscr{B}$ be the Borel σ-algebra of subsets in $\mathscr{T}'$ generated by $\sigma(\mathscr{T}', \mathscr{T})$ topology.

Let U be a $\mathscr{C}^\infty(\mathbb{R})$, real function bounded from below. For $\Lambda \in \mathscr{F}$ and $\eta \in \mathscr{T}'$, by $\Phi_\eta^{\partial\Lambda}$ we denote a (weak) solution of the Dirichlet problem, which is formally given by:

$$(-\Delta + m^2)\,\Phi_\eta^{\partial\Lambda}(x) + U^{(1)}(\Phi_\eta^{\partial\Lambda}(x)) = 0 \quad , \quad \text{for} \quad x \in \Lambda\,, \tag{13}$$

$$\Phi_\eta^{\partial\Lambda}(x) = \eta(x), \quad \text{for} \quad x \in \Lambda^c\,, \tag{14}$$

where $U^{(1)}$ denotes first derivative of U. We assume that U is such that the classical problem (13), (14) has a unique solution. If a Φ fulfills (13) for all $x \in R^d$ we will say that Φ is a global solution of this equation.

Let us assume that there is a Borel set $\Omega \subset \mathscr{T}'$ such that for each $\eta \in \Omega$ the problem (13) has a unique solution $\Phi_\eta^{\partial\Lambda} \in \Omega$. Let us define

$$\Sigma := \mathscr{B} \cap \Omega\,. \tag{15}$$

The following proposition follows easily from the above definitions: (for U considered in Sects. 3 and 6)

Proposition 1. *The family*

$$\mathscr{E} := \{E_{\Lambda^c}^{\cdot}\}_{\Lambda \in \mathscr{F}}\,,$$

where

$$E_{\Lambda^c}^{\eta}(\,\cdot\,) := \delta_{\Phi_\eta^{\partial\Lambda}}(\,\cdot\,) \tag{16}$$

is a local Markov specification on the standard Borel space (Ω, Σ)*. The set of all extremal Gibbs measures for this* $\mathscr{E}$ *is given by*

$$\partial\mathscr{G}(\mathscr{E}) = \{\mu = \delta_\Phi : \Phi \text{ is a global solution of (13)}\}\,, \tag{17}$$

and we have $\mu \in \partial\mathscr{G}(\mathscr{E}) \Rightarrow \mu \in \mathrm{GMP}$. □

We call the specification $\mathscr{E}$ defined in Proposition 1 the ground specification and the elements of $\partial\mathscr{G}(\mathscr{E})$ the ground states for the system with local interaction U (we will also say shortly: for interaction U).

Remark. All the above is obviously fulfilled if we take as Ω the set $\mathscr{C}$, but for our pourposes it is not sufficient. Note also that in general if Ω contains elements which are not simply the functions, then here (14) has symbolic sense. We will consider this case more precisely in Sects. 3 and 6. Further it will be also clear why we have chosen such names for $\mathscr{E}$ defined by (16) and elements of $\partial\mathscr{G}(\mathscr{E})$. □

3. The Free Euclidean Fields: Their Ground Specification and Ground Gibbs States

Let us consider the Classical Euclidean Field Theory given by the equation

$$(-\Delta + m^2)\,\Psi(x) = 0 \tag{18}$$

in the space R^d (with $m > 0$ if $d \leqq 2$). It has been proven in [14] (extending works [1, 2]), that the associated Dirichlet problem

$$\begin{aligned} (-\Delta + m^2)\,\Psi_\eta^{\partial\Lambda}(x) &= 0 \quad , \quad x \in \Lambda\,, \\ \Psi_\eta^{\partial\Lambda}(x) &= \eta(x), \quad x \in \Lambda^c \end{aligned} \tag{19}$$

has, defined in a unique way, a solution in any $\Lambda \subset \mathscr{F}$ open with a boundary condition η from some Borel set $\Omega \subset \mathscr{D}'$. (See [14] for details. Note also, that in this work the case of more general elliptic operators is considered.) Hence we have

Proposition 2. *There exist the ground (Markov) specifications $\mathscr{E}^0$ for Classical Euclidean Free Field Theories. The extremal ground Gibbs states for these specifications are given by all solutions of* (18) *through the definition in* (17). *All these extremal Gibbs measures for $\mathscr{E}^0$ have the global Markov property.* □

Let for $\Lambda \in \mathscr{F}$, $\Delta^{\partial\Lambda}$ be the [selfadjoint in $L_2(\Lambda)$] Laplacian with Dirichlet boundary condition on $\partial\Lambda$ (e.g. [6]).

Let

$$G^{\partial\Lambda} := (-\Delta^{\partial\Lambda} + m^2)^{-1} \tag{20}$$

with $m>0$ if $d \leqq 2$. For $\beta > 0$ we define the measures $\mu_{0,\beta}^{\partial\Lambda}$ by

$$\mu_{0,\beta}^{\partial\Lambda} e^{i\varphi(f)} := \exp(-\tfrac{1}{2}\beta^{-1}\|f\|_{-1,\partial\Lambda}^2), \tag{21}$$

where

$$\|f\|_{-1,\partial\Lambda}^2 := \int f(x)\, G^{\partial\Lambda}(x,y)\, f(y)\, dx\, dy \tag{22}$$

for $\operatorname{supp}(f) \subset \Lambda$.

We also assume that on Σ_{Λ^c} the measure (21) coincides with $\delta_{\varphi\equiv 0}$. For all $\eta \in \mathscr{D}'$, for which the Dirichlet problem (19) has well defined solution $\Psi_\eta^{\partial\Lambda}$, we define the measure

$$E_{\Lambda^c}^{\eta}(F) := \delta_\eta \mu_{0,\beta}^{\partial\Lambda}(F(\varphi + \Psi_{\eta'}^{\partial\Lambda})). \tag{23}$$

One can prove (e.g. [14, 2]) that the family $\mathscr{E}_\beta^0 := \{E_{\Lambda^c}^{\cdot}\}_{\Lambda \in \mathscr{F}}$, where $E_{\Lambda^c}^{\cdot}$ are given by (23), forms the local Markov specification on some standard Borel space (Ω, Σ), with $\Omega \subset \mathscr{D}'$. We call $\mathscr{E}_\beta^0$ the free (local) specification at inverse temperature β. [We would like to add an obvious remark, that if one has constructed the local specification by (23) on some Borel set $\Omega_1 \subset \mathscr{S}'$, then we can also define by (23) the specification on the set

$$\{\Psi + \Omega_1\} \subset \mathscr{D}',$$

where Ψ fulfill (18), and so we have no á priori restrictions on the growth of η at infinity.]

Let

$$G := (-\Delta + m^2)^{-1} \tag{24}$$

where Δ is the selfadjoint Laplacian in $L_2(R^d)$ with $D(\Delta) \supset \mathscr{C}_0^\infty(R^d)$ and $m>0$ if $d \leqq 2$.

Let

$$\|f\|_{-1}^2 := \int f(x)\, G(x,y)\, f(y)\, dx\, dy. \tag{25}$$

Then for any global solution Ψ of classical free field equation (18) the measure defined by

$$\mu_{\Psi,\beta} e^{i\varphi(f)} := \exp(-\tfrac{1}{2}\beta^{-1}\|f\|_{-1}^2 + i\Psi(f)) \tag{26}$$

is the extremal Gibbs measure for free specification $\mathscr{E}^0_\beta$, which has the global Markov property. The set of the measures $\mu_{\Psi,\beta}$ given by (26) is exactly equal to $\partial\mathscr{G}(\mathscr{E}^0_\beta)$, see [8].

