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New Proofs of the Existence of the Feigenbaum Functions

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Abstract. A new proof of the existence of analytic, unimodal solutions of the Cvitanović-Feigenbaum functional equation $\lambda g(x) = -g(g(-\lambda x))$, $g(x) \approx 1 - \text{const}|x|^r$ at 0, valid for all λ in (0, 1), is given, and the existence of the Eckmann-Wittwer functions [8] is recovered. The method also provides the existence of solutions for certain given values of r, and in particular, for $r=2$, a proof requiring no computer.

0. Notations

If $z \in \mathbf{C}$, we denote z^* its complex conjugate, and reserve the notation $\bar{S}$ to denote the closure of a set S.

Let J be an open, possibly empty interval in $\mathbf{R}$. We denote

$$\mathbf{C}(J) = \{z \in \mathbf{C}: \operatorname{Im} z \neq 0 \text{ or } z \in J\}.$$

In particular, $\mathbf{C}(\emptyset) = \mathbf{C}_+ \cup \mathbf{C}_-$, where

$$\mathbf{C}_+ = -\mathbf{C}_- = \{z \in \mathbf{C}: \operatorname{Im} z > 0\}.$$

$\mathbf{F}(J)$ is the real Fréchet space of functions f, holomorphic on $\mathbf{C}(J)$, with $f(z^*)^* = f(z)$, equipped with the topology of uniform convergence on compact subsets of $\mathbf{C}(J)$. $\mathbf{P}(J)$ is the subset of $\mathbf{F}(J)$ consisting of the functions f such that $f(\mathbf{C}_+) \subset \bar{\mathbf{C}}_+$, and $f(\mathbf{C}_-) \subset \bar{\mathbf{C}}_-$. These functions are often called Herglotz or Pick functions.

$\mathbf{P}_0(J)$ is the subset of $\mathbf{P}(J)$ consisting of the functions f such that $|f(z)/z| \to 0$ as $z \to \infty$ in non-real directions.

1. Introduction

The functional equation

$$g(x) = -\frac{1}{\lambda} g(g(-\lambda x)) \tag{1.1}$$

was formulated by Cvitanović and Feigenbaum [11] (see also [5]) to explain universal features of period doubling in maps of the interval. This section enumerates a set of constraints for the solutions whose existence is proved in the subsequent sections. These properties are suggested by the accumulated literature.

C1. *g is an even C^1 map of $[-1,1]$ into itself with a unique critical point at* 0, *and* $g(0)=1$.

C2. *There is a real $r>1$, a complex neighborhood of $[0,1]$ in $\mathbf{C}$, and a function f, holomorphic in this neighborhood, with $f'(0)\neq 0$, such that*

$$g(x)=f(x^r) \quad \textit{for} \quad 0\leqq x\leqq 1. \tag{1.2}$$

In particular, near 0, $g(x)\approx 1-\mathrm{const}\,|x|^r$. If g satisfies **C1** and **C2**, then (1.1) is equivalent to

$$g(x)=-\frac{1}{\lambda}g(g(\lambda x)) \tag{1.3}$$

which implies $g(1)=-\lambda$. Since g is decreasing on $(0,1)$, so is $x\to g(|\lambda|x)-x$, which takes the value 1 at 0 and $g(|\lambda|)-1\leqq 0$ at 1. Hence it vanishes at a unique $x_0\in(0,1)$, where $g(x_0)=-g(x_0)/\lambda$. Thus if $\lambda\neq -1$, $g(x_0)=0$, so $g(1)\leqq 0$. It is easily checked that $\lambda=-1$ is incompatible with our hypotheses, as well as $\lambda=0$ or 1, and so we impose:

C3. $0<\lambda<1$.

Note that this implies $x_0>\lambda$ (for otherwise $g(x_0/\lambda)\in[-1,1]$, but $g(x_0/\lambda)=-1/\lambda<-1$). Similarly, from $g(g(\lambda))=\lambda^2$, it follows $g(\lambda)>\lambda$ since otherwise $g(\lambda)/\lambda\in[-1,1]$, but

$$g(g(\lambda)/\lambda)=-g(\lambda^2)/\lambda<-g(\lambda x_0)/\lambda=-x_0/\lambda<-1.$$

For each $r>1$, it appears, from numerical experimentation and existing rigorous results [11, 3, 13, 1, 14, 10], that a locally unique solution exists, with the above properties, and depends smoothly on r, with λ an increasing function of r.

The study by Eckmann and Wittwer [8] of the asymptotic behavior of the problem as $r\to\infty$, has shown that, in that limit, g and f degenerate but $f(t)^r$, x_0^r, λ^r have non-trivial limits. It is therefore useful to consider:

$$G(t)=f(t)^r=g(t^{1/r})^r, \qquad y_0=x_0^r, \qquad \tau=\lambda^r. \tag{1.4}$$

Since $f(y_0)=0$, G is, in general, only analytic on $(0,y_0)$. We also introduce, with [8] and [3],

$$a(t)=G(\tau t)=g(\lambda t^{1/r})^r, \tag{1.5}$$

which is analytic and decreasing on $[0,y_0/\tau)$.

A straightforward generalization of the facts known from [9] in the case $r=2$ leads to the requirement:

C4. *The inverse function of $g|(0,1)$ extends to a function $u\in-\mathbf{P}((-1/\lambda,1))$. The inverse function on of $f|(0,1)$ extends to a function $U\in-\mathbf{P}((-\lambda^{-1},\lambda^{-2}))$.*

Clearly this implies, for all ζ in $\mathbf{C}((-1/\lambda, 1))$,

$$U(\zeta)=u(\zeta)^r,$$

and that u sends $\mathbf{C}_-$ into $\{\zeta: 0<\operatorname{Arg}\zeta<\pi/r\}$. Moreover,

$$U(0)=y_0, \quad U(1)=0, \quad U(x_0)=\tau y_0.$$

Taking the inverse function of a, and rescaling it by $\frac{1}{y_0}$, we define:

$$V(\zeta)=\frac{1}{y_0}a^{-1}(y_0\zeta), \tag{1.6}$$

$$\psi(z)=\frac{1}{y_0}U(z), \tag{1.7}$$

and we obtain the conditions:

C5.

$$V\in -\mathbf{P}((0,1/y_0\tau^2)), \quad V(1)=1, \quad V'(1)=-\frac{1}{\lambda}, \tag{1.8}$$

$$\psi\in -\mathbf{P}((-\lambda^{-1},\lambda^{-2})), \quad \psi(0)=1, \quad \psi(1)=0, \tag{1.9}$$

$$\psi(z)=V(\psi(-\lambda z)), \tag{1.10}$$

$$V(\zeta)=\frac{1}{\tau}\psi((y_0\zeta)^{1/r}). \tag{1.11}$$

The conditions **C1**, **C2** define a particular class of solutions of (1.1). There are many others, which are not at all considered in this paper. Some are analytic but have additional critical points in (0, 1), and it is likely that they correspond to a bifurcation, in function space, of codimension >1. Some are less regular (see e.g. [4]) and may be expected to play a less prominent (or more complicated) role in the dynamics of maps of the interval. Note, in particular, that, inasmuch as the fixed point g is conjectured to attract maps of many one-parameter families, it will attract, in particular, many analytic ones, e.g. $1-\mu|x|^r$, whose inverse functions have the properties corresponding to **C4**. Since these properties are very stable under limits, g itself can be expected to inherit them.

Several proofs of the existence of solutions satisfying **C1–C4** already exist for particular values of r and λ [3, 13, 1, 14, 9, 8, 10]. Except for [3] and [8], they do little to reveal the branch of special function theory which probably underlies the subject. Nor will this paper shed much light on this, but it is, hopefully, a small step in the right direction. The method of this paper is to look for solutions as fixed points of a map suggested by **C5**, and to apply the Schauder-Tikhonov theorem [7] by taking advantage of the normality properties of Herglotz functions. Section 3 uses a version M_λ of this map defined for a fixed value of λ, and proves the existence of solutions satisfying **C1–C5** for every $\lambda\in(0,1)$. Moreover it is possible to reobtain the existence of the Eckmann-Wittwer functions in the limit $\lambda\to 1$. In fact M_λ is essentially identical to the map used (and proved to be contractive) in

[8]; however it is used here in different function spaces. Appendix 2 owes much to [8] and to the ideas of Ecalle reviewed there. It essentially shows that, when suitably reinterpreted, M_λ has a limit when $\lambda \to 1$, and gives a direct proof of the existence of the Eckmann-Wittwer functions. But since the Schauder-Tikhonov theorem does not assert any kind of uniqueness, the proof in Sect. 3 implies no continuous dependence of r on λ, although it is intuitively obvious, and proved in [3] for small λ, that this dependence is, in fact, analytic; and it is likely that the map M_λ used in Sect. 3 is, in fact, a contraction. Section 4 describes a version of the method where r is fixed. It is, unfortunately, much less successful, although it does prove the existence of solutions for $r \leqq 14$. In particular, for $r=2$, it provides a proof that requires no other computing machinery than paper and pen. It would be much more interesting to be able to define and solve a fixed point problem for ψ or V regarded as a function of two complex variables, e.g. z and λ. This remains a possibility for the future. To a certain extent the methods of this paper can be applied to the case of circle maps. This will be described in a paper in preparation by J.-P. Eckmann and myself.

The literature concerning Feigenbaum's theory is very extensive, and only a small part of it appears in the list of references. The reader is referred, in particular, to [11, 12, 2, 8, 16] for more detailed scientific as well as bibliographical information.

2. Classical Results About P(J)

The properties recalled below can be found e.g. in [6, 15].

2.1. Integral Representation

Any $f \in \mathbf{P}(J)$ has a unique integral representation:

$$f(z) = az + b + \int d\mu(t) \left[\frac{1}{t-z} - \frac{t}{t^2+1} \right], \tag{2.1}$$

valid for all $z \in \mathbf{C}(J)$. Equivalently, for any $z_0 \in \mathbf{C}(J)$,

$$f(z) - f(z_0) = a(z - z_0) + \int d\mu(t) \left[\frac{1}{t-z} - \frac{1}{t-z_0} \right]. \tag{2.2}$$

Here μ is a positive measure on $\mathbf{R}$ with support in $\mathbf{R} - J$, such that $\int d\mu(t)(|t|+1)^{-2} < \infty$. For any continuous ϕ on $\mathbf{R}$, sufficiently decreasing at ∞,

$$\int \phi(t) d\mu(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int \phi(t) \operatorname{Im} f(t + i\varepsilon) dt. \tag{2.3}$$

The constant $a \geqq 0$ is called the angular derivative of f at infinity. Uniformly in any closed angle contained in $\mathbf{C}_\pm$,

$$\lim_{|z| \to \infty} |(f(z) - az)/z| = 0.$$

We denote $\mathbf{P}_0(J)$ the subset of $\mathbf{P}(J)$ consisting of functions for which $a=0$. It is dense in $\mathbf{P}(J)$: if, e.g., f belongs to $\mathbf{P}(J)$ with $J=(0,1)$ or $(0,\infty)$, then, for $0<s<1$, $f_s(z)=f(z^s)$ defines an element f_s of $\mathbf{P}_0(J)$ which tends to f as $s\to 1$. This remark can simplify the verification of inequalities such as (3.4), (3.5), etc.

2.2. Positivity Conditions on Derivatives

Suppose that $J\neq\emptyset$. Then, for every $z\in J$, and every finite complex sequence $v_0,\dots,v_N$,

$$\sum_{j,k=0}^{N}\frac{f^{(j+k+1)}(z)}{(j+k+1)!}v_j^* v_k \geqq 0.$$

In particular $f^{(n)}$ is positive for all odd n, and:

$$Sf(z)\equiv\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left[\frac{f''(z)}{f'(z)}\right]^2\geqq 0. \tag{2.4}$$

2.3. Special Case of $J=(-\infty,0)$

If $f\in\mathbf{P}((-\infty,0))$, then $f^{(n)}$ is positive for all $n\in\mathbf{N}_*$. If, moreover, $f(x)\to 0$ when $x\to-\infty$ in $\mathbf{R}$, then $\int(|t|+1)^{-1}d\mu(t)<\infty$, and

$$f(z)=\int\frac{d\mu(t)}{t-z}.$$

2.4. Normality

$\mathbf{P}(\emptyset)$ is a normal family. The same is true of the subset of functions in $\mathbf{P}(J)$ which, on J, are bounded in modulus by some fixed $M<\infty$.

