The CP_{N-1} 1/N-Action by Inverse Scattering Transformation in Angular Momentum

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Abstract. The effective action which generates $1/N$ expansion of the CP_{N-1} model in two dimensions is studied here by inverse-problem methods. The action contains a functional determinant, in which auxiliary scalar and vector fields are assumed to have a spherical symmetry. This leads to the introduction, as an associated linear problem, of a radial Schrödinger equation with two potentials v and θ, and a potential-dependent centrifugal term $\{(\ell - r\theta)^2/r^2 - 1/4r^2\}$. The full inverse scattering formalism is developed here for this diffusion problem. It is formulated in terms of two-component Jost solutions, and leads to a matricial Gel'fand-Levitan-Marchenko equation. The scattering data associated to the potentials by this IST are then used to obtain a closed local form for the whole effective action. This is indeed possible for the CP_{N-1} model, owing to the classical integrability. Moreover it is found that no spherically symmetric instanton exists in this case. However the absence of supplementary informations on the $1/N$ series, due to the non-integrability at quantum level, does not allow safe quantitative conclusions on the general behaviour of the $1/N$ series at large orders.

Introduction

The inverse problem methods have undergone important developments in the past years. They have been used to solve, first of all classically, then quantum-mechanically, a quantity of models in field theory and statistical mechanics. The central idea of those methods is the existence of a link (one-to-one correspondence) between a potential $v(x)$ and a set of scattering data, conveniently defined [1]. One can in this way solve non-linear equations that admit a Lax pair, i.e. a set of linear partial differential equations, the compatibility condition of which is the non-linear equation [2]. One of the linear differential equations is a diffusion problem, with the field(s) as potential; the other one indicates the time evolution of the

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scattering data, thereby allowing a complete solution of the non-linear equation by inverse scattering transform. The quantum version of this procedure has been developed later [3] and was applied to a variety of models, known as "integrable models." Another use of the inverse scattering method was developed in the past few years. The one-to-one link between a potential $v(x)$ and a set of S.D. makes it possible to study non-local actions of the form:

$$S_{\text{eff}} = \ln \det \frac{\mathscr{O}(v)}{\mathscr{O}(0)} + \int_{-\infty}^{+\infty} \mathscr{P}(v) d^\nu x, \tag{0.1}$$

where $\mathscr{O}(v)$ is a differential operator, and $\mathscr{P}(v)$ is a local, polynomial function of the fields. Such objects often occur in quantum field theory, in particular as effective actions arising after integrating over some field variables in the functional integral formalism; for instance $1/N$-expansion-inducing actions always have this shape [4]; also effective bosonic actions obtained by integration over anticommuting fermionic variables [5]. It is a very interesting problem to study non-constant saddle-points of these actions, either to get information about the behaviour of the perturbative expansion, [6], or to study the possible instanton-induced unstability of the models [7]. The inverse scattering method provides us with a powerful tool to investigate these questions for non-local actions (0.1). Instead of dealing with non-local expressions in the fields ($\ln \det (d^2 + m^2 + v)$, for instance), one can show that, at least when the fields depend on one single variable, the non-local part of S_{eff} has a local form as a function of the scattering data, associated to these fields by the diffusion problem [8].

$$\mathscr{O}(v) \cdot \Psi = 0. \tag{0.2}$$

Recently, the I.S.T. has been developed for the radial Schrödinger equation [9], and was subsequently applied to the problem of solving the instanton-equation of $1/N - \Phi^4$ model in 2, 3, and 4 dimensions [10]. It was also developed for the radial Dirac equation [11] and applied to the problem of $1/N$-expansion for quartic-coupled fermions in two dimensions [11] and (Yukawa $+ \Phi^4$) coupled fermions in four dimensions [5c]. As a continuation of this program, we develop here the study, by inverse problem methods, of the effective action that generates the $1/N$ expansion for the CP_{N-1} model [12]. The associated linear problem is given by a generalized form of the radial Schrödinger equation. It reads:

$$\left\{ -\frac{d^2}{dr^2} + \left(\frac{\ell - r\theta}{r} \right)^2 - \frac{1}{4r^2} + m^2 + v \right\} \Phi(r) = 0, \tag{0.3}$$

where θ and v are potentials depending on r; ℓ is the angular momentum (spectral variable). In Sect. I, we introduce the model, and recall the formulation of its $1/N$ expansion [13]. The effective action reads:

$$S_{\text{eff}} = -N \left\{ \ln \det(-D_\mu D_\mu - \alpha) + \int_{-\infty}^{+\infty} \frac{\alpha}{2f} d^2 x \right\}, \tag{0.4}$$

where $D_\mu \equiv \partial_\mu + iA_\mu$; A_μ and α are the auxiliary fields of the CP_{N-1} model. After dimensional regularization of the theory, one gets:

$$S_{\text{eff}} = -N \ln \det \left(\frac{-\partial_\mu^2 + 2iA_\mu \partial^\mu + i\partial_\mu A^\mu + A_\mu A^\mu + m^2 + v}{-\partial_\mu^2 + m^2} \right)$$

$$+ \frac{\Gamma\left(1 - \dfrac{v}{2}\right)}{(4\pi)^{v/2}} m^{(v-2)/2} \int_{-\infty}^{+\infty} d^v x \cdot v(x). \qquad (0.5)$$

In Sect. II we give the expression of the effective action for a spherically symmetric 2-dimensional configuration of the fields. This means here

$$v = v(x_1, x_2) \equiv v(r), \qquad (0.6a)$$

$$A_\mu = \varepsilon_{\mu\nu} \partial^\nu \tilde{\theta}(r) = \varepsilon_{\mu\nu} \frac{x^\nu}{r} \theta(r), \qquad (0.6b)$$

where

$$r = (x_1^2 + x_2^2)^{1/2}. $$

In this way one resets S_{eff} as a partial-wave expansion with generic term:

$$\ln \det \left(\frac{-d_r^2 + m^2 + \left(\dfrac{\ell - r\theta}{r}\right)^2 + v - \dfrac{1}{4r^2}}{-d_r^2 + m^2 + \dfrac{\ell^2 - 1/4}{r^2}} \right) \qquad (0.7)$$

and the linear problem (0.3) now appears. The whole scheme of I.S.T. for this problem is given in Sect. III. There are two main differences with respect to the standard problem associated to radial Schrödinger equation [9]. First of all, the Jost function $F(\ell)$, defined as:

$$F(\ell) = \lim_{r \to 0} r^{\pm \ell + 1/2} \Phi(r, \ell), \qquad \text{Re}(\ell) \begin{array}{c} > 0 \\ < 0 \end{array}, \qquad (0.8)$$

where Φ is the regular solution at $+\infty$, $(\Phi(r) = e^{-r}(1 + O(1/r)))$ does not go to $F_0(\ell, v = 0)$ when $|\ell|$ goes to infinity, but rather behaves like:

$$(\text{Re}\,\ell \gtrless 0)\, F(\ell, v, \theta) \underset{(|\ell| \to +\infty)}{=} F_0(\ell, 0, 0) \exp \int_0^{+\infty} \mp \theta \, dr. \qquad (0.9)$$

This is due to the $(\ell\theta/r)$ term in (0.3). The second and most important difference is that, owing to this coupling between the potential and spectral parameter, one has to use the following object:

$$\underset{\sim}{\Phi} \equiv \begin{pmatrix} \Phi \\ (\ell - r\theta)\Phi \end{pmatrix} \qquad (0.10)$$

to formulate the orthogonality, closure, and I.S.T. relations; the kernel of the associated Gel'fand-Levitan equation is a 2×2 matrix. Once these qualitative changes have been done, the procedure follows as usual. Note that a similar I.S.T. has been defined for the s-wave Klein-Gordon equation [14, 15] with the same

characteristics. It was used for computing functional determinants with background electric fields in [16].

Section IV is devoted to the computation of the effective action as a function of the scattering data. One finds:

$$S_{\text{eff}} = \frac{1}{\pi^2} \oint_{-\infty}^{+\infty} \oint \frac{\ln D(\tau) \ln D(\tau')}{(\tau - \tau')^2} \, d\tau \, d\tau' + \ln \frac{D(0)}{D_0(0)}$$

$$- \frac{1}{\pi} \int_{-\infty}^{+\infty} \left(\sum_K \frac{\text{sgn} \, \text{Re} \ell_K}{\ell_K - i\tau} \ln D(\tau) \right) d\tau + 2 \sum_{K \neq L} (\text{sgn} \, \text{Re} \ell_K)$$

$$\times (\text{sgn} \, \text{Re} \ell_L) \ln(\ell_K - \ell_L) + 2 \sum_K \ln(4\ell_K^2 \cdot \sin(\pi \ell_K \, \text{sgn} \, \text{Re} \ell_K) \cdot c_K^{-1}). \quad (0.11)$$

The effective action is separable in terms of the scattering data $\{\ln D(\tau), \ell_K, c_K, K = 1 \ldots N_B\}$. This feature was already found in two-dimensional integrable models: $O(N)$ non-linear sigma model [17, 27], Gross-Neveu and Chiral Gross-Neveu [11, 28]. Here the CP_{N-1} model is classically integrable [12] but not quantum-mechanically [18]. The link between classical integrability and separability of the effective action for two-dimensional models appears again here, at least at leading order in $(1/N)$. Quantum non-integrability cannot indeed be seen at this order. The second conclusion is that no instanton (at least spherically symmetric) exists in this model. However the $1/N$ expansion is not known for the S-matrix (owing to non-integrability); hence no certain quantitative conclusion on the $1/N$ series can be obtained. One can however conjecture that it has a more convergent behaviour than the usual Borel-summable series; but this remains an hypothesis.

I. The CP_{N-1} Model and its $1/N$ Expansion

We shall briefly introduce the two-dimensional model which is studied here, and derive the effective action which generates the $1/N$ expansion for it [4, 13]. The fundamental fields are complex vectors $Z = (z_1 \ldots z_N)$, where two vectors Z and Z' are identified if $Z = \alpha Z'$, $\alpha \in \mathbb{C}$. Hence $Z \in \mathbb{C}P_{N-1}$. This is equivalent to considering fields of fixed norm (for instance $|Z|^2 = 1$) with an arbitrary phase $Ze^{i\Lambda(x)}$, $\Lambda(x)$ being a real field. In fact we shall consider rescaled fields with a norm $|Z|^2 = N/2f$ so as to generate the $1/N$ expansion, N being the number of components; f is a coupling constant. The action of the model reads [12]

$$S = \int_{-\infty}^{+\infty} d^2x \left(\partial_\mu Z \partial_\mu \bar{Z} + \frac{f}{2N} (\bar{Z} \overleftrightarrow{\partial_\mu} Z)(\bar{Z} \overleftrightarrow{\partial_\mu} Z) \right) \quad \text{(in euclidean space)}, \quad (1.1)$$

where Z is defined up to a phase $\Lambda(x)$. This invariance reflects itself into a gauge invariance of the action S, when one introduces the composite vector field A_μ as:

$$A_\mu(x) = \bar{Z} \overleftrightarrow{\partial_\mu} Z.$$

S is invariant under

$$\begin{cases} Z \rightarrow \exp(i\Lambda(x)) \cdot Z \\ A_\mu \rightarrow A_\mu + \frac{2N}{f} \partial_\mu \Lambda. \end{cases} \quad (1.2)$$

This will be useful later on, when we define the "spherically symmetric vector potential" A_μ. The factor f is the coupling constant of the model, which will later on undergo dimensional transmutation.