For any unbounded open set $Q \subset R^d$ the conditional expectation with respect to Σ_{Q^c} associated to $\mu_{\Psi,\beta}$ is given by the following measure:

$$E_{\Psi,\beta}(e^{i\varphi(f)}|\Sigma_{Q^c})(\eta) = E^\eta_{Q^c,\beta}(e^{i\varphi(f)}) := \exp(-\tfrac{1}{2}\beta^{-1}\|f\|^2_{-1,\partial Q} + i\Psi^{\partial Q}_\eta(f)), \quad (27)$$

where $f \in \mathscr{D}$, $\Psi^{\partial Q}_\eta$ is a unique in a set of $\mu_{\Psi,\beta}$-measure one solution of the Dirichlet problem in Q with boundary data η. We have that

$$\exists \tilde{\eta} \in \mathscr{S}' \quad \eta = \tilde{\eta} + \Psi, \quad \mu_{\Psi,\beta}\text{-a.e.}, \quad (28)$$

hence

$$\Psi^{\partial Q}_\eta = \Psi^{\partial Q}_{\tilde{\eta}} + \Psi, \quad (29)$$

where $\Psi^{\partial Q}_{\tilde{\eta}}$ is defined in [14] as a solution of (19) which is unique in the sense that for μ_0-a.a. $\tilde{\eta} \in \mathscr{S}'$,

$$\Psi^{\partial Q}_{\tilde{\eta}} \equiv \lim_{\mathscr{F}_0} \Psi^{\partial Q}_{\chi_\Lambda \tilde{\eta}} = \lim_{\mathscr{F}_0} \Psi^{\partial(Q\cap\Lambda)}_{\tilde{\eta}} \quad (30)$$

(with χ_Λ the characteristic function of Λ). This essentially means that $\Psi^{\partial Q}_{\tilde{\eta}}$ is independent of behavior of $\tilde{\eta}$ outside ∂Q.

We can easily see, using (21)–(27), that the following proposition holds (in which convergence of measures means convergence of the characteristic functional):

Proposition 3.

$$\mathscr{E}^0_\beta \xrightarrow[\beta\to\infty]{} \mathscr{E}^0 \quad (31)$$

in the sense that for any $\Lambda \in \mathscr{F}$, $\eta \in \Omega$,

$$E^\eta_{\Lambda^c,\beta} \xrightarrow[\beta\to\infty]{} \delta_{\Psi^{\partial\Lambda}_\eta}, \quad (32)$$

where $E^\eta_{\Lambda^c,\beta}$ *is given by* (23), *and*

$$\mu_{\Psi,\beta} \xrightarrow[\beta\to\infty]{} \delta_\Psi, \quad (33)$$

as also for any unbounded open set $Q \subset R^d$

$$E^\eta_{Q^c,\beta} \xrightarrow[\beta\to\infty]{} \delta_{\Psi^{\partial Q}_\eta}, \quad (34)$$

i.e. in the limit as the temperature β^{-1} *goes to zero, the local structure of Euclidean Free Field Theory is exactly prescribed by the free ground specification* $\mathscr{E}^0$ *and all possible Gibbs measures (as also their properties) are exactly prescribed by the set of ground states for* $\mathscr{E}^0$ *(and their properties).* □

Now it is "obvious" why the states $\mu_{\Psi,\beta}$ are extremal (for $\mathscr{E}^0_\beta$) and why they have the global Markov property, since it is obvious for elements of $\partial\mathscr{G}(\mathscr{E}^0)$. By this remark we want only to stress the importance of further investigations of connections of the theory of Gibbs measures and theory of partial differential equations, particularly of the connections of (extremality and) the global Markov property with the locality property for the equations of more general type (13).

Furthermore we will investigate these problems: The present paper is concerned with the investigation of local aspects of the connections between theories mentioned in its title in the context of Euclidean Field Theory.

In Sect. 4 we define and investigate the local specifications $\mathscr{E}_\beta^U$ with interaction U at temperature β^{-1}, in the case of models of two dimensional Euclidean Field Theory. We will show that [in the sense as in (31)] $\mathscr{E}_\beta^U \xrightarrow[\beta\to\infty]{} \mathscr{E}^U$. Using this result we propose in Sect. 5 a definition of Euclidean Field Theory with interaction U associated to Classical Euclidean Field Theory given locally by the family of corresponding Dirichlet problems (13) or equivalently by the ground local specification $\mathscr{E}^U$. Section 6 is devoted to investigation of ground specifications $\mathscr{E}^U$ in d-dimensional Euclidean space. We expect that our investigations can give some new light on the existence problem of Euclidean (Scalar) Field Theory in space of dimension $d=4$.

In the second part of this work we will also show the existence of Gibbs measures $\mu_{\Phi,\beta}$ for the specifications $\mathscr{E}_\beta^U$, their extremality and the global Markov property [the proof of the later property is not finished in the $P(\varphi)_2$ case] proving validity of a conjecture we state in [20].

It is very natural from the physical point of view and very important to consider the families of local specifications $\mathscr{E}_\beta^U$ at different β with interaction U and the associated Gibbs states. In particular our investigations make more clear the similarities of theory of phase transitions in statistical mechanics (e.g. [17]) and in Euclidean Field Theory (e.g. [9, 6]).

4. The Specifications and the Ground Specifications for the Euclidean Field Theory in Two Dimensions

Let $\mathscr{M}_r$ be the space of regular measures μ on $(\mathscr{S}',\mathscr{B})$ in the sense that

$$\mu e^{\varphi(f)} \leqq C e^{|||f|||} \tag{35}$$

for some $C>0$ and

$$|||f|||:=a\|G*f\|_{L_1}+b\|G*f\|_{L_p}^p+\tfrac{1}{2}\|f\|_{-1}^2 \tag{36}$$

with arbitrary constants $a,b\geqq 0$ and some $p\geqq 4$. Note that a,b,C, and p can depend on μ.

It can be proven (see e.g. [1, 20]) that for any $\mu\in\mathscr{M}_r$ there exist the solutions $\Psi_\eta^{\partial\Lambda}(x)$ of Dirichlet problem (19) for μ-a.a. $\eta\in\mathscr{S}'$ such that $\Psi_\eta^{\partial\Lambda}(x)\in L_r(\mu)\otimes L_s(\Lambda,dx)$ for any $1\leqq r$, $s<\infty$ for all bounded open $\Lambda\subset R^2$ with piecewise $\mathscr{C}^1$ boundary, i.e. $\Lambda\in\mathscr{F}$.

Let for $\Lambda\in\mathscr{F}$

$$U_\Lambda(\varphi+\Psi_\eta^{\partial\Lambda}):=\int_\Lambda :U(\varphi+\Psi_\eta^{\partial\Lambda}):_{0,\beta}(x)\,dx, \tag{37}$$

where : $:_{0,\beta}$ means the normal ordering with respect to a free measure $\mu_{0,\beta}$ with some mass $m>0$, and a real function U of the following types:

$$\begin{aligned} &- \text{polynomial } \lambda \sum_{i=1}^{2n} a_i q^i, \quad n \in N, \quad a_{2n}>0, \\ &- \text{trigonometric } \lambda \int \cos(\alpha q + \vartheta(\alpha)) d\varrho(\alpha), \\ &- \text{exponential } \lambda \int \exp(\alpha q) d\varrho(\alpha). \end{aligned} \tag{38}$$

[$d\varrho(\alpha)$ means a nonnegative finite – or only finite in trigonometric case – measure supported in $(-2\sqrt{\pi}, 2\sqrt{\pi})$ and $\lambda>0$.]

The function $U_\Lambda(\varphi+\Psi_\eta^{\partial\Lambda})$ is well defined in $L_r(\mu_{0,\beta}^{\partial\Lambda})\otimes L_s(\mu)$ – for any $\mu\in\mathcal{M}_r$ and any $1\leqq r,s<\infty$ in the polynomial case, and at least for $r,s=2$ in the last two cases – as the limit (in adequate space) of a sequence of functions $U_\Lambda(\varphi_\varepsilon+\Psi_{\eta_\varepsilon}^{\partial\Lambda})$ defined on all $\mathcal{S}'\times\mathcal{S}'$ $\left[\text{where } \varphi_\varepsilon(x):=\varphi(h_\varepsilon(\cdot-x)) \text{ with some } h_\varepsilon\in\mathcal{D}, h_\varepsilon \xrightarrow[\varepsilon\to 0]{} \delta\right]$.