2.5. Iteration of Functions in $\mathbf{P}(\emptyset)$

Denote f_+ the restriction of $f\in\mathbf{P}(\emptyset)$ to $\mathbf{C}_+$. Then (see [15]), either f_+ is an isomorphism of $\mathbf{C}_+$, or f_+^n converges, uniformly on any compact subset of $\mathbf{C}_+$, to a constant C. There are three possible cases:

1) $C=\infty$: this can happen only if $a>1$.
2) $C\in\mathbf{R}$.
3) $C\in\mathbf{C}_+$: then C is an attractive fixed point of f_+, i.e. $|f_+'(C)|<1$.

2.6. Final Remark

Let $f\in\mathbf{P}((b,c))$, not identically 0, with $-\infty<b<0<c<\infty$ and suppose that $f(0)=0$. Then, on (b,c), $f(x)/x$ is a strictly positive, convex function and, for all $z=x+iy$ such that $b<x<c$,

$$\frac{|f(z)|}{|z|}\leqq\frac{f(x)}{x}.$$

3. The Fixed λ Method

For any fixed $\lambda \in (0,1)$, this method sets up a map M_λ which we first describe informally. Throughout this section, λ is fixed in $(0,1)$.

1) Start with a given $\psi_0 \in -\mathbf{P}((-\lambda^{-1}, \lambda^{-2}))$, satisfying $\psi_0(0)=1, \psi_0(1)=0$, and other conditions to be stated later.

2) Define:

$$V(\zeta) = \frac{1}{\tau}\psi_0((\zeta/\alpha)^{1/r}),$$

the real numbers $\tau > 0$, $\alpha > 0$, $r > 1$ being adjusted so that

$$V(1)=1, \quad V'(1) = -\frac{1}{\lambda}, \quad \tau = \lambda^r.$$

3) Find ψ such that

$$\psi(z) = V(\psi(-\lambda z)), \quad \psi(0)=1, \quad \psi(1)=0,$$

and verify that ψ satisfies all the conditions imposed on ψ_0. Then define

$$M_\lambda \psi_0 = \psi.$$

To carry out step 3), it will be convenient to introduce auxiliary functions, in particular W:

$$W(\zeta) = V(V(\zeta)).$$

We now study the map M_λ in detail. Recall that, in this section, λ is chosen once and for all in $(0,1)$. In the remainder of this paper, we denote:

$$A \equiv A(\lambda) = -\frac{1}{\lambda \log \lambda}, \quad B \equiv B(\lambda) = (1-\lambda^2)A(\lambda). \tag{3.0}$$

Note that $A \geqq e$, and B is a decreasing function of λ tending to 2 as $\lambda \to 1$.

3.1. Determination of τ, α, and r

We start from a fixed $\psi_0 = 1 - \hat{\psi}_0 \in -\mathbf{P}((-\lambda^{-1}, \lambda^{-2}))$, with $\psi_0(0)=1$ and $\psi_0(1)=0$. As recalled in Sect. 2, there is a constant $a_0 \geqq 0$, and a positive measure ξ_0, with support in

$$\Sigma(\lambda) = \mathbf{R} - (-\lambda^{-1}, \lambda^{-2}), \tag{3.1}$$

such that, for all $z \in \mathbf{C}((-\lambda^{-1}, \lambda^{-2}))$,

$$\frac{\psi_0(z)}{1-z} = a_0 + \int \frac{d\xi_0(t)}{(t-z)(t-1)}, \tag{3.2}$$

and:

$$0 \leqq \int \frac{d\xi_0(t)}{t(t-1)} = 1 - a_0. \tag{3.3}$$

Rewriting the integrand of (3.2) in the form:

$$\frac{d\xi_0(t)}{t(t-1)}\frac{t}{(t-z)},$$

and noting that, for $0 \leqq z \leqq \lambda^{-2}$, and t in $\Sigma(\lambda)$,

$$(1+\lambda z)^{-1} \leqq t(t-z)^{-1} \leqq (1-\lambda^2 z)^{-1},$$

we obtain:

$$\frac{1}{1+\lambda z} \leqq \frac{\psi_0(z)}{1-z} \leqq \frac{1}{1-\lambda^2 z} \qquad (0 \leqq z \leqq \lambda^{-2}). \tag{3.4}$$

Similarly, from

$$-\psi_0'(z) = a_0 + \int \frac{d\xi_0(t)}{(t-z)(t-1)}\frac{(t-1)}{(t-z)},$$

it follows that, for $-\lambda^{-1} \leqq z \leqq 1$,

$$\frac{(1-\lambda^2)}{(1-\lambda^2 z)(1-z)} \leqq \frac{-\psi_0'(z)}{\psi_0(z)} \leqq \frac{(1+\lambda)}{(1+\lambda z)(1-z)}, \tag{3.5}$$

and from

$$-\psi_0''(z) = 2\int \frac{d\xi_0(t)}{(t-z)^2}\frac{1}{t-z},$$

it follows that, for $-\lambda^{-1} \leqq z \leqq \lambda^{-2}$,

$$-\frac{2\lambda}{1+\lambda z} \leqq \frac{\psi_0''(z)}{\psi_0'(z)} \leqq \frac{2\lambda^2}{1-\lambda^2 z}. \tag{3.6}$$

Suppose now that, for some positive τ, α, and $r > 1$, the function

$$V(\zeta) = \frac{1}{\tau}\psi_0((\zeta/\alpha)^{1/r}) \tag{3.7}$$

is defined and differentiable at $\zeta = 1$, and satisfies $V(1) = 1$, $V'(1) = -\lambda^{-1}$. Then we must have:

$$\psi_0(z_1) = \tau, \qquad z_1 = \alpha^{-1/r}, \tag{3.8}$$

and:

$$\frac{z_1\psi_0'(z_1)}{\psi_0(z_1)} = -\frac{r}{\lambda}. \tag{3.9}$$

If we also require $\tau = \lambda^r$, i.e. $r = \log\tau/\log\lambda$, we must have:

$$q(z_1) = 0, \qquad q(z) \equiv \frac{\psi_0'(z)}{\psi_0(z)} - \frac{A}{z}\log\psi_0(z). \tag{3.10}$$

The function q is smooth on $[0,1)$, with $q(0)=\psi_0'(0)(1-A)>0$. When $z\to 1$, it behaves like $-(1-z)^{-1}-A\log(1-z)$, and tends to $-\infty$. It therefore has zeroes in $(0,1)$, and we take z_1 as any one of them. Actually it will shortly be seen that there is only one such zero.

Having chosen z_1 in this way, we define:

$$\tau=\psi_0(z_1), \qquad r=\frac{\log\tau}{\log\lambda}, \qquad \alpha=z_1^{-r}. \tag{3.11}$$

Since ψ_0 is strictly decreasing on $[-\lambda^{-1},\lambda^{-2}]$, we have: $0<\tau<1$, $r>0$, $\alpha>1$.

3.2. Lower Bounds on z_1 and $1/\tau$

Since ψ_0 is negative on $(1,\lambda^{-2})$, the function $\log\psi_0$ belongs to $-\mathbf{P}((-\lambda^{-1},1))$. When $z\in\mathbf{C}_-$, clearly $0<\operatorname{Im}\log\psi_0(z)<\pi$, so that the angular derivative of this function at infinity vanishes. It has, therefore, an integral representation, for $z\in\mathbf{C}((-\lambda^{-1},1))$,

$$\log\psi_0(z)=-\int\sigma(t)dt\left[\frac{1}{t-z}-\frac{1}{t}\right], \tag{3.12}$$

where $\sigma\in L^\infty$ has support in $\mathbf{R}-(-\lambda^{-1},1)$, and $0\leqq\sigma\leqq 1$. Moreover, for $t\in(1,\lambda^{-2})$, $\log\psi_0(t+i0)=\log[-\psi_0(t)]-i\pi$, so that $\sigma(t)=1$ there. The function q has the integral representation, in $\mathbf{C}((-\lambda^{-1},1))$,

$$q(z)=\int\frac{\sigma(t)dt}{t-z}\left[\frac{A}{t}-\frac{1}{t-z}\right]. \tag{3.13}$$

Let $0<z<1$. For $t\leqq-\lambda^{-1}$, the integrand is positive. For $t\geqq 1$, it has the sign of $(A-1)t-Az\geqq(A-1)-Az$, and is certainly >0 if $z<1-A^{-1}$. We conclude:

$$z_1>1-\lambda\log\frac{1}{\lambda}>\lambda. \tag{3.14}$$

The last inequality follows from the usual inequality $\log x\leqq x-1$ for all $x>0$, strict for $x\neq 1$. Since $r>0$, it follows that:

$$\alpha<\frac{1}{\tau}. \tag{3.15}$$

To get stronger bounds on z_1, we separate, in the integral in (3.13), the contributions from $[1,\lambda^{-2}]$ and from the rest of the support of σ. For $0<z<1$, as already noted, the contribution from $(-\infty,-\lambda^{-1})$ is positive. For $t\geqq\lambda^{-2}$, the integrand has the sign of $(A-1)t-Az$, which is minorized by:

$$(A-1)\lambda^{-2}-A=A\lambda^{-2}\left(1-\lambda\log\frac{1}{\lambda}-\lambda^2\right)>0.$$

Therefore, for $0<z<1$, $q(z)\geqq q_2(z)$, where:

$$q_2(z)=\int_1^{1/\lambda^2}dt\left[\frac{A}{z}\left(\frac{1}{t-z}-\frac{1}{t}\right)-\frac{1}{(t-z)^2}\right]=\frac{A}{z}\log\left[\frac{1-\lambda^2 z}{1-z}\right]-\frac{1-\lambda^2}{(1-z)(1-\lambda^2 z)}. \tag{3.16}$$

It is convenient to use the variable

$$\xi = \frac{1-\lambda^2 z}{1-z}, \qquad z = \frac{\xi - 1}{\xi - \lambda^2}. \tag{3.17}$$

This gives:

$$(1-\lambda^2) z q_2(z) = \chi(\xi) - \xi, \tag{3.18}$$

$$\chi(\xi) = B \log \xi + 1 + \lambda^2 - \frac{\lambda^2}{\xi}. \tag{3.19}$$

The function χ is increasing and concave on $(0, \infty)$, and $\chi(\xi) - \xi$ vanishes at $\xi = 1$ and at a unique $\hat{\xi} > 1$. Since $1 \leqq \xi \leqq \hat{\xi}$ is equivalent to $\chi(\xi) \geqq \xi$, it follows that $\hat{\xi}$ is a lower bound for $\xi_1 = (1 - \lambda^2 z_1)/(1 - z_1)$. Applying the bounds (3.4) shows that:

$$\frac{1}{\tau} \geqq \xi_1 \geqq \hat{\xi}. \tag{3.20}$$

It is immediate to verify that:

$$\chi\left(\frac{1}{\lambda}\right) - \frac{1}{\lambda} = 1 - \lambda + \lambda^2 - \lambda^3 > 0,$$

from which it follows that $\tau < \lambda$ and hence $r > 1$. But we need the more precise bound:

$$\hat{\xi} > \lambda^{-(1+y)}, \qquad y = \frac{2\lambda^2}{1-\lambda^2}. \tag{3.21}$$

Inserting $\xi = \lambda^{-(1+y)}$ into $\chi(\xi) - \xi$ gives:

$$\frac{1}{\lambda}\left[1 + \lambda + \lambda^2 - \exp\left(\frac{2\lambda^2}{1-\lambda^2} \log \frac{1}{\lambda}\right)\right] + \lambda^2(1 - \lambda^{1+y}).$$

The positivity of the first bracket follows from the

Lemma 1. *For* $0 \leqq x \leqq 1$, *the quantity*

$$(1-x^2) \log(1 + x + x^2) + 2x^2 \log x$$

is non-negative, and vanishes only at 0 *and* 1.

The straightforward and tedious proof of this is sketched in Appendix 1.

As a consequence of (3.21),

$$\frac{1}{\tau} > \left(\frac{1}{\lambda}\right)^{1+y}, \qquad r > 1 + \frac{2\lambda^2}{1-\lambda^2}. \tag{3.22}$$

We also note:

$$\frac{1}{\tau} > 3. \tag{3.23}$$

Indeed, $\chi(3) - 3 > 1 + B - 3 > 0$.

