

# A Phase Cell Approach to Yang-Mills Theory

## I. Modes, Lattice-Continuum Duality\*

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**Abstract.** For the abelian Yang-Mills theory, a one-to-one correspondence is established between continuum gauge potentials and compatible lattice configurations on an infinite sequence of finer and finer lattices. The compatibility is given by a block spin transformation determining the configuration on a lattice in terms of the configuration on any finer lattice. Thus the configuration on any single lattice is not an “approximation” to the continuum field, but rather a subset of the variables describing the field.

It is proven that the Wilson actions on the lattices monotonically increase to the continuum action as one passes to finer and finer lattices. Configurations that minimize the continuum action, subject to having the variables fixed on some lattice, are studied.

### 0. Introduction

We consider an infinite sequence of finer and finer lattices. To each bond of each lattice there is assigned a group element, in the additive group of real numbers. There is a compatibility requirement to these assignments; the assignments to any one of the lattices are determined in terms of the assignments to any finer lattice, by an averaging procedure due to Balaban. Given a continuously differentiable gauge potential,  $A_\mu(x)$ , one can define compatible assignments to the lattices, as above, such that, in a suitable sense, the lattice “fields” approach  $A_\mu(x)$  as one passes to finer and finer lattices. The Wilson actions likewise approach the continuum action  $\frac{1}{2} \int (dA)^2$ . A particularly useful feature of the above duality between lattice fields and continuum fields is the following: let  $p$  be a plaquette in any of the lattices, then the group assignment to  $p$ ,  $A_{\partial p}$ , is given in terms of an integral with the continuum field

$$A_{\partial p} = \int \chi_p(x) \cdot \mathbf{A}(x), \quad (0.1)$$

where  $\chi_p(x)$  is a function associated to  $p$ .

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We consider fixing the group assignments on one of the lattices. We then seek a continuum field  $A_\mu(x)$  compatible with this assignment, and minimizing the continuum action subject to this constraint. [This does not determine  $A_\mu(x)$  uniquely.] To be specific, let the assignments be fixed on a lattice of length scale  $L$  (edge size). Assume these assignments are zero (the zero element) for bonds at distance greater than  $cL$  from some point  $z$ . We will find an  $A_\mu(x)$  compatible with this assignment, minimizing the continuum action, and “smooth” enough, so that the following results hold for the induced assignments to the finer lattices, and for  $A_\mu(x)$ . We let  $|A(m)|$  be the largest assignment, in absolute value, at the length scale  $L$ .

*Estimate 0.1.*

$$|A_\mu(x)| < c \frac{1}{L} e^{-\gamma|x-z|/L} |A(m)|. \quad (0.2)$$

*Estimate 0.2.*

$$|DA_\mu(x)| < \frac{c}{L^2} e^{-\gamma|x-z|/L} |A(m)|, \quad (0.3)$$

where  $D$  indicates any first partial.

*Estimate 0.3.*

$$\frac{1}{|x-y|^{1-\varepsilon}} |DA_\mu(x) - DA_\mu(y)| < \frac{c_\varepsilon}{L^{3-\varepsilon}} e^{-\gamma|x-z|/L} |A(m)| \quad \text{each } \varepsilon > 0, \quad (0.4)$$

where

$$|x-y| < cL. \quad (0.5)$$

*Estimate 0.4.* Let  $e$  be an edge at length scale  $l$ , and  $A(e)$  the corresponding assigned group element

$$|A(e)| < c \frac{l}{L} e^{-\gamma d(e,z)/L} |A(m)|. \quad (0.6)$$

*Estimate 0.5.* Let  $e_1$  and  $e_2$  be parallel (oriented) edges at length scale  $l$ ,

$$|A(e_1) - A(e_2)| < c \frac{l}{L} \cdot \frac{d(e_1, e_2)}{L} e^{-\gamma d(e_1, z)/L} |A(m)|, \quad (0.7)$$

where

$$d(e_1, e_2) < cL. \quad (0.8)$$

[We understand here parallel edges to have same orientation, and  $d(e_1, e_2)$  is measured between corresponding vertices.]