It is important to note that a topological invariant can be added to S, describing the winding number of the configuration A_μ as:

$$Q = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d^2x \, \varepsilon_{\mu\nu} \partial_\mu A_\nu(x) \, . \tag{1.3}$$

Hence the possibility of defining a θ-vacuum theory described by the action $S + \theta Q$ [19]. We shall not study here θ-vacuum models and restrict ourselves to the pure action given in (1.1). In fact, we shall restrict to configurations where the supplementary term induced by θQ would be zero. This is necessary for the consistency of Inverse Scattering (A_μ must decrease fast enough). We also recall that classical instantons (for the field Z, $\bar{Z}$) exist in this model, owing to particular topological properties of the space CP_{N-1} (see [19, 20]). Let us now formulate the $1/N$ expansion of the model [13]. The generating functional reads:

$$\mathscr{L}(J, \bar{J}, K_\mu) = \int \mathscr{D}Z \mathscr{D}\bar{Z} \cdot \prod_x \delta\left(Z\bar{Z} - \frac{N}{2f}\right)$$

$$\times \exp(-s(Z, \bar{Z})) \cdot \exp \int_{-\infty}^{+\infty} d^2x (\bar{J}(x)Z(x) + J(x)\bar{Z}(x) + K_\mu(x)A_\mu(v)) \, , \tag{1.4}$$

K_μ being the source coupled to the composite vector field A_μ. We shall now introduce Lagrange multipliers to implement the conditions

$$\begin{cases} |Z|^2 = \dfrac{N}{2f} \, , \\[2mm] A_\mu = (f/2N)^{1/2} \cdot (\bar{Z} \overleftrightarrow{\partial}_\mu Z) \, . \end{cases}$$

One uses the following identities

$$\int \mathscr{D}\alpha \exp \int_{-\infty}^{+\infty} d^2x \, \alpha(x) \left(Z\bar{Z} - \frac{N}{2f}\right) = \prod_x \delta\left(Z\bar{Z} - \frac{N}{2f}\right), \tag{1.5a}$$

$$\int \mathscr{D}A_\mu \exp \int_{-\infty}^{+\infty} - \left(A_\mu + i\sqrt{\frac{f}{2N}} \, \bar{Z} \overleftrightarrow{\partial}_\mu Z\right)^2 = \mathscr{L}_0 \, . \tag{1.5b}$$

The constant $\mathscr{L}_0$ is reabsorbed in the normalization of the path integral. Hence the new generating functional reads:

$$\mathscr{L} = \int \mathscr{D}Z \mathscr{D}\bar{Z} \mathscr{D}\alpha \mathscr{D}A_\mu \exp - \left\{ \int_{-\infty}^{+\infty} d^2x \cdot \left(\partial_\mu Z \cdot \partial^\mu \bar{Z}\right.\right.$$

$$\left.\left. + 2i\sqrt{\frac{f}{2N}} \, A_\mu \bar{Z} \overleftrightarrow{\partial}_\mu Z + A_\mu A^\mu - \alpha \bar{Z} Z + \frac{N}{2f}\alpha\right)\right\} + (J, \bar{J}, K_\mu) \, . \tag{1.6}$$

We can now integrate over Z and $\bar{Z}$ in (1.6), after having replaced $A_\mu A^\mu$ by $\bar{Z} A_\mu A^\mu Z \cdot \dfrac{2f}{N}$, and rescaled A_μ as $A_\mu \rightarrow \sqrt{\dfrac{2f}{N}} A_\mu$. This leads to the following model,

described by the generating functional:

$$\mathscr{Z} = \int \mathscr{D}\alpha \mathscr{D}A_\mu \exp - N\Bigg(\ln \det(-D_\mu D_\mu - \alpha)$$

$$+ \int\limits_{-\infty}^{+\infty} \frac{\alpha}{2f} d^2x \Bigg) + \int\limits_{-\infty}^{+\infty} \bar{J} \cdot (D_\mu D_\mu + \alpha)^{-1} \cdot J d^2 x d^2 y$$

$$+ \int\limits_{-\infty}^{+\infty} (1/(2f)^{1/2} \cdot K_\mu A^\mu + f/2N K_\mu K^\mu) d^2 x, \qquad (1.7)$$

where the operator D_μ is the "covariant derivative":

$$D_\mu = \partial_\mu + iA_\mu.$$

The gauge invariance of this model is reflected here by the invariance of

$$S_{\text{eff}} = -N\left\{ \ln \det(-D_\mu D_\mu - \alpha) + \int \frac{\alpha}{2f} d^2x \right\}$$

under the transformation:

$$\alpha \to \alpha; \ A_\mu \to A_\mu + \partial_\mu \Lambda(x).$$

There is no anomaly here, so the determinant is invariant under a gauge transformation. We have now in (1.7) a manifestly $1/N$-expandable theory; $1/N$ has the role of a coupling constant, or of $\hbar$ in semi-classical expansions. The purpose of the whole paper is now to put the effective action $S_{\text{eff}} = -N\left(\ln \det(-D_\mu D_\mu + v) - \int\limits_{-\infty}^{+\infty} d^2x \frac{v}{2f} \right)$ under a tractable form so as to study its characteristics, and especially its (possible) saddle-points. First of all we must find the perturbative vacuum around which the $1/N$ expansion is to be done. The saddle-point equations read:

$$\frac{\delta S}{\delta v} = \langle x | \frac{1}{(-D_\mu D_\mu + v)} | x \rangle - \frac{1}{2f} = 0, \qquad (1.8\text{a})$$

$$\frac{\delta S}{\delta A_\mu} = \langle x | \frac{-2i\partial_\mu + 2A_\mu}{(-D_\mu D_\mu + v)} | x \rangle = 0. \qquad (1.8\text{b})$$

It is clear that $A_\mu = 0$ is the correct solution for (1.8b): indeed, we are looking for constant background fields, and any constant A_μ is gauge-equivalent to $A_\mu = 0$. Moreover, covariance properties show that $A_\mu = 0$ indeed solves (1.8b). If we put this value back into (1.8a), we see that this equation contains a divergent term $\left(\langle x | \frac{1}{(\partial^2 + \sigma)} | x \rangle \right.$ is not defined in exactly two dimensions$\left. \right)$. We shall now regularize the theory, and Eq. (1.8a) will give us the relation (at leading order in $1/N$) between bare (infinite) and renormalized (finite) parameters. We shall use dimensional regularization, where (until precised), the fields A_μ, $(\mu = 1 \ldots v)$ and v, will depend on v variables $(x_1 \ldots x_v)$. Here we need not worry about the "dimensional extension" of A_μ since in any dimension v, rotational invariance of the theory defined by S_{eff} holds (S_{eff} clearly contains only *scalars*), and therefore *all*

components of A_μ will be zero in the vacuum. This regularization will moreover be easy to deal with when one expands in partial waves the ln det (see next part). Now Eq. (1.8a) reads:

$$\langle x| \frac{1}{-\partial_\mu \partial_\mu + v_0} |x\rangle - \frac{1}{2f} = 0 = \frac{\Gamma(1-v/2)}{(4\pi)^{v/2}} v_0^{(v-2)/2} - \frac{1}{2f}. \tag{1.9}$$

Equation (1.9) provides us with a renormalization of the coupling constant f as a function of the "mass parameter" v_0 (vacuum). One finds here the same features as for $NL\sigma$ model or Gross and Neveu [21], namely a dimensional transmutation. As we shall see later, there is also a mass generation at this level ("classical" with respect to $1/N$), since v will appear as the (mass)^2 for the Z-particles (see also [23] for dynamical mass generation in the $O(N)$ σ-model). We end up with the following effective action:

$$S_{\text{eff}} = -N \ln \det \left(\frac{-D_\mu D_\mu + m^2 + v}{-\partial_\mu^2 + m^2} \right) + \frac{\Gamma(1-v/2)}{(4\pi)^{v/2}} m^{(v-2)} \int_{-\infty}^{+\infty} d^v x \cdot v(x)$$

$$+ \text{source terms} \left(\sim \bar{J}(D_\mu D_\mu + \alpha)^{-1} J \quad \text{and} \quad \sim K_\mu A^\mu \right). \tag{1.10}$$

As we have indicated before, this effective action is finite now that the coupling constant has been dimensionally transmuted according to (1.9) and mass generation has occurred. This can be checked directly by expanding the ln det in inverse powers of $(-\partial_\mu \partial^\mu + m^2)$. This expansion reads:

$$S_{\text{eff}} = -N \ln \det \left(\frac{-\partial_\mu^2 + 2iA_\mu \partial^\mu + i\partial_\mu A^\mu + A_\mu A^\mu + m^2 + v}{-\partial_\mu^2 + m^2} \right)$$

$$+ \frac{\Gamma(1-v/2)}{(4\pi)^{v/2}} m^{v-2} \int_{-\infty}^{+\infty} d^v x \, v(x). \tag{1.11}$$

We shall use here the gauge invariance in v dimensions of the theory and impose Lorentz-gauge condition to the configurations in (1.11): $\partial_\mu A^\mu = 0$. Hence S_{eff} reads, at one-loop level:

$$\int_{-\infty}^{+\infty} (A_\mu A^\mu + v) d^v x \cdot \int_{-\infty}^{+\infty} d^v k \cdot \frac{1}{(k^2 + m^2)} + \frac{\Gamma(1-v/2)}{(4\pi)^{v/2}} \cdot m^{v-2} \cdot \int_{-\infty}^{+\infty} d^v x \, v(x). \tag{1.12}$$

The divergence of v cancels exactly between the two terms in (1.12) (this is equivalent to the saddle-point Eq. (1.8a)). The $(A_\mu A_\mu)$-divergence will be cancelled by the two-loops contribution, the *divergent part of* which is given by:

$$2 \int_{-\infty}^{+\infty} (\tilde{A}_\mu(0) \cdot k^\mu) \frac{1}{k^2 + m^2} \cdot (\tilde{A}_v(0) \cdot k^v) \frac{1}{k^2 + m^2} \cdot d^v k. \tag{1.13}$$

It can be checked that (1.13) cancels the divergence of (1.12), leaving only a finite term, quadratic in A_μ. This allows us to obtain the propagators of the $1/N$ expansion. The Z-propagator at leading order directly follows from (1.10):

$$\langle \overline{\tilde{z}_a, \tilde{z}_b} \rangle = \frac{\delta_{ab} \delta^{(2)}(p-q)}{p^2 + m^2} \tag{1.14}$$

(the $\sim$ denotes Fourier-transform). The Z-particles are N ordinary massive charged bosons in the fundamental representation of $SU(N)$ (at leading order in $1/N$).

The v and A_μ propagators were computed in [13] using a Pauli-Villars regularization to make the cancellation of $(A_\mu A^\mu)$ divergence explicit: Since the model has one single independent parameter at $(1/N)^0$ level. (that is, the mass of the partons) (plus of course the number N itself) and since our $\bar{Z} - Z$ propagator is the same at leading order as the one in [13], provided one identifies the masses, one will get the same propagators at leading order in $1/N$, *in the limit $v = 2$* which is unambiguous since S_{eff} is finite. Namely we have: $(v = 2)$

$$\langle \widetilde{v,v} \rangle \equiv A(p) = \frac{1}{2\pi} \, p^2 (p^2 + 4m^2)^{-1/2} \ln \frac{(p^2 + 4m^2)^{1/2} + (p^2)^{1/2}}{(p^2 + 4m^2)^{1/2} - (p^2)^{1/2}}. \qquad (1.15)$$

The inverse v-propagator contains no zero at any p^2; therefore there is no associated physical particle. It induces a short-range force between the Z-particles. $(v = 2)$ (no Lorentz-gauge condition)

$$D_{\mu\nu}(p) = \langle \widetilde{A_\mu, A_\varrho} \rangle = \left(\delta_{\mu\varrho} - \frac{p_\mu p_\varrho}{p^2} \right) \left((p^2 + 4m^2) A(p) - \frac{1}{\pi} \right). \qquad (1.16)$$

The exchange of A_μ-field yields a long-range (Coulombian) interaction between the particles Z. It indicates a *confinement* of the partons Z, owing to the pole at $p^2 = 0$ of $D_{\mu\nu}(p)$ in (1.16). No physical particle is associated to this field, either. All these features are discussed and derived in [13]. See also [4, 19].

We shall now concentrate on our problem, which is to treat the non-local effective action (1.10) so as to give it a local form, and get information about saddle-points, and more generally about the properties of S_{eff}. As we shall see, this leads to a non-trivial inverse scattering problem with very specific and interesting features.

We shall restrict ourselves, first of all, to configurations of A_μ with zero "supplementary components"; since we finally want to get saddle-points in *exactly* two dimensions, it is not necessary to worry about A_μ, $\mu > 2$. Moreover, having in mind the procedure used in the previously studied cases of two-dimensional fermionic models, we shall restrict ourselves to configurations depending only on x_1 and x_2. This is in fact natural, since we shall work in the end with exactly two-dimensional euclidean space. Note that, owing to covariance properties, the solution $A_\mu = 0$, $\mu > 2$, is then compatible with the general saddle-point condition in v dimensions:

$$-2i \langle x | \frac{-\partial_\mu + iA_\mu}{(-D_\mu D_\mu + v)} |x \rangle = 0. \qquad (1.17)$$

This restriction on the space of configuration will be useful when we derive the expression of the renormalized effective action; the reduction of the eigenvalue problem, associated to the ln det, from v to 2 dimensions, will be very simple. In fact, we are simply doing a trivial extension of any two-dimensional configuration $(v(x_1, x_2); A_\mu(x_1, x_2); \mu = 1, 2)$ to v-dimensional action, and going back to two dimensions. This will give us the two-dimensional action, since after renormalization according to (1.9), S_{eff} is a continuous function of (v, v, A_μ) without poles: the procedure is consistent. (We shall again impose Lorentz-gauge condition to the configurations: $\partial_\mu A_\mu = 0$.)