3.3. Uniqueness of z_1

The derivative of $zq(z)$ in $(0,1)$ satisfies:

$$\frac{\psi_0(z)}{\psi_0'(z)}\frac{d}{dz}(zq(z)) = -z\frac{\psi_0'(z)}{\psi_0(z)} - A + 1 + z\frac{\psi_0''(z)}{\psi_0'(z)} \geqq -z\frac{\psi_0'(z)}{\psi_0(z)} - A + \frac{1-\lambda z}{1+\lambda z}$$

[by (3.6)]. At $z=z_1$, this gives:

$$|(zq(z))'|_{z=z_1} \geqq A\log\frac{1}{\tau}\left[A\left(\log\frac{1}{\tau}-1\right)+\frac{1-\lambda}{1+\lambda}\right]. \tag{3.24}$$

Thus, the derivative of $zq(z)$ at every zero it has in $(0,1)$ is strictly negative. Therefore $zq(z)$ has only one zero in $(0,1)$.

3.4. Lower Bound on τ

It is clear that $\tau > 0$, since $z_1 \neq 1$. In this subsection, we prove the existence of a (strictly positive) lower bound for τ, which depends only on λ. In Subsect. 3.11, using additional constraints on ψ_0, we shall prove the existence of a lower bound uniform in λ as $\lambda \to 1$.

Separating, in the integral representation of $\log\psi_0$, the contribution of $[1, \lambda^{-2}]$ from the rest gives:

$$\log\psi_0(z) + \log\frac{1-\lambda^2 z}{1-z} = -\int_\Sigma \sigma(t)dt\frac{z}{t(t-z)}.$$

[Recall that $\Sigma = \mathbf{R} - (-\lambda^{-1}, \lambda^{-2})$.] Letting z tend to 1 from below gives:

$$k \equiv \int_\Sigma \frac{\sigma(t)dt}{t(t-1)} = \log\left[\frac{-1}{\psi_0'(1)(1-\lambda^2)}\right]. \tag{3.25}$$

Letting z tend to 1 in the inequalities (3.4) we find:

$$\frac{1}{1+\lambda} \leqq -\psi_0'(1) \leqq \frac{1}{1-\lambda^2}, \tag{3.26}$$

and hence:

$$k \leqq \log\frac{1}{1-\lambda} \leqq \frac{\lambda}{1-\lambda}. \tag{3.27}$$

The function $q_1(z) = q(z) - q_2(z)$ has been shown to be positive in $(0,1)$. To majorize it, we write it in the form:

$$q_1(z) = \int_\Sigma \frac{\sigma(t)dt}{t(t-1)} I(t,z), \qquad I(t,z) = \frac{(A-1)t - Az}{(t-z)^2}(t-1).$$

It is easy to see that, when $t \in \Sigma$ and $z \in (0,1)$,

$$I(t,z) \leqq (A-1)(1+\lambda),$$

and so, for $z \in (0,1)$,

$$q_1(z) \leqq (A-1)(1+\lambda)k,$$

$$(1-\lambda^2)zq(z) \leqq (1-\lambda^2)zq_2(z) + (1-\lambda^2)(A-1)(1+\lambda)k.$$

Assume that

$$(1-\lambda^2)(A-1)(1+\lambda)k + 1 + \lambda^2 \leqq K. \tag{3.28}$$

Then, using again the variable ξ defined in (3.17), we get:

$$\chi(\xi) - \xi \leqq (1-\lambda^2)zq(z) \leqq \chi(\xi) - \xi - 1 - \lambda^2 + K,$$

with χ as in (3.19). In particular,

$$(1-\lambda^2)zq(z) \leqq K + B\log\xi - \xi \equiv S(\xi). \tag{3.29}$$

To obtain an upper bound for the root of the right-hand side and hence for ξ_1, we may e.g. note that $S'(\xi) = B/\xi - 1$, so that, for $\xi > 2B$,

$$S(\xi) < S(2B) - \tfrac{1}{2}(\xi - 2B).$$

This becomes negative if $\xi > 2S(2B) + 2B$, and a fortiori if $\xi > 2(K + B\log B)$, and this is then an upper bound for ξ_1. The bounds (3.4) give:

$$\frac{1}{\tau} \leqq \frac{1+\lambda z_1}{1-z_1} = \frac{\xi_1 - \lambda}{1-\lambda}. \tag{3.30}$$

Remark. Inserting (3.27) into the left-hand side of (3.28) leads to:

$$S(\xi) \leqq \frac{(1+\lambda)^2}{\log 1/\lambda} + 1 + B\log\xi - \xi.$$

For $\lambda \leqq e^{-2}$, this is negative when ξ is given the value

$$\xi = \lambda^{-(1+Y)}, \qquad Y = \frac{4\lambda}{\log 1/\lambda}.$$

By (3.30), this implies:

$$r < 1 + \frac{1}{\log 1/\lambda}[4\lambda - \log(1 - e^{-2})],$$

which confirms that $r \to 1$ as $\lambda \to 0$.

3.5. Definition of the Functions V and W

We can now define:

$$V(\zeta) = \frac{1}{\tau}\psi_0((\zeta/\alpha)^{1/r}). \tag{3.31}$$

Since $r > 1$, $-V$ is a Herglotz function. It is defined, real and analytic at the real points in $(0, \alpha\tau^{-2})$. In particular:

$$V(1) = 1, \qquad V'(1) = -\frac{1}{\lambda}, \qquad V(\alpha) = 0. \tag{3.32}$$

This function satisfies the following identities, where we denote $z=(\zeta/\alpha)^{1/r}$:

$$V'(\zeta)=\frac{1}{\tau r}\frac{z}{\zeta}\psi_0'(z), \tag{3.33}$$

$$\frac{V''(\zeta)}{V'(\zeta)}=-\frac{1}{\zeta r}\left[r-1-z\frac{\psi_0''(z)}{\psi_0'(z)}\right], \tag{3.34}$$

$$SV(\zeta)=S\psi_0(z)\left[\frac{z}{r\zeta}\right]^2+\frac{1}{2\zeta^2}(1-r^{-2}). \tag{3.35}$$

(Recall that $S\psi_0$ and SV denote the Schwarzian derivatives of ψ_0 and V, respectively: see 2.2.) From (3.34) and the inequalities (3.6), it follows, when $\zeta\in(0,\alpha\tau^{-2})$,

$$r-1-\frac{2\lambda^2 z}{1-\lambda^2 z}\leqq -r\zeta\frac{V''(\zeta)}{V'(\zeta)}\leqq r-1+\frac{2\lambda z}{1+\lambda z}. \tag{3.36}$$

In particular when $0<\zeta\leqq\alpha$, i.e. $0<z\leqq 1$, using the first inequality in (3.36), and the lower bound on r given by (3.22), we see that V is convex. Since $V'(1)=-1/\lambda$, this implies $V(\zeta)\geqq 1-(\zeta-1)/\lambda$ in $(0,\alpha)$, and so:

$$\alpha>1+\lambda. \tag{3.37}$$

We also define

$$W(\zeta)=V(V(\zeta)), \tag{3.38}$$

and it will be convenient to introduce:

$$\hat{V}(\zeta)=1-V(1-\zeta),\qquad \hat{W}(\zeta)=1-W(1-\zeta)=\hat{V}(\hat{V}(\zeta)). \tag{3.39}$$

Both W and $\hat{W}$ are Herglotz functions. On the reals, W is defined, real and holomorphic on $(0,\alpha)$: $W(\alpha)=V(0)=1/\tau$, and $W(0)=V(1/\tau)$ is defined since $1/\tau<\alpha/\tau^2$.

It is clear that $W(0)<0$, i.e. $\hat{W}(1)>1$, because we have shown that $1/\tau>\alpha$, but we need more precise bounds.

3.6. Lower Bounds on $\hat{W}(1)$

We start by applying the inequalities (3.36) to the case $\zeta=1$, and we find:

$$0<-\frac{V''(1)}{V'(1)}<1. \tag{3.40}$$

Further, by (3.35) and the positivity of $S\psi_0$,

$$SV(\zeta)\geqq\frac{1-r^{-2}}{2\zeta^2},\qquad 0<\zeta<\alpha\tau^{-2}. \tag{3.41}$$

Since $W=V\circ V$,

$$W'(\zeta)=V'(V(\zeta))V'(\zeta), \tag{3.42}$$

$$\frac{W''(\zeta)}{W'(\zeta)}=\frac{V''(V(\zeta))}{V'(V(\zeta))}V'(\zeta)+\frac{V''(\zeta)}{V'(\zeta)}, \tag{3.43}$$

$$SW(\zeta)=SV(V(\zeta))V'(\zeta)^2+SV(\zeta). \tag{3.44}$$

When $\zeta=1$, combining (3.40) and (3.43) gives:

$$0 \leqq \frac{W''(1)}{W'(1)} = -\frac{\hat{W}''(0)}{\hat{W}'(0)} \leqq \frac{1}{\lambda} - 1. \tag{3.45}$$

We now apply (3.44) and (3.41) to get:

$$SW(\zeta) \geqq \frac{V'(\zeta)^2(1-r^{-2})}{2V(\zeta)^2} + \frac{(1-r^{-2})}{2\zeta^2}. \tag{3.46}$$

For $0<\zeta<1$, the convexity of V implies:

$$-V'(\zeta) \geqq \frac{V(\zeta)-1}{1-\zeta},$$

hence

$$-\frac{V'(\zeta)}{V(\zeta)} \geqq \frac{1}{1-\zeta-\dfrac{1}{V'(\zeta)}} \geqq \frac{1}{1-\zeta+\lambda}.$$

It follows that

$$2SW(\zeta) \geqq (1-r^{-2})\left[\frac{1}{(1-\zeta+\lambda)^2} + \frac{1}{\zeta^2}\right],$$

and hence

$$2S\hat{W}(\zeta) \geqq (1-r^{-2})\left[\frac{1}{(\zeta+\lambda)^2} + \frac{1}{(1-\zeta)^2}\right].$$

In (0, 1), the right-hand side has a minimum at $\zeta=(1-\lambda)/2$, and, using the lower bound on r in (3.22), we get

$$S\hat{W}(\zeta) \geqq s(\lambda) \equiv \frac{16\lambda^2}{(1+\lambda^2)^2(1+\lambda)^2}. \tag{3.47}$$

It follows that, for $0<\zeta<1$,

$$\frac{d}{d\zeta}\left[\frac{\hat{W}''(\zeta)}{\hat{W}'(\zeta)}\right] = S\hat{W}(\zeta) + \frac{1}{2}\left[\frac{\hat{W}''(\zeta)}{\hat{W}'(\zeta)}\right]^2 \geqq s(\lambda),$$

and so, using (3.45),

$$\frac{\hat{W}''(\zeta)}{\hat{W}'(\zeta)} \geqq -\left(\frac{1}{\lambda}-1\right) + s\zeta,$$

$$\log \hat{W}'(\zeta) \geqq \log \lambda^{-2} - \left(\frac{1}{\lambda}-1\right)\zeta + s\zeta^2/2.$$

The function $\lambda \to -2\log\lambda - \lambda^{-1} + 1$ has a unique maximum at 1/2 in (0, 1), vanishes at 1, and takes the value $3-e$ at $\lambda = 1/e$. It is thus positive for $\lambda \geqq 1/e$, and we conclude that for $\lambda \geqq 1/e$ and $0<\zeta<1$,

$$\hat{W}'(\zeta) \geqq \exp(s\zeta^2/2) \geqq 1 + s\zeta^2/2, \qquad \hat{W}(\zeta) \geqq \zeta\left(1+\frac{s\zeta^2}{6}\right). \tag{3.48}$$

We now remark that, since $-V(\zeta)$ and $W(\zeta)$ have been defined as Herglotz functions of $\zeta^{1/r}$, with $r>1$, they have zero angular derivative at infinity. In particular, for $\zeta \in \mathbf{C}((1-\alpha, 1))$,

$$\frac{\hat{W}(\zeta)}{\zeta} = \int \frac{du(t)}{t(t-\zeta)} = \int \frac{du(t)}{t^2} \frac{t}{t-\zeta}, \tag{3.49}$$

where u is a positive measure with support in $\mathbf{R}-(1-\alpha, 1)$, and $\int du(t)t^{-2} = \lambda^{-2}$. (Recall that $1-\alpha < -\lambda$.)