*Estimate 0.6.* Let  $p$  be a plaquette at length scale  $l$ , and  $A_{\partial p}$  the corresponding group assignment,

$$|A_{\partial p}| < c \left(\frac{l}{L}\right)^2 e^{-\gamma d(p,z)/L} |A(m)|. \quad (0.9)$$

Estimate 0.7. Let  $p_1$  and  $p_2$  be parallel (oriented) plaquettes at length scale  $l$ ,

$$|A_{\partial p_1} - A_{\partial p_2}| < c_\varepsilon (d(p_1, p_2))^{1-\varepsilon} \frac{l^2}{L^{3-\varepsilon}} e^{-\gamma d(p_1, z)/L} |A(m)| \quad \text{each } \varepsilon > 0, \quad (0.10)$$

where  $d(p_1, p_2)$  is measured between corresponding vertices and

$$d(p_1, p_2) < cL. \quad (0.11)$$

One has an elegant stability theorem.

**Abelian Stability Theorem.** *For any set of compatible assignments to the lattices, the Wilson Actions are a monotonically decreasing function of the length scale. That is, averaging decreases the action. We will write  $S_0^r$  for the action at length scale  $l_r = 1/2^r$ . Then we have*

$$S_0^r \geq S_0^s \quad \text{if } r > s. \quad (0.12)$$

Finally we note that we have not used any results of Balaban. He has some similar results, though none that include ours, since he always works with a fixed finest lattice on which the action is minimized. He has discrete analogs of Estimates 0.4–0.7, that at the early stages of our research, provided guidelines of what results to expect. (See Proposition 1.3 of [1].) He did not have our stability theorem, (but a weaker result with a constant  $c$  in (0.12)). (See Theorem 7.1.1 of [2].) In each case our technique of proof is different.

### 1. Averaging

We deal with compatible lattices,  $\mathcal{L}^0, \mathcal{L}^1, \mathcal{L}^2, \dots$ , with the edge size of  $\mathcal{L}^r$ ,  $l_r = 1/2^r$ . (It is a trivial modification to deal with edge size  $1/N^r$ .)  $\mathcal{L}^r$  is viewed as  $(1/2^r \mathbb{Z})^4$  in  $R^4$ , and we often identify points of the lattices in this way. To an edge  $e$  we have associated the real number  $A(e)$ , and to  $e$  with the opposite orientation,  $-e$ , we have  $A(-e) = -A(e)$ . To each vertex in  $\mathcal{L}^r$ , there is associated a vertex in  $\mathcal{L}^{r+1}$  (the same point in  $R^4$ ), the “base point” of a “block” of  $2^4$  vertices in  $\mathcal{L}^{r+1}$ . (Here the base point will be a corner of the block, for  $l_r = 1/N^r$  other locations of base points are possible, which leads again to trivial modifications.) In a  $2-d$  situation we picture a bond in  $\mathcal{L}^r$  and the two blocks in  $\mathcal{L}^{r+1}$  associated to its vertices

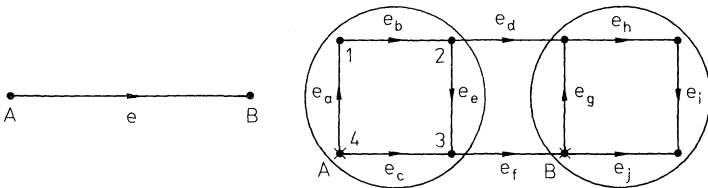


Fig. 1

The points  $A$  and  $B$  in  $\mathcal{L}^{r+1}$  are base points of the circled blocks. Averaging (we refer to as Balaban averaging) is defined in Eq. (1.8) of [1]. We establish a maximal tree in each block. We let  $x$  be a point in one of the blocks,  $x'$  the corresponding point in the other block.  $\Gamma_x$  is a path between base points, along portions of the

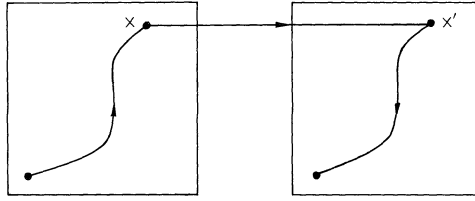


Fig. 2

maximal trees and a straight line segment joining  $x$  and  $x'$ . If we pick the maximal trees in Fig. 1 to be composed of  $e_a, e_c, e_e,$  and  $e_g, e_j, e_i$  in the two blocks, we may write in a natural notation

$$A(e) = \frac{1}{4}(A_{\Gamma_1} + A_{\Gamma_2} + A_{\Gamma_3} + A_{\Gamma_4}), \tag{1.1}$$

$$A_{\Gamma_1} = A(e_a) + A(e_b) + A(e_d) - A(e_g),$$

$$A_{\Gamma_2} = A(e_c) - A(e_e) + A(e_d) + A(e_h) + A(e_i) - A(e_j), \tag{1.2}$$

$$A_{\Gamma_3} = A_{\Gamma_4} = A(e_c) + A(e_f).$$

In general,  $A(e)$  is the average over the  $A_{\Gamma_x}$ . (For any oriented path,  $\Gamma$ ,  $A_\Gamma = \sum_\alpha A(e_\alpha)$ , where  $\Gamma$  is composed of the  $e_\alpha$ , with proper orientations.)