We define the local Markov specification $\mathcal{E}_\beta^U=\{E_{\Lambda^c}^{\cdot}\}_{\Lambda\in\mathcal{F}}$ for a field with interaction U and at temperature β^{-1} by

$$E_{\Lambda^c}^\eta(F) := \frac{\mu_{0,\beta}^{\partial\Lambda} e^{-\beta U_\Lambda(\varphi+\Psi_\eta^{\partial\Lambda})} F(\varphi+\Psi_\eta^{\partial\Lambda})}{\mu_{0,\beta}^{\partial\Lambda} e^{-\beta U_\Lambda(\varphi+\Psi_\eta^{\partial\Lambda})}} \tag{39}$$

on a standard Borel space (Ω_1,Σ_1), where Ω_1 is a Borel set such that

$$\Omega_1 \subseteqq \bigcup_{\mu\in\mathcal{M}_r} \Omega_\mu, \tag{40}$$

where for any $\mu\in\mathcal{M}_r$, Ω_μ is a Borel set of μ-measure one, on which the function $\Psi_\eta^{\partial\Lambda}$ is defined (at each point $\eta\in\Omega_\mu$).

We assume, and it can be realized, that Ω_1 is chosen sufficiently large, i.e. such that the compatibility condition (10) for a specification is fulfilled.

Having $\mathcal{E}_\beta^U$ defined on some $\Omega_1\subset\mathcal{S}'$ we can define it also on the sets $\{\Psi+\Omega_1\}\subset\mathcal{D}'$, where Ψ fulfills (18).

Let Ω be a Borel set $\{\Psi+\Omega_1\}\subset\mathcal{D}'$, with Ψ fulfilling (18), and let Σ denote the σ-algebra of Borel sets in Ω. From now on we consider the specification $\mathcal{E}_\beta^U$ on the standard Borel space (Ω,Σ). (For mathematically precise construction of specifications for random fields cf. [14, 15].)

Let $\Phi_\eta^{\partial\Lambda}$ be a solution of the Dirichlet problem (13) in a volume $\Lambda\in\mathcal{F}$. We can write (13) in the form of the Hammerstein equation

$$\Phi_\eta^{\partial\Lambda}(x) = \Psi_\eta^{\partial\Lambda}(x) - \int_\Lambda G^{\partial\Lambda}(x,y) U^{(1)}(\Phi_\eta^{\partial\Lambda}(y)) dy \equiv \Psi_\eta^{\partial\Lambda}(x) - G^{\partial\Lambda} * U^{(1)}(\Phi_\eta^{\partial\Lambda})(x). \tag{41}$$

(We will show in Sect. 6 that for interesting U Eq. (41) has a solution.)

Now let us change the integration variables in (39) as follows

$$\varphi = \tilde{\varphi} - G^{\partial\Lambda} * U^{(1)}(\Phi_\eta^{\partial\Lambda}), \tag{42}$$

then using the fact that

$$\frac{d\mu_{0,\beta}^{\partial\Lambda}(\tilde{\varphi} - G^{\partial\Lambda} * U^{(1)}(\Phi_\eta^{\partial\Lambda}))}{d\mu_{0,\beta}^{\partial\Lambda}(\tilde{\varphi})} = \exp\left(\beta\tilde{\varphi}(U^{(1)}(\Phi_\eta^{\partial\Lambda})) - \frac{\beta}{2}\|U^{(1)}(\Phi_\eta^{\partial\Lambda})\|_{-1,\partial\Lambda}^2\right), \tag{43}$$

we have

$$E^{\eta}_{\Lambda^c}(F)=\frac{\mu^{\partial\Lambda}_{0,\beta}e^{-\beta(U_\Lambda(\varphi+\Psi^{\partial\Lambda}_\eta-G^{\partial\Lambda}*U^{(1)}(\Phi^{\partial\Lambda}_\eta))-\varphi(U^{(1)}(\Phi^{\partial\Lambda}_\eta)))}F(\varphi+\Psi^{\partial\Lambda}_\eta-G^{\partial\Lambda}*U^{(1)}(\Phi^{\partial\Lambda}_\eta))}{\mu^{\partial\Lambda}_{0,\beta}e^{-\beta(U_\Lambda(\varphi+\Psi^{\partial\Lambda}_\eta-G^{\partial\Lambda}*U^{(1)}(\Phi^{\partial\Lambda}_\eta))-\varphi(U^{(1)}(\Phi^{\partial\Lambda}_\eta)))}}, \tag{44}$$

and using our assumption (41) we can write (44) in the form

$$E^{\eta}_{\Lambda^c}(F)=\frac{\mu^{\partial\Lambda}_{0,\beta}e^{-\beta\mathscr{V}_\Lambda(\varphi,\Phi^{\partial\Lambda}_\eta)}F(\varphi+\Phi^{\partial\Lambda}_\eta)}{\mu^{\partial\Lambda}_{0,\beta}e^{-\beta\mathscr{V}_\Lambda(\varphi,\Phi^{\partial\Lambda}_\eta)}}, \tag{45}$$

where we have written

$$\mathscr{V}_\Lambda(\varphi,\Phi^{\partial\Lambda}_\eta)\equiv\int_\Lambda :\mathscr{V}(\varphi,\Phi^{\partial\Lambda}_\eta)\,:_{0,\beta}(x)dx$$
$$:=\int_\Lambda :U(\varphi+\Phi^{\partial\Lambda}_\eta(x))-U(\Phi^{\partial\Lambda}_\eta(x))-\varphi(U^{(1)}(\Phi^{\partial\Lambda}_\eta(x)))\,:_{0,\beta}(x)dx\,. \tag{46}$$

In order to investigate the β dependence of (45) let us observe the identity

$$\mu^{\partial\Lambda}_{0,\beta}e^{i\varphi(f)}=\mu^{\partial\Lambda}_{0,\beta=1}e^{i\beta^{-1/2}\varphi(f)}\equiv\mu^{\partial\Lambda}_0e^{i\beta^{-1/2}\varphi(f)}, \tag{47}$$

which is the simple consequence of definition (21) of $\mu^{\partial\Lambda}_{0,\beta}$ for any $\beta>0$.

Using (47) we can write (45) in the form

$$E^{\eta}_{\Lambda^c}(F)=\frac{\mu^{\partial\Lambda}_0e^{-\beta\mathscr{V}_\Lambda(\beta^{-1/2}\varphi,\Phi^{\partial\Lambda}_\eta)}F(\beta^{-1/2}\varphi+\Phi^{\partial\Lambda}_\eta)}{\mu^{\partial\Lambda}_0e^{-\beta\mathscr{V}_\Lambda(\beta^{-1/2}\varphi,\Phi^{\partial\Lambda}_\eta)}}. \tag{48}$$

It is sufficient to consider the functions F of the form

$$F(\varphi)=e^{i\varphi(f)},\qquad f\in\mathscr{D}. \tag{49}$$

Then

$$F(\beta^{-1/2}\varphi+\Phi^{\partial\Lambda}_\eta)=(e^{i\beta^{-1/2}\varphi(f)})\,e^{i\Phi^{\partial\Lambda}_\eta(f)} \tag{50}$$

and because $\beta^{-1/2}\varphi(f)\xrightarrow[\beta\to\infty]{}0$ as the function on $\mathscr{D}'$ pointwise, so

$$F(\beta^{-1/2}\varphi+\Phi^{\partial\Lambda}_\eta)\xrightarrow[\beta\to\infty]{}F(\Phi^{\partial\Lambda}_\eta) \tag{51}$$

pointwise and so in $L_p(\mu^{\partial\Lambda}_0)$, for $1\leqq p<\infty$.

Now heuristically, we have

$$\beta\mathscr{V}_\Lambda(\beta^{-1/2}\varphi,\Phi^{\partial\Lambda}_\eta)=\tfrac{1}{2}:\varphi^2\,:_0(U^{(2)}(\Phi^{\partial\Lambda}_\eta))+\sum_{k>2}\frac{\beta^{-k/2+1}}{k!}:\varphi^k\,:_0(U^{(k)}(\Phi^{\partial\Lambda}_\eta))$$
$$\xrightarrow[\beta\to\infty]{}\tfrac{1}{2}:\varphi^2\,:_0(U^{(2)}(\Phi^{\partial\Lambda}_\eta)), \tag{52}$$

hence we can expect, that at least in $L_1(\mu^{\partial\Lambda}_0)$

$$e^{-\beta\mathscr{V}_\Lambda(\beta^{-1/2}\varphi,\Phi^{\partial\Lambda}_\eta)}\xrightarrow[\beta\to\infty]{}e^{-\frac{1}{2}:\varphi^2:_0(U^{(2)}(\Phi^{\partial\Lambda}_\eta))} \tag{53}$$

if the right-hand side is $\mu^{\partial\Lambda}_0$ integrable.