II. The Functional Determinant and Partial Wave Expansion

Since no potential (A_μ, v) depends on x_ν, $\nu > 2$, the momentum $\mathbf{K}^\perp$ $(\equiv K_\mu, \mu > 2)$ is a good quantum number for the operator $(-\partial_\mu \partial^\mu + m^2 + 2iA_\mu \partial^\mu + A_\mu A^\mu + v)$. Actually the functional determinant is a "product" of the eigenvalues of the linear problem:

$$(-\partial_\mu^2 + m^2 + 2iA_\mu \partial^\mu + A_\mu A^\mu + v)\varphi = \varepsilon\varphi. \tag{2.1}$$

Extracting $\mathbf{K}_\perp$ as $\varphi = \varphi(x_1, x_2)\exp(i\mathbf{K}_\perp \cdot \mathbf{x}_\perp)$ leads to a purely two-dimensional eigenvalue problem with a parameter $k_\perp$:

$$(-\partial_1\partial^1 - \partial_2\partial^2 + (m^2 + k_\perp^2) + 2iA_\mu \partial^\mu - A_\mu A^\mu + v)\varphi = \varepsilon\varphi(x_1, x_2). \tag{2.2}$$

Now we recall that we are interested, in the end, in the search of saddle-points for the effective action containing the ln det of operator (2.1). It is a natural assumption to say that dominant saddle-points, in two dimensions, should have the largest possible symmetry; that is, the 2-dimensional rotational symmetry described by the angular momentum $\mathscr{L}$:

$$\mathscr{L} \equiv i(x_1\partial_2 - x_2\partial_1). \tag{2.3}$$

Note that the procedure was different for scalar fields (ϕ^4 in 2, 3, and 4 dimensions [10]). There we had assumed a ν-dimensional rotational invariance *and* taken the limit $\nu \to 2$ of the effective action. However, in the case of spinor fields ([5c, 11]) and here vector fields, we assume a cylindrical-like symmetry (2-dimensional spherical symmetry, times a trivial behaviour in the $\nu - 2$ other dimensions): this leads to easier computations.

We shall then assume that the fields have a spherical symmetry. This means that:

$$v(x_1, x_2) \equiv v(r), \qquad r = (x_1^2 + x_2^2)^{1/2}. \tag{2.4}$$

It is less clear to see what A_μ must be. In fact, we must have:

$$[\mathscr{L}, 2i\mathbf{A} \cdot \partial + \mathbf{A} \cdot \mathbf{A}] = 0, \tag{2.5}$$

so as to get a spherically symmetric hamiltonian. $\mathscr{L}$ is the angular momentum in two dimensions. Introducing polar coordinates $\begin{cases} x_1 = r\cos\chi \\ x_2 = r\sin\chi \end{cases}$, the operator $\mathscr{L}$ reads:

$$\mathscr{L} = i\frac{d}{d\chi}. \tag{2.6}$$

Now the "spherical symmetry equation" (2.5) reads:

$$\left[d_\chi, 2i(A_1\cos\chi + A_2\sin\chi)d_r + 2i\left(-A_1\frac{\sin\chi}{r} + A_2\frac{\cos\chi}{r} \right) \right.$$
$$\left. \times d_\chi + A_1 A_1 + A_2 A_2 \right] = 0. \tag{2.7}$$

A necessary condition is that the coefficients of ∂_r and ∂_χ in the development of (2.7) cancel. One gets:

$$A_1 = \cos\chi \cdot f(r) + \sin\chi \cdot \theta(r), \tag{2.8a}$$

$$A_2 = \sin\chi \cdot f(r) - \cos\chi \cdot \theta(r), \tag{2.8b}$$

which are also sufficient conditions for (2.7) to be true. The gauge choice $\partial_\mu A^\mu = 0$ gives:

$$f'(r) + \frac{f(r)}{r} = 0, \tag{2.9}$$

which has no regular solution other than $f \equiv 0$. One could alternatively note that the "$f(r)$" part in A_μ (2.8) is a pure gauge, and that the "$\theta(r)$" part can be reset as:

$$A_\mu(\mu = 1, 2) = -\varepsilon_{\mu\nu}\partial^\nu \tilde{\theta}(r), \tag{2.10}$$

where $\tilde{\theta}(r) = \int^r \theta(r)dr$, $\varepsilon_{12} = 1$. It is in fact a well-known result that any 2-dimensional gauge field A_μ can be written as

$$A_\mu = \partial_\mu \tilde{\Phi}(\mathbf{x}) + \varepsilon_{\mu\nu}\partial^\nu \tilde{\theta}(\mathbf{x}).$$

Putting back (2.4) and (2.10) in (2.2) leads to the following eigenvalue problem:

$$\left\{ -d_r^2 - \frac{1}{r}d_r - \frac{1}{r^2}(d_\chi)^2 - 2i\frac{\theta}{r}d_\chi + \theta^2(r) + (m^2 + k_\perp^2) + v \right\} \varphi = \varepsilon\varphi. \tag{2.11}$$

Introducing now $\Psi(r) \equiv \sqrt{r}\, \varphi(r)$, so as to eliminate the "$(1/r)d_r$" in (2.11), and using the commutation of the hamiltonian in (2.11) with the angular momentum id_χ, we rewrite (2.11) with an explicit eigenvector of id_χ as Ψ:

$$\Psi(x_1, x_2) = \Psi(r)\exp(-i\ell\chi). \tag{2.12}$$

The eigenvalue ℓ is here a positive or negative integer. This is necessary to get the consistency condition $\Psi(\chi + 2\pi) = \Psi(\chi)$. The degeneracy associated to the angular momentum $\mathcal{L}$ in two dimensions is 1, therefore the full degeneracy of any eigenvalue ℓ in (2.10) is given by the degeneracy associated to $\mathbf{k}_\perp$, times an "intrinsic degeneracy" of the purely radial spectral problem. We shall only have to integrate over $\mathbf{k}_\perp$ when computing the ν-dimensional determinant. Afterwards the remaining degeneracy will be taken into account by the radial determinant. Equation (2.10) now has a manifest spectral parameter: ℓ

$$\left\{ -d_r^2 + \left(\frac{\ell - r\theta}{r}\right)^2 - \frac{1}{4r^2} + (m^2 + k_\perp^2) + v \right\} \Psi(r) = \varepsilon\Psi(r). \tag{2.13}$$

Now we can reexpress the ln det in the effective action in the way indicated above. Replacing ln det by tr log, using the form $\varphi = e^{iK_\perp x_\perp}\varphi(x_1, x_2)$, and taking the trace on the supplementary dimensions gives:

$$\ln\det\left(\frac{-D_\mu D_\mu + v}{-\partial_\mu^2 + m^2}\right)_{(\nu)}$$

$$= L^{\nu-2}\int\frac{d^{\nu-2}k_\perp}{(2\pi)^{\nu-2}}\ln\det\left(\frac{-d_\mu^2 + m^2 + k_\perp^2 + 2iA\cdot\partial + A^2 + v}{-d_\mu^2 + m^2 + k_\perp^2}\right)_{(II)}. \tag{2.14}$$

L is the length associated to the supplementary components. In fact, one has simply done a partial Fourier transform on the supplementary components, using the fact that the (A, v) configuration does not depend on x_μ, $\mu > 2$.

Now it is possible to expand in partial waves the two-dimensional functional determinant in (2.14). One gets the final result:

$$\ln \det \left(\frac{-D_\mu D_\mu + v}{-\partial_\mu^2 + m^2} \right)_{(v)} = \frac{L^{v-2}}{(2\pi)^{v-2}} \int_{-\infty}^{+\infty} d^{v-2} k_\perp \left(\sum_{\ell=0}^{\pm\infty} \right).$$

$$\ln \det \left(\frac{-d_r^2 + m^2 + k_\perp^2 + \left(\dfrac{\ell - r\theta}{r} \right)^2 - \dfrac{1}{4r^2} + v}{-d_r^2 + m^2 + k_\perp^2 + (\ell^2 - 1/4)/r^2} \right). \qquad (2.15)$$

It is clear on (2.15) that one must now express the radial determinant as a function of the spectral parameter ℓ, keeping $(k_\perp^2 + m^2)$ as a mass parameter. This shall be done in the same way as for the radial Schrödinger equation, [9] or radial Dirac equation [11].

Note that the expression under the integral in (2.15) is formally divergent. We shall comment later on the exact meaning of Eq. (2.15). For the moment, we shall concentrate on the individual determinant on the right-hand side, and we shall try to reexpress:

$$\ln \det \left(\frac{-d_r^2 + m^2 + k_\perp^2 + \left(\dfrac{\ell - r\theta}{r} \right)^2 - \dfrac{1}{4r^2} + v}{-d_r^2 + m^2 + k_\perp^2 + (\ell^2 - 1/4)/r^2} \right)$$

in a local form. The full scheme of inverse scattering transform, already given in [22], will also work here. However, quantitative and even qualitative differences with respect to the standard Schrödinger equation occur now, owing to the coupling of the angular momentum (spectral parameter) to the potential $\theta(r)$ through the term $\ell\theta(r)/r$ (coming from the term $2iA_\mu\partial^\mu$ in the effective action). We shall see these modifications in the next section.

III. The Inverse Scattering Transform of "Coupled" Schrödinger Equation

This "coupled" Schrödinger equation reads:

$$\left(-d_r^2 - \frac{1}{4r^2} + \left(\frac{\ell - r\theta}{r} \right)^2 + m^2 + k_\perp^2 + v \right) \varphi = 0. \qquad (3.1)$$

To begin with, we define the Jost solutions and regular solutions of Eq. (3.1). The Jost solution is regular at $r \to 0$:

$$\cdot \ell \in \mathbb{C},$$

$$\cdot f(r) = (\mu r)^{\ell + 1/2}(1 + O(r)) \qquad \mathrm{Re}\,\ell > 0, \qquad (3.2a)$$

$$\cdot f(r) = (\mu r)^{-\ell + 1/2}(1 + O(r)) \qquad \mathrm{Re}\,\ell < 0, \qquad (3.2b)$$

$$\cdot \mu = (m^2 + k_\perp^2)^{1/2}.$$

All through this paper, we admit that θ and v obey the necessary requirements for "standard" IST, namely:

$$\lim_{r\to 0} r^2 v(r) = \lim_{r\to 0} r\theta(r) = 0, \qquad \lim_{r\to\infty} r^2 v(r) = \lim_{r\to\infty} r\theta(r) = 0,$$

so that the centrifugal barrier dominates at $r \to 0$ and $r \to +\infty$, enabling us to do "perturbative" expansions around free solutions. This is less evident here, as we shall see later, than in the standard Schrödinger equation. The regular solution is defined for both signs of ℓ in the same way:

$$\varphi(r) = 1 \cdot e^{-\mu r} (r \to \infty).   \tag{3.3}$$

Here we use the mass-scale $\mu^2 = (m^2 + k_\perp^2)$. We shall now omit this mass scale, but one should remember that there is a dependence of this scale μ^2 on the transverse momentum $\mathbf{k}_\perp$, which is integrated over in the full determinant. We shall come back to it later. We must here give some precisions about (3.1). The spectral variable ℓ is complex, while the potentials v and θ are normally real. We shall (in this section) consider generally v and θ as complex, and indicate the properties of Scattering Data that arise if they are real. The whole IST holds for complex potentials (see [2]), but the SD then lose many of their properties; moreover the physical interpretation of complex saddle-points is not obvious.