We now make a number of related observations all having to do with the fact that in many situations the contributions, by averaging, of contributions along the maximal trees cancel.

*Plaquette Averaging – Closed Loop Averaging*

Let  $v$  be a vertex in  $\mathcal{L}^r$ ; in  $\mathcal{L}^{r+1}$  there are  $2^4$  vertices in the block with base point  $v$ ; in  $\mathcal{L}^{r+2}$  there are  $2^8$  vertices in the  $2^4$  blocks with base points the  $2^4$  vertices above in  $\mathcal{L}^{r+1}$ . By such a “cascade process” there are  $2^{4(s-r)}$  vertices in  $\mathcal{L}^s$ ,  $s > r$ , associated to  $v$ , the vertices in a “superblock” with base point  $v$ . Let  $\Gamma$  be a closed path in  $\mathcal{L}^r$ , with one vertex  $v$ . We consider  $\Gamma_\alpha$ , a parallel translation of  $\Gamma$ , in  $\mathcal{L}^s$ , where the vertex corresponding to  $v$  has been translated to the vertex  $\alpha$ , one of the vertices in the superblock. There are  $2^{4(s-r)}$  such  $\Gamma_\alpha$ , one of them the original  $\Gamma$ , viewed in the lattice  $\mathcal{L}^s$ . If  $\Gamma$  has  $n$  edges in  $\mathcal{L}^r$ , then each  $\Gamma_\alpha$  has  $2^{(s-r)}n$  edges in  $\mathcal{L}^s$ . We then have

$$A_\Gamma = \frac{1}{2^{4(s-r)}} \cdot \sum_\alpha A_{\Gamma_\alpha}. \tag{1.3}$$

(Here it is understood that on the left side of the equation  $A_\Gamma$  is calculated in  $\mathcal{L}^r$ , and on the right side, each  $A_{\Gamma_\alpha}$  in  $\mathcal{L}^s$ .) The particular case that  $\Gamma$  is a plaquette is of greatest interest. If we label plaquettes at level  $r$ , by  $P_i$ , and plaquettes at level  $r + 1$  by  $p_j$  we find from (1.3)

$$A_{\partial P_i} = \sum_j \alpha(i)_j A_{\partial p_j} \tag{1.4}$$

for non-negative numbers  $\alpha(i)_j$ . We illustrate this in a  $2-d$  situation. We consider two levels  $r$  and  $r + 1$ .  $ABCD$  are the vertices of a plaquette at the  $r$  level. The integers name nine plaquettes at the  $r + 1$  level. There are four squares,  $s_a, s_b, s_c, s_d$ ;

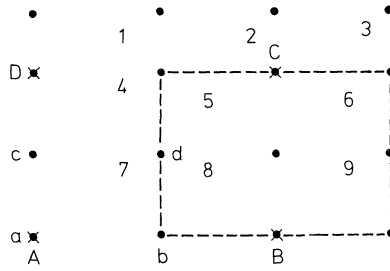


Fig. 3

$s_b$  is illustrated in dotted lines, the others are translations of  $s_b$  with lower left vertices  $a, c, d$ . The group element assigned to the plaquette  $ABCD$  at the  $r$  level is the average of the “integral” around the four squares taken at the  $r + 1$  level. In terms of plaquettes we may see this is equivalent to the following.

$$A_{\partial ABCD} = \frac{1}{4} [A_{\partial 1} + A_{\partial 3} + A_{\partial 7} + A_{\partial 9} + 2(A_{\partial 2} + A_{\partial 4} + A_{\partial 6} + A_{\partial 8}) + 4A_{\partial 5}].$$