Therefore, for $\zeta \in [0, 1)$,

$$\hat{W}(\zeta) \geqq \frac{\zeta}{\lambda(\lambda+\zeta)} \geqq \zeta + \zeta\left[\frac{1}{\lambda(\lambda+1)} - 1\right]. \tag{3.50}$$

(In fact this bound would hold even if the angular derivative of $\hat{W}$ at infinity did not vanish.) The last bracket in (3.50) is positive for $\lambda < (\sqrt{5}-1)/2 \sim 0.61$ and, in particular for $0 < \lambda \leqq 1/2$,

$$\hat{W}(\zeta) \geqq \zeta\left(1 + \frac{\zeta^2}{3}\right).$$

On the other hand,

$$4[s(\lambda)]^{-1/2} = \frac{1}{\lambda} + 1 + \lambda + \lambda^2$$

is a convex function of λ in $(0, 1)$ and so for $\lambda \in [1/2, 1]$,

$$4[s(\lambda)]^{-1/2} \leqq \max(4, \tfrac{15}{4}) = 4,$$

so that $s(\lambda) \geqq 1$, and, by (3.48), $\hat{W}(\zeta) > \zeta(1 + \zeta^2/6)$. Thus:

Lemma 2. *There exists a number $a \geqq \frac{1}{6}$ such that, for all $\lambda \in (0, 1)$, $\zeta \in [0, 1]$,*

$$\hat{W}(\zeta) \geqq \zeta(1 + a\zeta^2). \tag{3.51}$$

3.7. Upper Bound on $\hat{W}(1)$

Recall that

$$W(0) = V\left(\frac{1}{\tau}\right) = \frac{1}{\tau}\psi_0(z_2),$$

where

$$1 < z_2 = (\tau\alpha)^{-1/r} = z_1\lambda^{-1} < \lambda^{-2}.$$

Using the bound (3.4) on ψ_0, we find:

$$-\psi_0(z_2) \leqq \frac{1}{\lambda} \frac{z_1 - \lambda}{1 - \lambda z_1} \leqq \frac{1}{\lambda},$$

and so:

$$W(0) \geqq -\frac{1}{\tau\lambda}, \quad \hat{W}(1) \leqq 1 + \frac{1}{\tau\lambda} = 1 + \left(\frac{1}{\lambda}\right)^{r+1}. \tag{3.52}$$

Recall that we have proved the existence of an upper bound for $\frac{1}{\tau}$ depending only on λ.

We also note that $\hat{W}(1-\alpha)=1-1/\tau$, and since $1-\alpha<-\lambda$, $\hat{W}(1-\alpha)/(1-\alpha) \leqq 1/\tau\lambda$. Together with (3.52) and the remarks in Subsect. 2.6, this shows that for all ζ with $1-\alpha \leqq \mathrm{Re}\,\zeta \leqq 1$,

$$\frac{|\hat{W}(\zeta)|}{|\zeta|} \leqq 1+\frac{1}{\tau\lambda}. \tag{3.53}$$

3.8. Definition of ψ

Our purpose is now to construct a function $\psi=1-\hat{\psi}$ in $-\mathbf{P}((-\lambda^{-1},\lambda^{-2}))$, satisfying:

$$\begin{aligned} &\psi(z)=V(\psi(-\lambda z))=W(\psi(\lambda^2 z)),\\ &\psi(0)=1, \quad \psi(1)=0, \end{aligned} \tag{3.54}$$

or, equivalently,

$$\begin{aligned} &\hat{\psi}(z)=\hat{V}(\hat{\psi}(-\lambda z))=\hat{W}(\hat{\psi}(\lambda^2 z)),\\ &\hat{\psi}(0)=0, \quad \hat{\psi}(1)=1. \end{aligned} \tag{3.55}$$

Recall that:

$$\begin{aligned} &\hat{V}\in -\mathbf{P}((1-\alpha\tau^{-2},1)), \quad \hat{V}(0)=0, \quad \hat{V}'(0)=-\lambda^{-1},\\ &\hat{W}=\hat{V}\circ\hat{V}\in \mathbf{P}((1-\alpha,1)), \quad \hat{W}(0)=0, \quad \hat{W}'(0)=\lambda^{-2}. \end{aligned}$$

As it is well-known, because $\lambda<1$, it is always possible to construct a unique function Ψ, holomorphic in a small disk around 0, and satisfying there:

$$\Psi(z)=\hat{V}(\Psi(-\lambda z))=\hat{W}(\Psi(\lambda^2 z)), \quad \Psi(0)=0, \quad \Psi'(0)=1. \tag{3.56}$$

For the sake of definiteness, we state the following lemma, which it is straightforward to verify (and by no means the best possible estimate):

Lemma 3. *Let f be a function holomorphic in $\{z\in\mathbf{C}\colon |z|<T\}$ and satisfying there, for some $M>0$, $\omega\in\mathbf{C}$, $0<|\omega|=s<1$,*

$$\left|f'(z)-\frac{1}{\omega}\right| \leqq 2M|z|, \quad \left|f(z)-\frac{z}{\omega}\right| \leqq M|z^2|.$$

Let $0<s<\kappa<1$ and $0<sR<\min[T/2,(\kappa-s)/4Ms^2]$. Then the mapping

$$K_f h(z)=f(h(\omega z))$$

is well defined on the class of the functions h which are holomorphic on $\{z\in\mathbf{C}\colon |z|<R\}$ and satisfy there

$$|h(z)-z|<|z^2|/sR.$$

It sends this class into itself and is a contraction with ratio κ in the distance

$$\|h_1-h_2\|=\sup\{|z^{-2}[h_1(z)-h_2(z)]|\colon |z|<R\}.$$

To apply this lemma, we derive from (3.49), first for real ζ, by the usual estimates, then for complex ζ by the remark in Subsect. 2.6, that, for $|\zeta|<3\lambda/4$, $|\hat{W}(\zeta)/\zeta|<4\lambda^{-3}$, hence by the Cauchy inequalities,

$$|\hat{W}''(\zeta)|<24\lambda^{-4} \quad \text{for} \quad |\zeta|<\lambda/2\,.$$

We can take, in the lemma, $f=\hat{W}$, $T=\lambda/2$, $M=12\lambda^{-4}$, $s=\lambda^2$, $\kappa=(1+\lambda^2)/2$, and $R=\min\{(1-\lambda^2)/96\lambda^2, 1/4\lambda\}$.

In the disk $\{z\colon |z|<R\}$, Ψ is the limit of a uniformly convergent sequence:

$$\Psi_n(z)=\hat{W}^n(\lambda^{2n}z)\,,$$

which satisfies $|\Psi_n(z)|<T$ for $|z|<R$.

We can now proceed to extend Ψ outside of the small disk by using the functional equation in (3.56). Note that, inside the disk, $\operatorname{Im} z>0$ implies $\operatorname{Im}\Psi(z)\geqq 0$, the equality being possible only if Ψ is a constant: but this is excluded by $\Psi'(0)=1$. Hence, in $\mathbf{C}_+\cup\mathbf{C}_-$, there never is any obstruction to extending Ψ, which is thus a Herglotz function (this also follows from Vitali's theorem). On R_+, we claim that Ψ continuously extends to a segment $[0,\gamma\lambda^{-2}]$ on which it assumes all the values in $[0,\hat{W}(1)]$. Indeed Ψ is clearly increasing as far as it can be extended. If its value never reaches $\hat{W}(1)$, then it can never reach 1. But then $\hat{W}(\Psi(z))$ is always defined, so Ψ extends to all of $\mathbf{R}_+$. It also extends to all of $\mathbf{R}_-$ by $\Psi(z)=\hat{V}(\Psi(-\lambda z))$. This implies that $\Psi(z)\equiv z$, absurd since $\Psi(z)<1$ by assumption for $z>0$. Therefore $\Psi(\mathbf{R}_+)$ contains $(0,\hat{W}(1))$. In particular there is a $\gamma>0$ such that $\Psi(\gamma)=1$, and hence $\Psi(\gamma\lambda^{-2})=\hat{W}(1)$, and $\Psi(-\gamma\lambda^{-1})=\hat{V}(1)=1-1/\tau$.

We can now define $\hat{\psi}(z)=\Psi(\gamma z)$, which satisfies the requirements in (3.55), and $\psi=1-\hat{\psi}$. Note that these functions are continuous at the ends of their real interval of analyticity, $(-\lambda^{-1},\lambda^{-2})$, with:

$$\begin{aligned}\hat{\psi}(-\lambda^{-1})&=1-1/\tau\,, \qquad \psi(-\lambda^{-1})=1/\tau\,,\\ \hat{\psi}(\lambda^{-2})&=\hat{W}(1)\,, \qquad \psi(\lambda^{-2})=W(0)\,.\end{aligned} \tag{3.57}$$

The map M_λ is defined by $M_\lambda\psi_0=\psi$.

Note that (3.57) and (3.52) imply that $|\psi(z)|\leqq 1/\tau\lambda$ for all $z\in(-\lambda^{-1},\lambda^{-2})$, and hence there is a constant $C_1(\lambda)>0$, depending only on λ, such that $|\psi(z)|\leqq C_1(\lambda)$ in $(-\lambda^{-1},\lambda^{-2})$.

3.9. Continuity of the Map M_λ

Recall that z_1 was defined as the unique zero of the function q in $(0,1)$. The bounds obtained in the preceding sections, in particular those of Subsect. 3.3, make it clear that z_1 is a continuous function of ψ_0; recall that we are using the topology of the Fréchet space $\mathbf{F}((-\lambda^{-1},\lambda^{-2}))$. [For example z_1 is the integral of $(2\pi i)^{-1}tf'(t)f(t)^{-1}dt$ on a small contour surrounding it, with $f(z)=zq(z)$.] Hence τ, r, α, V, W all depend continuously on ψ_0.

The restriction of Ψ to $\{z\colon |z|\leqq R\lambda^2\}$ is also continuous in ψ_0, since e.g. its Taylor series converges uniformly, and its coefficients depend continuously on ψ_0.

By the very construction of Ψ, the domain of Ψ is the union of an increasing sequence of compacts $\{K_n\}$, with $K_{n-1}\subset\subset K_n$, $-\lambda K_n\subset\subset K_{n-1}$, $K_0=\{z\colon |z|\leqq\lambda^2 R\}$, such that $\Psi(K_n)$ is contained in the domain of $\hat{V}$. We prove inductively the continuous dependence on ψ_0 of the restriction of Ψ to K_n, assuming it to hold on K_{n-1}. Let ψ_1 be a function close to ψ_0 in $-\mathbf{P}((-\lambda^{-1},\lambda^{-2}))$, and $\hat{V}_1$, Ψ_1 the functions obtained from it in the same way as $\hat{V}$, Ψ from ψ_0. For a given $\varepsilon>0$, ψ_1 can be chosen so close to ψ_0 that, for all $\zeta\in\Psi(K_{n-1})+\Delta(\varepsilon)$, $\hat{V}(\zeta)$ and $\hat{V}_1(\zeta)$ are both defined with $|\hat{V}(\zeta)-\hat{V}_1(\zeta)|<\varepsilon/2$, $\Delta(\varepsilon)=\{z\colon |z|\leqq\varrho_\varepsilon\}$. Let S be an upper bound for $|\hat{V}'|$ on $\Psi(K_{n-1})+\Delta(\varepsilon)$. Choose ψ_1 so close to ψ_0 that, for all $z\in K_{n-1}$, $\Psi_1(z)$ is defined and $|\Psi_1(z)-\Psi(z)|<\min(\varrho_\varepsilon,\varepsilon/2S)$. Then for $z\in K_n$, $\Psi_1(-\lambda z)\in\Psi(K_{n-1})+\Delta(\varepsilon)$, and

$$\begin{aligned}|\Psi_1(z)-\Psi(z)|\leqq&|\hat{V}(\Psi_1(-\lambda z))-\hat{V}_1(\Psi_1(-\lambda z))|\\&+|\hat{V}(\Psi_1(-\lambda z))-\hat{V}(\Psi(-\lambda z))|\leqq\varepsilon.\end{aligned}$$

Thus Ψ depends continuously on ψ_0. It remains to check that γ is also continuous in ψ_0. This follows from $\gamma=-\psi'(0)$, and the fact that the same bounds, obtained for $\psi_0'(z)$ in (3.5), also hold for $\psi'(z)$. In particular, $1-\lambda^2\leqq\gamma\leqq1+\lambda$. Since the domain of analyticity of Ψ around γ keeps a finite size, and $[\Psi'(\gamma)]^{-1}$ remains bounded, $\gamma=\Psi^{-1}(1)$ depends continuously on ψ_0.