Let us now define the Jost function. From (3.2) and (3.3), we know that φ is a linear combination of the regular solution $r^{\ell+1/2}(\mathrm{Re}\,\ell > 0)$ and the irregular solution $r^{-\ell+1/2}(r \to 0)$. We define the Jost function as:

$$F(\ell) = \lim_{r\to 0} \ell \cdot \varphi(r,\ell) \cdot r^{\ell-1/2}, \qquad \mathrm{Re}\,\ell > 0,   \tag{3.4a}$$

$$F(\ell) = \lim_{r\to 0} -\ell \cdot \varphi(r,\ell) \cdot r^{-\ell-1/2}, \qquad \mathrm{Re}\,\ell < 0.   \tag{3.4b}$$

It is equivalent to set:

$$\varphi(r,\ell) = \frac{F^+(\ell)}{\ell} f^-(r,\ell) - \frac{F^-(\ell)}{\ell} f^+(r,\ell),   \tag{3.5}$$

where

$$f^-(r) = r^{-\ell+1/2}; \qquad f^+(r) = r^{\ell+1/2}; \qquad r \to 0,$$

and

$$F(\ell) = F^+(\ell), \qquad \mathrm{Re}\,\ell > 0,   \tag{3.6a}$$

$$F(\ell) = F^-(\ell), \qquad \mathrm{Re}\,\ell < 0.   \tag{3.6b}$$

It is clear on these two definitions that F is analytic in $(\mathrm{Re}\,\ell > 0)$ and $(\mathrm{Re}\,\ell < 0)$. It can be shown that an analytic continuation exists for F^+ and F^- on each side of the imaginary axis $(\mathrm{Re}\,\ell = 0)$. This is necessary to get a consistent meaning of (3.5) in a band containing the axis. This property, and the linked property of analyticity of φ-solutions in the plane (1), will be used implicitly each time we shall speak of

$$F^+(i\tau) \equiv \lim_{\varepsilon \to 0^+} F^+(i\tau + \varepsilon), \qquad F^-(i\tau) \equiv \lim_{\varepsilon \to 0^-} F^-(i\tau + \varepsilon).$$

However, the Jost function itself will always be defined as $F^+(\ell)$ when $\mathrm{Re}\,\ell > 0$, $F^-(\ell)$ when $\mathrm{Re}\,\ell < 0$. In fact, we shall never have to use $F(i\tau)$, but always $F(i\tau \pm \varepsilon)$ in

the following, so the definition of the Jost function remains unambiguous. The identity (3.5) will be used as a limit when $|\mathrm{Re}\,\ell|$ goes to zero.

Another useful definition of the Jost function is given by the Wronskian of two solutions (φ, f). Defining W as:

$$W(f, g) = d_r f \cdot g - d_r g \cdot f, \tag{3.7}$$

one can prove that

$$F^\pm(\ell) = -\tfrac{1}{2} W(\varphi, f^\pm). \tag{3.8}$$

This follows from the property that $\dfrac{d}{dr} W(f(r, \ell), g(r, \ell)) = 0$ for two solutions f and g with same spectral parameter ℓ, a consequence of which is that $W(f^+, f^-) = 2\ell$ (from (3.2)).

We shall briefly recall that in the free case ($\theta = 0$, $v = 0$), the φ- and f-solutions are modified Bessel functions:

$$\varphi_0(r, \ell) = \sqrt{\frac{2r}{\pi}} K_\ell(r), \tag{3.9}$$

$$f_0(r, \ell) = 2^\ell \Gamma(\ell + 1) \sqrt{r} I_\ell(r), \tag{3.10}$$

$$F(\ell) = (2\pi)^{-1/2} 2^{\pm \ell} \Gamma(1 \pm \ell); \qquad \mathrm{Re}\,\ell \begin{matrix} >0 \\ <0 \end{matrix}. \tag{3.11}$$

Let us now define the scattering data. In this more general case (compared to the standard Schrödinger equation), the Jost function is no more even. Moreover, a particular feature of F/F_0 is that the limit when $|\ell|$ goes to infinity is *no more* 1, owing to the coupling $\ell\theta/r$ in the Schrödinger equation. One can obtain this behaviour by solving in powers of $1/\ell$ the associated Ricatti equation; this gives the following behaviour for f and φ, which will be very useful in the following: Starting from

$$\left(-d_r^2 + \frac{\ell^2 - 1/4}{r^2} - \frac{2\ell\theta}{r} + 1 + \theta^2 + v\right)\Phi = 0, \tag{3.12}$$

and setting

$$\varphi = \varphi_0 \exp \int_{+\infty}^{r} \Psi(r)\,dr,$$

so as to keep the boundary conditions on φ as $(\varphi e^{-r}, r \to \infty)$, we get the following Ricatti equation:

$$-\Psi' - \Psi^2 - 2\frac{\varphi_0'}{\varphi_0}\Psi + \theta^2 + v - \frac{2\ell\theta}{r} = 0, \tag{3.13}$$

and a similar one for f, parametrized as $f \equiv f_0 \exp \int_{0}^{r} \Psi^* \, dr$, so as to get correct normalizations at $r \to 0$ (one should replace φ_0'/φ_0 by f_0'/f_0). This equation can be solved by expanding in series of $1/\ell$ Ψ and φ_0'/φ_0 (respectively Ψ^*, f_0'/f_0): this was

the standard way to obtain trace-identities in the previous studies [5c, 10, 11]. Here one gets the following solutions: $(\text{Re}\,\ell > 0)$

$$\varphi = \varphi_0 \cdot \exp \int_{+\infty}^{r} \theta(r)\,dr \cdot \exp \frac{r\theta}{2\ell} \cdot \exp - \int_{+\infty}^{r} \frac{rv}{2\ell}\,dr \cdot (1 + O(1/\ell^2)), \qquad (3.14)$$

$$f = f_0 \cdot \exp \int_{0}^{r} -\theta(r)\,dr \cdot \exp \frac{r\theta}{2\ell} \cdot \exp \int_{0}^{r} \frac{rv}{2\ell} \cdot (1 + O(1/\ell^2)). \qquad (3.15)$$

Similar expressions hold for $\text{Re}\,\ell < 0$, except that $(\ell \leftrightarrow -\ell,\ \theta \leftrightarrow -\theta)$. These expansions will be very useful in the next computations. Here we can see that:

$$\lim_{|\ell| \to +\infty} \frac{F^{\pm}(\ell)}{F_0^{\pm}(\ell)} = \exp \int_{0}^{+\infty} \mp \theta(r)\,dr, \qquad \pm : [\text{depending on (sign } \text{Re}\,\ell)] \quad (3.16)$$

which is a qualitative difference with respect to the standard case where $\lim F/F_0$ is 1 when $|\ell| \to +\infty$. Hence the analyticity studies have to be done with $\frac{F^{\pm}}{F_0^{\pm}} \exp \pm \int_{0}^{+\infty} \theta \cdot dr$ so as to get a convenient large $|\ell|$ behaviour $(O(1/\ell))$ for the functions involved in dispersion relations. The scattering data can now be defined. They consist of:

a) a discrete spectrum $\{\ell_K\}$ such that $F(\ell_K) = 0$. They are the only values for which $\varphi_K(r)$ (or $f_K(r)$ which is then proportional to $\varphi_K(r)$) is square-integrable. The corresponding normalization coefficients $\{c_K\}$ will be given later, since we shall see that one has to modify the definition of scalar product.

b) a continuum spectrum $\{\ell = i\tau, \tau \in \mathcal{R}\}$. The corresponding SD will be seen to be $D(\tau) = \frac{F^+(i\tau)F^-(i\tau)}{F_0^+(i\tau)F_0^-(i\tau)}$. The non-unit limit $(|\ell| \to \infty)$ of F/F_0 does not modify this definition, since the sign of the exponential depends on whether one considers F^+ or F^-, in (3.16); therefore $\ln D(\tau)$ has a convenient behaviour $(O(1/\tau^2))$ when $|\tau| \to +\infty$. The properties of the scattering data are not so clear as in the case of the "pure" Schrödinger equation. When the potentials are real, the general property holds that

$$F(\ell^*) = F^*(\ell), \qquad (3.17)$$

by complex-conjugating the equation. Hence the zeroes are either real or coupled by pairs (ℓ_K, ℓ_K^*), and $D(+\tau)$ is equal to $D^*(-\tau)$. One cannot however say in this general case that the zeroes are *real*. It is only possible to prove that, given a zero $\ell_K = \ell_R^K + i\ell_I^K$:

$$\ell_I^K(2\alpha^K \ell_R^K + \beta^K) = 0, \qquad (3.18)$$

where

$$\alpha^K = \int_{0}^{+\infty} \frac{\varphi_K \varphi_K^*}{r^2}\,dr, \qquad \beta^K = \int_{0}^{+\infty} \frac{\varphi_K \theta \varphi_K}{r}\,dr. \qquad (3.19)$$

Hence, if $\theta = 0$, $\beta^K = 0$, and since $\alpha^K \neq 0$, $\ell_R^K \neq 0$ (imaginary zeroes are forbidden since the integral $\int_{\varepsilon}^{+\infty} \frac{\varphi_K \varphi_K^*}{r^2}\,dr$ would become divergent as $\ln\varepsilon$), one gets the "standard" result for Schrödinger equation:

$$\ell_I = \text{Im}\,\ell^K = 0. \qquad (3.20)$$

Moreover if $\theta = 0$, and v real, the equation is invariant under $\ell \to -\ell$, and $F^+(\ell)$ $= F^-(-\ell)$ for $\mathrm{Re}\,\ell > 0$ (the Jost function is even). Here

◆ all zeroes are real, and $(\ell_K \leftrightarrow -\ell_K)$.

◆ normalization constants are real: $c_K^{-1} \equiv \int\limits_0^{+\infty} \dfrac{\varphi_K^2(r)}{r^2}\, dr$.

◆ $D(\tau) = \dfrac{F^+(i\tau)\,F^-(i\tau)}{\tau \sinh \pi\tau} \in \mathbb{R}^+$.

More generally $\theta = 0$ and v complex lead to

$$F^+(\ell) = F^-(-\ell),$$

$$\ell_K = -\ell_{(-K)}; \qquad c_K = c_{(-K)}; \qquad D(\tau) = D(-\tau).$$

Let us now go back to the general case, θ and v generically complex. We shall see later on that this is a consistent set of S.D. That is, one can reconstruct v and θ from these S.D. only. Let us introduce now the scalar product between two solutions. We reach here the most capital difference between the standard radial Schrödinger equation and this "coupled" radial Schrödinger equation. In this case, it is necessary to introduce the following two-components object:

$$\underset{\sim}{\varphi} = \begin{pmatrix} \varphi \\ (\ell - r\theta)\varphi \end{pmatrix}$$

for *all* solutions φ of the Schrödinger equation with spectral parameter ℓ and potential (θ, v). In the whole scheme of the inverse scattering transform which now follows, we shall deal with such objects, and *not* with the "one-dimensional" solutions φ.

In particular we must define the scalar product as

$$\langle \varphi_1, \varphi_2 \rangle = \int\limits_0^{+\infty} \frac{dr}{r^2}\, \underset{\sim}{\varphi}_1^T \sigma_1 \underset{\sim}{\varphi}_2 . \tag{3.21}$$

This definition only has a meaning if φ_1 and φ_2 are solutions of the equation with equal (or different) spectral parameters, while the usual definition of the scalar product could be extended to any couple of functions. Moreover, this product is not positive-definite in the general case. However, it has the property that:

$$\frac{1}{r^2}\, \underset{\sim}{\varphi}_1^T \sigma_1 \underset{\sim}{\varphi}_2 = \frac{1}{r^2}\, (\varphi_1(\ell_1 + \ell_2 - 2r\theta)\varphi_2), \tag{3.22}$$

or using the equation of φ_1 and φ_2:

$$\frac{1}{r^2}\, \underset{\sim}{\varphi}_1^T \sigma_1 \underset{\sim}{\varphi}_2 = \frac{1}{\ell_1 - \ell_2} \frac{d}{dr}\, W(\varphi_1, \varphi_2). \tag{3.23}$$

Therefore the expression (3.21) will be easy to compute once one knows the asymptotic and $(r \to 0)$ behaviour of the functions involved in it. In particular, one proves the following "pseudo-orthogonality" relations:

a) The "scalar product" of two discrete solutions is given by: (For $\mathrm{Re}\ell_1 > 0$)

$$\langle \underset{\sim}{\varphi}(\ell_1), \underset{\sim}{\varphi}(\ell_2) \rangle = \delta_{\ell_1, \ell_2} \frac{F^-(\ell_1) F^{+\prime}(\ell_1)}{\ell_1} \equiv c_K^{-1}, \tag{3.24}$$

and $(+) \leftrightarrow (-)$ for $\mathrm{Re}\ell_1 < 0$.

b) The scalar product of two φ-functions (regular at $r \to +\infty$) belonging to the continuous spectrum is given by

$$\langle \underset{\sim}{\varphi}(i\tau), \underset{\sim}{\varphi}(i\tau') \rangle = \frac{1}{i\tau} F^+(i\tau) F^-(i\tau) \delta(\tau - \tau'). \tag{3.25}$$

This formula can easily be obtained from the formula in the case of the "standard Schrödinger equation," which reads

$$\langle \varphi(i\tau), \varphi(i\tau') \rangle = \frac{1}{\tau^2} F^+(i\tau) F^-(i\tau) \left(\delta(\tau + \tau') + \delta(\tau - \tau') \right). \tag{3.26}$$

The difference is that one obtains (3.25) by using a definition of the scalar product $\langle \varphi, \varphi \rangle$ which, when $\theta \equiv 0$, differs from the definition in (3.16) by a factor $(\ell + \ell')$. Hence the disappearance of $\delta(\tau + \tau')$ and the (2τ) factor. Apart from these details, the formulae giving $W(\varphi_1, \varphi_2)$ lead to identical expressions in terms of F^+, F^-, r and τ in the two cases, hence the similar results for the scalar products defined by (3.21).