*Proof of the Abelian Stability Theorem*

The  $\alpha(i)_j$  above satisfy

$$\sum_i \alpha(i)_j = 1/2^2, \tag{1.5}$$

$$\sum_j \alpha(i)_j = 2^2. \tag{1.6}$$

By the Schwarz inequality and (1.4) and (1.6) it follows that

$$(A_{\partial p_i})^2 \leq 2^2 \sum_j \alpha(i)_j (A_{\partial p_j})^2, \tag{1.7}$$

and summing over  $i$  using (1.5)

$$\sum_i (A_{\partial p_i})^2 \leq \sum_j (A_{\partial p_j})^2, \tag{1.8}$$

and thus

$$S_0^r \leq S_0^{r+1}. \tag{1.9}$$

*Path Averaging*

Let  $\Gamma$  be an oriented path in  $\mathcal{L}^r$  joining vertices  $A$  and  $B$ . Let  $\Gamma_\alpha$  be a translation of  $\Gamma$  in  $\mathcal{L}^{r+1}$ , carrying the vertex corresponding to  $A$  to  $\alpha$  a vertex in the block with base point  $A$ . Let  $\hat{\Gamma}_\alpha$  be a path in  $\mathcal{L}^{r+1}$  joining  $A$  and  $B$  consisting of  $\Gamma_\alpha$  and portions of the maximal trees in the blocks with base points  $A$  and  $B$ . We then have

$$A_\Gamma = \frac{1}{2^4} \sum_\alpha A_{\hat{\Gamma}_\alpha} \tag{1.10}$$

(where on the left side assignments are from  $\mathcal{L}^r$ , on the right side from  $\mathcal{L}^{r+1}$ ). By iterating we obtain a similar formula for  $A_\Gamma$  as an average over paths in  $\mathcal{L}^s, s > r$ ,

$$A_\Gamma = \frac{1}{2^{4(s-r)}} \sum_\alpha A_{\hat{\Gamma}_\alpha}. \tag{1.11}$$

Here  $\alpha$  is summed over vertices in a superblock with base point  $A$ .  $\hat{\Gamma}_\alpha$  is a union of  $\Gamma_\alpha$  and paths in the two superblocks associated to  $A$  and  $B$ . The paths in the superblocks are pleasant to analyze, we leave this to the reader. The only fact we here need is that if  $\Gamma$  has  $n$  edges in  $\mathcal{L}^r$ , then  $\hat{\Gamma}_\alpha$  in  $\mathcal{L}^s$  has  $\leq (n+c)2^{s-r}$  edges. This may be shown by induction from the considerations leading to (1.10).

*The Continuum Limit of Plaquette Averaging*

From the statement of plaquette averaging above, Eq. (1.3), one is led to the following construction of plaquette assignments to a continuum field  $A_\mu(x)$ .

On  $[0, 2] \times [0, 2]$  we define  $A(x, y), B(x, y)$

$$\begin{pmatrix} A(x, y) \\ B(x, y) \end{pmatrix} = \begin{cases} \begin{pmatrix} x \\ -y \end{pmatrix} & 0 \leq x \leq 1 & 0 \leq y \leq 1 \\ \begin{pmatrix} 2-x \\ y \end{pmatrix} & 1 \leq x \leq 2 & 0 \leq y \leq 1 \\ \begin{pmatrix} -x \\ -2+y \end{pmatrix} & 0 \leq x \leq 1 & 1 \leq y \leq 2 \\ \begin{pmatrix} -2+x \\ 2-y \end{pmatrix} & 1 \leq x \leq 2 & 1 \leq y \leq 2 \end{cases} \quad (1.12)$$

We consider the plaquette,  $p$ , in  $\mathcal{L}^r$ , parallel to the  $i-j$  direction, with vertices as drawn below



**Fig. 4**



We set

$$\begin{aligned} A_{\hat{p}} = & \int_{x_t}^{x_t+2^{-r}} dx_t \int_{x_s}^{x_s+2^{-r}} dx_s \int_{x_i}^{x_i+2^{-r+1}} dx \int_{x_j}^{x_j+2^{-r+1}} dy \\ & \times [A(2^r(x-x_i), 2^r(y-y_j))A_i(\cdot) + B(\cdot)A_j(\cdot)]2^{3r}. \end{aligned} \quad (1.13)$$

Here  $B(\cdot)$  has the same arguments as  $A$  in the first term, and  $A_i(\cdot)$  and  $A_j(\cdot)$  have  $i$  and  $j$  coordinates  $x$  and  $y$  respectively, and  $t$  and  $s$  coordinates dummy  $x_t$  and  $x_s$ . The assignments via Eq. (1.13) automatically satisfy the consistency requirements, of Balaban averaging. We introduce a vector function  $\chi_p(x)$  for each plaquette  $p$  such that we may write (1.13) as follows:

$$A_{\hat{p}} = \int d^4x \chi_p(x) \cdot \mathbf{A}(x) = (\chi_p, A). \quad (1.14)$$

The right side of (1.13) is the average value of  $\int A \cdot ds$  over translates of a square of edge size  $2^{-r}$  and parallel to the  $i-j$  coordinate axes. The position of one corner of the square varies over the cube  $x_k \leq \bar{x}_k \leq x_k + 2^{-r}$  with equal weighting during averaging.