Combining the relations (48)–(53) we can conclude that (writing $E^{\eta}_{\Lambda^c,\beta}$ for $E^{\eta}_{\Lambda^c}$ to show explicitly the dependence on β):

$$E^{\eta}_{\Lambda^c,\beta}(F)\xrightarrow[\beta\to\infty]{}\delta_{\Phi^{\partial\Lambda}_\eta}(F)\,. \tag{54}$$

From that we can write that

$$\mathscr{E}_\beta^U \xrightarrow[\beta \to \infty]{} \mathscr{E}^U, \tag{55}$$

i.e. in the limit as the temperature β^{-1} goes to zero the local specification $\mathscr{E}_\beta^U$ with an interaction U converges to the ground local specification $\mathscr{E}^U$ (for the same interaction U).

We will investigate now the limits (52) and (53) in the particular models. We will see that (52) is easy and we have convergence in $L_p(\mu_0^{\partial\Lambda})$ with at least $p=2$ (and arbitrary $1 \leqq p < \infty$ in the polynomial case). The harder, but more interesting case is (53). We will see that there are polynomial interactions, for which this limit cannot exist in the form (54) for all boundary conditions (as should be expected from the presence of phase transitions).

The Case of Polynomial Interactions. Let

$$U(q) = \lambda \sum_{k=1}^{2n} a_k q^k \equiv \lambda P(q), \qquad a_{2n} > 0, \qquad \lambda > 0. \tag{56}$$

In order to prove that

$$\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda}) \xrightarrow[\beta \to \infty]{} \tfrac{1}{2} : \varphi^2 :_0 (U^{(2)}(\Phi_\eta^{\partial\Lambda})) \tag{57}$$

in $L_p(\mu_0^{\partial\Lambda})$, for any $1 \leqq p < \infty$, using representation of polynomial as in (52), we need only to show that

$$\left\| : \varphi^k :_0 (U^{(k)}(\Phi_\eta^{\partial\Lambda})) \right\|_2 < \infty \tag{58}$$

for all $1 \leqq k \leqq 2n$ (it follows from hypercontractive estimates, see e.g. [6]). But by our assumption (41) (we will verify this in Sect. 6) the functions $U^{(k)}(\Phi_\eta^{\partial\Lambda}(x))$, $1 \leqq k \leqq 2n$, are in $L_r(\Lambda, dx)$ with $1 < r < \infty$, hence (58) holds (e.g. [6]).

Now let us consider the convergence (53). Because of (57) we need only to show that

$$\mu_0^{\partial\Lambda} e^{-p\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda})} \leqq C < \infty \tag{59}$$

for a constant $C > 0$ independent of $\beta \geqq 1$, and some $p > 1$. However we cannot expect that (59) will be fulfilled in general since

$$\mu_0^{\partial\Lambda} e^{-\frac{1}{2} : \varphi^2 :_0 (U^{(2)}(\Phi_\eta^{\partial\Lambda}))}$$

cannot be finite for an arbitrary semibounded polynomial $P(q)$ and an arbitrary boundary condition η. We now show that if

$$\lambda P^{(2)}(q) > -m^2, \tag{60}$$

then (59) holds independently of $\Phi_\eta^{\partial\Lambda}$. [Let us note that (60) can be fulfilled for any semibounded polynomial if λ is sufficiently small.]

Let $\tilde{m}^2 \equiv m^2 - 2\varepsilon > 0$ (with some $0 < \varepsilon < 1$) be such that

$$U^{(2)}(q) = \lambda P^{(2)}(q) > -\tilde{m}^2. \tag{61}$$

Since

$$\mu_0^{\partial\Lambda} \exp(r\tfrac{1}{2}(\tilde{m}^2 + \varepsilon) : \varphi^2 :_0 (\chi_\Lambda)) < \infty \tag{62}$$

for $r \geqq 1$, $r(\tilde{m}^2+\varepsilon)<m^2$, so it is sufficient to prove that for any $s \geqq 1$, there is a constant C_1 independent of $\beta \geqq 1$ such that

$$\mu_0^{\partial\Lambda} \exp(-s\beta v_{\varepsilon,\Lambda}(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda})) < C_1 < \infty\,, \tag{63}$$

where

$$v_\varepsilon(q,p) := \mathscr{V}(q,p) + \tfrac{1}{2}(\tilde{m}^2+\varepsilon)q^2\,. \tag{64}$$

Let us denote $\varphi_\kappa(x) := \varphi(h_\kappa(\cdot - x))$ with $\hat{h}_\kappa \in \mathscr{D}'$, $\hat{h}_\kappa(z)=1$ for $z \leqq \kappa$ and $\hat{h}_\kappa \xrightarrow[\beta\to\infty]{} 1$, and $c_\kappa := \mu_0 \varphi_\kappa(x)^2$. For simplicity let $N \equiv 2n = \deg P(q)$. We need the following lemma:

Lemma 1. *There are positive constants γ, $D(\Phi_\eta^{\partial\Lambda})$, $C(\Phi_\eta^{\partial\Lambda})$ all independent of $\beta \geqq 1$ such that*

$$\beta v_{\varepsilon,\Lambda}(\beta^{-1/2}\varphi_\kappa, \Phi_\eta^{\partial\Lambda}) > -D(\Phi_\eta^{\partial\Lambda})c_\kappa^{N/2}\,, \tag{65}$$

and for any $s \geqq 2$

$$\|\beta v_{\varepsilon,\Lambda}(\beta^{-1/2}\varphi_\kappa, \Phi_\eta^{\partial\Lambda}) - \beta v_{\varepsilon,\Lambda}(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda})\|_{L_s(\mu_0^{\partial\Lambda})} \leqq C(\Phi_\eta^{\partial\Lambda}) s^{N/2} \cdot \kappa^{-\gamma}\,. \tag{66}$$

From this lemma the bound (63) follows by Nelson's original arguments (see [11]). This ends the proof of (59) in the case of polynomial interactions.

Proof of Lemma 1. The second statement easily follows from the fact that $v_\varepsilon(q,p)$ does not contain terms of degree less than two and the fact that $\Phi_\eta^{\partial\Lambda}(x) \in L_p(\Lambda, dx)$ for $\Lambda \in \mathscr{F}$ and any $1 \leqq p < \infty$ (see Sect. 6). The proof of the first statement is based on the ideas of [3]. Under the assumption (61), we have that the function

$$v_0 := v_\varepsilon(q,p)_{\varepsilon\equiv 0} \equiv \sum_{k=2}^{N} a_k q^k \tag{67}$$

with $a_k = a_k(p)$, fulfills

$$v_0(q,p)_{q=0} = 0\,, \qquad \frac{d}{dq} v_0(q,p)_{q=0} = 0\,, \qquad \frac{d^2}{dq^2} v_0(q,p) = U^{(2)}(q+p) + m^2 > 0\,. \tag{68}$$

Writing

$$v_0(q,p) = \frac{1}{N-2} a_N q^N + \sum_{k=3}^{N-1} \left(\frac{1}{N-2} a_N q^N + a_k q^k \right) + a_2 q^2\,, \tag{69}$$

and observing that

$$\frac{1}{N-2} a_N q^N + a_k q^k \geqq |q|^k \tag{70}$$

for

$$|q| \geqq \frac{N-2}{a_N} (|a_k|+1)^{\frac{1}{N-k}}\,, \tag{71}$$

we have

$$v_0(q,p) = \frac{1}{N-2} a_N q^N + \sum_{k=3}^{N-1} |q|^k + a_2 q^2 \tag{72}$$

for

$$|q| \geqq \max_{3 \leqq k \leqq N-1} (N-2) \frac{1}{a_N} (|a_k|+1)^{\frac{1}{N-k}} \equiv C(p) \geqq 1 . \tag{73}$$