We shall see now that a general closure relation, dual of the orthogonality relations, holds for the $\underset{\sim}{\varphi}$ objects. This follows from introducing the Green function of the differential operator, defined as usual:

$$D \cdot G(r, r') = \delta(r - r').$$

This Green function is given, for a spectral parameter ℓ, by:

$$G(r, r', \ell) = \frac{f(r_<, \ell) \varphi(r_>, \ell)}{2F(\ell)}, \tag{3.27}$$

where f is the Jost solution for $\mathrm{Re}\ell > 0$ or $\mathrm{Re}\ell < 0$.

Now we proceed as usual, and compute not *one*, but *three* integrals of $G(\Gamma, \Gamma', \ell)$. Let $\Gamma = \Gamma^+ \cup \Gamma^-$ be the integration contour given by $\Gamma^\pm =]-i\infty \pm \varepsilon$, $+i\infty \pm \varepsilon[\cup ($right (left) semi-circle of radius $\to +\infty)$. We integrate now $(1)G$, $(2)\ell G$ and $(3)\ell^2 G$ over this contour. The poles ℓ_K such that $F(\ell_K) = 0$ will lead to residues contributing as:

(1) $\dfrac{\ell_K}{2F^-(F^+)'} \; \varphi_K \varphi_K,$

(2) $\dfrac{\ell_K^2}{2F^-(F^+)'} \; \varphi_K \varphi_K,$

(3) $\dfrac{\ell_K^3}{2F^-(F^+)'} \; \varphi_K \varphi_K \cdot (F^+)' \equiv \dfrac{dF^+}{d\ell}\bigg|_{\ell = \ell_K} \cdot \text{``}\varphi_K \varphi_K\text{''} \equiv \varphi_K(r) \varphi_K(r').$

● The straight lines $-i\infty$ to $+i\infty$ contribute as

$$(1) \quad \int_{-\infty}^{+\infty} \varrho(\tau)\varphi(i\tau)\varphi(i\tau)d\tau,$$

$$(2) \quad \int_{-\infty}^{+\infty} i\tau \cdot \varrho(i\tau)\varphi(i\tau)\varphi(i\tau)d\tau,$$

$$(3) \quad \int_{-\infty}^{+\infty} \tau^2 \varrho(i\tau)\varphi(i\tau)\varphi(i\tau)d\tau,$$

where $\varrho(i\tau) = \dfrac{i\tau}{F^+(i\tau)F^-(i\tau)}$ is the non-normalized continuous scattering data,

$$(\text{``}\varphi(i\tau)\varphi(i\tau)\text{''} \equiv \varphi(r,i\tau)\varphi(r',i\tau)).$$

● The two semi-circles contribute as:

$$(1) \quad 0,$$

$$(2) \quad r^2\delta(r-r'),$$

$$(3) \quad 2r\theta\,\delta(r-r')\cdot r^2.$$

This follows from using the asymptotic formulae (3.14) and (3.15). One can now rewrite this whole integral, with its different contributions, as a closure relation on the $\underset{\sim}{\varphi}$ objects; namely

$$\int d\lambda \varrho(\lambda)\underset{\sim}{\varphi}(r,\lambda)\underset{\sim}{\varphi}^T(r',\lambda) = \sigma_1 r^2\delta(r-r'), \tag{3.28}$$

where $\int d\lambda\varrho(\lambda)F(\lambda)$ is to be understood here and in all the following computations as

$$\int_{-\infty}^{+\infty} d\tau\varrho(i\tau)F[\lambda=i\tau] + \sum_K c_K F(\lambda=\ell_K). \tag{3.29}$$

This relation generalizes the closure relation of the solutions of the standard Schrödinger equation ($\theta \equiv 0$). This already shows that the S.D. F^+F^- and $\{\ell_K, c_K\}$ are convenient objects to describe a set of potentials $\{\theta, v\}$, in that their associated φ form a complete set of functions. Proceeding as usual, we now derive a dispersion relation for the Jost function. Owing to its behaviour at $|\ell| \to +\infty$, we define:

$$V^+(\ell) = \frac{F^+(\ell)}{F_0^+(\ell)} \cdot \exp \int_0^{+\infty} \theta dr \cdot \prod_{\text{Re}\,\ell_K > 0} \left(\frac{\ell + \ell_K}{\ell - \ell_K}\right)$$

for $\text{Re}\,\ell > 0$ (and a similar expression for V^-, $\text{Re}\,\ell < 0$). Hence the following dispersion relations hold for F^+ and F^-, as a direct consequence of dispersion relations on the function $\ln V^\pm(\ell)$ $\left(\text{for } \text{Re}\,\ell \begin{array}{c} > 0 \\ < 0 \end{array}\right)$

$$\frac{F^\pm(\ell)}{F_0^\pm(\ell)} = \exp \mp \int_0^{+\infty} \theta \cdot dr \cdot \prod_K \frac{\ell - \ell_K^\pm}{\ell - \ell_K^\mp} \cdot \exp \int_{-\infty}^{+\infty} \left(\mp \frac{\ln D(\tau)}{(i\tau - \ell)}\right) \frac{d\tau}{2\pi}. \tag{3.30}$$

In the case of the standard Schrödinger equation with real potential:

$$\theta \equiv 0; \quad \ell_{-K} \equiv -\ell_K; \quad \ln D(\tau) = \ln D(-\tau),$$

and one gets back the standard dispersion relation:

$$\frac{F(\ell)}{F_0(\ell)} = \prod_{K=1}^{N_B} \frac{\ell - \ell_K}{\ell + \ell_K} \exp \int_0^{+\infty} 2\ell \cdot \frac{\ln D(\tau)}{(\tau^2 + \ell^2)} \frac{d\tau}{\pi}. \tag{3.31}$$

It is now time to derive the fundamental trace identity, i.e. to obtain the relation between the Jost function (3.31) and the functional determinant. This follows from taking the derivative of the determinant with respect to ℓ, that is:

$$\frac{d}{d\ell} \ln \det \left(\frac{d_r^2 + 1 + v + \frac{(\ell - r\theta)^2 - 1/4}{r^2}}{d_r^2 + 1 + \frac{\ell^2 - 1/4}{r^2}} \right)$$

$$= \int_0^{+\infty} dr \left(\frac{2\ell}{r^2} - \frac{2\theta}{r} \right) G(r, r, \ell) - \frac{2\ell}{r^2} G_0(r, r, \ell). \tag{3.32}$$

Using the definition of G (3.27), the wronskian property (3.23), and the asymptotic behaviours (3.4), (3.5), enables one to compute (3.32), and to get:

$$\frac{d}{d\ell} \ln \det \left(\frac{D(\ell)}{D_0(\ell)} \right) = \frac{d}{d\ell} \ln \frac{F(\ell)}{F_0(\ell)}. \tag{3.33}$$

Moreover it is possible to compute $\ln \det(O)$ as a series in $1/\ell$ for $|\ell|$ going to $\pm \infty$; one gets in this way, using the expression of $G(r, r, \ell)$ (3.27),

$$\ln \det \left(\frac{D}{D_0} \right) = \mp \int_0^{+\infty} \theta(r) dr + O\left(\frac{1}{\ell} \right). \tag{3.34}$$

Since we know that such is the behaviour of F/F_0 when $\ell \to \pm \infty$ (see (3.16)), we can identify:

$$\ln \det \frac{D}{D_0} = \ln \frac{F(\ell)}{F_0(\ell)}. \tag{3.35}$$

From expanding both sides of (3.35) in series of $1/\ell$, or from the resolution of the Ricatti Eq. (3.13) in powers of $(1/\ell)$, we get the trace identities associated to this linear problem.

TR 0:
$$\int_0^{+\infty} \theta(r) dr = \int_0^{+\infty} \theta(r) dr$$

trivially. This trace identity will be obtained as a non-standard tr-id. in the end.

TR 1:
$$\int_0^{+\infty} rv(r) dr = 2 \sum_K (-\operatorname{sgn} \operatorname{Re} \ell_K) \ell_K + \int_{-\infty}^{+\infty} \cdot \ln D(\tau) \frac{d\tau}{\pi},$$

TR 2:
$$\int_0^{+\infty} (r^2 \theta(r) - 2r\theta(r)v(r)) dr = \sum_K \frac{\operatorname{sgn} \operatorname{Re} \ell_K}{2} \ell_K^2 + \oint_{-\infty}^{+\infty} \frac{\ln D(\tau) \cdot d\tau}{2\pi} i\tau.$$

Note that in the case of standard Schrödinger equation [9], (TR 2) reduces to $0 = 0$.

All trace identities are obtained by a recurrent construction:

$$\Psi^{(n+1)} = \frac{\Psi^{(n)'} + \sum\limits_{p=1}^{n} \Psi^{(n-p)} \Psi^{(p)} - 2 \frac{\theta}{r} \Psi^{(n)} + \sum\limits_{p=1}^{n} Q^{(p)} \Psi^{(n-p)}}{Q^{(-1)}}, \qquad (3.36)$$

where $Q^{(n)}$ are the coefficients of the expansion in $(1/\ell)$ of φ_0'/φ_0, and trace identities are:

$$\int\limits_{0}^{+\infty} \Psi^{(n)}(r) dr = \sum\limits_{K} (\text{sgn Re} \ell_K) \frac{(-\ell_K)^n}{n} + \int\limits_{-\infty}^{+\infty} (i\tau)^{n-1} \frac{\ln D(\tau)}{2\pi} d\tau. \qquad (3.37)$$

Now the final procedure is to show that the scattering data defined above are in one-to-one correspondence with a set of potentials (v, θ) and to formulate the inverse problem. Let us introduce the kernel K (2×2 matrix) defined as:

$$K(r, r') = \int\limits_{-\infty}^{+\infty} \varrho_0(i\tau) \left(\underline{\varphi}(r, i\tau) - \mathscr{R}(r) \underline{\varphi}_0(r, i\tau) \right) \underline{\varphi}_0^T(r', i\tau) \cdot d\tau \cdot \sigma_1, \qquad (3.38)$$

where $\mathscr{R}(r)$ compensates the asymptotic behaviour of φ compared to φ_0 when the spectral parameter ℓ goes to infinity. We shall see that the only consistent definition of K is with:

$$\mathscr{R}(r) = \cosh \int\limits_{+\infty}^{r} \theta(r) dr. \qquad (3.39)$$

It is possible to show that

$$\int\limits_{0}^{+\infty} \frac{1}{r'^2} K(r, r') \underline{\varphi}_0(r', \ell) dr' = (\underline{\varphi} - \mathscr{R} \underline{\varphi}_0)(r, \ell) \qquad (3.40)$$

for any $\ell (\text{Re} \ell > 0 \text{ or } < 0)$. This follows essentially from the wronskian property, that converts $\int\limits_{0}^{+\infty} \frac{dr}{r^2} \underline{\varphi}_0^T(i\tau) \underline{\varphi}_0(\ell)$ into a "$\delta(\ell - i\tau)$" whenever applied to a regular function of $i\tau$. The mechanism is exactly the same as in the standard ($\theta = 0$) case, once the wronskian property is used. Hence this kernel interpolates between free (φ_0) and interacting (φ) solutions. It is possible to define general kernels K that interpolate between solutions $\underline{\varphi}(v', \theta')$ and $\underline{\varphi}(v, \theta)$. They look exactly like (3.38), once one replaces $\underline{\varphi}_0$ by $\underline{\varphi}(v', \theta')$, ϱ_0 by ϱ (interpreted as in (3.29) as continuum + discrete measure) and θ by $\theta - \theta'$ in (3.39).

A general property of these kernels is that:

$$K(r, r') = 0, \qquad r' < r. \qquad (3.41)$$

This is obtained by deforming the integration contour in (3.38) as a reunion of two semi-circles C_1 and C_2 (left and right of the complex plane) with radius going to infinity, using the fact that $\varrho(i\tau) = \frac{i\tau}{F^+(i\tau) F^-(i\tau)}$ and $\underline{\varphi}^T = \frac{F^+}{\ell} \underline{f}^{-T} - \frac{F^-}{\ell} \underline{f}^{+T}$. This leads to an integral $\oint\limits_{C_1 \cup C_2} (\underline{\varphi}' - \mathscr{R} \underline{\varphi}) \underline{f}^T d\lambda$, where f is always the Jost solution for $\text{Re} \ell > 0$ or < 0. Asymptotic behaviour of φ, φ_0 and f leads to $\oint\limits_{C_1 \cup C_2} \left(\frac{r'}{r} \right)^{(\text{sgn Re} \ell) \cdot \ell} d\ell$,

which is *zero* when $r' < r$, and must be defined more precisely when $r' \geqq r$. We shall see this later, for $r' = r$.