The information collected at this point suffices to apply the Schauder-Tikhonov theorem. Before we do so, we shall devote the two next subsections to obtaining bounds uniform in λ in the limit $\lambda\to1$.

3.10. The Functions H and H_0

We define:

$$H(w)=\psi(e^{\beta w}),\qquad H_0(w)=\psi_0(e^{\beta w}),\qquad \beta=\log\frac{1}{\lambda},\tag{3.58}$$

and

$$\hat{H}=1-H,\qquad \hat{H}_0=1-H_0.\tag{3.59}$$

These functions are holomorphic and periodic with period $2\pi i/\beta$ in $\mathbf{C}$ minus the cuts:

$$2+\frac{2mi\pi}{\beta}+\mathbf{R}_+,\qquad 1+\frac{(2m+1)i\pi}{\beta}+\mathbf{R}_+,\qquad m\in\mathbf{Z}.$$

In particular, H and H_0 map the strips $\{w\colon 0<\pm\operatorname{Im}w<\pi/\beta\}$ into $-\mathbf{C}_\pm$, respectively. They tend to 1 when w tends to infinity in the negative real direction. When w is real and increases from $-\infty$ to 2, they decrease from 1 to $\psi(\lambda^{-2})$ and $\psi_0(\lambda^{-2})$, respectively. They satisfy the functional equations:

$$H(w)=W(H(w-2)),\qquad \hat{H}(w)=\hat{W}(\hat{H}(w-2)),\tag{3.60}$$

and

$$H(0)=H_0(0)=0,\qquad \hat{H}(0)=\hat{H}_0(0)=1.\tag{3.61}$$

For real $w<2$,

$$H_0'(w)=\beta z\psi_0'(z),$$

$$\frac{H_0''(w)}{H_0'(w)}=\beta\left[1+z\frac{\psi_0''(z)}{\psi_0'(z)}\right],$$

where $z=e^{\beta w}$. By (3.6) it follows:

$$\frac{H_0''(w)}{H_0'(w)} \geqq \beta \frac{1-\lambda z}{1+\lambda z},$$

and this is positive for $z<1/\lambda$, i.e. $w<1$. Thus H_0 is concave decreasing on $(-\infty, 1)$. Since ψ obeys the same bounds (3.6) as ψ_0, H is also concave decreasing on $(-\infty, 1)$.

We now denote:

$$w_1 = -\zeta_1 = -\log z_1/\log\lambda, \quad \text{i.e.} \quad z_1 = \exp\beta w_1, \qquad \zeta_1 = \frac{\log\alpha}{\log(1/\tau)}. \tag{3.62}$$

Then $0<\zeta_1<1$, and:

$$H_0(w_1)=\tau, \qquad H_0'(w_1) = -\frac{\tau}{\lambda}\log\frac{1}{\tau}. \tag{3.63}$$

Since H_0 is decreasing concave, and vanishes at 0,

$$|H_0'(w_1)| < \frac{H_0(w_1)}{\zeta_1},$$

and so:

$$\log\alpha \leqq \lambda, \qquad \alpha \leqq e^{\lambda}. \tag{3.64}$$

We now turn to some consequences of the functional equations (3.60). They imply:

$$\hat{H}(-2n) = \hat{W}^{-n}(1) \tag{3.65}$$

for all $n\in\mathbf{N}$. Let $w\leqq 0$, and denote temporarily $x=\hat{H}(w)$, $y=\hat{H}(w-2)$. Then $0<y<x\leqq 1$, and $x=\hat{W}(y)$, so that, by (3.51),

$$ay^3+y-x\leqq 0.$$

To verify that this implies $y\leqq x(1-a'x^2)$, for a certain $a'>0$, it suffices to check that, for all $x\in(0,1)$,

$$ax^3(1-a'x^2)^3-a'x^3\geqq 0,$$

i.e.

$$a(1-a')^3-a'\geqq 0.$$

Since the left-hand side is $\geqq a(1-3a')-a'$, this inequality is satisfied by

$$a'=\frac{a}{1+3a}.$$

For $a=1/6$, this gives $a'=1/9$. Thus, for $w\leqq 0$, by the convexity of $\hat{H}$,

$$\hat{H}'(w)\geqq \frac{1}{2}[\hat{H}(w)-\hat{H}(w-2)], \qquad \hat{H}'(w)\geqq \frac{a'}{2}\hat{H}(w)^3. \tag{3.66}$$

In particular,

$$\hat{H}'(0) \geqq \frac{a'}{2}, \quad \text{i.e.} \quad \hat{\psi}'(1) \geqq \frac{a'}{2\log(1/\lambda)}. \tag{3.67}$$

Integrating (3.66) with the initial condition $\hat{H}(0)=1$, we find:

$$\hat{H}(w) \leqq (1-a'w)^{-1/2}, \qquad w \leqq 0. \tag{3.68}$$

The inequality (3.66) is equivalent to

$$\beta z \hat{\psi}'(z) \geqq \frac{a'}{2} \hat{\psi}(z)^3, \quad \text{for all} \quad z \in [0,1]. \tag{3.69}$$

For any $z \in [0,1]$, $\hat{\psi} \to (\hat{\psi}(z), \beta z \hat{\psi}'(z))$ is a continuous linear map of $\mathbf{P}((-\lambda^{-1}, \lambda^{-2}))$ into $\mathbf{R}^2$, and (3.69) requires its image to be contained in the closed convex set $\{(X,Y)\colon 2Y \geqq a'X^3 \geqq 0\}$. Therefore:

Lemma 4. *The map M_λ sends into itself the compact convex set*

$$\begin{aligned} \mathbf{E}_1(\lambda) = \{\psi_0 \in -\mathbf{P}((-\lambda^{-1}, \lambda^{-2}))\colon & \psi_0(0)=1,\ \psi_0(1)=0, \\ & |\psi_0(z)| \leqq C_1(\lambda) \text{ for all } z \in (-\lambda^{-1}, \lambda^{-2}), \\ & (2z\log\lambda)\psi_0'(z) \geqq a'[1-\psi_0(z)]^3 \text{ for } z \in [0,1]\}. \end{aligned} \tag{3.70}$$

To prove the compactness of $\mathbf{E}_1(\lambda)$, note that every function belonging to it maps $\mathbf{C}((-\lambda^{-1}, \lambda^{-2}))$ into $\mathbf{C}((-2C_1(\lambda), 2C_1(\lambda)))$, which can be conformally mapped into the unit disk by an obvious transformation, so that $\mathbf{E}_1(\lambda)$ is a normal family, and that every limit of functions in $\mathbf{E}_1(\lambda)$ is in $\mathbf{E}_1(\lambda)$.

From now on, we assume that ψ_0 is chosen in $\mathbf{E}_1(\lambda)$, and that, therefore, the inequalities (3.66)–(3.69) hold with $\hat{H}$ and $\hat{\psi}$ replaced by $\hat{H}_0$ and $\hat{\psi}_0$ respectively. In particular:

$$-\psi_0'(1) \geqq \frac{a'}{2\log 1/\lambda}. \tag{3.71}$$

To conclude this subsection, we note the formula:

$$V(\zeta) = \frac{1}{\tau} H_0\left(\frac{\log(\zeta/\alpha)}{\log(1/\tau)}\right). \tag{3.72}$$

3.11. Uniform Lower Bound on τ

Using the fact that ψ_0 is now supposed to belong to $\mathbf{E}_1(\lambda)$, we can improve the estimates in Subsect. 3.4 so as to get a lower bound on τ uniform as $\lambda \to 1$. First by inserting (3.71) into (3.25), we get:

$$k \leqq \log \frac{2\log\lambda^{-1}}{(1-\lambda^2)a'} \leqq \log \frac{2}{a'\lambda(1+\lambda)}. \tag{3.73}$$

From this and (3.27), it follows that there is a constant $K_1 > 0$, independent of λ, such that, for all $\lambda \in (0,1)$, $k \leqq K_1 \lambda$. Since

$$\frac{1-\lambda^2}{\log 1/\lambda} \leqq 2,$$

the constant K of (3.28) can be taken equal to $4K_1+2$, i.e. independent of λ. Thus ξ_1 has an upper bound $\xi_{\max}$ which tends to a finite limit as $\lambda\to 1$, in particular:

$$\xi_{\max} \leqq 2K+2B\log B.$$

Second, we note that the bound

$$\frac{1}{\tau} \leqq \frac{\xi_{\max}-\lambda}{1-\lambda} \tag{3.74}$$

is ineffective as λ tends to 1. We therefore use the bound (3.68), as applied to $\hat{H}_0$ rather than $\hat{H}$, to get:

$$\tau = 1-\hat{H}_0(-\zeta_1) \geqq 1-(1+a'\zeta_1)^{-1/2}, \tag{3.75}$$

where ζ_1, defined in (3.62), verifies

$$\zeta_1 \geqq \frac{1}{\log\lambda}\log\frac{\xi_{\max}-1}{\xi_{\max}-\lambda^2} \geqq \frac{\lambda(1+\lambda)}{\xi_{\max}-\lambda^2}. \tag{3.76}$$

It follows from (3.75) that

$$\frac{1}{\tau} \leqq \frac{3}{2}+\frac{2}{a'\zeta_1} \leqq \frac{3}{2}+\frac{2(\xi_{\max}-\lambda^2)}{a'\lambda(1+\lambda)}. \tag{3.76}$$

This bound is well behaved as $\lambda\to 1$.

3.12. Existence of Fixed Points

As a result of the last two subsections, we have:

Lemma 5. *There exists a continuous function* $\lambda\to C(\lambda)$ *on* $(0,1]$ *such that, for each* λ *in* $(0,1)$, *the map* M_λ *sends into itself the compact convex set*

$$\begin{aligned}\mathbf{E}(\lambda) = \{\psi_0 \in -\mathbf{P}((-\lambda^{-1},\lambda^{-2})) \colon \psi_0(0)=1,\ \psi_0(1)=0,\\ |\psi_0(z)| \leqq C(\lambda) \text{ for all } z\in(-\lambda^{-1},\lambda^{-2}),\\ (2z\log\lambda)\psi_0'(z) \geqq a'[1-\psi_0(z)]^3 \text{ for } z\in[0,1]\}.\end{aligned} \tag{3.78}$$

Remarks. 1. $\mathbf{E}(\lambda)$ is compact and convex for the same reasons as $\mathbf{E}_1(\lambda)$ (which contains it).

2. It has actually been proved that:

$$\mathbf{E}(\lambda) \subset M_\lambda(\mathbf{E}_1(\lambda)), \qquad \mathbf{E}_1(\lambda) \subset M_\lambda(\mathbf{E}_0(\lambda)),$$

where

$$\mathbf{E}_0(\lambda) = -\mathbf{P}((-\lambda^{-1},\lambda^{-2})) \cap \{\psi_0 \colon \psi_0(0)=1,\ \psi_0(1)=0\}.$$

3. It is not difficult, but not very enlightening, to obtain, along the lines suggested at various places in the preceding subsections, an explicit version of $C(\lambda)$.

Applying the Schauder-Tikhonov theorem, we obtain:

Theorem 6. *There exists, for each* $\lambda\in(0,1)$, *at least one fixed point of* M_λ *in* $\mathbf{E}(\lambda)$. *Every fixed point of* M_λ *in* $\mathbf{E}_0(\lambda)$ *is in* $\mathbf{E}(\lambda)$.