On the other hand, for $|q| < C(p)$, we have

$$\tfrac{1}{2}\varepsilon q^2 \geqq \tfrac{1}{2} \frac{\varepsilon}{C(p)^{k-2}} |q|^k , \qquad 3 \leqq k \leqq N . \tag{74}$$

Hence for any $q \in R$ we have

$$v_\varepsilon(q, p) \geqq \sum_{k=2}^{N} b_k |q|^k , \tag{75}$$

with

$$b_N = a_N , \qquad b_2 = a_2 , \qquad b_k = \frac{\varepsilon}{2(N-3)C(p)^{k-2}} , \qquad 3 \leqq k < N . \tag{76}$$

Let now $q = \beta^{-1/2} \varphi_\kappa(x)$, then we have

$$\beta : v_\varepsilon(q, p) \underset{0}{:} = \beta \left(v_\varepsilon(q, p) + \left(\sum_{k=2}^{N} a_k \sum_{j=1}^{[k/2]} \alpha_{k,j} \beta^{-j} c_\kappa^j q^{k-2j} \right) - \frac{\varepsilon}{2} \beta^{-1} c_\kappa \right) \tag{77}$$

with $\alpha_{k,j}$ some combinatorial factors. From that, using (75), we obtain

$$\beta : v_\varepsilon(\beta^{-1/2} \varphi_\kappa, p) \underset{0}{:} \geqq \beta \sum_{k=2}^{N} \beta^{-k/2} \left(b_k |\varphi_\kappa|^k + a_k \sum_{j=1}^{[k/2]} \alpha_{k,j} c_\kappa^j \varphi_\kappa^{k-2j} \right) - \tfrac{1}{2} \varepsilon c_\kappa . \tag{78}$$

We have also

$$b_k |\varphi_\kappa|^k + a_k \sum_{j=1}^{[k/2]} \alpha_{k,j} c_\kappa^j \varphi_\kappa^{k-2j} \geqq -D \max_{k,j} \left\{ \frac{|a_k|^{\frac{k}{2j}}}{(b_k)^{\frac{k-2j}{2j}}} \right\} (c_\kappa + 1)^{N/2} \tag{79}$$

with D a positive constant independent of k and c_κ.

Since from definition (67) $|a_k| = \mathcal{O}((|p|+1)^{N-k})$, and from (73) $C(p) = \mathcal{O}(|p|+1)$, so using (76) we have

$$\max_{k,j} \frac{|a_k|^{\frac{k}{2j}}}{(b_k)^{\frac{k-2j}{2j}}} \leqq D_1 (|p|+1)^{\frac{(N-2)^2}{2}} , \qquad N \geqq 4 \tag{80}$$

$\left(\text{where } 2 \leqq k \leqq N \text{ and } 1 \leqq j \leqq \left[\frac{k}{2} \right] \text{ and } D_1 \text{ is a positive constant} \right)$. Combining (78)–(80) with $p = \Phi_\eta^{\partial \Lambda}(x)$ and $\beta \geqq 1$, we obtain

$$\beta v_{\varepsilon, \Lambda}(\beta^{-1/2} \varphi_\kappa, \Phi_\eta^{\partial \Lambda}) \geqq -N D_1 \int_\Lambda (|\Phi_\eta^{\partial \Lambda}(x)| + 1)^{\frac{(N-2)^2}{2}} dx \cdot c_\kappa^{N/2} . \tag{81}$$

Here it is important that the left-hand side of (81) is independent of $\beta \geqq 1$ and finite, since $\Phi_\eta^{\partial \Lambda}(x) \in L_p(\Lambda, dx)$ for any $1 \leqq p < \infty$. $\square$

The Case of Trigonometric Interactions. Let

$$U(q)=\lambda\int\cos(\alpha q+\vartheta(\alpha))d\varrho(\alpha) \tag{82}$$

with $d\varrho(\alpha)$ a finite measure supported in $(-2\sqrt{\pi},2\sqrt{\pi})$. We have

$$\beta\mathcal{V}_\Lambda(\beta^{-1/2}\varphi,\Phi_\eta^{\partial\Lambda})=\tfrac{1}{2}:\varphi^2:_0(U^{(2)}(\Phi_\eta^{\partial\Lambda}))+\sum_{k>2}\frac{\beta^{-k/2+1}}{k!}:\varphi^k:_0(U^{(k)}(\Phi_\eta^{\partial\Lambda})), \tag{83}$$

where the series converges in any $L_p(\mu_0^{\partial\Lambda})$, $1\leqq p<\infty$ for sufficiently big $\beta\geqq 1$. Hence one can see that (83) goes in $L_p(\mu_0^{\partial\Lambda})$ (for any $1\leqq p<\infty$) as $\beta\to\infty$ to $\frac{1}{2}:\varphi^2:_0(U^{(2)}(\Phi_\eta^{\partial\Lambda}))$.

As before, in order to prove (53) in the case under consideration it is sufficient to prove the bound

$$\mu_0^{\partial\Lambda}e^{-r\beta\mathcal{V}_\Lambda(\beta^{-\frac{1}{2}}\varphi,\Phi_\eta^{\partial\Lambda})}<C<\infty \tag{84}$$

for some $r>1$ and a positive constant independent of $\beta\geqq 1$. We need to assume that

$$\lambda\int d|\varrho(\alpha)|\alpha^2<\tilde{m}^2<m^2, \tag{85}$$

because only in this case

$$\mu_0^{\partial\Lambda}e^{-\frac{1}{2}:\varphi^2:_0(U^{(2)}(\Phi_\eta^{\partial\Lambda}))}<\infty. \tag{86}$$

Since

$$\mu_0^{\partial\Lambda}e^{r\frac{1}{2}\tilde{m}^2:\varphi^2:_0(\chi_\Lambda)}<\infty \tag{87}$$

for any $r\geqq 1$, $r\cdot\tilde{m}^2<m^2$, so for (84) it is sufficient to prove that

$$\mu_0^{\partial\Lambda}e^{-s\beta v_\Lambda(\beta^{-1/2}\varphi,\Phi_\eta^{\partial\Lambda})}<C<\infty, \tag{88}$$

where

$$\beta v_\Lambda(\beta^{-1/2},\Phi_\eta^{\partial\Lambda}):=\beta\mathcal{V}_\Lambda(\beta^{-1/2}\varphi,\Phi_\eta^{\partial\Lambda})+\tfrac{1}{2}\tilde{m}^2:\varphi^2:_0(\chi_\Lambda), \tag{89}$$

and $s=\dfrac{r}{r-1}$, and a constant $C>0$ is independent of $\beta\geqq 1$. The estimation (88), and so the convergence (53), follows by [10, Theorem 3] with the use of the following lemma:

Lemma 2. *Under the condition* (85) *we have*

$$\beta v_\Lambda(\beta^{-1/2}\varphi_\kappa,\Phi_\eta^{\partial\Lambda})\geqq-\tfrac{1}{2}\tilde{m}^2|\Lambda|c_\kappa-\tfrac{1}{2}\lambda\int d|\varrho(\alpha)|\alpha^2|\Lambda|c_\kappa(1+e^{+\beta^{-1}\alpha^2c_\kappa/2}), \tag{90}$$

and $\forall 1\leqq p<\infty\ \exists\beta_0\geqq 1\ \forall\beta\geqq\beta_0$,

$$\|\beta v_\Lambda(\beta^{-1/2}\varphi_\kappa,\Phi_\eta^{\partial\Lambda})-\beta v_\Lambda(\beta^{-1/2}\varphi,\Phi_\eta^{\partial\Lambda})\|_{L_p(\mu_0^{\partial\Lambda})}<C(\Phi_\eta^{\partial\Lambda})p^\delta\kappa^{-\gamma} \tag{91}$$

with the positive constants $C(\Phi_\eta^{\partial\Lambda})$, δ *and* γ *all independent of* $\beta\geqq\beta_0$. □