This triangularity property holds for all kernels K. Now let us compute in two ways the quantity

$$\int_{-\infty}^{+\infty} \varrho(i\tau) \underset{\sim}{\varphi}_0(r) \underset{\sim}{\varphi}^T(r') d\tau \equiv A(r, r') . \qquad (3.42)$$

Using the kernel $\tilde{K} : \underset{\sim}{\varphi} \to \underset{\sim}{\varphi}_0$, we get:

$$A(r, r') = \int \varrho(\lambda) \underset{\sim}{\varphi}(r) \tilde{\mathscr{R}}(r) \underset{\sim}{\varphi}^T(r') d\lambda + \int_0^{+\infty} dr'' \int d\lambda \varrho(\lambda) \tilde{K}(r, r'') \underset{\sim}{\varphi}(r'') \underset{\sim}{\varphi}^T(r') dr'' ,$$

$$(3.43)$$

where $\tilde{K}$ is the $(v, \theta) \to (0, 0)$ kernel and $\tilde{\mathscr{R}}$ is the "asymptotic compensation" that is equal to $\mathscr{R}$, since

$$\cosh \int_0^{+\infty} \theta(r) dr = \cosh \int_0^{+\infty} -\theta(r) dr .$$

Inversely, using the transformation $\underset{\sim}{\varphi}_0 \to \underset{\sim}{\varphi}$,

$$A(r, r') = \int_{-\infty}^{+\infty} \varrho(\lambda) \underset{\sim}{\varphi}_0(r) \mathscr{R}(r') \underset{\sim}{\varphi}_0^T(r') d\lambda$$

$$+ \int_0^{+\infty} dr'' \int_{-\infty}^{+\infty} d\lambda \varrho(\lambda) \underset{\sim}{\varphi}_0(r) \underset{\sim}{\varphi}_0^T(r'') K^T(r', r'') dr'' . \qquad (3.44)$$

Closure relations lead to

$$r^2 \cdot \delta(r - r') \cdot \mathscr{R}(r) \cdot \sigma_1 + \tilde{K}(r, r') \cdot \sigma_1 = r^2 \delta(r - r')$$

$$\times \mathscr{R}(r') \sigma_1 + \sigma_1 K^T(r', r) + \mathscr{R}\Omega(r, r') + \int_0^{+\infty} \Omega(r, r'') K^T(r', r'') dr'' ,$$

where

$$\Omega(r, r') = \int (\varrho(\lambda) - \varrho_0(\lambda)) \underset{\sim}{\varphi}_0(r) \underset{\sim}{\varphi}_0^T(r') d\lambda . \qquad (3.45)$$

When $r' < r$, $K(r, r') = 0$; hence the kernel K obeys the following equations (generalized Gel'fand-Levitan-Marchenko equations):

$$K(r', r) \sigma_1 + \mathscr{R}(r) \Omega^T(r, r') + \int_{r'}^{+\infty} K(r', r'') \Omega^T(r, r'') \frac{dr''}{r''^2} = 0 , \qquad (3.46)$$

or

$$\sigma_1 \cdot \frac{K(r', r)}{\mathscr{R}(r')} \cdot \sigma_1 + \sigma_1 \Omega(r', r) + \int_{r''}^{+\infty} \sigma_1 \frac{K(r', r'')}{\mathscr{R}(r')} \sigma_1 \cdot \sigma_1 \Omega(r'', r) \frac{dr''}{r''^2} = 0 . \quad (3.47)$$

Since Ω is completely given in terms of the scattering data, we see on Eq. (3.47) that the kernel $K/\mathscr{R}$ is completely determined in a unique way once the scattering data are given and provided they have a regular behaviour [1.c]. We now have to obtain the potentials (v, θ) from the solution $K/\mathscr{R}$ of (3.47).

Introducing $K/\mathscr{R} = \begin{pmatrix} K_1/\mathscr{R} & K_2/\mathscr{R} \\ K_3/\mathscr{R} & K_4/\mathscr{R} \end{pmatrix}$, and expressing (3.38) when $(r' \to r)$ by means of the wronskian property and ℓ-asymptotic behaviour of f and φ gives the following results:

$$\frac{K_2(r,r)}{\mathscr{R}(r)} = r \tanh \int_0^r \theta \cdot dr, \tag{3.48}$$

$$\frac{K_1(r,r)}{\mathscr{R}(r)} = r \int_0^r rv(r)dr + r \cdot \theta(r) \tanh \int_0^r \theta \cdot dr. \tag{3.49}$$

Hence

$$\theta(r) = \frac{d}{dr} \operatorname{Arg} \tanh \frac{K_2(r,r)}{r\mathscr{R}(r)}, \tag{3.50}$$

$$v(r) = \frac{1}{r} \frac{d}{dr} \left(\frac{1}{r} \frac{K_1(r,r)}{\mathscr{R}(r)} + \frac{K_2(r,r)}{r\mathscr{R}(r)} \frac{d}{dr} \operatorname{Arg} \tanh \frac{K_2(r,r)}{r\mathscr{R}(r)} \right). \tag{3.51}$$

These relations also hold when both φ and φ_0 are interacting solutions, and K interpolates between them. The generalization of (3.48) is straightforward; however (3.49) requires a non-trivial cancellation between the contribution

of $\oint (\varphi - \mathscr{R}\varphi_0) \cdot r\theta_0 \cdot \dfrac{f_0}{F(\lambda)} \cdot d\lambda$ and the part of the contribution of

$\oint \lambda \cdot (\varphi - \mathscr{R}\varphi_0) \cdot \dfrac{f_0}{F(\lambda)} \cdot d\lambda$ induced by the asymptotic behaviour of $\varphi_0 f_0/F(\lambda)$ as

$(\varphi^* f^*/F^*) \left(1 + \dfrac{r\theta_0}{\lambda} \right)$, where the $(*)$ index denotes free solutions. A careful

computation shows that this is so. Hence one can generalize (3.48) and (3.49), replacing θ by $(\theta - \theta_0)$ and v by $(v - v_0)$. Note that this scheme only holds when the "compensating asymptotic term" $\mathscr{R}$ is chosen to be $\cosh \int_0^{+\infty} \theta(r)dr$, i.e. the mean value of the two asymptotic behaviours of $\underset{(|\ell| \to +\infty)}{\varphi/\varphi_0}$. In this sole case the infinite contributions to $K_1(r,r)$ coming from integration over the circle at infinity in (3.38) cancel between the two semi-circles. Otherwise one gets a divergence for $K_1(r,r)$ as $K_1(r,r) \propto (\mathscr{R}' - \mathscr{R}) \cdot R$, where $\mathscr{R}'$ is any "wrong" compensating term, and R the radius of the circle. This is also the case for s-wave Klein-Gordon equation ($\equiv$energy-dependent-potential Schrödinger equation), see [14, 15].

The relations (3.30), (3.51) are exactly similar to the one obtained in [15] for s-wave Klein-Gordon equation. Moreover, when $\theta \equiv 0$, one sees directly from (3.38) that K_1 is equal to the kernel interpolating between free and interacting solutions of the radial Schrödinger equation [9] and that it obeys in (3.47) the usual GLM equation with integral kernel Ω_2 (we define:

$$\Omega \equiv \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{pmatrix},$$

and when $\theta \equiv 0$, the SD are such that $\Omega_1 = \Omega_4 = 0$, and $K_2 = K_3 = 0$). Finally when the SD have the properties of real-potentials-scattering data, namely ($\ell_K \leftrightarrow \ell_K^*$, $c_K \leftrightarrow c_K^*$, $D^*(\tau) = D(-\tau)$), one clearly gets a real kernel $\Omega(r,r')$; therefore, (using the

hypothesis that the SD are regular enough to guarantee the unicity of the solution of the GLM equation), K is real, and so are v and θ; this shows that the whole scheme is self-consistent. Finally it is possible to obtain a differential expression of the Gel'fand-Levitan equation; by infinitesimal change of the SD, and using (3.50), (3.51), one ends with the following equations:

$$\delta\theta(r) = \frac{d}{dr}\left(\frac{1}{r}\delta(c_K\varphi_K(r)\varphi_K(r)) + \int_{-\infty}^{+\infty}\frac{1}{D^2(\tau)}\cdot\sinh\pi\tau\cdot\delta D(\tau)\cdot\varphi(r,i\tau)\varphi(r,i\tau)\cdot d\tau\right),$$

$$(3.52)$$

$$\delta v(r) = \frac{1}{r}\frac{d}{dr}\left(\frac{1}{r}\delta(c_K\cdot(\ell_K - r\theta_0)\cdot\varphi_K(r)\varphi_K(r))\right.$$

$$\left. + \int_{-\infty}^{+\infty}d\tau\cdot\frac{(i\tau - r\theta)\sinh\pi\tau}{(D(\tau))^2}\delta D(\tau)\varphi(r,i\tau)\varphi(r,i\tau)\right). \quad (3.53)$$

This will be used now to obtain two non-standard trace identities. First of all: $Q_1 \equiv \int_0^{+\infty} \theta(r)dr$ can be evaluated by differentiating with respect to c_K, ℓ_K and $D(\tau)$, computing the resulting integral, thanks to the asymptotic behaviour (3.2), in terms of the SD, and integrating back to obtain a closed expression in terms of the scattering data,

$$\int_0^{+\infty}\theta(r)dr = 2\sum_K(\operatorname{sgn}\operatorname{Re}\ell_K)\ln(\ell_K)\cdot + \fint_{-\infty}^{+\infty}\frac{\ln D(\tau)}{\tau}d\tau. \quad (3.54)$$

It was to be expected that the "asymptotic behaviour" $\int_0^{+\infty}\theta(r)dr$ was not an independent SD, since it did not appear in the closure relation, nor in the inverse scattering kernel Ω. Note that the continuum term in TR 0, TR 1, TR 2 ... goes in increasing powers: $1/\tau, 1, \tau, \ldots$; in fact, TR(0) is almost a standard trace identity in this respect.

A second identity can be obtained in the same way:

$$Q_2 \equiv \int_0^{+\infty}rv(r)\ln r\,dr$$

will appear in the next section as a result of the renormalization of the functional determinant. This trace identity was already derived in the case of the Schrödinger equation. One has to use the wronskian identities (3.22), (3.23) to compute the resulting integrals which have the form:

$$\int_0^{+\infty}\frac{1}{r^2}(\ell - r\theta)\varphi(\ell,r)\varphi(\ell,r)dr(\ell = \ell_K \quad \text{or} \quad \ell = i\tau). \quad (3.55)$$

[For the previous trace identity (TR 0), the differentiation with respect to the SD lead to an exact integral directly.] In particular, when differentiating with respect to c_K, we obtain, after part integration, the above expression, which is exactly $-1/c_K$. Hence $Q_2 = \sum_K -\ln c_K + f(\lambda_K, D(\tau))$. Differentiation with respect to the

scattering data ℓ_K leads, after part-integration, to the following expression:

$$\frac{\delta Q}{\delta \ell_K} = \int_0^{+\infty} \frac{1}{r^2} c_K \frac{(\ell_K + \delta\ell_K - r\theta)\varphi^2(\ell_K + \delta\ell_K) - (\ell_K - r\theta)\varphi^2(\ell_K)}{\delta\ell_K} \, dr, \quad (3.56)$$

which by wronskian properties, can be recast as:

$$\frac{\delta Q}{\delta \ell_K} = \int_0^{+\infty} c_K \cdot \frac{1}{\delta\ell_K} \left\{ \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \left(\frac{d}{dr} W(\varphi(\ell_K + \delta\ell_K + \varepsilon), \varphi(\ell_K + \delta\ell_K)) \right. \right.$$

$$\left. \left. - \frac{d}{dr} W(\varphi(\ell_K + \varepsilon), \varphi(\ell_K)) \right) \right\}, \quad (3.57)$$

and finally, at leading order in ε and δ, as:

$$\frac{\delta Q}{\delta\ell_K} = \int_0^{+\infty} \left[2c_K\ell_K \cdot \frac{1}{\delta\ell_K} \cdot \left\{ \lim_{\varepsilon \to 0} \frac{\frac{d}{dr} W(\varphi(\ell_K + \delta\ell_K + \varepsilon), \varphi(\ell_K + \delta\ell_K))}{(2\ell_K + 2\delta\ell_K + \varepsilon) \cdot 2\varepsilon} \right. \right.$$

$$\left. \left. - \frac{\frac{d}{dr} W(\varphi(\ell_K + \varepsilon), \varphi(\ell_K))}{(2\ell_K + \varepsilon) \cdot 2\varepsilon} \right\} + 4 \frac{\frac{d}{dr} W(\varphi(\ell_K + \varepsilon), \varphi(\ell_K)) \cdot c_K \cdot \ell_K}{2\varepsilon(2\ell_K + \varepsilon)(2\ell_K + \varepsilon + \delta\ell_K)} \right] dr.$$

$$(3.58)$$

The first two terms are formally identical to the two terms one gets when computing this trace identity for the standard Schrödinger equation (see [10b, c]); hence the result will be the same, written in terms of the Jost functions at $\ell_K = \ell$. The third term finally gives $\frac{1}{\ell_K}$, which adds a factor $\ln \ell_K$ to the expression of Q: this merely compensates the normalization of the factor c_K here, when one compares it to the normalization chosen in the standard case [9, 10].