Equations (1.9) and (1.14) are beautiful features that attest to the correctness of the Balaban averaging procedure. Each would seem to almost uniquely require this averaging procedure (among the class of gauge invariant procedures).

### 2. Bond Assignments Corresponding to a Given $A_\mu(x)$

This section is properly a continuation of the considerations of the previous section. In Eqs. (1.13)–(1.14) we have the correct plaquette assignments associated to the continuum field  $A_\mu(x)$ , in a very simple form. The bond assignments are not so easy to come by, they depend on the particular (arbitrary) choice of maximal tree in each block. (Amazingly the plaquette assignments are independent of these choices – an infinite number of choices may be made.) Not surprisingly the bond assignments are derived by a limiting procedure. For each  $r$ , we define an “ $r$ -approximate” bond assignment on the  $\mathcal{L}^s$  for  $s \leq r$ . The actual bond assignment will be the limit of these  $r$ -approximate assignments as  $r \rightarrow \infty$ . [We always assume  $A_\mu(x)$  is continuously differentiable.]

#### *r-Approximate Bond Assignments*

We first define the assignments to  $\mathcal{L}^r$  of the  $r$ -approximate procedure. We consider a bond (edge) at level  $r$ , joining  $(x_t, x_j, x_t, x_s)$  and  $(x_i + 2^{-r}, x_j, x_t, x_s)$ . We associate to this bond (with an orientation in the  $+x_i$  direction) the group element

$$2^{3r} \int_{x_t}^{x_t+2^{-r}} dx_t \int_{x_s}^{x_s+2^{-r}} dx_s \int_{x_i}^{x_i+2 \cdot 2^{-r}} dx \int_{x_j}^{x_j+2^{-r}} dy c(x) A_t(\cdot), \tag{2.1}$$

where

$$c(x) = \begin{cases} 2^r(x - x_i) & (x - x_i) \leq 2^{-r} \\ 2 - 2^r(x - x_i) & (x - x_i) \geq 2^{-r} \end{cases} \tag{2.2}$$

It is easy to ferret out that this assignment yields the correct plaquette variables. At levels  $s < r$  the bond assignments are obtained from the level  $r$  assignments by the use of Balaban averaging.

#### *Convergence of the r-Approximate Assignments*

We assume the following bounds on the  $A_\mu(x)$ :

$$|A_\mu(x)| \leq B_1, \tag{2.3}$$

$$|DA_\mu(x)| \leq B_2. \tag{2.4}$$

We denote by  $A(e, r_0)$  the assignment to edge  $e$  by the  $r_0$ -approximate assignments due to  $A_\mu(x)$ . If  $e$  is at level  $r_0$ , we easily see from (2.1), (2.2), and (2.3) that

$$|A(e, r_0)| \leq B_1 \ell_{r_0}. \tag{2.5}$$

Using the bound on the number of edges of averaged paths, the  $\Gamma_w$ , given shortly after (1.11), we see that (2.5) implies

$$|A(e, r_0)| \leq c B_1 \ell(e), \tag{2.6}$$

where  $\ell(e) = \ell_s$  if  $e$  is at (length) scale  $s$ , i.e.  $\ell(e)$  is the length of  $e$ . Similarly if  $e_1$  is parallel to  $e_2$  and at the same scale

$$|A(e_1, r_0) - A(e_2, r_0)| \leq cd(e_1, e_2)B_2\ell(e_1). \tag{2.7}$$

[Here to obtain this estimate we compare the assignments to the  $\hat{f}_\alpha$  at scale  $r_0$  that are averaged to get  $A(e_1, r_0)$  with those averaged to get  $A(e_2, r_0)$ , using (2.1), (2.2), and (2.4). Again we need the bound on the number of edges in  $\hat{f}_\alpha$ .] Note that the bounds in (2.6) and (2.7) are independent of  $r_0$ .