Remark. On the right-hand side of (90) we have the lower bound, which contains the term proportional to $(\ln\kappa)\kappa^z$ with $z=\dfrac{\beta^{-1}\alpha^2}{4}$. Since $\beta\to\infty$, so $z\to 0$. This, together with the fact that $C(\Phi_\eta^{\partial\Lambda})$, δ and γ on the right-hand side of (91) are independent of $\beta \geqq \beta_0$, makes possible to use the arguments of [10, Theorem 3]. □

Proof. Under the condition (85), the function $v(q,p)$ has the properties (68), and so

$$v(q,p) \geqq 0 . \tag{92}$$

Since

$$:\beta v(\beta^{-1/2}\varphi_\kappa(x), \phi_\eta^{\partial\Lambda}(x)):_0 = \beta v(\beta^{-1/2}\varphi_\kappa(x), \phi_\eta^{\partial\Lambda}(x))$$
$$+\beta\lambda \int d\varrho(\alpha)\,(e^{\frac{\beta^{-1}\alpha^2 c_\kappa}{2}} - 1)\cos(\beta^{-1/2}\alpha\varphi_\kappa(x) + \alpha\Phi_\eta^{\partial\Lambda}(x) + \vartheta(\alpha)) - \tfrac{1}{2}\tilde{m}^2 c_\kappa , \tag{93}$$

so using (92), we have

$$:\beta v(\beta^{-1/2}\varphi_\kappa(x), \Phi_\eta^{\partial\Lambda}(x)):_0 \geqq -\tfrac{1}{2}\tilde{m}^2 c_\kappa - \tfrac{1}{2}\lambda \int d|\varrho(\alpha)|\alpha^2 c_\kappa (e^{\frac{\beta^{-1}\alpha^2 c_\kappa}{2}} + 1) . \tag{94}$$

Hence by integration in volume Λ we get (90).

In order to prove (91) we use the elementary formulas

$$\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda}) = \beta \int_0^1 ds_1 \int_0^{s_1} ds_2 \frac{d^2}{ds_2^2} U_\Lambda(\beta^{-1/2}s_2\varphi + \Phi_\eta^{\partial\Lambda}) , \tag{95}$$

and

$$\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda}) - \beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi_\kappa, \Phi_\eta^{\partial\Lambda}) = \int_0^1 dt \frac{d}{dt} \beta \mathscr{V}_\Lambda(\beta^{-1/2}(t\varphi + (1-t)\varphi_\kappa), \Phi_\eta^{\partial\Lambda} \tag{96}$$

together with the fact that the integrations as well as differentiations with respect to s_i and t can be interchanged with integration with the measure $\mu_0^{\partial\Lambda}$, as well as the fact that the integral with the measure $\mu_0^{\partial\Lambda}$ of the products of $U_\Lambda(\beta^{-1/2}s_i\varphi + \Phi_\eta^{\partial\Lambda})$ can be explicitly computed. The proof can be carried out exactly as the proof of Theorem 3.1 in [10]. □

The Case of Exponential Interactions. Let

$$U(q) = \lambda \int d\varrho(\alpha) e^{\alpha q} \tag{97}$$

with $\lambda > 0$ and $d\varrho(\alpha)$ a probability measure supported in $(-2\sqrt{\pi}, 2\sqrt{\pi})$. The convergence (52) can be proven (as in the trigonometric case) in any $L_p(\mu_0^{\partial\Lambda})$, $1 \leqq p < \infty$ (see Lemma 3 below). Note that now we have

$$U^{(2)}(\Phi_\eta^{\partial\Lambda}(x)) = \lambda \int d\varrho(\alpha)\alpha^2 e^{\alpha\Phi_\eta^{\partial\Lambda}(x)} > 0 . \tag{98}$$

We need only to prove that

$$\mu_0^{\partial\Lambda} e^{-2\beta\mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda})} < C < \infty \tag{99}$$

with a constant $C > 0$ independent of $\beta \geqq 1$. This inequality follows, by use – as in [4, p. 394] – of Nelson's original arguments [11], from the following lemma:

Lemma 3. *For interaction* (97) *we have*

$$\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi_\kappa, \Phi_\eta^{\partial\Lambda}) \geqq -(\tfrac{1}{2}\lambda \int_\Lambda dx\, d\varrho(\alpha)\alpha^2 e^{\alpha\Phi_\eta^{\partial\Lambda}(x)}) c_\kappa , \tag{100}$$

and for any $1 \leqq p < \infty$ *there is* $\beta_p \geqq 1$ *such that for any* $\beta \geqq \beta_p$,

$$\mu_0^{\partial\Lambda} |\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi_\kappa, \Phi_\eta^{\partial\Lambda}) - \beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda})|^p \leqq C(\Phi_\eta^{\partial\Lambda}) \kappa^{-p\gamma}, \tag{101}$$

with the positive constants $C(\Phi_\eta^{\partial\Lambda})$, γ *independent of* $\beta \geqq \beta_p$. □

Proof. Since for any $q \in R$,

$$e^q - 1 - q \geqq 0, \tag{102}$$

so with

$$q = \alpha\beta^{-1/2}\varphi_\kappa(x) - \frac{\beta^{-1}\alpha^2 c_\kappa}{2}, \tag{103}$$

we have

$$:\beta \mathscr{V}(\beta^{-1/2}\varphi_\kappa(x), \Phi_\eta^{\partial\Lambda}(x)):_0 = \beta\lambda \int d\varrho(\alpha) e^{\alpha\Phi_\eta^{\partial\Lambda}(x)}\Big(e^{-\frac{\beta^{-1}\alpha^2 c_\kappa}{2}} e^{\beta^{-1/2}\alpha\varphi_\kappa(x)}$$
$$+1 - \beta^{-1/2}\alpha\varphi(x)\Big) = \beta\lambda \int d\varrho(\alpha) e^{\alpha\Phi_\eta^{\partial\Lambda}(x)}(e^q - 1 - q) - \left(\tfrac{1}{2}\lambda \int d\varrho(\alpha) e^{\alpha\Phi_\eta^{\partial\Lambda}(x)}\alpha^2\right) c_\kappa$$
$$\geqq -\left(\tfrac{1}{2}\lambda \int d\varrho(\alpha) e^{\alpha\Phi_\eta^{\partial\Lambda}(x)}\alpha^2\right) c_\kappa\,. \tag{104}$$

After integration over $x \in \Lambda$, we get (100).

Let us now prove (101). We have

$$\mu_0^{\partial\Lambda} |\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi_\kappa, \Phi_\eta^{\partial\Lambda}) - \beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda})|^p$$
$$= \mu_0^{\partial\Lambda} \left| \sum_{n\geqq 2} \frac{\beta^{-n/2+1}}{n!} :\varphi_n^n - \varphi^n:_0 (U^{(n)}(\Phi_\eta^{\partial\Lambda})) \right|^p$$
$$\leqq \left(\sum_{n\geqq 2} \frac{\beta^{-n/2+1}}{n!} (p-1)^{n/2} \left\| :\varphi_\kappa^n - \varphi^n:_0 (U^{(n)}(\Phi_\eta^{\partial\Lambda})) \right\|_2 \right)^p, \tag{105}$$

where in the last step we used the triangle inequality together with hypercontractive estimates. Using

$$|U^{(n)}(\Phi_\eta^{\partial\Lambda}(x))| \leqq (2\sqrt{\pi})^n U(\Phi_\eta^{\partial\Lambda}(x)), \tag{106}$$

we have (e.g. [6]), that

$$\left\| :\varphi_\kappa^n - \varphi^n:_0 (U^{(n)}(\Phi_\eta^{\partial\Lambda})) \right\|_2 \leqq c^n n! \, \|U(\Phi_\eta^{\partial\Lambda})\|_{L_s(dx)} \kappa^{-\gamma} \tag{107}$$

with the positive constants c, and some $s > 1$. From (105) and (107) we get for all $\beta \geqq \beta_p > pc^2$:

$$\mu_0^{\partial\Lambda} |\beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi_\kappa, \Phi_\eta^{\partial\Lambda}) - \beta \mathscr{V}_\Lambda(\beta^{-1/2}\varphi, \Phi_\eta^{\partial\Lambda})|^p \leqq (C(\beta_p) \, \|U(\Phi_\eta^{\partial\Lambda})\|_{L_s(dx)})^p \kappa^{-p\gamma} \tag{108}$$

with a constant $C(\beta_p) > 0$. This ends the proof of (101). □

We end the proof of convergence (55) of the sequences of local specifications for Euclidean fields with interactions given by the real $\mathscr{C}^2$ functions U. In each case we assumed that

$$U^{(2)}(q) > -m^2\,. \tag{109}$$

As we will see in Sect. 6 this condition is also connected with uniqueness of the solution $\Phi_\eta^{\partial\Lambda}$ for a given boundary condition η.