Similar features appear for the continuum contribution. One finally ends with a similar expression for Q, in terms of the scattering data, as in the standard case, with the important differences that: the zeroes are no more real and do not go by pairs $(-\ell_K, \ell_K)$; similarly $\ln D(\tau)$ is no more real, nor even. This generalized expression reads:

$$Q_2 = \frac{1}{\pi^2} \oint_{-\infty}^{+\infty} \oint_{-\infty}^{+\infty} d\tau \, d\tau' \frac{\ln D(\tau) \ln D(\tau')}{(\tau - \tau')^2} + \frac{1}{\pi} \int_{-\infty}^{+\infty} \ln D(\tau)$$

$$\times (\Psi(i\tau) + \Psi(-i\tau) + \ln 2) d\tau - \frac{1}{\pi} \int_{-\infty}^{+\infty} d\tau \cdot \left(\sum_K \frac{\operatorname{sgn} \operatorname{Re} \ell_K}{i\tau - \ell_K} \ln D(\tau) \right)$$

$$+ \sum_K \ln \left(\frac{\pi\ell_K \cdot 2^{2(1 - \ell_K \cdot \operatorname{sgn} \operatorname{Re} \ell_K)}}{(\Gamma(\ell_K \cdot \operatorname{sgn} \operatorname{Re} \ell_K))^2} \right)$$

$$+ \sum_{K \neq L} (\operatorname{sgn} \operatorname{Re} \ell_K)(\operatorname{sgn} \operatorname{Re} \ell_L) \ln(\ell_K - \ell_L) - \sum_K \ln \tilde{c}_K. \quad (3.59)$$

Note that the $\tilde{c}_K$ here is equal to $(\ell_K \times c_K)$ to make the link with the standard case clearer. Now that we have developed the whole scheme of inverse scattering for Eq. (3.1), we can come back to our problem and express the CP_N effective action as a local function of the SD. We shall see that this is possible for all terms.

IV. The Effective Action of CP_n

We recall that the effective action reads as a "renormalized series" plus a counterterm:

$$S_{\text{eff}} = L^{\nu-2} \int_{-\infty}^{+\infty} \frac{d^{\nu-2}k_\perp}{(2\pi)^{\nu-2}} \sum_{\ell=-\infty}^{+\infty} \ln \det \left(\frac{-d_r^2 + \mu^2 + \dfrac{(\ell-r\theta)^2 - 1/4}{r^2} + v}{-d_r^2 + \mu^2 + \dfrac{\ell^2 - 1/4}{r^2}} \right)$$

$$- 2\pi L^{\nu-2} \frac{\Gamma(1-\nu/2)}{(4\pi)^{\nu/2}} m^{\nu-2} \int_0^{+\infty} rv(r)\,dr. \tag{4.1}$$

We shall now replace the radial functional determinant by its expression in terms of the scattering data:

$$S_{\text{eff}} = L^{\nu-2} \int_{-\infty}^{+\infty} \frac{d^{\nu-2}k_\perp}{(2\pi)^{\nu-2}} \left(\sum_{\ell=-\infty}^{+\infty} \ln \frac{F(\ell,k_\perp)}{F_0(\ell,k_\perp)} \right) - 2\pi \frac{\Gamma(1-\nu/2)}{(4\pi)^{\nu/2}} \int_0^{+\infty} rv(r)\,dr, \tag{4.2}$$

ℓ being any positive or negative integer.

Let us now use the "partial wave expansion" of the renormalization coefficient

$$\frac{2\pi}{(4\pi)^{\nu/2}} \Gamma(1-\nu/2) m^{\nu-2} = \int_{-\infty}^{+\infty} \frac{d^{\nu-2}k}{(2\pi)^{\nu-2}} \sum_{\ell=-\infty}^{+\infty} I_\ell K_\ell(r\sqrt{m^2+k^2}), \tag{4.3}$$

where ℓ is any positive or negative integer. This follows from the Gegenbauer sum rule: [24]

$$K_0(w) = I_0 K_0(w) + 2 \sum_{\ell=1}^{+\infty} I_\ell K_\ell(w) \cos \ell\varphi, \qquad w \equiv \sqrt{2} \cdot r \cdot (1-\cos\varphi)^{1/2}, \tag{4.4}$$

and from the limit behaviour of $K_0(w)$ when $w \to 0$ as:

$$K_0(w) \sim \ln(w) + O(1), \tag{4.5}$$

once one assumes the following rules of dimensionally regularized integration [25]:

$$\int_{-\infty}^{+\infty} d^\nu k \, \mathscr{P}(k) = 0 \quad \text{for any polynomial } P, \nu \in \mathbb{R}^+/\mathbb{N}, \tag{4.6}$$

$\int_{-\infty}^{+\infty} d^\nu k \, f(k) = f(0) + O(\nu)$, $\nu \to 0$, when f is a test function, decreasing fast enough for $|k| \to +\infty$, and well-defined when $|k| \to 0$. $\tag{4.7}$

The identity (4.3) can be obtained in another, more formal, way. Using the identity between the functional determinant expressed as a sum of partial waves, and the original expression, which reads

$$\ln \det \frac{-D_\mu D_\mu + \sigma}{-\partial_\mu^2 + m^2} = L^{\nu-2} \int_{-\infty}^{+\infty} \frac{d^{\nu-2}k_\perp}{(2\pi)^{\nu-2}} \cdot \sum_{\ell=-\infty}^{+\infty} \ln \det$$

$$\times \left(\frac{-d_r^2 + (\ell^2 - 1/4)/r^2 + v + m^2 + k_\perp^2}{-d_r^2 + (\ell^2 - 1/4)/r^2 + m^2 + k_\perp^2} \right).$$

(when $\theta \equiv 0$), one can write in two ways the linear part in v; the left-hand side yields a

$$2\pi \frac{\Gamma(1-v/2)}{(4\pi)^{v/2}} L^{v-2} m^{v-2} \int\limits_0^{+\infty} r v(r) dr,$$

and the right-hand side yields a

$$L^{v-2} \int\limits_{-\infty}^{+\infty} \frac{d^{v-2}k_\perp}{(2\pi)^{v-2}} \sum_{\ell=-\infty}^{+\infty} \int\limits_0^{+\infty} I_\ell K_\ell(\mu r) \cdot r v(r) dr.$$

Since this equality is true for any function $v(r)$, one obtains (4.3) as an immediate consequence. This "partial wave expansion" was already used in [5c] and [11], together with its four-dimensional analogue, to obtain the renormalized effective action for fermionic models in two and four dimensions. Using now the expression (4.3), we can rewrite the complete effective action as an integral over $d^{v-2}k_\perp$:

$$S_{\text{eff}} = L^{v-2} \int\limits_{-\infty}^{+\infty} \frac{d^{v-2}k_\perp}{(2\pi)^{v-2}} \cdot \sum_{\ell=-\infty}^{+\infty} \left\{ \ln \frac{F(\ell,k_\perp)}{F_0(\ell,k_\perp)} - \int\limits_0^{+\infty} I_\ell K_\ell(r\sqrt{m^2+k_\perp^2})rv(r)dr \right\}.$$
(4.8)

We shall now recall another useful sum rule. Using the Gegenbauer sum rule, together with the following equality

$$\sum_{\ell>0} \frac{\cos\ell\varphi}{\ell} = \ln\varphi,$$
(4.9)

we get

$$I_0 K_0 + \left(\sum_{\ell>0} + \sum_{\ell<0} \right) \left(I_\ell K_\ell - \frac{1}{2|\ell|} \right) (r\sqrt{m^2+k_\perp^2}) = -\ln(\sqrt{m^2+k_\perp^2} \cdot r) - \gamma + 2\ln 2,$$
(4.10)

by taking the limit $\varphi \to 0$. Going back to (4.8), we can take directly the limit $v \to 2$ of this expression, since all functions of $k_\perp$ in (4.8), are test-functions.

We shall make some comments about this regularization scheme now: As we have just mentioned, both $\ln F(\ell,k_\perp)/F_0(\ell,k_\perp)$ and $I_\ell(\mu r)K_\ell(\mu r)$ are test-functions, since they behave as $1/|k_\perp|$ when $|k_\perp|$ goes to $+\infty$. This is evident for $I_\ell K_\ell$ using its asymptotic expansion (ℓ-finite, $m^2+k_\perp^2 \to +\infty$), and also for the ln det, which is formally obtained as a loop expansion containing powers of $(I_\ell K_\ell)$ multiplied by potentials $(\theta^2, \theta/r, v)$. However one cannot apply this scheme to the separated terms of the series. Indeed, neither $\sum_\ell \ln \frac{F}{F_0}$ nor $\left(\sum_\ell IK\right)$ are defined when $k_\perp \to 0$. They can be considered as defined $under\ the\ integration\ sign$, and when $v \neq 2$. Let us explain this more precisely: We have mentioned in Sect. II that the series $\sum_\ell \ln F/F_0(\ell)$ was divergent. Rigorously speaking, we assume that these divergent undefined series $\left(\sum_\ell IK\right)$ and $\left(\sum_\ell \ln F/F_0\right)$ have been regularized everywhere as $\sum_\ell I_\ell K_\ell \cdot \cos\ell\varphi$ and $\sum_\ell \ln \frac{F(\ell)}{F_0(\ell)}\cos\ell\varphi$, with φ going to zero. In this case, the two series are convergent when $\varphi \neq 0$, and the subtraction of the "asymptotic" behaviour given by the series (constant) $\times \sum_{\ell>0} \cdot \frac{1}{\ell}\cos\ell\varphi$ (which is also convergent

for $\varphi \neq 0$), gives a finite result for both $\sum\limits_{\ell=-\infty}^{+\infty} \left(IK - \dfrac{1}{2|\ell|} \right)$ and $\sum\limits_{\ell=-\infty}^{+\infty} \left(\ln \dfrac{F}{F_0} - \dfrac{TR\,1}{2|\ell|} \right)$ (no more "φ" regularization is needed then). All this must be done *under* the $\int\limits_{-\infty}^{+\infty} \cdot\, d^{\nu-2}k_\perp$ sign so as to use (4.6) and formally be able to speak of $\int \sum\limits_{\ell} \ln \dfrac{F}{F_0}$ and $\int \sum\limits_{\ell} I_\ell K_\ell$ as being well defined, since this integration over $d^{\nu-2}k_\perp$ enables one to add any constant or polynomial (in $k_\perp$) quantity. (This is how one demonstrates (4.3).) However, once the two series are thus "made convergent," the use of (4.7) to get the limit $\nu \to 2$ is forbidden for each separately, since these two objects: $\sum\limits_{\ell} \ln \dfrac{F}{F_0} -$ linear part in $1/\ell$ or $\sum\limits_{\ell} I_\ell K_\ell - \dfrac{1}{2|\ell|}$ are *no more* test functions. The expression (4.8) is the only one that, at the same time, is entirely written in terms of test functions, *and* subtracts its divergent term to each of the series $\sum\limits_{\ell} \ln F/F_0$ and $\sum\limits_{\ell} I_\ell K_\ell$. This *mutual* subtraction clearly appears when one writes $\left(\text{under } \int\limits_{-\infty}^{+\infty} d^{\nu-2}k_\perp \right)$:

$$\sum\limits_{\substack{\ell \neq 0 \\ \ell > 0}}^{+\infty} \left\{ \ln \frac{F(\ell, k_\perp)}{F_0(\ell, k_\perp)} + \ln \frac{F(-\ell, k_\perp)}{F_0(-\ell, k_\perp)} + \frac{2}{\ell} \sum\limits_{K} (\text{sgn Re}\,\ell_K) \cdot \ell_K \right.$$

$$\left. + \frac{1}{\ell} \int\limits_{0}^{+\infty} rv(r)\,dr - 2 \int\limits_{0}^{+\infty} I_\ell K_\ell (r\sqrt{m^2 + k_\perp^2}) \cdot r \cdot v(r) \cdot dr \right\}. \qquad (4.11)$$

Using the trace identity (TR 1) shows that (4.11) and (4.8) are identical. Now both ($\ln$ det) and $\int\limits_{0}^{+\infty} \cdot\, I_\ell K_\ell rv(r)\,dr$ in (4.11) are test functions. Moreover this series is convergent: the last two terms give back Eq. (4.10) (when sum is taken over $\ell \neq 0$ and $\ell = 0$) and the first two terms lead to a series behaving asymptotically as $1/\ell^2$ (the term of order 1 exactly cancels in $\sum\limits_{\ell \neq 0}$ owing to its dependence on the sign of ℓ, as $\pm \int\limits_{0}^{+\infty} \theta(r)\,dr$.

