Commun. Math. Phys. 106, 211–239 (1986)

Communications in
Mathematical
Physics

Generalized Solutions of the Radiative Transfer Equations in a Singular Case

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Abstract. This paper is devoted to the study of the radiative transfer equations:

$$\text{(TR)}\qquad \begin{cases} \dfrac{\partial \mathscr{E}}{\partial t} + \displaystyle\int_{\nu,\Omega} \sigma_\nu(\mathscr{E})\,(B_\nu(\mathscr{E}) - I)\,\frac{d\Omega}{|S^N|}\,d\nu = 0\,, \\ \dfrac{\partial I}{\partial t} + \Omega \cdot \nabla_x I + \sigma_\nu(\mathscr{E})\,(I - B_\nu(\mathscr{E})) = \mathscr{D} I\,. \end{cases}$$

First, we prove a global existence theorem, which allows a blow-up of the opacity $\sigma_\nu(\mathscr{E})$ when $\mathscr{E} \to 0$. Thus, it extends Mercier's previous result [13]. This proof relies mainly on a nonlinear version of Hille-Yosida theorem: see Crandall-Ligett [9].

Then, we prove the uniqueness of the semigroup solving (TR), and some regularity results (in the class of functions with bounded variation).

Finally, we prove the convergence of some splitting algorithms associated to (TR).

Introduction

We are interested in a system of two nonlinear PDEs which can be actually regarded as a perturbation of the well-known transport equation. These equations are classical in astrophysics and represent the evolution of a stellar atmosphere in the absence of hydrodynamical motion and heat conduction. The photons in the medium will be ruled by a classical transport equation involving terms describing emission, true absorption and Thomson scattering. On the other hand we shall assume local thermodynamical equilibrium for the matter. It means that a local temperature T [and energy $\mathscr{E}(T)$] can be defined at each point of the medium. Moreover, the emission coefficient at each point is proportional to the true

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absorption coefficient through the following factor (Planck's function):

$$B_\nu(T)=\frac{2h\nu^3}{c^2}\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1}.^{1}$$

This assumption enforces the emission and true absorption terms to be coupled in an additional thermodynamical equation (energy balance), giving rise to a perturbation of the kinetic equation. The scattering will be assumed to be isotropic, and conservative. More details about the physical framework of this work can be found in [2, 8, 15]. In the sequel we assume that $\mathscr{E}(T)$ is proportional to T, so we write $B_\nu(\mathscr{E})$ instead of $B_\nu(T)^2$.

Now, we present the system describing this phenomena. We are given a smooth convex open subset X of $\mathbb{R}^{N+1}$. For each point $x\in\partial X$, $n(x)$ will denote the unit outward normal to ∂X at point x. The unknowns of our problem are:

- $I(x,t,\Omega,\nu)$, where $x\in X$, $t\geqq 0$, $\Omega\in S^N$, $\nu\in\mathbb{R}_\nu^{+*}$, which represents the density of photons located at x at time t, with frequency ν and velocity with direction Ω,
- $\mathscr{E}(x,t)$, where $x\in X$, $t\geqq 0$, which is the material energy at position x at time t.

Now, the radiative transfer equations are written as:

$$\frac{\partial\mathscr{E}}{\partial t}+\iint_{\mathbb{R}_\nu^{+*}\times S^N}[q_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E})I]\,\delta\Omega\,d\nu=0\,, \tag{0.1}$$

$$\frac{\partial I}{\partial t}+\Omega\cdot\nabla_x I+\sigma_\nu(\mathscr{E})I-q_\nu(\mathscr{E})=\mathscr{D}I\,, \tag{0.2}$$

$$I(x,0,\Omega,\nu)=I_0(x,\Omega,\nu),\ \mathscr{E}(x,0)=\mathscr{E}_0(x)\,, \tag{0.3}$$

$$I|_{(\partial X\times S^N)_-}=h\,. \tag{0.4}$$

In the monodimensional case ($N=0$) S^N is replaced by $[-1,1]$ and Ω by μ (the incidence parameter). The rest remains unchanged.

The function $\sigma_\nu(\mathscr{E})$ is the opacity of the matter, usually very complicated: it contains all the difficulties from quantum mechanics for this problem. In the physical problem, $q_\nu(\mathscr{E})=\sigma_\nu B_\nu(\mathscr{E})$, where $B_\nu(\mathscr{E})$ is Planck's function (as defined above). Here, we will only need few assumptions on σ_ν and B_ν (see Sect. I). $\mathscr{D}$ is an integral operator describing the scattering, and may be written:

$$\mathscr{D}I(x,\Omega,\nu)=\int_{S^N}\kappa_d(\Omega,\Omega')\,(I(\Omega')-I(\Omega))\,\delta\Omega'\,. \tag{0.5}$$

Finally, $\delta\Omega$ is the normalized measure on S^N

$$\delta\Omega=d\Omega/|S^N|\,, \tag{0.6}$$

and we have denoted

$$\partial(X\times S^N)_-=\{(x,\Omega)\in\partial X\times S^N\,|\,\Omega\cdot n(x)<0\}\,. \tag{0.7}$$

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[1] c denotes the velocity of the light, h the Planck's constant, and k the Boltzmann constant

[2] However, when $\mathscr{E}(T)$ is an increasing smooth function of T, the same methods may be applied [if the assumptions (H1), (H2)...(H9) are satisfied]

The main purpose of this paper is to study the Cauchy problem (0.1)–(0.4). The existence result (of Sect. II) relies on Crandall-Ligett generation theorem [9]. Indeed, we treat the general case where $\sigma_\nu(\mathscr{E})$ may blow up when $\mathscr{E}\to 0$. Thus (0.1)–(0.4) are no longer a Lipschitz perturbation of the transport equation. Nevertheless, Crandall-Ligett theorem asserts the existence of a generalized solution of (0.1)–(0.4), even knowing very few assumptions on $\sigma_\nu(\mathscr{E})$ and very few a priori estimates on the solution. The counterpart of this general theory is that the equations hold only in a semigroup sense and that it does not provide uniqueness results. To prevent these difficulties, we prove that, for uniformly positive initial data $\mathscr{E}_0$, the generalized solution is classical. The uniqueness of the semigroup easily follows from this remark.

Then we are able to prove the regularity of the generalized solutions in the singular case (in the class of functions with bounded variation).

Finally, we prove the convergence of some splitting algorithms associated with (0.1)–(0.4). This result relies on a splitting formula, well-known for Hilbert spaces, that we adapt to the case of general Banach spaces.

Much of this work is inspired by Mercier [13] (at least, Sects. II and V). The equations and notations are close to [13]. Our idea was to extend [13] to the case of initially cold area (i.e. $\mathscr{E}_0 \geqq 0$ instead of $\mathscr{E}_0 \geqq \alpha > 0$) and to deal with very general opacities $\sigma_\nu(\mathscr{E})$, in order to fit with the physical difficulties.

This paper is organized as follows: the assumptions and main results are stated in Sect. I. We give also heuristic derivations for the theorems. Section II is devoted to the existence proof, and the maximum principle. It consists in proving the maximal accretiveness for the operator associated to (0.1)–(0.4). In Sect. III, we prove that the generalized solution of (0.1)–(0.4) is classical when the initial energy $\mathscr{E}_0$ is uniformly positive. We deduce from this the uniqueness of the semigroup solving (0.1)–(0.4).

In Sect. IV, we shall prove a BV regularity theorem when the temperature is allowed to vanish. Finally Sect. V is devoted to the study of some splitting algorithms for (0.1)–(0.4): we prove their convergence.