Remark. Let us also note that under the condition (109) it is easy to prove the analogue, as in the present paragraph, (as well as the other) results in the case of Euclidean fields on the lattice $\mathbb{Z}^d$ for any $d \geqq 1$ (see [21]).

We summarize these ideas in

Proposition 4. *In two dimensional Euclidean space the local specifications $\mathscr{E}_\beta^U$ of Euclidean Field Theory are homotopically equivalent to corresponding ground specification $\mathscr{E}^U$ of Classical Euclidean Field Theory.* □

Remark. Here the homotopic equivalence means that there is a path in the space of specifications (defined by the continuous paths (54)), which connects $\mathscr{E}_\beta^U$ and $\mathscr{E}^U$. We see that the Markov property of specifications is a homotopic invariant in the considered cases.

5. Theory of Gibbs Measures and Euclidean Field Theory

It is known (see [16, Sect. 4.3]), that any probability measure μ on the space $(\mathscr{D}', \mathscr{B})$ is a Gibbs measure, i.e. there is a local specification $\mathscr{E}$ on $(\mathscr{D}', \mathscr{B})$ such, that $\mu \in \mathscr{G}(\mathscr{E})$.

From our considerations in preceding sections, it follows, that the following definition is natural:

Definition 1. A field theory defined in Euclidean region by a probability measure μ on the space of distributions $(\mathscr{D}', \mathscr{B})$ is associated to the Classical Euclidean Field Theory with local interaction U if and only if $\mu \in \mathscr{G}(\mathscr{E}_\beta^U)$ for some $\beta > 0$, and

$$\mathscr{E}_\beta^U \xrightarrow[\beta \to \infty]{} \mathscr{E}^U,$$

where $\mathscr{E}^U$ is the ground specification for U. □

Remark. It is expected that from the existent theories also φ_3^4 fulfills this definition with adequate families $\mathscr{E}_\beta^U$ and $\mathscr{E}^U$. □

Let us note that à priori there can exist totally different families $\mathscr{E}_\beta^U$ homotopically equivalent to the same ground specification $\mathscr{E}^U$. This is connected with the analogous phenomenon as in classical mechanics, where there can exist nontrivially different Lagrangians (and so Hamiltonians and other conserved quantities), which give the same equations of motion. This fact – interesting in itself – can also have important application in the construction of a nontrivial model of Field Theory.

If we accept Definition 1, then the first question on the way to finding any field theory representable by a probability measure is: does there exist any ground specification on $(\mathscr{D}', \mathscr{B})$? Note that we have defined a free ground specification $\mathscr{E}^0$ or equivalently the set of solutions $\Psi_\eta^{\partial A}$ of the Dirichlet problems for free field equation (19) for a set $\Omega \subset \mathscr{D}'$ of boundary dates η, such that $\mathscr{C}^\infty \subset \Omega$ and $\mu(\Omega) = 1$ for any regular probability measure μ on $(\mathscr{D}', \mathscr{B})$. [A more restrictive definition of the regularity of measure than (35) is needed in higher dimensions.] If we want to give an answer to the above stated question given $\mathscr{E}^0$ we should consider the Hammerstein equation

$$\Phi_\eta^{\partial A}(x) = \Psi_\eta^{\partial A}(x) - \int G^{\partial A}(x, y)\, U^{(1)}(\Phi_\eta^{\partial A}(y))\, dy\,.$$

We will discuss this equation in the next sections.

6. The Dirichlet Problems for Equations of Classical Euclidean Field Theory

Let the dimension of Euclidean space be $d=2$. (The case $d>2$ will be discussed elsewhere.)

Let $\Lambda \in \mathscr{F}$. Let $\Psi_\eta^{\partial\Lambda}(x)$ be a unique solution of the Dirichlet problem in Λ for the free Euclidean field theory:

$$(-\Delta+m^2)\Psi_\eta^{\partial\Lambda}(x)=0\,, \qquad x\in\Lambda\,; \qquad \Psi_\eta^{\partial\Lambda}(x)=\eta(x)\,, \qquad x\in\Lambda^c \tag{110}$$

with $\eta\in\mathscr{D}'$. Recall that the function

$$\mathscr{D}'\times\Lambda\ni(\eta,x)\to\Psi_\eta^{\partial\Lambda}(x) \tag{111}$$

is well defined for μ-a.a. $\eta\in\mathscr{D}'$ for any regular probability measure μ on $(\mathscr{D}',\mathscr{B})$ and

$$\|\,\|\Psi_\eta^{\partial\Lambda}(x)\|_{L_s(\Lambda,dx)}\|_{L_r(\mu)}<\infty \tag{112}$$

for any $1\leqq r,s<\infty$.

In this paragraph we consider the Dirichlet problems in $\Lambda\in\mathscr{F}$ for the classical Euclidean field theory with interaction:

$$(-\Delta+m^2)\Phi_\eta^{\partial\Lambda}(x)+U^{(1)}(\Phi_\eta^{\partial\Lambda}(x))=0\,, \qquad x\in\Lambda\,; \qquad \Phi_\eta^{\partial\Lambda}(x)=\eta(x)\,, \qquad x\in\Lambda^c\,, \tag{113}$$

where U is a real at least $\mathscr{C}^2$ function bounded from below. We consider (113) in the sense that for a given solution $\Psi_\eta^{\partial\Lambda}$ of (110), we look for $\Phi_\eta^{\partial\Lambda}$ such that

$$\xi_\eta^{\partial\Lambda}(x)\equiv\Phi_\eta^{\partial\Lambda}(x)-\Psi_\eta^{\partial\Lambda}(x) \tag{114}$$

is a function on $\mathbb{R}^2$ identically equal to zero on Λ^c and $\mathscr{C}^\infty$ on Λ.

We will be interested in the case, when $U^{(1)}(\Phi_\eta^{\partial\Lambda}(x))\in L_2(\Lambda,dx)$. Then (113) is equivalent to the following Hammerstein equation:

$$\Phi_\eta^{\partial\Lambda}(x)=\Psi_\eta^{\partial\Lambda}(x)-\int_\Lambda G^{\partial\Lambda}(x,y)\,U^{(1)}(\Phi_\eta^{\partial\Lambda}(y))\,dy\,, \tag{115}$$

we can write (115) in the form

$$\xi_\eta^{\partial\Lambda}(x)=-\int_\Lambda G^{\partial\Lambda}(x,y)\,U^{(1)}(\xi_\eta^{\partial\Lambda}(y)+\Psi_\eta^{\partial\Lambda}(y))\,dy\equiv T(\xi_\eta^{\partial\Lambda})\,(x)\,. \tag{116}$$

(Note that the operator T depends on $\Psi_\eta^{\partial\Lambda}$ as also on other parameters of U. If it will be needed, we will denote this dependence explicitly.)

Before solving (116), let us make some remarks on the uniqueness of its solution. Suppose that there exist two solutions $\xi_\eta^{\partial\Lambda}$ and $\zeta_\eta^{\partial\Lambda}$ of (116). Then their difference

$$\delta\xi:=\xi_\eta^{\partial\Lambda}-\zeta_\eta^{\partial\Lambda} \tag{117}$$

fulfills the equation

$$\left((-\Delta+m^2)+\int_0^1 ds\,U^{(2)}(s\xi_\eta^{\partial\Lambda}(x)+(1-s)\zeta_\eta^{\partial\Lambda}(x)+\Psi_\eta^{\partial\Lambda}(x))\right)\delta\xi(x)\equiv L_\eta\delta\xi(x)=0. \tag{118}$$

From that we see that if

$$\forall q\in R\,, \qquad U^{(2)}(q)>-m^2\,, \tag{119}$$

then $\delta\xi(x)=0$, since in this case the selfadjoint linear operator L_η in $L_2(\Lambda,dx)$ has no zero as an eigenvalue.