3.13. Some Properties of the Fixed Points

We now assume that a fixed point has been chosen in $\mathbf{E}(\lambda)$ for each λ in $(0,1)$, and denote the corresponding functions ψ_λ, V_λ, W_λ, H_λ, etc., keeping the preceding meaning for τ, α, r. These numbers depend, of course, on the choice of the fixed point. Note that:

$$\begin{aligned} &V_\lambda(0)=\frac{1}{\tau}=\psi_\lambda(-\lambda^{-1}), \qquad V_\lambda(\alpha)=0, \\ &\psi_\lambda(\lambda^{-2})=W_\lambda(0)=H_\lambda(2), \\ &V_\lambda(\alpha\tau^{-2})=\frac{1}{\tau}W_\lambda(0). \end{aligned} \tag{3.79}$$

We define $y_0=1/\alpha$, $x_0=\alpha^{-1/r}$, and:

$$\begin{aligned} &U_\lambda(z)=y_0\psi_\lambda(z), \qquad U_\lambda\in -\mathbf{P}((-\lambda^{-1},\lambda^{-2})), \\ &u_\lambda(z)=U_\lambda(z)^{1/r}, \qquad u_\lambda\in -\mathbf{P}((-\lambda^{-1},1)). \end{aligned} \tag{3.80}$$

These functions satisfy:

$$\begin{aligned} &u_\lambda(z)=\frac{1}{\lambda}u_\lambda(u_\lambda(-\lambda z)) \quad \text{for all} \quad z\in\mathbf{C}((-\lambda^{-1},1)), \\ &U_\lambda(1)=u_\lambda(1)=0, \qquad u_\lambda(0)=x_0, \qquad U_\lambda(0)=y_0. \end{aligned} \tag{3.81}$$

A straightforward generalization of the results of [9] is possible. We enumerate some salient facts without going into details.

1) The Feigenbaum function restricted to $[0,x_0/\lambda]$ is the inverse function g_λ of the restriction of u_λ to $[-\lambda^{-1},1]$. It satisfies all the conditions **C1–C5**.

2) The function u_λ has continuous boundary values at the border of $\mathbf{C}((-\lambda^{-1},1))$ and its values there are always non-real except at $-\lambda^{-1}$ and 1. Its only singularities on the real axis are simple branch points at $(-\lambda)^{-n}$, $n=0,1,2,3,\ldots$. Its continuation across its regularity segments on $\mathbf{R}$ can be studied by the method of [9]. (Note that when r is not an integer, the "domain of analyticity" of g_λ becomes a ramified Riemann surface.)

3) The functions ψ_λ, V_λ, W_λ, H_λ are also continuous at the boundaries of the cut planes where they have been defined, and they are bounded there. In fact $u_\lambda(-i\infty)=c(\lambda)\in\mathbf{C}_+$ is a periodic point of period 2 for u_λ/λ: $u_\lambda(c(\lambda))=\lambda c(\lambda)^*$.

3.14. Existence of the Eckmann-Wittwer Functions

Recall (see Subsect. 3.6) that the function $\hat{W}_\lambda$ is in $\mathbf{P}((1-\alpha,1))$ and is bounded in modulus on the real interval $(1-\alpha,1)$, by $1+1/\tau\lambda$, with $\alpha>1+\lambda$. On that interval, we have therefore $|\hat{W}_\lambda(\zeta)|\leqq C(\lambda)$. In particular there is a constant $\hat{C}$ such that for all $\lambda\geqq 0.5$, $|\hat{W}_\lambda(\zeta)|<\hat{C}$ on $(1-\alpha,1)$. As a result, the functions $\{\hat{W}_\lambda,\ \lambda\geqq 0.5\}$ form a normal family, and we can find a sequence $\{\lambda_n\}$, tending to 1, such that the $\hat{W}_{\lambda_n}$ converge to $\hat{W}_1\in\mathbf{P}((-1,1))$. This function is finite and non-constant, since it must satisfy

$$\hat{W}_1(0)=0, \qquad \hat{W}_1'(0)=1. \tag{3.82}$$

For $0.5 \leqq \lambda < 1$, the function $\hat{H}_\lambda$ is holomorphic in the domain

$$\left\{w:\ 0<|\mathrm{Im}\,w|<\frac{\pi}{-\log\lambda} \text{ or } \mathrm{Im}\,w=0 \text{ and } \mathrm{Re}\,w<2\right\}, \tag{3.83}$$

which it maps into $\mathbf{C}_+ \cup \mathbf{C}_- \cup \{|w|<\hat{C}\}$. Hence $\{\hat{H}_\lambda\}$ is also a normal family and, changing to a subsequence if necessary, we can arrange that the $\hat{H}_{\lambda_n}$ converge, uniformly on every compact subset of $\mathbf{C}((-\infty, 2))$, to a Herglotz function $\hat{H}_1$. The subsequence can also be chosen such that the corresponding τ and α also have limits τ_1 and α_1. In the limit, we have:

$$\hat{H}_1(w)=\hat{W}_1(\hat{H}_1(w-2)), \quad \hat{H}_1(0)=1, \tag{3.84}$$

and, on the real axis,

$$0<\hat{H}_1(w) \leqq \frac{1}{\sqrt{1-w/9}}, \quad w \leqq 0. \tag{3.85}$$

Since, for $\lambda<1$, Eq. (3.72) holds, the functions V_{λ_n} also converge to a function $V_1 \in -\mathbf{P}((0, \alpha_1\tau_1^{-2}))$. This function satisfies $V_1(1)=1$, $V_1'(1)=-1$, and, at least near 1, $V_1(V_1(\zeta))=W_1(\zeta) \equiv 1-\hat{W}_1(1-\zeta)$. This relation extends analytically in $\mathbf{C}((0, \alpha_1))$.

It is interesting to ask about the fate of the function ψ_{λ_n} when $n \to \infty$. Since $\psi_\lambda(z) = H_\lambda(\log z/\log 1/\lambda)$, and since $\hat{H}_\lambda(w) \leqq (1-w/9)^{-1/2}$ on $(-\infty, 0)$, we see that, if e.g. z is fixed in $(0,1)$, $\psi_{\lambda_n}(z) \to 1$. This is not in contradiction with $\psi_\lambda(1)=0$, because 1 is not interior to the limit (intersection in this case) of the domains $\mathbf{C}((-\lambda^{-1}, \lambda^{-2}))$ as λ tends to 1, but on the boundary of this limit.

4. The Fixed r Method

It is tempting to apply the preceding method, with its quasi-tautological estimates, to prove the existence of the Feigenbaum functions for given values of r, instead of λ. This section describes the very limited extent to which this can be carried out, at least in a straightforward way. In this section, $r>1$ is fixed once and for all.

We start again from a function $\psi_0 = 1-\hat{\psi}_0$ belonging to $-\mathbf{P}((-\lambda_0^{-1}, \lambda_0^{-2}))$, with $\psi_0(0)=1$ and $\psi_0(1)=0$, where $\lambda_0 \in (0,1)$ depends on the choice of ψ_0. We then attempt to define a function V by the same formula (3.7) as in the fixed-λ method, the constants $\tau>0$ and $\alpha>1$ being determined by requiring that:

There must exist $\lambda \in (0,1)$ such that:

$$V(1)=1, \quad V'(1)=-\frac{1}{\lambda}, \quad \lambda^r=\tau. \tag{4.1}$$

This implies that $z_1=\alpha^{-1/r}$ must satisfy

$$\psi_0(z_1)=\tau=\lambda^r, \quad \frac{z_1}{r}\frac{\psi_0'(z_1)}{\psi_0(z_1)}=-\frac{1}{\lambda}, \tag{4.2}$$

and hence

$$-\frac{z_1}{r}\psi_0'(z_1)\psi_0(z_1)^{1/r-1} \equiv -z\frac{d}{dz}\psi_0(z)^{1/r}|_{z=z_1}=1. \tag{4.3}$$

There clearly exists a $z_1 \in (0,1)$ which satisfies (4.3), and it has to be estimated as well as possible. As in Sect. 3,

$$\frac{1}{1+\lambda_0 z} \leqq \frac{\psi_0(z)}{1-z} \leqq \frac{1}{1-\lambda_0^2 z} \quad (0 \leqq z < \lambda_0^{-2}), \tag{4.4}$$

and, for $-\lambda_0^{-1} < z < 1$,

$$\frac{(1-\lambda_0^2)}{(1-\lambda_0^2 z)(1-z)} \leqq \frac{-\psi_0'(z)}{\psi_0(z)} \leqq \frac{(1+\lambda_0)}{(1+\lambda_0 z)(1-z)}. \tag{4.5}$$

The similar bound $\psi_0''(z)/\psi_0'(z) \geqq -2\lambda_0/(1+\lambda_0 z)$ implies that the left-hand side of (4.3) has a strictly positive derivative in $(0,1)$, so that (4.3) has a unique solution there. Moreover (4.3), (4.4), and (4.5) imply:

$$\begin{gathered} \frac{z_1(1-\lambda_0^2)}{r(1-z_1)^{1-1/r}(1-\lambda_0^2 z_1)(1+\lambda_0 z_1)^{1/r}} \leqq 1, \\ 1 \leqq \frac{z_1(1+\lambda_0)}{r(1-z_1)^{1-1/r}(1+\lambda_0 z_1)(1-\lambda_0^2 z_1)^{1/r}}. \end{gathered} \tag{4.6}$$

The first and last expressions in (4.6) are increasing in z_1. For a fixed λ_0, they take the value 0 at $z_1 = 0$, and $+\infty$ at $z_1 = 1$. Hence the first and last inequalities respectively express $z_1 \leqq z_{\max}(\lambda_0)$ and $z_1 \geqq z_{\min}(\lambda_0)$. Moreover the first (respectively last) expression in (4.6) is, for a fixed z_1, decreasing (respectively increasing) in λ_0. Hence *$z_{\max}(\lambda_0)$ is an increasing function of λ_0 and $z_{\min}(\lambda_0)$ is a decreasing function of λ_0.*

From (4.5) it now follows that

$$\lambda_{\min}(\lambda_0) \leqq \lambda \leqq \lambda_{\max}(\lambda_0), \tag{4.7}$$

where

$$\lambda_{\min}(\lambda_0) = \left[\frac{1-z_{\max}(\lambda_0)}{1+\lambda_0 z_{\max}(\lambda_0)}\right]^{1/r}, \tag{4.8}$$

$$\lambda_{\max}(\lambda_0) = \left[\frac{1-z_{\min}(\lambda_0)}{1-\lambda_0^2 z_{\min}(\lambda_0)}\right]^{1/r}. \tag{4.9}$$

It is easy to verify that [because $z_{\max}'(\lambda_0) \geqq 0$] *$\lambda_{\min}(\lambda_0)$ is a decreasing function of λ_0*, so that $\lambda_0 \leqq b$ implies $\lambda \geqq \lambda_{\min}(\lambda_0) \geqq \lambda_{\min}(b)$.

To obtain an upper bound on λ, we can use some of the work expended on the fixed-λ method as follows. We note that z_1 is a zero of the same function q as in (3.10), with *the same expression for A*, viz. $A = A(\lambda) = 1/(-\lambda \log \lambda)$. Since $\log \psi_0$ has an integral representation analogous to (3.12), we immediately get:

$$z_1 \geqq 1 + \lambda \log \lambda \geqq \lambda. \tag{4.10}$$

The difference with the case of Sect. 3 is that the real domain of analyticity of ψ_0 is now $(-\lambda_0^{-1}, \lambda_0^{-2})$ and that σ is known to be equal to 1 on $[1, \lambda_0^{-2}]$, instead of $[1, \lambda^{-2}]$.