I. Assumptions and Main Results

The purpose of the present section is to state precisely the assumptions we need on the parameters of (0.1)–(0.4). We also introduce some notations and finally give some heuristic proofs for the main results of this paper.

1. Assumptions

One of the physical constraints about the opacity $\sigma_\nu(\mathscr{E})$ is that it may blow up when $\mathscr{E}\to 0_+$. Moreover, the frequency dependance of $\sigma_\nu(\mathscr{E})$ is not known very accurately. Therefore, it is rather hopeless to look for some good mathematical models for $\sigma_\nu(\mathscr{E})$. Thus, we work with some very general assumptions on the opacity, coping with the fact that $\lim\limits_{\mathscr{E}\to 0_+} \sigma_\nu(\mathscr{E}) = +\infty$.

We assume that:

(H1) $\sigma_\nu(\mathscr{E})$, $q_\nu(\mathscr{E})$ are positive, defined on $(\mathbb{R}^{+*})^2$, and, for a.e. $\nu>0$, belong to $C^1(\mathbb{R}_{\mathscr{E}}^{+*})$;

(H2) for a.e. $\nu>0$, $\mathscr{E}\to\sigma_\nu(\mathscr{E})$ [respectively $\mathscr{E}\to q_\nu(\mathscr{E})$] is nonincreasing (respectively nondecreasing);

(H3) for a.e. $\nu>0$, $q_\nu(\mathscr{E})\to 0$ when $\mathscr{E}\to 0_+$;

(H4) $\forall\mathscr{E}>0,\ K>0,\ \sigma_\nu(\mathscr{E})\,B_\nu(K)\in L^1(\mathbb{R}_\nu^{+*})$, where $B_\nu=\dfrac{q_\nu}{\sigma_\nu}$;

(H5) the operator $\mathscr{E}\to q_\nu(\mathscr{E})$ is continuous from $L^1(X)^+$ to $L^1(X\times S^N\times\mathbb{R}_\nu^{+*})^+$;[3]

(H6) $\kappa_d(\Omega,\Omega')\in L^\infty(S^N\times S^N)^+$, is symmetric, and $\int\limits_{S^N}\kappa_d(\Omega,\Omega')\,d\Omega'=1$, $\forall\Omega\in S^N$;

(H7) $\forall\mathscr{E}>0,\ \sigma_\nu(\mathscr{E})\in L^\infty(\mathbb{R}_\nu^{+*})$;

(H8) $\forall 0<a<b,\ \operatorname*{Sup}\limits_{\substack{\nu>0\\ \mathscr{E}\in]a,b[}}|\sigma_\nu'(\mathscr{E})\,B_\nu(\mathscr{E})|<+\infty$;

(H9) $\forall 0<a<b,\ \operatorname*{Sup}\limits_{\substack{\nu>0\\ \mathscr{E}\in]a,b[}}|q_\nu(\mathscr{E})|<+\infty$;

Formally, we are given a function $h=h(x,\Omega,\nu)$ defined on $(\partial X\times S^N)_-\times\mathbb{R}^{+*}$ (the incoming density at the boundary) which satisfies:

$$0\leqq h(x,\Omega,\nu)\leqq B_\nu(M) \tag{1.1}$$

for some positive constant M. We shall need some regularity assumptions on h for the BV regularity theorem. This point will be discussed in Sect. IV.

Remark. It would be enough to assume that $h\in L^1((\partial X\times S^N)_-\times\mathbb{R}_\nu^{+*};\ \Omega\cdot n(x)\,d\Gamma\,\delta\Omega\,d\nu)^+$ for the existence theorem. However, the bound (1.1) will be of constant use later (see Cessenat [6 and 7] for a complete treatment of the boundary conditions and trace theorems in transport theory).

2. *Setting of the Problem*

Let us begin with an energy estimate for the problem (0.1)–(0.4); by integrating (0.1) on X, (0.2) on $X\times S^N\times\mathbb{R}_\nu^{+*}$, by adding up the obtained equalities we get:

$$\frac{d}{dt}\left(\int\mathscr{E}\,dx+\iiint I\,dx\,\delta\Omega\,d\nu\right)\leqq-\iiint\limits_{(\partial X\times S^N)_-\times\mathbb{R}^+}\Omega\cdot n(x)h\,d\Gamma(x)\,\delta\Omega\,d\nu<\infty \tag{1.2}$$

[because $\int\mathscr{D}I\,\delta\Omega=0$, κ_d being symmetric according to (H6)]. The a priori estimate (1.2) leads us to the following abstract formulation of (0.1)–(0.4). According to (1.2), $(\mathscr{E},I)$ belongs to the Banach space: $E=L^1(X)\times L^1(X\times S^N\times\mathbb{R}_\nu^{+*})$.

Then, we define on E an operator Q by:

$$Q(\mathscr{E},I)=\left(\iint[q_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E})I]\,\delta\Omega\,d\nu;\ \Omega\cdot\nabla_x I+\sigma_\nu(\mathscr{E})I-q_\nu(\mathscr{E})-\mathscr{D}I\right),$$

$$D(Q)=\{(\mathscr{E},I)\in E^+\,|\,\Omega\cdot\nabla_x I\in L^1(X\times S^N\times\mathbb{R}_\nu^{+*}),\ I_{|(\partial X\times S^N)_-}=h,\quad\text{and}\quad \sigma_\nu(\mathscr{E})I\in L^1(X\times S^N\times\mathbb{R}_\nu^{+*})\},$$

taking as a convention $0\cdot\sigma_\nu(0)=0$.

[3] If E is a real ordered vector space, E^+ denotes its positive cone

Now, (0.1)–(0.4) can be written as the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}(\mathscr{E}, I) + Q \cdot (\mathscr{E}, I) = 0\,, \\ (\mathscr{E}, I)|_{t=0} = (\mathscr{E}_0, I_0) \end{cases} \tag{1.3}$$

[the boundary condition being contained in the definition of $D(Q)$].

3. *Main Results*

Written as (1.3), Eqs. (0.1)–(0.4) allow a treatment by the nonlinear version of Hille-Yosida theory (cf. Crandall and Ligett [9]). Mercier ([13, 14]) has already noticed that some kind of truncation of the operator Q is accretive. Namely, we are going to prove that Q is, in some sense, m-accretive (cf. part II for a precise statement). Thus, we have

Theorem A (Global existence). *Under assumptions* (H1)–(H6), *there exists a strongly continuous contraction semigroup on* E^+, *denoted by* $\exp(-tQ)$ *and defined as:*

$$\exp(-tQ) \cdot Z = \lim_{n \to +\infty} \left(I + \frac{t}{n} Q\right)^{-n} \cdot Z, \ \forall Z \in E^+, t \geqq 0\,. \tag{1.4}$$

From now on, $(\mathscr{E}, I)(t) = \exp(-tQ) \cdot (\mathscr{E}_0, I_0)$ [with $t \geqq 0$ and $(\mathscr{E}_0, I_0) \in E^+$] will be called the generalized solution of (0.1)–(0.4). The main difficulty is to understand in which sense Eqs. (0.1)–(0.2) hold for generalized solutions – we do not even know whether they hold in $\mathscr{D}'$. However, we can state that the generalized solution is a classical solution when $\mathscr{E}_0$ is uniformly positive. The first step in this direction is the following

Theorem B (Maximum and Minimum Principle). *Let* $(\mathscr{E}_0, I_0) \in E^+$ *such that:* $\exists K \geqq 0$ *such that:* $h \leqq B_\nu(K)$, $I_0 \leqq B_\nu(K)$, *and* $\mathscr{E}_0 \leqq K$ *(respectively* $\exists k \geqq 0$ *such that* $B_\nu(k) \leqq h$, $B_\nu(k) \leqq I_0$, *and* $k \leqq \mathscr{E}_0$*).*