Now we will show in each interesting case of U separately, that a solution of (113) exists.

The Case of Trigonometric Interactions. Let

$$U(q)=\lambda\int d\varrho(\alpha)\cos(\alpha q+\vartheta(\alpha)) \tag{120}$$

with $\lambda>0$ and $d\varrho(\alpha)$ a finite measure (not necessarily nonnegative). Since $U^{(1)}$ in the considered case is a bounded function, so a solution $\xi_\eta^{\partial\Lambda}$ of (115) if one exists, is a bounded continuous function. Therefore we can consider (115) in the space of bounded continuous functions $\mathscr{C}(\bar{\Lambda})$ on $\bar{\Lambda}$. Since the continuous function

$$\xi\to T(\xi) \tag{121}$$

maps $\mathscr{C}(\bar{\Lambda})$ into a bounded convex subset of equicontinuous functions (what follows from the definition of T in (116) and properties of U given by (120)), so from Schauder fixed point theorem (e.g. [18, Theorem 4.1.1]), we conclude that (116) and so (115) and (113) has a solution. We have the uniqueness of the solution under the condition

$$\lambda\int\alpha^2 d|\varrho(\alpha)|<m^2. \tag{122}$$

The Case of Exponential Interactions. Let now

$$U(q)=\lambda\int d\varrho(\alpha)e^{\alpha q} \tag{123}$$

with $\lambda>0$ and $d\varrho(\alpha)$ a probability measure supported in $(-2\sqrt{\pi},2\sqrt{\pi})$. [We assume that $d\varrho(\alpha)$ has no positive mass at the point $\alpha=0$.)]

We are interested in such a solution of (115) for which for every regular measure μ,

$$\|\,\|U(\Phi_\eta^{\partial\Lambda}(x))\|_{L_2(\Lambda,dx)}\|_{L_2(\mu)}<\infty. \tag{124}$$

In this case (116) is equivalent to (113). Let us first consider the case when $d\varrho(\alpha)$ is supported in a halfline, i.e. in $[0.2\sqrt{\pi})$ or $(-2\sqrt{\pi},0]$, since we will use the same methods in analysis of global aspects. We consider only the case $\operatorname{supp} d\varrho(\alpha)\subset[0,2\sqrt{\pi})$, because the second case is almost the same. Now under the fixed conditions, we have that a solution $\xi_\eta^{\partial\Lambda}$ of (115) – if it exists – is nonpositive and bounded as follows

$$-\int_\Lambda G^{\partial\Lambda}(x,y)U^{(1)}(\Psi_\eta^{\partial\Lambda}(y))dy\leqq\xi_\eta^{\partial\Lambda}(x)\leqq 0. \tag{125}$$

[This is the consequence of properties of $G^{\partial\Lambda}(x,y)$ and U.] Moreover we see that the bounded closed and convex set of functions which fulfill (125) is mapped by T [defined in (116)] into itself. Hence by analogous arguments as in the trigonometric case based on the Schauder's fixed point theorem, we have the existence of solution $\xi_\eta^{\partial\Lambda}$ of (115). The uniqueness, for each $\lambda>0$, in the case of U given by (123) follows from the fact that $U^{(2)}(q)>0$.

The case of general exponential interaction will be considered together with a polynomial case in the following point:

The Case of Interactions Monotonous at Infinity. For simplicity let us assume that

$$U^{(2)}(q)>-\tilde{m}^2>-m^2, \tag{126}$$

hence we may and do assume that $U^{(2)}$ is nonnegative. [Because if (126) is fulfilled we can redefine interaction by taking $U^{(1)}+m^2q$ instead of $U^{(1)}$ and the mass $m^2-\tilde{m}^2>0$ instead of m^2 in (113). By this manner we can obtain an analogous equation as (113), but now with an increasing first derivative of interaction.]

Using the notation (114), let us write our Dirichlet problem (113) in the form

$$(-\Delta+m^2)\xi_\eta^{\partial\Lambda}(x)+F(\xi_\eta^{\partial\Lambda})(x)=f_\eta^{\partial\Lambda}, \quad x\in\Lambda; \quad \xi_\eta^{\partial\Lambda}(x)=0, \quad x\in\Lambda^c \tag{127}$$

with

$$F(\xi)(x):=U^{(1)}(\xi(x)+\Psi_\eta^{\partial\Lambda}(x))-U^{(1)}(\Psi_\eta^{\partial\Lambda}(x)) \tag{128}$$

and

$$f_\eta^{\partial\Lambda}(x):=-U^{(1)}(\Psi_\eta^{\partial\Lambda}(x)). \tag{129}$$

Since we have assumed that $U^{(1)}$ is nondecreasing, so for any x and any real function $\xi(x)$, we have

$$\xi(x)F(\xi)(x)\geqq 0. \tag{130}$$

Moreover since we assumed that

$$U^{(1)}(\Psi_\eta^{\partial\Lambda}(x))\in L_s(\Lambda,dx), \quad \mu\text{-a.e.} \tag{131}$$

for some $s>1$ and any $\mu\in\mathscr{M}_r$, so we have

$$\|f_\eta^{\partial\Lambda}(\cdot)\|_{-1,\partial\Lambda}<\infty. \tag{132}$$

Hence we can prove the existence of a weak solution of (113) (in the sense that $F(\xi_\eta^{\partial\Lambda})(x)\in L_{1,\mathrm{loc}}(\Lambda)$ and (113) is satisfied in the sense of distributions on $\mathscr{C}_0^\infty(\Lambda)$ exactly by the method of [13] (proof of Theorem 1). In fact in the interesting cases we have $U\in\mathscr{C}^\infty$, and since from [14] the functions $\Psi_\eta^{\partial\Lambda}(x)$ are harmonic in Λ (μ-a.e. for any $\mu\in\mathscr{M}_r$), so $f_\eta^{\partial\Lambda}(x)\in\mathscr{C}^\infty(\Lambda)$. Hence and from the ellipticity of our problem we have $\xi_\eta^{\partial\Lambda}\in\mathscr{C}^\infty(\Lambda)$, and so $\Phi_\eta^{\partial\Lambda}(x)\in\mathscr{C}^\infty(\Lambda)$ and fulfill (131). Under our assumption (126) we have also uniqueness.[1]

This ends the discussion of ground local specifications in two dimensional Euclidean space (leaving the other cases for further investigation). □

By this we would like to close the investigation of local aspects of the connections of the theory of Gibbs measures of Euclidean Field Theory and the theory of partial differential equations of Classical Euclidean Field Theory. The global aspects of these connections will be studied in the second part of this work: We will prove the existence of measures $\mu_{\Phi,\beta}\in\partial\mathscr{G}(\mathscr{E}_\beta^U)$ for all global solutions Φ of (13) and show that they have the global Markov property as well as that they have the representation (3.8) of [20]. We also prove that $\mu_{\Phi,\beta}\xrightarrow[\beta\to\infty]{}\delta_\Phi$ and for any unbounded $Q\subset R^2$ with smooth boundary, for $\mu_{\Phi,\beta'}$-a.a. $\eta\in\mathscr{D}'$, $\beta'\geqq 1$, we have $E_{Q^c,\beta}^\eta(\cdot)\xrightarrow[\beta\to\infty]{}\delta_{\Phi_\eta^{\partial Q}}$, where $\Phi_\eta^{\partial Q}$ is the unique – in the set $\mu_{\Phi,\beta'}$ measure one – solution of Dirichlet problem in Q with boundary data η, and $E_{Q^c,\beta}^\eta(\cdot)=E_{\Phi,\beta}(\cdot|\Sigma_{Q^c})(\eta)$. (For analogous results in the lattice case, see [21].)

[1] For a more detailed and more general investigation of the non-linear Dirichlet problem with distributional boundary data (113) see [22]

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