We now note that if the assignments to edges at scale  $r_0$  are changed with a bound  $\varepsilon$

$$|\delta A(e)| < \varepsilon, \tag{2.8}$$

we find that the corresponding assignments at higher scales, as determined by averaging, are changed with bound

$$|\delta A(e)| < c(\ell(e)/\ell_{r_0})\varepsilon, \tag{2.9}$$

again by consideration of path averaging and a bound on the edges in  $\hat{f}_\alpha$ .

Now let  $e$  be an edge at level  $r_0$ . We view the difference  $A(e, r_0) - A(e, r_0 + 1)$ , and find

$$|A(e, r_0) - A(e, r_0 + 1)| < cB_2\ell_{r_0}^2. \tag{2.10}$$

[ $A(E, r_0)$  is expressed as an average over  $A_{\hat{f}_\alpha}$ , where  $A_{\hat{f}_\alpha} = A_{\Gamma_\alpha} +$  contributions along the maximal trees. The contributions along the maximal trees, at the two ends of  $\hat{f}_\alpha$ , try to cancel, with a difference bounded in (2.10); the average of the  $A_{\Gamma_\alpha}$  actually add up to  $A(e, r_0)$ .]

Using (2.9) and (2.10) we get for any edge at level  $s < r_0$

$$|A(e, r_0) - A(e, r_0 + 1)| \leq c \cdot \left(\frac{\ell(e)}{\ell_{r_0}}\right) \cdot B_2\ell_{r_0}^2. \tag{2.11}$$

This proves that the limit of  $r$ -approximates exists and we get the edge assignments corresponding to  $A_\mu(x)$ ,

$$A(e) = \lim_{r_0 \rightarrow \infty} A(e, r_0). \tag{2.12}$$

The limit assignments satisfy (2.6) and (2.7) and in fact we may write

$$|A(e)| < c\ell(e) \sup_{d(x, e) < c\ell(e)} |A_\mu(x)|, \tag{2.13}$$

and for  $e_1 \parallel e_2$  at the same level, and satisfying

$$d(e_1, e_2) < cL, \quad L > \ell(e_1), \tag{2.14}$$

$$|A(e_1) - A(e_2)| < cd(e_1, e_2)\ell(e_1) \sup_{d(x, e) < cL} |DA_\mu(x)|. \tag{2.15}$$

Equations (2.13) and (2.14) follow from (2.6) and (2.7) and the nice observation that  $A(e)$  depends only on  $A_\mu(x)$  for  $d(x, e) < c\ell(e)$ ; as some reflection on the averaging procedure shows.

It is immediate that a given compatible set of assignments to the lattices arises from (is associated to) at most one continuously differentiable  $A_\mu(x)$ .

### 3. Modes

For the purposes of the present paper a level  $r$  mode is a compatible assignment of bond averages, to the  $\mathcal{L}^r$ , arising from an  $A_\mu(x)$  (continuously differentiable) such



that for some point  $z$ , and for level  $r$  edges  $e_i$ ,

$$A(e_i) = 0 \quad \text{if} \quad d(e_i, z) > cl_r. \tag{3.1}$$

We also require  $A_\mu(x)$  to minimize the continuum action, subject to the constraint of having the bond assignments at level  $r$  fixed.

By scaling arguments it is enough to study level 0 modes, and to prove Estimates 0.1–0.7 with  $L=1$ . By translation invariance and linearity we may restrict our study to the single configuration of bond assignments at level 0 having exactly one non-zero value, for a bond at the origin. We must find  $A_\mu(x)$ , and associated bond assignments to the  $\mathcal{L}^r$ , such that Estimates 0.1–0.7 hold with  $L=1$ ,  $z=0$ .

In this section (and Part II of this paper) we construct a potential,  $A_\mu^N(x)$ , satisfying Estimates 0.1–0.3, yielding the correct *plaquette* assignments at level 0, and minimizing the continuum action subject to this constraint. But  $A_\mu^N(x)$  does not yield the correct bond assignments at level 0. (From our previous discussions it is not surprising that it is harder to satisfy bond assignments than *plaquette* 3-2 assignments.) In the next section we will construct  $A_\mu(x)$  from  $A_\mu^N(x)$  and prove Estimates 0.1–0.7 for this  $A_\mu(x)$  and associated bond assignments.  $A_\mu(x)$  will be a gauge transformation of  $A_\mu^N(x)$ ,

$$A_\mu(x) = A_\mu^N(x) + \partial_\mu A. \tag{3.2}$$

From this it easily follows  $A_\mu(x)$  minimizes the continuum action subject to the constraint of bond assignments at level 0. We postpone to Part II of this paper the proof that  $A_\mu^N(x)$  satisfies Estimates 0.1–0.3, and here prove all else assuming this.