Then, for each $t \geqq 0$, $(\mathscr{E}, I)(t) = \exp(-tQ)$. $(\mathscr{E}_0, I_0)$ *satisfies:* $\mathscr{E}(t) \leqq K$ *and* $I(t) \leqq B_\nu(K)$ *(respectively* $\mathscr{E}(t) \geqq k$ *and* $I(t) \geqq B_\nu(k)$*).*

Theorem B can be derived from Eqs. (0.1)–(0.4) by a straightforward computation. We notice that:

$$\begin{aligned} &\iint \sigma_\nu(\mathscr{E}) [B_\nu(\mathscr{E}) - I] \operatorname{sgn}^+(\mathscr{E} - K)\, \delta\Omega\, d\nu \\ &= \iint \sigma_\nu(\mathscr{E}) [B_\nu(\mathscr{E}) - B_\nu(K)] \operatorname{sgn}^+(\mathscr{E} - K)\, \delta\Omega\, d\nu \\ &\quad + \iint \sigma_\nu(\mathscr{E}) [B_\nu(K) - I] \operatorname{sgn}^+(\mathscr{E} - K)\, \delta\Omega\, d\nu \\ &= \iint \sigma_\nu(\mathscr{E}) [B_\nu(\mathscr{E}) - B_\nu(K)]^+ \delta\Omega\, d\nu + \iint \sigma_\nu(\mathscr{E}) [B_\nu(K) - I] \operatorname{sgn}^+(\mathscr{E} - K) \delta\Omega\, d\nu\,. \end{aligned} \tag{1.5}$$

In the same way, we have

$$\begin{aligned} \iint \sigma_\nu(\mathscr{E}) [I - B_\nu(\mathscr{E})] \operatorname{sgn}^+(I - B_\nu(K))\, \delta\Omega\, d\nu &= \iint \sigma_\nu(\mathscr{E}) [I - B_\nu(K)]^+ \delta\Omega\, d\nu \\ &\quad + \iint \sigma_\nu(\mathscr{E}) [B_\nu(K) - B_\nu(\mathscr{E})] \operatorname{sgn}^+(I - B_\nu(K))\, \delta\Omega\, d\nu\,. \end{aligned} \tag{1.6}$$

By adding (1.5) to (1.6) we obtain:

$$\begin{aligned}&\iint \sigma_\nu(\mathscr{E})\,[B_\nu(\mathscr{E})-I]\,\mathrm{sgn}^+(\mathscr{E}-K)\,\delta\Omega\,d\nu\\ &\quad+\iint \sigma_\nu(\mathscr{E})\,[I-B_\nu(\mathscr{E})]\,\mathrm{sgn}^+(I-B_\nu(K))\,\delta\Omega\,d\nu\\ &=\iint \sigma_\nu(\mathscr{E})\,[(B_\nu(\mathscr{E})-B_\nu(K))^+-(B_\nu(\mathscr{E})-B_\nu(K))\,\mathrm{sgn}^+(I-B_\nu(K))]\,\delta\Omega\,d\nu\\ &\quad+\iint \sigma_\nu(\mathscr{E})\,[(I-B_\nu(K))^+-(I-B_\nu(K))\,\mathrm{sgn}^+(\mathscr{E}-K)]\,\delta\Omega\,d\nu\geqq 0\,. \qquad (1.7)\end{aligned}$$

The same analysis on the scattering term is standard, κ_d being symmetric and positive. In (1.5)–(1.7) we have only used the fact that B_ν is nondecreasing, and that σ_ν is positive.

Now, we multiply (0.1) by $\mathrm{sgn}^+(\mathscr{E}-K)$ and we integrate it on X: we multiply (0.2) by $\mathrm{sgn}^+(I-B_\nu(K))$ and we integrate it on $X\times S^N\times\mathbb{R}_\nu^{+*}$. We add the two obtained equalities, and using (1.7) and (1.8), we obtain:

$$\begin{aligned}&\frac{d}{dt}\left(\int_X (\mathscr{E}-K)^+\,dx+\iiint_{X\times S^N\times\mathbb{R}^{+*}}(I-B_\nu(K))^+\,dx\,\delta\Omega\,d\nu\right)\\ &\leqq-\iiint_{(\partial X\times S^N)_-\times\mathbb{R}^{+*}}(h-B_\nu(K))^+\,\Omega\cdot n\,d\Gamma\,\delta\Omega\,d\nu=0\,. \qquad (1.8)\end{aligned}$$

And (1.8) is precisely the expected result. But, according to the previous remark, this is only a formal proof, since $(\mathscr{E},I)(t)$ has not been proved to be a solution of (0.1)–(0.4) in $\mathscr{D}'$.

Theorem B can be used to prevent $\sigma_\nu(\mathscr{E})$ from blowing up, and to consider the nonlinear terms in Q as a Lipschitz perturbation of the transport operator.

If we were dealing with reflexive Banach spaces, the linear semigroup theory would ensure that $(\mathscr{E},I)$ is a strong solution of (0.1)–(0.4). In fact, we have the:

Theorem C (Regularity for Positive $\mathscr{E}$). *Assume* (H1)–(H9) *and, moreover, assume that* $\exists k, K>0$ *such that*

$$k\leqq\mathscr{E}_0\leqq K;\qquad B_\nu(k)\leqq I_0\leqq B_\nu(K);\qquad B_\nu(k)\leqq h\leqq B_\nu(K);$$
$$\Omega\cdot\nabla_x I_0\in L^\infty(X\times S^N;M^1(\mathbb{R}_\nu^{+*}))\,.$$

Then, $(\mathscr{E},I)(t)=\exp(-tQ)\cdot(\mathscr{E}_0,I_0)$ *is the unique classical solution of* (0.1)–(0.4) *(for a precise definition of the usual "classical" – see Sect. III).*

Remark. In the one-dimensional case, Theorem C will provide continuity of the temperature $\mathscr{E}(t)$, for all $t>0$, under the same assumptions as in Theorem C.

Another easy consequence of Theorem C is

Theorem D (Uniqueness). $\mathrm{Exp}(-tQ)$ *is the only contraction semigroup on* E^+, *solving* (1.3) *in the classical sense for* $(\mathscr{E}_0,I_0)$ *satisfying the assumptions of Theorem C.*

We end the present section with a regularity result which is still valid when $\sigma_\nu(\mathscr{E})$ can become infinite, for vanishing $\mathscr{E}$.

Theorem E. *Assume* (H1)–(H9), *and*

$h\in C^1(\bar{X};L^\infty(S^N;L^1(\mathbb{R}_\nu^{+*})))$, $0\leqq h\leqq B_\nu(M)$ *for some positive constant* M,

$0\leqq(\mathscr{E}_0,I_0)\leqq(M,B_\nu(M))$, $(\mathscr{E}_0,I_0)\in BV(X)\times BV(X;L^1(S^N\times\mathbb{R}_\nu^{+*}))$.