Assume now that $\lambda \geqq \lambda_0$. Then we can repeat all the calculations of Subsect. 3.2, giving lower bounds on z_1 and $1/\tau$: denoting

$$\xi = \frac{1-\lambda^2 z}{1-z}, \qquad z = \frac{\xi - 1}{\xi - \lambda^2}, \tag{4.11}$$

we find again:

$$(1-\lambda^2)zq(z) \geqq (1-\lambda^2)zq_2(z) = \chi(\xi) - \xi, \tag{4.12}$$

$$\chi(\xi) = B\log\xi + 1 + \lambda^2 - \frac{\lambda^2}{\xi}. \tag{4.13}$$

Recall that χ is a concave increasing function and that $\chi(\xi)-\xi$ vanishes at 1 and at a unique $\hat{\xi} > 1$, which is a lower bound for $\xi_1 = (1-\lambda^2 z_1)/(1-z_1)$, itself a lower bound for $1/\tau$. Since $0<\lambda<1$, we can make the change of variable $\xi = \lambda^{-x}$, with $x>0$. Then if $\hat{\xi} = \lambda^{-\hat{x}}$, it has been seen in Subsect. 3.2 that

$$\lambda^2 \leqq \frac{\hat{x}-1}{1+\hat{x}}, \qquad \hat{x} \equiv \hat{x}(\lambda) > \frac{1+\lambda^2}{1-\lambda^2}. \tag{4.14}$$

Moreover, $0 < x < \hat{x}(\lambda)$ is equivalent to $\xi^{-1}[\chi(\xi)-\xi] > 0$, which, in terms of x, reads:

$$x\lambda^{x-1}(1-\lambda^2) - (1-\lambda^x)(1-\lambda^{x+2}) \equiv Z(\lambda, x) > 0. \tag{4.15}$$

Since this is a decreasing function of ξ for $\xi \geqq \hat{\xi}$, it is (for fixed λ) decreasing in x for $x \geqq \hat{x}(\lambda)$. It is easily checked that $\partial Z(\lambda, x)/\partial\lambda > 0$ when $\lambda^2 \leqq (x-1)/(1+x)$. Hence $\hat{x}(\lambda)$ is an increasing function of λ. The last inequality in (4.14), and an upper bound on $\hat{x}(\lambda)$ which is easy to get (by the same argument as in the remark at the end of Subsect. 3.4) show that $\hat{x}(\lambda) \to 1$ as $\lambda \to 0$. We denote $x \to b(x)$ the function equal to 0 for $0 \leqq x \leqq 1$, and to the inverse function of $\hat{x}$ for $x>1$. The inequality (4.15) is then equivalent to: $\lambda > b(x)$. Since $1/\tau \geqq \hat{\xi}$, we must have $r \geqq \hat{x}(\lambda)$, and hence

$$r\lambda^{r-1}(1-\lambda^2) - (1-\lambda^r)(1-\lambda^{r+2}) \leqq 0. \tag{4.16}$$

This means that we must have

$$\lambda \leqq b(r), \qquad b(r)^2 < \frac{r-1}{1+r}. \tag{4.17}$$

This proves:

If $b \in [b(r), 1)$, then $\lambda_0 \leqq b$ implies that $\lambda \leqq b$ and also, as already seen, $\lambda \geqq \lambda_{\min}(b)$.

Choosing λ_0 in $[\lambda_{\min}(b(r)), b(r)]$, we can take z_1 as the solution of (4.3) in $(0,1)$ and define $\tau = \psi_0(z_1)$, $\lambda = \tau^{1/r}$, $\alpha = z_1^{-r} < \tau^{-1}$, and V by (3.7). The function V then belongs to $-\mathbf{P}((0, \alpha\lambda_0^{-2r}))$, vanishes at α and is convex on $(0,\alpha)$ for the same reasons as in Sect. 3. In particular, again, $\alpha \geqq 1+\lambda$. The function $W = V \circ V$ is certainly defined near 1, and also at α, since $W(\alpha) = V(0) = 1/\tau$.

However, in order for W to be defined at 0, it is necessary that $V(1/\tau)$ be defined, i.e. that $0 < (\alpha\tau)^{-1/r} \leqq \lambda_0^{-2}$, or, equivalently,

$$z_1\lambda_0^2 \leqq \lambda. \tag{4.18}$$

If so, then $W(0)<0$, since $1/\tau>\alpha$. A sufficient condition for (4.18) to hold is that, for some $b \geqq b(r)$,

$$1-z_{\max}(b)-(1+bz_{\max}(b))[b^2 z_{\max}(b)]^r>0. \tag{4.19}$$

Let us assume that (4.19) holds. Then $W \in \mathbf{P}((0,\alpha))$. The estimates in Sect. 3, leading to

$$\hat{W}(\zeta) \equiv 1-W(1-\zeta) \geqq \zeta(1+a\zeta^2), \tag{4.20}$$

with $a \geqq 1/6$, for $\zeta \in (0,1)$, extend to the present case. Defining, as before,

$$H_0(w)=\psi_0(\exp \beta_0 w), \qquad \beta_0=-\log \lambda_0,$$

we again find that H_0 is concave on $(-\infty, 1)$. The same calculations as in Subsect. 3.7 give

$$W(0) \geqq -\frac{1}{\tau}\frac{z_1-\lambda}{\lambda-\lambda_0^2 z_1},$$

and hence

$$\hat{W}(1) \leqq 1+\frac{1-b^2}{\lambda_{\min}^r(\lambda_{\min}-b^2 z_{\max})}. \tag{4.21}$$

The hypothesis that (4.19) holds implies that this bound is finite for the considered value of r. The final steps of the method are the same as in the fixed-λ method. To conclude:

Lemma 7. *For a fixed value of $r>1$, a sufficient condition for the existence of a fixed point is that there exist a pair (b,z) of numbers in $(0,1)$ satisfying the three following conditions:*

$$rb^{r-1}(1-b^2)-(1-b^r)(1-b^{r+2})>0, \tag{4.22}$$

$$z(1-b^2)-r(1-z)^{1-1/r}(1+bz)^{1/r}(1-b^2 z)>0, \tag{4.23}$$

$$1-z-(1+bz)(b^2 z)^r>0. \tag{4.24}$$

Recall that (4.22) implies $b>b(r)$, and that $b(r)^2<(r-1)/(1+r)$.

For a given r, it is easy to verify numerically whether or not the conditions in the lemma can be satisfied, and it appears that the method works for all $r \leqq 14.4$, although we have no proof of the (numerically patent) fact that, if the conditions can be satisfied for a certain r_0, the same is true for all $r \in (1, r_0)$. The method, however, is particularly easy to apply for small integer values of r. For example, in the case $r=2$, it suffices to find b and z in $(0,1)$ such that:

$$b^4+2b-1>0, \tag{4.25}$$

$$z^2(1-b^2)^2-4(1-z)(1+bz)(1-b^2 z)^2>0, \tag{4.26}$$

$$1-z-(1+bz)z^2 b^4>0. \tag{4.27}$$

These conditions are satisfied by $b=1/2$ and $z=0.9$. This is obvious for (4.25), and, for (4.27), follows from:

$$(1-z)=0.1>(1+b)b^4=\tfrac{1.5}{16}.$$

To verify (4.26), note that

$$z^2(1-b^2)^2 > 0.8 \times \tfrac{9}{16} = 0.45,$$

and

$$4(1-z)(1+bz)(1-b^2z)^2 < 0.4 \times \tfrac{3}{2}(1-0.225)^2 < 0.4 \times \tfrac{3}{2} \times (0.8)^2 = 0.4 \times 0.96 < 0.4.$$

The cases $r=3$ ($b=0.65$, $z=0.9$) and $r=4$ ($b=0.7$, $z=0.9$) are still manageable. Higher values of r require the use of machines. Owners of a pocket calculator can verify, if so inclined, that Table 1 gives, for integer r up to 14, pairs (b, z) satisfying (4.22)–(4.24).

Table 1

r	b	z
2	0.5	0.9
3	0.65	0.9
4	0.7	0.9
5	0.75	0.92
6	0.78	0.93
7	0.81	0.93
8	0.83	0.94
9	0.85	0.94
10	0.86	0.945
11	0.87	0.95
12	0.88	0.95
13	0.888	0.954
14	0.8946	0.956

Appendix 1. Proof of Lemma 1

Denote, for $x \geqq 0$,

$$f(x) = (1-x^2)\log(1+x+x^2) + 2x^2 \log x.$$

To prove that $f(x) > 0$ for $0 < x < 1$, since $f(0) = f(1) = 0$, it suffices to prove that $f'' < 0$ on $(0, 1)$. Explicit calculations show that $f''(x) \to -\infty$ as $x \to 0$, $f''(1) = 2(1-\log 3) < 0$, and $f'''(x) = P(x)/Q(x)$, where

$$Q(x) = x(1+x+x^2)^3, \qquad P(x) = 2(1-x)^2(1+2x)(2+x).$$

The verification of this is facilitated by noting that $f(x) = -x^2 f(1/x)$, hence $f'''(x) = x^{-4} f'''(1/x)$, and, since $Q(1/x) = x^{-8} Q(x)$, $P(x) = x^4 P(1/x)$.

Appendix 2. Direct Proof of Existence for $\lambda = 1$

It is possible to define a fixed point problem directly for $\lambda = 1$ (i.e. $r = \infty$) by working with the function H_0 instead of ψ_0. Let

$$\mathbf{S} = \{\hat{H} \in \mathbf{P}((-\infty, 2)) \colon \hat{H}(0) = 1,\ 0 \leqq \hat{H}(w) \leqq 10 \text{ for } w \in (-\infty, 2),$$
$$\hat{H}'(w) \geqq \tfrac{1}{18} \hat{H}(w)^3 \text{ for } w \leqq 0\}. \qquad \text{(A.1)}$$

The last condition implies:

$$0<\hat{H}(w)\leqq(1-w/9)^{-1/2} \quad \text{for} \quad w\leqq 0. \tag{A.2}$$

Let $\hat{H}_0\in\mathbf{S}$ and $H_0=1-\hat{H}_0$. Then, in $\mathbf{C}((-\infty,2))$,

$$\hat{H}_0(w)=\int\frac{d\varrho_0(t)}{t-w}, \qquad \frac{H_0(w)}{w}=-\int\frac{d\varrho_0(t)}{t(t-w)}, \tag{A.3}$$

where ϱ_0 is a positive measure with support in $[2,\infty)$, such that $\int d\varrho_0(t)/t=1$.

The function V will be defined by

$$V(\zeta)=\frac{1}{\tau}H_0\left(\frac{\log(\zeta/\alpha)}{\log(1/\tau)}\right). \tag{A.4}$$

The constants $\tau>0$, $\alpha>1$ must be such that $V(1)=1=-V'(1)$. This implies $Q(\zeta_1)=0$, where

$$Q(\zeta)=\frac{H_0'(-\zeta)}{H_0(-\zeta)}-\log H_0(-\zeta), \tag{A.5}$$

$$\zeta_1=-w_1=-\frac{\log\alpha}{\log\tau}. \tag{A.6}$$

The functions $\log H_0$ and Q have integral representations:

$$\log H_0(-\zeta)=-\int_0^\infty\frac{\sigma(t)dt}{t+\zeta}, \qquad Q(\zeta)=\int_0^\infty\frac{\sigma(t)dt}{t+\zeta}\left[1-\frac{1}{t+\zeta}\right] \tag{A.7}$$

(for $\zeta\in\mathbf{C}-\mathbf{R}_-$), where $\sigma\in L^\infty(\mathbf{R})$ has support in $\mathbf{R}_+$, equals 1 on $[0,2]$, and takes values in $[0,1]$ everywhere. It follows from (A.7) that $Q(\zeta)>0$ when $\zeta\geqq 1$. It is clear from (A.5) that $Q(\zeta)\to-\infty$ when $\zeta\to 0$. Thus ζ_1 exists in $(0,1)$. For more precise bounds, just as in Sect. 3, we split Q as $Q=Q_1+Q_2$, where Q_2, the contribution of $[0,2]$ in the integral in (A.7), is given by:

$$\begin{gathered}2Q_2(\zeta)=2\log\left(1+\frac{2}{\zeta}\right)-\frac{4}{\zeta(\zeta+2)}=\chi(\xi)-\xi,\\ \xi=1+\frac{2}{\zeta}, \qquad \chi(\xi)=2\log\xi+2-\frac{1}{\xi}.\end{gathered} \tag{A.8}$$

The function $\chi(\xi)-\xi$, just as in Sect. 3, is concave and vanishes at 1 and at a unique $\bar{\xi}>1$, and $\xi_1=1+2/\zeta_1>\bar{\xi}$. Since $\chi(5)-5>0.01$, it follows that $\bar{\xi}>5$, and $\zeta_1<1/2$. Defining $\tau=H_0(-\zeta_1)$, and using (A.3), we get:

$$\frac{1}{\tau}\geqq\xi_1>5. \tag{A.9}$$

The uniqueness of ζ_1 follows as in Sect. 3, from:

$$Q'(\zeta_1)\geqq\log\frac{1}{\tau}\left(\log\frac{1}{\tau}-1\right)>\frac{24}{25}. \tag{A.10}$$

To obtain a lower bound on τ, we denote

$$k=\int_2^\infty \frac{\sigma(t)dt}{t}=-\log[-2H_0'(0)], \tag{A.11}$$

and observe that

$$2Q(\zeta)\leqq\chi(\xi)-\xi+2k\equiv S(\xi). \tag{A.12}$$

Any $\xi_{\max}>1$ such that $S(\xi_{\max})<0$ is an upper bound for ξ_1, for example $\xi_{\max}=4\log 4+4k$. Using (A.1) this gives $\xi_{\max}<15$. From (A.2) it then follows:

$$\frac{1}{\tau}\leqq\frac{3}{2}+\frac{18}{\zeta_1}<128. \tag{A.13}$$

We now define:

$$\log\alpha=\zeta_1\log\frac{1}{\tau}, \tag{A.14}$$

so that $1<\alpha<1/\tau$.