We proceed to find  $A(x) = A_\mu^N(x)$  minimizing the continuum action and having prescribed *plaquette* averages at level 0. (We do not immediately impose the condition that these assignments are due to a single non-zero bond assignment.) We thus have prescribed

$$(\chi_p, A) = \beta_p, \quad p \in \mathcal{L}^0. \tag{3.3}$$

The  $\beta_p$  are not arbitrary but must arise from a possible assignment of group elements to the bonds (edges) in  $\mathcal{L}^0$ . We seek a minimum of the action  $S$ , for a Landau gauge  $A'$ , a gauge transformation of  $A$ .

$$S = \frac{1}{2} \int \sum_{i,j} \left( \frac{\partial A'_i}{\partial x_j} \right)^2 + \frac{1}{2} \alpha^2 \sum_{p \in \mathcal{L}^0} ((\chi_p, A') - \beta_p)^2, \tag{3.4}$$

and then take the limit  $\alpha \rightarrow \infty$ . (Alternatively one could use Lagrange multipliers.) We let  $D$  be the differential-integral operator

$$D = -\Delta + \alpha^2 \sum_\gamma \chi_\gamma \chi_\gamma. \tag{3.5}$$

The sum over  $\gamma$  is understood to be over *plaquettes* in  $\mathcal{L}^0$ . We now exhibit the Fourier transform of  $C = D^{-1}$ ,

$$C(p, p') = \frac{1}{p^2} \delta(p-p') - \sum_n (2\pi)^4 \frac{1}{p^2} \tilde{P}(p) \frac{\alpha^2}{1 + \alpha^2 M(p)} \bar{\tilde{P}}(p') \frac{1}{p'^2} \delta(p-p'-2\pi n), \tag{3.6}$$

$$M = M_{ab} = \sum_n (2\pi)^4 \bar{\tilde{P}}_a(p+2\pi n) \frac{1}{(p+2\pi n)^2} \tilde{P}_b(p+2\pi n). \tag{3.7}$$

$n$  signifies a vector with integer coefficients.  $\tilde{P}$  (and  $M$ ) are a vector (and a matrix) in a 6 dimensional space, one dimension for each possible pair of spatial coordinate axes.  $\tilde{P}$  is also a four dimensional vector. In (3.7) the four dimensional vector components of  $\tilde{P}_a$  and  $\tilde{P}_b$  are understood to be contracted with each other. If  $a$  is the pair of axes,  $i$  and  $j$ , then  $\tilde{P}_a$  is the Fourier transform of the function

$$[A(x_i, x_j) \mathbf{i}_i + B(x_i, x_j) \mathbf{i}_j] \cdot \chi_0^{i,j}, \tag{3.8}$$

where  $\chi_0^{i,j}$  is the characteristic function of the set

$$\{0 \leq x_i \leq 2, 0 \leq x_j \leq 2, 0 \leq x_r \leq 1, 0 \leq x_s \leq 1\}. \tag{3.9}$$

We have taken an arbitrary fixed orientation of the axes to get this expression. In Appendix A of [3] a similar Fourier transform expression is employed in a discrete, non-matrix version, to get exponential fall-off of certain functions,

$$\begin{aligned} \tilde{P}_a(p) = & \left(\frac{1}{2\pi}\right)^2 \frac{i}{p_r} (e^{-ip_r} - 1) \frac{i}{p_s} (e^{-ip_s} - 1) \frac{i}{p_i^2 p_j^2} (1 - e^{-ip_i})^2 (1 - e^{-ip_j})^2 \\ & \times (p_j \mathbf{i}_i - p_i \mathbf{i}_j). \end{aligned} \tag{3.10}$$

From (3.4) we see the minimizing  $A'$  will satisfy

$$A' = \alpha^2 C \sum_{\gamma} \beta_{\gamma} \chi_{\gamma}. \tag{3.11}$$

Taking the limit  $\alpha \rightarrow \infty$  we find in terms of Fourier transforms,

$$A'_i(p) = \sum_{\gamma, a} \frac{1}{p^2} \tilde{P}_{i,a}(p) (\varepsilon + M(p))_{a,\gamma}^{-1} \beta_{\gamma} e^{-i\hat{\gamma} \cdot p}. \tag{3.12}$$

Here  $\varepsilon$  is a positive quantity that approaches zero (i.e. the  $\lim_{\varepsilon \rightarrow 0^+}$  is understood).  $\gamma$  is the pair of coordinate axes determined by the plaquette  $\gamma$ ;  $\gamma$  is parallel to these axes. For the plaquette drawn in Fig. 3,  $\hat{\gamma}$  is its lower left corner.