Then, $(\mathscr{E}(t), I(t)) = \exp(-tQ)\cdot(\mathscr{E}_0, I_0) \in BV(X)\times BV(X; L^1(S^N \times \mathbb{R}_\nu^{+*}))$, *with a uniform bound on each compact t-interval.*

Theorem E is a consequence of the following remark: differentiating (0.1) and multiplying it by $\operatorname{sgn}(\partial\mathscr{E}/\partial x_i)$, differentiating (0.2) and multiplying it by $(\partial I/\partial x_i)$ yields, after addition:

$$\frac{d}{dt}\left(\sum_{i=1}^{N+1}\left\{\int_X \left|\frac{\partial\mathscr{E}}{\partial x_i}\right| dX + \iiint_{X\times S^N\times\mathbb{R}_\nu^{+*}} \left|\frac{\partial I}{\partial x_i}\right| dx\,\delta\Omega\,dv\right\}\right)$$
$$\leqq -\sum_{i=1}^{N+1} \iiint_{(\partial X\times S^N)_-\times\mathbb{R}_\nu^{+*}} |\Omega\cdot n| \left|\frac{\partial I}{\partial x_i}\right| d\Gamma\,\delta\Omega\,dv \tag{1.9}$$

according to (H2). It remains to prove that the terms $\partial I/\partial x_i$ can be controlled on the boundary, by using Eq. (0.2). Unfortunately, such arguments cannot be used in the present situation, since the generalized solutions do not satisfy (0.1)–(0.2) in a suitable sense. However, this remark will be of great importance for the proof of Theorem E, see Sect. IV. Moreover, this remark seems to be nothing more than accretiveness for the operator Q, since it uses mainly the assumption (H2): see Sects. II and IV.

The splitting results will be stated later, in part V.

One of the goals of this article is to give rigorous proofs of all these results, since straightforward computations on the equations are no longer valid, when dealing with generalized solutions.

II. Global Existence and Maximum Principle

We are now going to prove:

Theorem A′. *Under assumptions* (H1)–(H6), *the operator* Q *is T-accretive, and satisfies the range condition:*

$$(\mathscr{R}) \qquad E^+ = \overline{D(Q)} \subset R(I+sQ), \quad \forall s>0\,.$$

According to Theorem 1 of Crandall and Ligett [9], Theorem A is a straightforward consequence of Theorem A′ and Theorem B is an easy corollary of:

Theorem B′. *(comparison)* (H1)–(H6) *are assumed in this theorem.*

1) *Let* $(\mathscr{E}_0, I_0)$ *and* $(\mathscr{E}'_0, I'_0) \in E^+$ *such that:* $(\mathscr{E}_0, I_0) \leqq (\mathscr{E}'_0, I'_0)$. *Then* $\forall t>0$, $\exp(-tQ)\cdot(\mathscr{E}_0, I_0) \leqq \exp(-tQ)\cdot(\mathscr{E}'_0, I'_0)$.

2) *Let* $(\mathscr{E}_0, I_0) \in E^+$; *let h and h′ be measurable functions on* $(\partial X\times S^N)_- \times \mathbb{R}_\nu^{+*}$ *such that:*

$$\exists M>0 \quad \text{such that} \quad 0 \leqq h \leqq h' \leqq B_\nu(M)\,.$$

Let $(\mathscr{E}, I)(t)$ *be the generalized solution of* (0.1)–(0.2) *with initial data* (0.4) *and boundary condition* (0.3) $I|_{(\partial X\times S^N)_-} = h$;

Let $(\mathscr{E}', I')(t)$ *be the generalized solution of* (0.1)–(0.2) *with initial data* (0.4) *and boundary condition* (0.3)′ $I|_{(\partial X\times S^N)_-} = h'$;

Then, for any positive t, $(\mathscr{E}, I)(t) \leqq (\mathscr{E}', I')(t)$.

This part will be organized as follows:
– in Subsection 1, we prove that Q is T-accretive; this is a mere extension of Mercier's proof (see [13]) of the accretiveness of some truncation (when $\tau \to 0$) of Q;
– in Subsection 2, we study the stationary problem which is actually the main part of the proof of Theorem A;
– in Subsection 3, we turn to the proof of Theorem B′ and Theorem B; actually this proof is slightly different from the "heuristic" one which has been given in Part I, since we use the T-accretiveness of Q.

For the functional analysis background of the proof of Theorem A′, we refer to the fundamental paper [9] by Crandall and Ligett, and to [4, 5 and 10].

1. T-Accretiveness of Q

Let us recall the definition of T-accretiveness: cf. [4] for a general statement. Let A be a single-valued operator on E, with domain $D(A)$, such that: $A(u,v)=(A_1(u,v), A_2(u,v))$. A is T-accretive if and only if, $\forall (u,v), (u',v') \in D(A)$,

$$\int_X [A_1(u,v)-A_1(u',v')]\,\mathrm{sgn}^+(u-u')\,dx + \iiint_{X\times S^N\times \mathbb{R}^{+*}} [A_2(u,v)-A_2(u',v')]\,\mathrm{sgn}^+(v-v')\,dx\,\delta\Omega\,dv \geqq 0\,,$$

where:

$$\forall \alpha \in \mathbb{R}, \mathrm{sgn}^+\alpha = 1 \Leftrightarrow \alpha > 0 \quad \text{and} \quad \mathrm{sgn}^+\alpha = 0 \Leftrightarrow \alpha \leqq 0\,.$$

Lemma 1. *Under assumptions* (H1)–(H6), *the operator Q is T-accretive.*

Proof of Lemma 1. We define on E the following operators

- $A(\mathscr{E}, I)=(0, \Omega\cdot\nabla_x I-\mathscr{D}I)$ with domain

$$D(A)=\{(\mathscr{E},I)\in E \,|\, \Omega\cdot\nabla_x I \in L^1(X\times S^N\times\mathbb{R}^{+*}),\ I|_{(\partial X\times S^N)_-}=h\}\,,$$

- $B(\mathscr{E},I)=\left(\iint_{S^N\times\mathbb{R}^{+*}} [q_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E})I]\,\delta\Omega\,dv,\ \sigma_\nu(\mathscr{E})I-q_\nu(\mathscr{E})\right)$, with domain:

$$D(B)=\{(\mathscr{E},I)\in E^+ \,|\, \sigma_\nu(\mathscr{E})I\in L^1(X\times S^N\times\mathbb{R}^{+*})\}$$

and $Q=A+B$, with $D(Q)=D(A)\cap D(B)$.

The fact that A is T-accretive is quite well-known. It remains to prove that B is T-accretive. Taking $(\mathscr{E},I)$ and $(\mathscr{E}',I')$ in $D(B)$, we can write the following equalities, where

$$\varphi=\mathrm{sgn}^+(\mathscr{E}-\mathscr{E}') \quad \text{and} \quad \psi=\mathrm{sgn}^+(I-I')\,,$$

$$\begin{aligned}
&\iiint [q_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E})I-(q_\nu(\mathscr{E}')-\sigma_\nu(\mathscr{E}')I')]\,\varphi\,dx\,\delta\Omega\,dv\\
&\quad+\iiint [\sigma_\nu(\mathscr{E})I-q_\nu(\mathscr{E})-(\sigma_\nu(\mathscr{E}')I'-q_\nu(\mathscr{E}'))]\,\psi\,dx\,\delta\Omega\,dv\\
&=\iiint [q_\nu(\mathscr{E})-q_\nu(\mathscr{E}')]\,(\varphi-\psi)\,dx\,\delta\Omega\,dv+\iiint [\sigma_\nu(\mathscr{E})I\\
&\quad-\sigma_\nu(\mathscr{E}')I']\,(\psi-\varphi)\,dx\,\delta\Omega\,dv\\
&\quad+\iiint [(q_\nu(\mathscr{E})-q_\nu(\mathscr{E}'))^+-(q_\nu(\mathscr{E})-q_\nu(\mathscr{E}'))\psi]\,dx\,\delta\Omega\,dv\\
&\quad+\iiint \sigma_\nu(\mathscr{E})\,[(I-I')^+-(I-I')\varphi]\,dx\,\delta\Omega\,dv\\
&\quad+\iiint I'[(\sigma_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E}'))\psi-(\sigma_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E}'))^-]\,dx\,\delta\Omega\,dv\geqq 0\,.
\end{aligned} \tag{2.1}$$