Hence the function under the integral sign in (4.8) is a well-defined test function, and the limit can be taken. One finally gets

$$S_{\text{eff}} = \sum\limits_{\ell=0}^{+\infty} \left(\ln \frac{F(\ell)}{F_0(\ell)} + \ln \frac{F(-\ell)}{F_0(-\ell)} + \frac{1}{\ell} \sum\limits_{K} \ell_K (\text{sgn Re}\,\ell_K) \right)$$

$$+ \frac{1}{2} \ln \frac{D(0)}{D_0(0)} + \int\limits_{0}^{+\infty} \left\{ I_0 K_0 + 2 \sum\limits_{\ell>0}^{+\infty} \left(I_\ell K_\ell - \frac{1}{2\ell} \right) \right\} rv(r)\,dr. \qquad (4.12)$$

We have replaced here $\ln F(0)/F_0(0)$ by $1/2 \ln \dfrac{F^+(0)F^-(0)}{F_0^+(0)F_0^-(0)} \equiv \ln \dfrac{D(0)}{D_0(0)}$. This is justified by noting that $F(0)$ is not a well-defined object, since the Jost function is F^+ for $\text{Re}\,\ell > 0$ and F^- for $\text{Re}\,\ell < 0$, and the limit $|\ell| \to 0$ is not well controlled (for interacting solutions). Actually, all through this paper we have used $F(i\tau) = F^+(i\tau + \varepsilon)$ or $F^-(i\tau - \varepsilon)$, $\varepsilon \to 0$, according to the side of the imaginary axis on

which the spectral parameter was to be taken. This prescription was never ambiguous. Here however it is clear that the full expression of S_{eff} has a manifest symmetry by exchanging $(\ell \to -\ell)$; therefore we understand that $\ln F(0)$ has the meaning of $1/2 \ln(F^+(0)F^-(0))$. Now we only have to apply the sum rules and series equalities so as to get the effective action in terms of the SD. We know that (3.30),

$$\ln \frac{F(\ell)}{F_0(\ell)} = \mp \int\limits_0^{+\infty} \theta(r)dr + \sum_{K\pm} \ln \frac{\ell - \ell_K^{\pm}}{\ell - \ell_K^{\mp}} \pm \int\limits_{-\infty}^{+\infty} \frac{\ln D(\tau)}{i\tau - \ell} \frac{d\tau}{2\pi} \left(\mathrm{Re}\,\ell \begin{array}{c} >0 \\ <0 \end{array} \right). \quad (4.13)$$

The first term cancels between $\sum\limits_\ell (\ell > 0)$ and $\sum\limits_\ell (\ell < 0)$ in (4.12). The following terms lead, together with the counterterms in (4.12), to the series:

$$\sum_{\ell \neq 0}^{+\infty} \sum_K (\mathrm{sgn}\,\mathrm{Re}\,\ell_K)\left(\ln \frac{\ell - \ell_K}{\ell + \ell_K} + \frac{2\ell_K}{\ell} \right) \pm \sum_{\ell \neq 0}^{\pm\infty} \int\limits_{-\infty}^{+\infty} \left(\frac{\ln D(\tau)}{i\tau - \ell} + \frac{\ln D(\tau)}{\ell} \right) \cdot \frac{d\tau}{2\pi},$$
$$(4.14)$$

which is obtained as [30]

$$\sum_K (\mathrm{sgn}\,\mathrm{Re}\,\ell_K)\left(2\gamma\ell_K + \ln \frac{\Gamma(1+\ell_K)}{\Gamma(1-\ell_K)} \right) + \int\limits_{-\infty}^{+\infty} (\psi(1-i\tau)$$

$$+ \psi(1+i\tau) - 2\gamma)\ln D(\tau)\frac{d\tau}{2\pi}. \quad (4.15)$$

Using now the sum rule (4.10) for $k_\perp = 0$, together with the trace identity for $\int\limits_0^{+\infty} r \cdot \ln mr \cdot v(r) \cdot dr$, (this term arises immediately when inserting (4.10) into (4.12)), we finally get the expression of the effective action as a function (in a completely closed form) of the scattering data:

$$S_{\mathrm{eff}} = \frac{1}{\pi^2} \mathop{\oint}\limits_{-\infty}^{+\infty} \mathop{\oint}\limits_{-\infty}^{+\infty} d\tau \, d\tau' \frac{\ln D(\tau)\ln D(\tau')}{(\tau - \tau')^2} + \frac{1}{2} \ln \frac{D(0)}{D_0(0)}$$

$$- \frac{1}{\pi} \int\limits_{-\infty}^{+\infty} \left(\sum_K \frac{\mathrm{sgn}\,\mathrm{Re}\,\ell_K}{\ell_K - i\tau} \cdot \right) \ln D(\tau) \cdot d\tau$$

$$+ 2 \sum_{K \neq L} (\mathrm{sgn}\,\mathrm{Re}\,\ell_K)(\mathrm{sgn}\,\mathrm{Re}\,\ell_L)\ln(\ell_K - \ell_L)$$

$$+ 2 \sum_K \ln(4\ell_K^2 \sin\pi(\ell_K \,\mathrm{sgn}\,\mathrm{Re}\,\ell_K)) - 2 \sum_K \ln \tilde{c}_K. \quad (4.16)$$

Note that when $\theta \equiv 0$, one can apply the particular properties of the scattering data derived in Sect. III; and when v is real, one recovers in that way the $1/N$-effective action of the non-linear sigma model [10]. The clearest property of this effective action is that it can be written in a closed form as a function of the scattering data. This feature was always met with whenever we studied $1/N$ expansion of 2-dimensional integrable models, NLσ or Gross-Neveu, or Chiral Gross-Neveu [27]. However there is a difference here. The CP_{N-1} model is *classically* integrable (the field equations are the compatibility conditions of an associated linear system, see [12]). This integrability is broken at quantum level by

renormalization effects [18]. However this does not seem to affect the separability of the $1/N$ action. Of course we have restricted ourselves to spherically symmetric fields; we have already seen effective actions that *were* separable in terms of the SD (while their "integrability" properties were not clear), once one had restricted the study to spherically symmetric fields: in particular, this was the case for the massless Φ^4-model *in four dimensions*, either classical or $1/N$ action, and also for the $1/N$ expansion of massless Φ^4 + Yukawa model *in four dimensions* with a convenient relation between the coupling constants [5c], but there is a more immediate reason. The terms inducing the non-conservation, at quantum level, of the first non-local charge, are obtained at the 1-loop level in the $1/N$ perturbation expansion [18]. Since we are dealing with a "classical" $1/N$-action, finite at tree level, but without any higher order effects (which should appear with higher orders of $1/N$), it is normal that we do not see any influence of the non-integrability of the model on the large-N effective action. Anyhow we believe that the link between classical integrability and/or quantum integrability, and separability of effective action, should be further investigated. In particular, one notes that in the case of the massive Thirring model, which *is* an integrable theory [28], the effective action obtained as a function of the auxiliary field:

$$S_{\text{eff}} = \ln \det(\partial\!\!\!/ + m + A\!\!\!/) + \int\limits_{-\infty}^{+\infty} A^2 \cdot d^2 x$$

does *not* seem separable in terms of the SD of the associated linear problem (taking $A_\mu = \varepsilon_{\mu\nu}\partial^\nu h(r)$, and making IST in angular momentum, $\int\limits_{-\infty}^{+\infty} A_\mu A^\mu d^2 x$ does not appear as a trace identity).

Let us go back to our specific problem of instantons. The second property of this action is that it does not have any (spherically symmetric) instanton, owing to the $(\ln C_K)$ dependence in (4.16). Any instanton would have to obey the following equations:

$$\frac{\delta S}{\delta c_K} = \frac{2}{c_K} = 0, \tag{4.17}$$

$$\frac{\delta S}{\delta \ell_K} = 0, \tag{4.18}$$

$$\frac{\delta S}{\delta D(\tau)} = 0. \tag{4.19}$$

Equation (4.17) implies that an instanton of S_{eff} cannot have a bound-state (ℓ_K, c_K) since any bound state would lead to a finite value of c_K. Equation (4.18) is then purposeless and we remain with (4.19), which merely implies (using S_{eff} without bound states)

$$\fint\limits_{-\infty}^{+\infty} \frac{\ln D(\tau)}{D(\tau')(\tau - \tau')^2} d\tau = 0$$

or equivalently:

$$\fint\limits_{-\infty}^{+\infty} \frac{\ln D(\tau)}{(\tau - \tau')^2} d\tau = 0$$

for any τ', which means that:

$$\frac{d}{d\tau}\left(\ln\frac{F^+ e^{-\int\theta dr}}{F_0^+} - \ln\frac{F^- e^{\int\theta dr}}{F_0^-}\right) = 0$$

(from the definition of $D(\tau)$ and the dispersion relations (3.13)). Hence the quantity $\delta(\tau) = \dfrac{F^+ e^{-\int\theta dr} \cdot F_0^-}{F_0^+ \cdot F^- e^{\int\theta dr}}$ is a constant for all τ, and since $\delta(\tau) = 1$ when $\tau \to +\infty$, it has a trivial value 1 for any τ. But we can write a dispersion relation for $\dfrac{F^+ e^{-\int\theta dr}}{F_0^+}$ and $\dfrac{F^- e^{\int\theta dr}}{F_0^-}$, using the quantity $\delta(\tau)$ instead of $D(\tau)$. We get

$$\ln\frac{F^\pm e^{\mp\int\theta dr}}{F_0^\pm}(\lambda) = \int_{-\infty}^{+\infty}\frac{\ln\delta(\tau)}{(i\tau - \lambda)}d\tau \qquad \left(\operatorname{Re}\lambda \begin{matrix}>0\\<0\end{matrix}\right). \tag{4.20}$$

Hence, since $\delta(\tau)$ is equal to 1, $\dfrac{F^\pm e^{\mp\int\theta dr}}{F_0^\pm}$ is also equal to 1, which means that $D(\tau)$ is equal to 1, and finally the configuration (θ, v) which has no eigenvalues ℓ_K and a unit continuous SD $D(\tau) \equiv 1$, is trivial. Hence no instanton exists with a finite action (at least spherically symmetric.)

However we do not know whether the $1/N$ series of the CP_{N-1} model is more than Borel-summable, or even convergent, as it seemed to be the case in the NLσ model [10] and the Chiral Gross-Neveu model [11]. Owing to the breaking of integrability by quantum effects, the S-matrix is not known, and therefore no indications, even indirect, exist about the $1/N$ series. Our general conclusion, therefore, is that the absence of spherically symmetric instantons is probably a proof of particular (more than Borel-summable) behaviour of the $1/N$ series, but that since no exact result is available, this conjecture is not supported by other facts.

However it must be emphasized that the inverse scattering method exposed in this paper (and also in all the other papers dealing with this subject, see [22]) can be used for other purposes than studying saddle-points of effective actions; in particular it is possible to use these methods for estimating the effective actions of *background* field configurations, thereby studying the structure of non-trivial vacua (see for instance [16]). In the particular case of CP_{N-1}, it should be possible to study the confinement of the field (A_μ) by computing directly by IST the action of a non-zero-A_μ configuration (for instance a soliton-like (θ, v) configuration with trivial continuum contribution $D(\tau) \equiv 1$, that can be obtained by explicitly solving the Gel'fand-Levitan equation with a degenerate kernel), and studying it when its range should go to infinity: therefore IST would provide us with a method for exact computation of the $1/N$ action for these "unconfined" configurations; we could in this way study how exactly they are suppressed in the functional integral. We shall leave this problem open here.

This type of study could also be extended to abelian, and possibly non-abelian gauge theories, but we have not formulated until now the proper inverse-scattering

problem, even for maximally symmetric gauge fields. In fact, with massless vector fields in four dimensions, it should be necessary to introduce two independent, angular-momentum-like spectral parameters (see [29]). This also remains a completely open problem.

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