Equation (3.12) is a delicate relation.  $M$  is a singular  $6 \times 6$  matrix. It has pitiable rank 3. The well-definedness of (3.12) depends on the vector  $(\varepsilon + M)^{-1}$  is applied to. The  $\beta_{\gamma}$  are picked to correspond to a single bond entering the origin with non-zero assignment.  $A_{\mu}^N(x)$  is the Fourier transform of  $A(p)$ , where  $A_i(p) = A'_i(p) + p_i X(p)$  for a suitable  $X(p)$  to be specified in Part II. We there show:

$$|A_{\mu}^N(x)| < c e^{-\gamma|x|}, \tag{3.13}$$

$$|DA_{\mu}^N(x)| < c e^{-\gamma|x|}, \tag{3.14}$$

$$\frac{1}{|x-y|^{1-\varepsilon}} |DA_{\mu}^N(x) - DA_{\mu}^N(y)| < c_{\varepsilon} e^{-\gamma|x|}, \tag{3.15}$$

for each  $\varepsilon > 0, |x-y| < 1$ .

#### 4. Some Proofs, A Basic Gauge Transformation

We begin with some easy implications.

Estimate 0.5  $\Rightarrow$  Estimate 0.6, immediate.

Estimate 0.2  $\Rightarrow$  Estimate 0.6, using (1.14).

(2.13) and (3.13)  $\Rightarrow$  Estimate 0.4, immediate.

(2.15) and (3.14)  $\Rightarrow$  Estimate 0.5, immediate.

Estimate 0.3  $\Rightarrow$  Estimate 0.7, using (1.14).

We are left with the necessity of showing that we can pick  $\Lambda$  in (3.2) such that  $A_\mu(x)$  as given in (3.2) leads to correct bond assignments at level 0, and such that  $A_\mu(x)$  satisfies the same bounds as  $A_\mu^N(x)$ , in (3.13)–(3.15).

### The Gauge Transformation, $\Lambda(x)$

For  $e$ , at level 0, we let  $A^N(e)$  be the bond assignment to  $e$  by  $A_\mu^N(x)$ , and  $A(e)$ , the correct bond assignment, the assignment due to  $A_\mu(x)$ , which we will soon specify.  $A_\mu(x)$  and  $A_\mu^N(x)$  are related by  $\Lambda$  in (3.2), we must find  $\Lambda$ . By (2.13) and (3.13) we have

$$|A^N(e)| < ce^{-\gamma d(e,0)}. \quad (4.1)$$

Clearly

$$|A(e)| < ce^{-\gamma d(e,0)}. \quad (4.2)$$

( $A(e)$  is zero almost everywhere.) We set up a gauge field  $h(x)$  on  $Z^4$  by

$$A(e) - A^N(e) = h(b) - h(a) \quad (4.3)$$

for  $e = \mathbf{ab}$ . Recall since  $A^N(e)$  and  $A(e)$  yield the same plaquette variables, they are related by a lattice gauge transformation. We require  $h(x) \xrightarrow{x \rightarrow \infty} 0$ . We deduce

$$|h(x)| < ce^{-\gamma|x|}. \quad (4.4)$$

We now find  $\Lambda(x)$  in  $\mathcal{C}^\infty(R^4)$  satisfying

$$\Lambda(x) = h(x), \quad x \in Z^4, \quad (4.5)$$

and

$$|D^\alpha \Lambda(x)| < c_\alpha e^{-\gamma|x|}, \quad (4.6)$$

as is trivial to accomplish. We let this be the  $\Lambda(x)$  of (3.2). Clearly  $A_\mu(x)$  given by (3.2) satisfies the bounds in (3.13)–(3.15). Mirabile dictu the bond assignments to  $e$  at level 0, due to this  $A_\mu(x)$ , are exactly the  $A(e)$ . As a final note we point out it is straightforward to show the lattice actions  $S_0^r$  approach the continuum action as  $r \rightarrow \infty$ .

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