This positiveness gives exactly the T-accretiveness for B, as defined before. Therefore Q is T-accretive as the sum of T-accretive operators. □

2. *Study of the Stationary Problem*

The range condition ($\mathscr{R}$) of Theorem A′ is obtained by introducing the stationary problem; we explain briefly what are the main difficulties: if we were dealing with a monodimensional problem, the stationary transport equation for (0.2) would be an ODE. But, as σ_ν blows up when $\mathscr{E}\to 0_+$, we have no Lipschitz regularity on the vector field defining this ODE. Now, Eq. (0.1) provides a L^1 a priori estimate on the nonlinearity:

$$\iiint_{X\times S^N\times \mathbb{R}^{+*}} \sigma_\nu(\mathscr{E}) I\, dx\, \delta\Omega\, d\nu \leqq C < +\infty\,. \tag{2.2}$$

The difficulty is to treat the convergence in this nonlinearity. The key is to use the monotonicity structure of the problem with (2.2) to deal with the convergence of the nonlinear term $\sigma_\nu(\mathscr{E})I$. This will lead to the:

Proposition 1. *Under assumptions* (H1)–(H6), $R(I+sQ)\supset \overline{D(Q)}=E^+$.

We will only consider the case $s=1$, the proof is similar for any $s>0$.

Proof of Proposition 1. Our proof will be divided in three steps: Treatment of a "regularized" system for "regular" data (in some sense we shall explain later), Proof of Proposition 1 for regular data, General case.

Step 1. Let $f, g\in E^+$; moreover, we assume the following:

$$\begin{cases} \exists K\geqq 0 \quad \text{such that } \forall x\in X, f(x)\leqq \dfrac{K}{2}, \\ \forall(x,\Omega,\nu)\in X\times S^N\times \mathbb{R}^{+*}, g(x,\Omega,\nu)\leqq B_\nu(K), \\ \forall(x',\Omega',\nu')\in (\partial X\times S^N)_-\times \mathbb{R}^{+*}, h(x',\Omega',\nu')\leqq B_\nu(K). \end{cases} \tag{2.3}$$

Since our proof is based upon an iterative scheme, we use (2.3) to provide the desired stability. The fact that the Planck functions are a privileged class as regards comparison of solutions is clear, when thinking of the formal proof of Theorem B we have given in Part I.

Lemma 2. *Under assumptions* (2.3) *and* (H1)–(H6), *there exists* $\alpha_0=\alpha_0(K)$ *such that:* $\forall\alpha\in\,]0,\alpha_0[$, *there exists a solution* $(\mathscr{E}_\alpha, I_\alpha)\in D(Q)$ *of*

$$\mathscr{E}_\alpha + \iint_{S^N\times\mathbb{R}^{+*}} (q_\nu(\mathscr{E}_\alpha)-\sigma_\nu(\mathscr{E}_\alpha)I_\alpha)\,\delta\Omega\, d\nu = f\,, \tag*{(2.4)$_\alpha$}$$

$$\begin{gathered} I_\alpha+\Omega\cdot\nabla_x I_\alpha+\sigma_\nu(\alpha+\mathscr{E}_\alpha)I_\alpha = g+q_\nu(\mathscr{E}_\alpha)+\mathscr{D}I_\alpha\,. \\ (I_\alpha|_{(\partial X\times S^N)_-}=h, \text{ since } (\mathscr{E}_\alpha, I_\alpha)\in D(Q)\,.) \end{gathered} \tag*{(2.5)$_\alpha$}$$

Moreover, we have the bounds:

$$0\leqq \mathscr{E}_\alpha\leqq K\,,\qquad 0\leqq I_\alpha\leqq B_\nu(K)\,. \tag*{(2.6)$_\alpha$}$$

Proof of Lemma 2. We shall exhibit a sequence $(\mathscr{E}^n, I^n)$ decreasing towards its limit $(\mathscr{E}_\alpha, I_\alpha) \in D(Q)$, solution of $(2.4)_\alpha$–$(2.5)_\alpha$. We take $I^0 = B_\nu(K)$ and $\mathscr{E}^0$ the solution of

$$\mathscr{E}^0 + \iint [q_\nu(\mathscr{E}^0) - \sigma_\nu(\mathscr{E}^0) B_\nu(K)] \, \delta\Omega \, d\nu = f. \tag{2.7}$$

Indeed, (2.7) has a unique solution, since its left-hand side defines a continuous function of $\mathscr{E}^0$ and K, increasing with respect to $\mathscr{E}^0$ and decreasing with respect to K. This solution is obviously positive, and because $f \leqq K/2$, we have:

$$\mathscr{E}^0 \leqq k < K, \text{ for some constant } k, \text{ which depends on } K \text{ only}. \tag{2.8}$$

Let us take $\alpha_0(K) = K - k$. For each $\alpha \in]0, \alpha_0[$, we have, according to (H2):

$$q_\nu(\mathscr{E}_0) \leqq q_\nu(\alpha + \mathscr{E}^0) \leqq \sigma_\nu(\alpha + \mathscr{E}^0) B_\nu(K), \tag{2.9}$$

for each $\nu \in \mathbb{R}_\nu^{+*}$. This inequality is the crucial point in the proof of Lemma 1. Now, we choose $\alpha \in]0, \alpha_0[$. The following iterative scheme will define the sequence: $(\mathscr{E}^n, I^n)$.

$$\left\{\begin{array}{l} I^n + \Omega \cdot \nabla I^n + \sigma_\nu(\alpha + \mathscr{E}^{n-1}) I^n = g + q_\nu(\mathscr{E}^{n-1}) + \mathscr{D} I^n \\ I^n|_{(\partial X \times S^N)_-} = h \end{array}\right\}, \tag{2.10$_n$}$$

$$\mathscr{E}^n + \iint_{S^N \times \mathbb{R}^{+*}} [q_\nu(\mathscr{E}^n) - \sigma_\nu(\mathscr{E}^n) I^n] \, \delta\Omega \, d\nu = f. \tag{2.11$_n$}$$

The existence and uniqueness of I^n when $\mathscr{E}^{n-1} \in L^\infty(X)^+$ is classical: cf. Bardos [3]. A straightforward induction will ensure that $\mathscr{E}^n$ and I^n are decreasing:
– using the maximum principle in $(2.10)_n$, if $\mathscr{E}^{n-1} \geqq \mathscr{E}^n$, $I^{n+1} \leqq I^n$;
– using the monotonicity properties of the left-hand side of $(2.11)_n$, if $I^{n+1} \leqq I^n$, $\mathscr{E}^{n+1} \leqq \mathscr{E}^n$.