The function V defined by (A.4) is in $-\mathbf{P}((0,\alpha\tau^{-2}))$. Its iterate $W=V\circ V$ is in $\mathbf{P}((0,\alpha))$, and $W(0)=V(1/\tau)<0$, $W(\alpha)=V(0)=1/\tau$. We also define $\hat{W}(\zeta)=1-W(1-\zeta)$. For $w<2$, it follows from (A.3) that:

$$\frac{H_0''(w)}{H_0'(w)}\leqq\frac{2}{2-w},\quad \frac{H_0'''(w)}{H_0'(w)}\leqq\frac{6}{(2-w)^2}. \tag{A.15}$$

The first of these bounds implies that, for $0<\zeta<\alpha\tau^{-2}$,

$$-\frac{V''(\zeta)}{V'(\zeta)}\geqq\frac{1}{\zeta}\left[1-\frac{2}{2\log(1/\tau)-\log(\zeta/\alpha)}\right]. \tag{A.16}$$

Hence V is convex on $(0,\alpha/\tau^2e^2)$, in particular on $(0,\alpha)$, hence $\alpha>2$. Also:

$$V''(1)\geqq 1-\frac{1}{\log(1/\tau)}>\frac{3}{8}. \tag{A.17}$$

Moreover:

$$SV(\zeta)\geqq\frac{1}{2\zeta^2},\quad 0<\zeta<\alpha\tau^{-2}, \tag{A.18}$$

$$SV(1)\leqq\frac{1}{2}+[\log(1/\tau)]^{-2}\frac{H_0'''(-\zeta_1)}{H_0'(-\zeta_1)}\leqq\frac{1}{2}+\frac{3}{2}[\log(1/\tau)]^{-2}, \tag{A.19}$$

$$W''(1)=0,\quad W'''(1)=SW(1)=2SV(1), \tag{A.20}$$

so that:

$$\frac{1}{6}\leqq\frac{W'''(1)}{6}\leqq 0.4. \tag{A.21}$$

Moreover:

$$\hat{W}^{(4)}(0)=-W^{(4)}(1)=2V''(1)SV(1)>\tfrac{3}{8}. \tag{A.22}$$

Since W is a Herglotz function, $\hat{W}^{(5)}(\zeta) \geqq 0$ on $(1-\alpha, 1)$, hence $\hat{W}^{(4)}(\zeta) > 3/8$ on $[0, 1)$. Thus

$$\hat{W}(\zeta) \geqq \zeta + \frac{\zeta^3}{6} + \frac{\zeta^4}{64}. \tag{A.23}$$

To obtain an upper bound on $\hat{W}(1) = 1 - W(0)$, we note that $W(0) = H_0(1-\zeta_1)/\tau$, and integrate the first inequality in (A.15) starting at $-\zeta_1$. This gives

$$\hat{W}(1) \leqq 2\log\frac{1}{\tau} < 10. \tag{A.24}$$

Definition of H and $\hat{H}$. The next step is to construct a function $\hat{H} = 1 - H \in \mathbf{S}$, such that, in $\mathbf{C}((-\infty, 2))$,

$$\hat{H}(w) = \hat{W}(\hat{H}(w-2)), \qquad \hat{H}(0) = 1. \tag{A.25}$$

To obtain, first, the existence of this function as an element of $\mathbf{P}((-\infty, 2))$, we denote, e.g. for $s \in [0.5, 1)$, $\tilde{W}_s = s^{-2}\hat{W}$, and construct a function Ψ_s satisfying:

$$\Psi_s(z) = \tilde{W}_s(\Psi_s(s^2 z)), \qquad \Psi_s(0) = 0, \qquad \Psi_s'(0) = 1. \tag{A.26}$$

As in Sect. 3, this function is defined as the limit of the convergent sequence $\tilde{W}_s^n(s^{2n}z)$ in a small disk around 0, then extended by using (A.26). In $\mathbf{C}_\pm$, this yields a Herglotz function. On the positive real axis Ψ_s extends to an increasing function on a certain interval $[0, L)$.

Recall that, on $[0, 1)$, $\tilde{W}_s$ satisfies $\tilde{W}_s(z) \geqq z + az^3$ with $1/6 \leqq a < 1$. This implies (see Subsect. 3.10) that, on $[0, 1]$, $\tilde{W}_s^{-1}$ is defined and $\tilde{W}_s^{-1}(x) \leqq x - a'x^3$, with $a' \geqq 1/9$. Since $\Psi_s^{-1}(x) = s^{-2n}\Psi_s^{-1}(\tilde{W}_s^{-n}(x))$, it is clear that $\Psi_s^{-1}(1) = \gamma > 0$ exists.

We denote

$$\tilde{H}_s(w) = \Psi_s(\gamma \exp(-w \log s)). \tag{A.27}$$

This function is holomorphic in a domain which contains

$$\Delta_s = \left\{ w \in \mathbf{C}((-\infty, 2)) \colon |\operatorname{Im} w| < \pi/\log\frac{1}{s} \right\}, \tag{A.28}$$

and maps $\Delta_s \cap \mathbf{C}_\pm$ into $\mathbf{C}_\pm$. On $(-\infty, 2)$, $\tilde{H}_s$ takes values in $(0, 10s^{-2})$. Hence $\{\tilde{H}_s \colon 0.5 < s < 1\}$ is a normal family, and the limit of a convergent subsequence (as $s \to 1$) yields the required $\hat{H}$. It belongs to $\mathbf{P}((-\infty, 2))$, takes values in $(0, 10)$ on $(-\infty, 2)$, satisfies (A.25), and on $(-\infty, 0]$,

$$\hat{H}(w-2) = \hat{W}^{-1}(\hat{H}(w)) \leqq \hat{H}(w) - \tfrac{1}{9}\hat{H}(w)^3,$$

so that $\hat{H} \in \mathbf{S}$. We define $H = 1 - \hat{H}$.

Uniqueness and Continuous Dependence of $\hat{H}$ on $\hat{H}_0$. It is clear that $\hat{W}$ depends continuously on $\hat{H}_0$ in the topology of their respective $\mathbf{F}(J)$. It will now be shown that $\hat{H}$ is unique and depends continuously on $\hat{W}$. Note that

$$\hat{H}(-2n) = \hat{W}^{-n}(1), \qquad n \in \mathbf{N}. \tag{A.29}$$

Because $\hat{H}$ is in **S**, it is the unique solution of a very simple and well-known interpolation (or moment) problem. In $\mathbf{C}((-\infty,2))$,

$$\hat{H}(w)=\int\frac{d\varrho(t)}{t-w},\quad \frac{H(w)}{w}=-\int\frac{d\varrho(t)}{t(t-w)},\quad \int\frac{d\varrho(t)}{t}=1, \tag{A.30}$$

where the positive measure ϱ has support in $[2,\infty)$. Let

$$\Phi(p)=\int_2^\infty e^{-pt}t^{-1}d\varrho(t). \tag{A.31}$$

This function is holomorphic for $\mathrm{Re}\,p>0$, continuous and bounded by $\exp(-2\mathrm{Re}\,p)$ for $\mathrm{Re}\,p\geqq 0$. For $\mathrm{Re}\,\zeta>-2$,

$$\frac{H(-\zeta)}{\zeta}=\int_0^\infty e^{-p\zeta}\Phi(p)dp=\int_{-\infty}^\infty x^{\frac{\zeta}{2}-1}\mu(x)dx, \tag{A.32}$$

where $\mu(x)$ is equal to $\frac{1}{2}\Phi(-\frac{1}{2}\log x)$ for $x\in[0,1]$ and to 0 elsewhere. Note that $0\leqq\mu(x)\leqq x/2$ for all $x\geqq 0$. For any integer $n\geqq 1$,

$$b_n\equiv\frac{H(-2n)}{2n}=\int x^{n-1}\mu(x)dx. \tag{A.33}$$

The Fourier transform $\tilde{\mu}$ of μ,

$$\tilde{\mu}(w)=\int e^{iwx}\mu(x)dx \tag{A.34}$$

extends to an entire function with modulus bounded by 1/2 on **R**, and satisfying, for all $w\in\mathbf{C}$,

$$|e^{-iw/2}\tilde{\mu}(w)|\leqq\exp\left|\frac{w}{2}\right|, \tag{A.35}$$

and

$$\tilde{\mu}(w)=\sum_{n=1}^{\infty}\frac{(iw)^{n-1}}{(n-1)!}b_n. \tag{A.36}$$

It follows that

$$\left|\tilde{\mu}(w)-\sum_{n=1}^{N}\frac{(iw)^{n-1}}{(n-1)!}b_n\right|=\left|\int\mu(x)\left[e^{iwx}-\sum_{n=0}^{N-1}\frac{(iwx)^n}{n!}\right]dx\right|\leqq\frac{|w|^N}{N!}e^{|w|}. \tag{A.37}$$

Let f be a C^2 function on **R**, with support in $[0,1]$, such that $\int f(x)x^k dx=\delta_{0k}$, $k=0,1$. Denote

$$\tilde{f}(w)=\int e^{iwx}f(x)dx,$$

so that $\tilde{f}(0)=1$, $\tilde{f}'(0)=0$, and

$$\tilde{v}(w)=\frac{1}{w^2}[\tilde{\mu}(w)-b_1\tilde{f}(w)-iwb_2\tilde{f}(w)],\quad v(x)=\frac{1}{2\pi}\int e^{-iwx}\tilde{v}(w)dw, \tag{A.38}$$

so that

$$-v''(x)=\mu(x)-b_1f(x)+b_2f'(x). \tag{A.39}$$

Then $\tilde{v}$ is also entire, v has support in $[0,1]$, and there exists a constant C, depending only on the choice of f, such that:

$$|e^{-iw/2}\tilde{v}(w)| \leqq C \exp\left|\frac{w}{2}\right|, \quad w \in \mathbf{C}; \quad |\tilde{v}(w)| < \frac{C}{w^2+1}, \quad w \in \mathbf{R}. \tag{A.40}$$

Given $\varepsilon > 0$, one can find $R > 0$ such that

$$\frac{1}{2\pi} \int_{|w|>R} \frac{C}{w^2+1} dw < \frac{\varepsilon}{2},$$

then N such that

$$\frac{R^{N-1}e^R}{\pi N!} < \frac{\varepsilon}{2}.$$

This ensures that

$$\left| v(x) - \frac{1}{2\pi} \int_{-R}^{R} e^{-iwx} \left[\sum_{n=1}^{N} \frac{(iw)^{n-1}}{(n-1)!} b_n - b_1 \tilde{f}(w) - iwb_2 \hat{f}(w) \right] \frac{dw}{w^2} \right| < \varepsilon.$$

Note that R and N are independent of $\hat{W}$. Finally the formula

$$\frac{H(-\zeta)}{\zeta} = \int -v(x)\left(\frac{\zeta}{2}-1\right)\left(\frac{\zeta}{2}-2\right) x^{\frac{\zeta}{2}-3} dx + \int [b_1 f(x) - b_2 f'(x)] x^{\frac{\zeta}{2}-1} dx \tag{A.41}$$

holds when $\operatorname{Re}\zeta > 6$, and shows that, in this half-plane, H depends continuously on H_0. Since H remains in the normal family $\mathbf{S}$, Vitali's theorem shows that H continuously depends on H_0 as an element of $\mathbf{S}$.

Existence of Fixed Points. The map T defined by $TH_0 = H$ is continuous on the compact convex set $\mathbf{S}$, which it maps into itself. Hence it has at least one fixed point in $\mathbf{S}$.

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