We only have to check the first step of this induction: we want to prove that $I^1 \leqq I^0$, knowing $\mathscr{E}^0$ to be less than or equal to k. We have

$$\begin{aligned} &(I^1 - B_\nu(K)) + \Omega \cdot \nabla_x (I^1 - B_\nu(K)) + \sigma_\nu(\mathscr{E}^0 + \alpha)(I^1 - B_\nu(K)) \\ &\quad = (g - B_\nu(K)) + \mathscr{D}(I^1 - B_\nu(K)) + [q_\nu(\mathscr{E}^0) - \sigma_\nu(\mathscr{E}^0 + \alpha) B_\nu(K)] \\ &\quad \leqq \mathscr{D}(I^1 - B_\nu(K)), \quad \text{according to (2.9)}; \end{aligned}$$

and $(I^1 - B_\nu(K))|_{(\partial X \times S^N)_-} \leqq 0$; therefore, using the maximum principle, we have

$$I^1 \leqq B_\nu(K) = I^0.$$

Now, it is clear that:

$$\mathscr{E}^n \to \mathscr{E}_\alpha \quad \text{in} \quad L^1(X)$$

$$I^n \to I_\alpha \quad \text{in} \quad L^1(X \times S^N \times \mathbb{R}^{+*})$$

$$q_\nu(\mathscr{E}^n) \to q_\nu(\mathscr{E}_\alpha) \quad \text{in} \quad L^1(X \times S^N \times \mathbb{R}_\nu^{+*}) \qquad \text{[see (H3)]}.$$

According to $(2.11)_n$, for a.e. $x \in X$: if $\mathscr{E}^n(x) \to 0$, then $f(x) = 0$ and

$$\int_{S^N} I^n(x) \delta\Omega \to 0 \text{ a.e. and in } L^1(X \times \mathbb{R}_\nu^{+*}).$$

Thus:

$$\iint_{S^N \times \mathbb{R}^{+*}} \sigma_\nu(\mathscr{E}^n(x)) I^n(x) \, \delta\Omega \, d\nu \to 0 = \iint_{S^N \times \mathbb{R}^{+*}} \sigma_\nu(\mathscr{E}_\alpha(x)) I_\alpha(x) \, \delta\Omega \, d\nu. \tag{2.12}$$

So, it is clear that $(\mathscr{E}_\alpha, I_\alpha)$ satisfies $(2.4)_\alpha$ a.e. in X, since the case when $\mathscr{E}_\alpha(x)>0$ is trivial.

As an easy consequence of the dominated convergence theorem, we know that $(2.5)_\alpha$ is satisfied in the distribution sense: $\forall 0<\alpha\leqq\alpha_0$,

$$0\leqq\sigma_\nu(\alpha+\mathscr{E}^{n-1})I^n\leqq\sigma_\nu(\alpha)\,B_\nu(K)\in L^1(X\times S^N\times\mathbb{R}^{+*}),$$

according to (H4), so that:

$$\sigma_\nu(\alpha+\mathscr{E}^{n-1})I^n\to\sigma_\nu(\alpha+\mathscr{E}_\alpha)I_\alpha \quad \text{in} \quad L^1(X\times S^N\times\mathbb{R}^{+*}),$$

and therefore

$$\Omega\cdot\nabla_x I^n\to\Omega\cdot\nabla_x I_\alpha \quad \text{in} \quad L^1(X\times S^N\times\mathbb{R}_\nu^{+*}).$$

As an easy consequence of $(2.4)_\alpha$, $\sigma_\nu(\mathscr{E}_\alpha)I_\alpha\in L^1(X\times S^N\times\mathbb{R}^{+*})$; therefore, $(\mathscr{E}_\alpha, I_\alpha)\in D(Q)$. □

Step 2. We keep the same assumptions on f, g, h as in Step 1. Now, we turn to the convergence of $(\mathscr{E}_\alpha, I_\alpha)$ for α going to zero. Since we have obviously a L^1 bound for:

$$\Omega\cdot\nabla_x I_\alpha+\sigma_\nu(\alpha+\mathscr{E}_\alpha)I_\alpha,$$

the key of our proof is to study the convergence of our sequences in the nonlinear terms to get a control on each term of the above expression. This is essential to have a solution of the stationary problem in $D(Q)$.

Lemma 3. *Under the same assumptions than in Lemma 2, there exists a unique* $(\mathscr{E}, I)\in D(Q)$ *such that:*

$$(I+Q)\cdot(\mathscr{E}, I)=(f,g). \tag{2.13}$$

Moreover, we have the bound:

$$0\leqq\mathscr{E}\leqq K \quad \textit{and} \quad 0\leqq I\leqq B_\nu(K). \tag{2.14}$$

Proof of Lemma 3. By coming back to $(2.10)_n^\alpha$ and $(2.11)_n^\alpha$ (where we have omitted the index α), a straightforward induction gives the fact that $(\mathscr{E}^\alpha, I^\alpha)$ is non-decreasing:
- let us assume that $\mathscr{E}_{n-1}^\alpha\leqq\mathscr{E}_{n-1}^{\alpha'}$ with $\alpha<\alpha'$; according to the maximum principle, and the fact that $\sigma_\nu(\alpha+\mathscr{E}_{n-1}^\alpha)\geqq\sigma_\nu(\alpha'+\mathscr{E}_{n-1}^{\alpha'})$ is a decay coefficient we obtain:

$$I_n^\alpha\leqq I_n^{\alpha'};$$

- then, because of the properties of $(2.11)_n$ underlined in Lemma 2, $I_n^\alpha\leqq I_n^{\alpha'}$ ensures that $\mathscr{E}_n^\alpha\leqq\mathscr{E}_n^{\alpha'}$.

This monotonicity will allow us to take the limit in the nonlinearities. First, we use the uniform bound $(2.6)_\alpha$ to obtain:

$$\mathscr{E}_\alpha\to\mathscr{E} \quad \text{in} \quad L^1(X) \text{ and a.e.},$$

$$I_\alpha\to I \quad \text{in} \quad L^1(X\times S^N\times\mathbb{R}^{+*}) \text{ and a.e.},$$

$$q_\nu(\mathscr{E}_\alpha)\to q_\nu(\mathscr{E}) \quad \text{in} \quad L^1(X\times S^N\times\mathbb{R}^{+*}) \text{ and a.e.}$$

when $\alpha\to 0$, and we get the bound:

$$0\leqq\mathscr{E}\leqq K, \qquad 0\leqq I\leqq B_\nu(K).$$

An argument similar to the one used in the proof of Lemma 2 yields:

$$\iint_{S^N\times\mathbb{R}^{+*}} \sigma_\nu(\mathscr{E}_\alpha)I_\alpha\,\delta\Omega\,dv \to \iint_{S^N\times\mathbb{R}^{+*}} \sigma_\nu(\mathscr{E})I\,\delta\Omega\,dv \tag{2.15}$$

in $L^1(X)$ and a.e., when $\alpha\to 0$. In particular, $\sigma_\nu(\mathscr{E})I\in L^1(X\times S^N\times\mathbb{R}^{+*})$.

Then, we write:

$$I\sigma_\nu(\mathscr{E})-I_\alpha\sigma_\nu(\mathscr{E}_\alpha)=I[\sigma_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E}_\alpha)]+\sigma_\nu(\mathscr{E}_\alpha)[I-I_\alpha]. \tag{2.16}$$

Using the dominated convergence theorem yields:

$$I[\sigma_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E}_\alpha)]\to 0 \quad \text{in} \quad L^1(X\times S^N\times\mathbb{R}^{+*}),$$

since $0\leqq I[\sigma_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E}_\alpha)]\leqq I\sigma_\nu(\mathscr{E})$.

Thus, we obtain, by using (2.15) that

$$\iint \sigma_\nu(\mathscr{E}_\alpha)(I-I_\alpha)\,\delta\Omega\,dv \text{ goes to zero in } L^1(X).$$

Since $I\leqq I_\alpha$, this implies that

$$\sigma_\nu(\mathscr{E}_\alpha)(I-I_\alpha) \text{ goes to zero in } L^1(X\times S^N\times\mathbb{R}_\nu^{+*}).$$

Thus, we have proved that:

$$\sigma_\nu(\mathscr{E}_\alpha)I_\alpha\to\sigma_\nu(\mathscr{E})I \quad \text{in} \quad L^1(X\times S^N\times\mathbb{R}_\nu^{+*})$$

when $\alpha\to 0$. After some details which are routine, we obtain the solution $(\mathscr{E},I)\in D(Q)$ of (2.13).

The uniqueness is clear, since Q is accretive. □

Step 3. We already know that $0\leqq h\leqq B_\nu(K)$ for some positive constant K (cf. Part I, 1-Assumptions). Then for each $\lambda>1$, let us take:

$$f_\lambda=\operatorname{Inf}(f,\lambda)$$

and

$$g_\lambda=\operatorname{Inf}(g,\lambda)\,\mathbb{1}_{\{(x,\Omega,\nu)\in X\times S^N\times\mathbb{R}^{+*}\mid \nu\in]1/\lambda,\lambda[\}}\,.$$

Thus, there exists, for each $\lambda>1$, a positive constant K_λ such that:

$$f_\lambda\leqq K_\lambda/2,\ g_\lambda\leqq B_\nu(K_\lambda) \quad \text{and} \quad K_\lambda\geqq M.$$

Using Lemma 3 there exists a unique solution $(\mathscr{E},I)\in D(Q)$ of:

$$\mathscr{E}_\lambda+\iint_{S^N\times\mathbb{R}^{+*}}(q_\nu(\mathscr{E}_\lambda)-\sigma_\nu(\mathscr{E}_\lambda)I_\lambda)\,\delta\Omega\,dv=f_\lambda, \tag{2.17$_\lambda$}$$

$$\begin{cases} I_\lambda+\Omega\cdot\nabla_x I_\lambda+\sigma_\nu(\mathscr{E}_\lambda)I_\lambda=g_\lambda+q_\nu(\mathscr{E}_\lambda)+\mathscr{D}I_\lambda \\ I_\lambda|_{(\partial X\times S^N)_-}=h\,[\text{since } (\mathscr{E}_\lambda,I_\lambda)\in D(Q)] \end{cases}. \tag{2.18$_\lambda$}$$

Since Q is T-accretive, and f_λ, g_λ are increasing with λ, $\mathscr{E}_\lambda$, and I_λ increase with λ [it is well-known that the resolvent $(I+Q)^{-1}$ is an order preserving mapping on its domain, when Q is T-accretive]. Integrating $(2.17)_\lambda$ over X, $(2.18)_\lambda$ over $X\times S^N\times\mathbb{R}_\nu^{+*}$ yields

$$\int_X \mathscr{E}_\lambda\,dx+\iiint_{X\times S^N\times\mathbb{R}^{+*}} I_\lambda\,dx\,\delta\Omega\,dv\leqq C, \tag{2.19}$$

and according to (H3) and (2.19), integrating (2.18) on $X\times S^N\times\mathbb{R}^{+*}$ yields:

$$\iiint_{X\times S^N\times\mathbb{R}^{+*}} \sigma_\nu(\mathscr{E}_\lambda)I_\lambda\,dx\,\delta\Omega\,dv\leqq C, \tag{2.20}$$

where C is a positive constant. We claim that, by using similar techniques (but easier since $\mathscr{E}_\lambda$ is nondecreasing) as in step 1 and step 2:

$$(\mathscr{E}_\lambda, I_\lambda) \to (\mathscr{E}, I) \quad \text{in } E \text{ when} \quad \lambda \to +\infty\,,$$

$(\mathscr{E}, I) \in D(Q)$ and $(I+Q)\cdot(\mathscr{E}, I) = (f, g)$. Now the proof of Proposition 1 is complete, together with the proof of Theorem A′. □

3. *Maximum Principle and Comparison of Generalized Solutions*

Let us denote by J_λ^Q the resolvent of Q; $\forall \lambda > 0$, $J_\lambda^Q = (I+\lambda Q)^{-1}$. According to Theorem A′, it is a contraction defined on E^+ with values in $D(Q)$. Moreover, since Q is T-accretive, J_λ^Q and $\exp(-tQ)$ are order preserving [4]; we can state it as a corollary of Theorem A′:

Corollary 1. *Under the same assumptions as in Theorem A′ for each* $\lambda > 0$, *and for each* (f, g) *and* $(f', g') \in E^+$ *such that* $(f, g) \leqq (f', g')$, *then:*

$$J_\lambda^Q \cdot (f, g) \leqq J_\lambda^Q \cdot (f' g') \quad \textit{and} \quad \exp(-\lambda Q)\cdot(f, g) \leqq \exp(-\lambda Q)\cdot(f' \cdot g')\,.$$

Thus, we have proved point 1) in Theorem B′. Now, we turn to the proof of point 2).

Let h' be another measurable function on $(\partial X \times S^N)_- \times \mathbb{R}^{+*}$ such that:

$$\exists M > 0 \quad \text{such that} \quad 0 \leqq h \leqq h' \leqq B_\nu(M)\,,$$

and let us define the operator Q' on E by:

$$Q'(\mathscr{E}, I) = (\textstyle\int\int [q_\nu(\mathscr{E}) - \sigma_\nu(\mathscr{E}) I]\, \delta\Omega\, d\nu,\ \Omega \cdot \nabla_x I + \sigma_\nu(\mathscr{E}) I - q_\nu(\mathscr{E}) - \mathscr{D} I)$$

with domain

$$D(Q') = \{(\mathscr{E}, I) \in E^+ \text{ such that } \sigma_\nu(\mathscr{E}) I \in L^1(X \times S^N \times \mathbb{R}_\nu^{+*}),$$
$$\Omega \cdot \nabla_x I \in L^1(X \times S^N \times \mathbb{R}_\nu^{+*}), \text{ and } I|_{(\partial X \times S^N)_-} = h'\}\,.$$

Obviously $\overline{D(Q')} = \overline{D(Q)} = E^+$ and Q' satisfies the range condition (R), and is T-accretive.

Lemma 4. *Let* $(\mathscr{E}_0, I_0) \in E^+$. *For each* $\lambda > 0$, *we have:*
(i) $J_\lambda^Q \cdot (\mathscr{E}_0, I_0) \leqq J_\lambda^{Q'} \cdot (\mathscr{E}_0, I_0)$ *and*
(ii) $\exp(-\lambda Q)\cdot(\mathscr{E}_0, I_0) \leqq \exp(-\lambda Q')\cdot(\mathscr{E}_0, I_0)$.

Proof of Lemma 4. (ii) is an easy consequence of (i) according to Crandall and Ligett [9]; indeed, since Q and Q' are T-accretive and satisfies (R):

$$\exp(-\lambda Q) = \lim_{n\to\infty} (J_{\lambda/n}^Q)^n \text{ (pointwise on } E^+),$$

and

$$\exp(-\lambda Q') = \lim_{n\to\infty} (J_{\lambda/n}^{Q'})^n\,.$$

Let us now prove point (i). We proceed exactly as in the proof of Theorem A′.

We first assume that $(\mathscr{E}_0, I_0)$ satisfies (2.3). According to Lemma 2, there exists $\alpha_0 > 0$ such that for each $\alpha \in]0, \alpha_0[$ we can build:
– $(\mathscr{E}_\alpha, I_\alpha) \in D(Q)$ solution of

$$\mathscr{E}_\alpha + \lambda \iint_{S^N \times \mathbb{R}^{+*}} [q_\nu(\mathscr{E}_\alpha) - \sigma_\nu(\mathscr{E}_\alpha) I] \, \delta\Omega \, d\nu = \mathscr{E}_0$$

$$\begin{cases} I_\alpha + \lambda\Omega \cdot \nabla_x I_\alpha + \lambda\sigma_\nu(\alpha + \mathscr{E}_\alpha) I_\alpha = I_0 + \lambda q_\nu(\mathscr{E}_\alpha) + \lambda \mathscr{D} I \\ I_\alpha|_{(\partial X \times S^N)_-} = h \end{cases}$$

– $(\mathscr{E}', I') \in D(Q')$ solution of

$$\mathscr{E}'_\alpha + \lambda \iint_{S^N \times \mathbb{R}^{+*}} [q_\nu(\mathscr{E}'_\alpha) - \sigma_\nu(\mathscr{E}'_\alpha) I'_\alpha] \, \delta\Omega \, d\nu = \mathscr{E}_0$$

$$\begin{cases} I'_\alpha + \lambda\Omega \cdot \nabla_x I'_\alpha + \lambda\sigma_\nu(\alpha + \mathscr{E}'_\alpha) I'_\alpha = I_0 + \lambda q_\nu(\mathscr{E}'_\alpha) + \lambda \mathscr{D} I'_\alpha \\ I'_\alpha|_{(\partial X \times S^N)_-} = h' . \end{cases}$$

Applying the maximum principle, and taking into account the properties of $(2.4)_\alpha$ (see the proof of Lemma 2), each step of the iterative schemes defined in the proof of Lemma 2 yields:

$$\forall \alpha \in]0, \alpha_0[, \quad (\mathscr{E}_\alpha, I_\alpha) \leqq (\mathscr{E}'_\alpha, I'_\alpha) .$$

Then, using Lemma 3 and taking the limits for $\alpha \to 0$ gives:

$$J^Q_\lambda \cdot (\mathscr{E}_0, I_0) \leqq J^{Q'}_\lambda \cdot (\mathscr{E}_0, I_0) .$$

The extension of this result for any $(\mathscr{E}_0, I_0) \in E^+$ can be performed as in step 3 of Proposition 1 and is routine.

This completes the proof of Theorem B′. □

We are going to end this section with the proof of Theorem B.

Let us take:

$(\mathscr{E}_0, I_0) \in D(Q)$, and $K > 0$ such that:

- $h \leqq B_\nu(K)$, and $(\mathscr{E}_0, I_0) \leqq (K, B_\nu(K))$.

Then we can define Q' as in the proof of Theorem B′, point 2), by choosing $h' = B_\nu(K)$.

According to Theorem B′, point 1):

$$\forall t > 0, \exp(-tQ) \cdot (\mathscr{E}_0, I_0) \leqq \exp(-tQ) \cdot (K, B_\nu(K)) ;$$

then, applying Theorem B′, point 2), we have for any positive t:

$$\exp(-tQ) \cdot (K, B_\nu(K)) \leqq \exp(-tQ') \cdot (K, B_\nu(K)) .$$

Thus, $\forall t > 0$, $\exp(-tQ) \cdot (\mathscr{E}_0, I_0) \leqq \exp(-tQ') \cdot (K, B_\nu(K))$;

and, $\forall \lambda > 0$, $J^{Q'}_\lambda \cdot (K, B_\nu(K)) = (K, B_\nu(K))$;

thus, $\forall t > 0$, $\exp(-tQ') \cdot (K, B_\nu(K)) = (K, B_\nu(K))$.

Remark. For the proof of Theorem B, we could have used the ideas of the "formal proof" in part I on the stationary problem. The one we have just given seems more interesting since it underlines the order-preserving properties of the problem. However, the "formal proof" provides an a priori estimate, without assuming accretiveness for Q (it requires only that B should be increasing). The open question is: can we expect any kind of global existence result without assuming accretiveness for Q?

III. The Case of a Uniformly Positive Temperature

The present section is devoted to the proof of Theorem C, Theorem D, and of some other results using the same kind of arguments. We are given two positive constants:

$$0<\mathscr{E}_{\min}<\mathscr{E}_{\max};$$

and, for the sake of simplicity, we assume that the incoming density h is the boundary value on $(\partial X\times S^N)_-\times\mathbb{R}_\nu^+$ of a function still denoted by h, such that:

$$\begin{cases} h\in C(\bar{X}, L^\infty(S^N\times\mathbb{R}_\nu^+)), \\ B_\nu(\mathscr{E}_{\min})\leqq h\leqq B_\nu(\mathscr{E}_{\max}). \end{cases} \tag{3.1}$$

We will use the Banach space:

$$E_1=L^\infty(X)\times L^\infty(X\times S^N; M^1(\mathbb{R}_\nu^+)),$$

where $M^1(\mathbb{R}_\nu^+)$ is the space of bounded Radon measures on $\mathbb{R}_\nu^+$. We know that E_1 is the dual space of $L^1(X)\times L^1(X\times S^N; C_0(\mathbb{R}_\nu^+))$, where $C_0(\mathbb{R}_\nu^+)$ is the space of all continuous functions on $\mathbb{R}_\nu^+$ going to zero at the infinity (see [16]). This remark will be used to obtain the compactness in the proof of Theorem C.

On E_1, we define the two following operators:

$A_1\cdot(\mathscr{E}, I)=(0, \Omega\cdot\nabla_x I-\mathscr{D}I)$ with domain

$$D(A_1)=\{(\mathscr{E}, I)\in E_1\,|\,\Omega\cdot\nabla_x I\in L^\infty(X\times S^N; M^1(\mathbb{R}^+)) \text{ and } I|_{(\partial X\times S^N)_-}=h\},$$

and

$$B_1\cdot(\mathscr{E}, I)=\left(\int_{\mathbb{R}^+} q_\nu(\mathscr{E})\,d\nu-\iint_{S^N\times\mathbb{R}^+}\sigma_\nu(\mathscr{E})I\,\delta\Omega;\ \sigma_\nu(\mathscr{E})I-q_\nu(\mathscr{E})\right)$$

with domain:

$$D(B_1)=\{(\mathscr{E}, I)\in E_1\,|\,\mathscr{E}_{\min}\leqq\mathscr{E}\leqq\mathscr{E}_{\max},\ B_\nu(\mathscr{E}_{\min})\leqq I\leqq B_\nu(\mathscr{E}_{\max})\}.$$

We are now able to state the main result of this section.

Theorem C′. *Under assumptions* (H1)–(H9) *and* (3.1), *if* $(\mathscr{E}_0, I_0)\in D(A_1)\cap D(B_1)$, *then* $(\mathscr{E}, I)(t)=\exp(-tQ)\cdot(\mathscr{E}_0, I_0)$ *is the unique classical solution of:*

$$\begin{cases} \dfrac{d}{dt}(\mathscr{E}, I)+A(\mathscr{E}, I)=-B(\mathscr{E}, I) \\ (\mathscr{E}, I)(0)=(\mathscr{E}_0, I_0). \end{cases} \tag{3.2}$$

This means that:

(i) $\forall t \geqq 0, (\mathscr{E}, I)(t) \in D(A_1) \cap D(B_1)$;

(ii) (3.2) holds with $\frac{d}{dt}(\mathscr{E}, I)$ taken in the sense of $\mathscr{D}'(\mathbb{R}_t^{+*} \times X) \times \mathscr{D}'(\mathbb{R}_t^{+*} \times X \times S^N \times \mathbb{R}_v^{+*})$;

(iii) since $\frac{d}{dt}(\mathscr{E}, I) \in L^\infty(]0, T[; E_1)$ for all $T > 0$, $(\mathscr{E}, I) \in C(\mathbb{R}_t^+; E_1)$, and this remark gives a sense to the initial condition.

The main consequences of Theorem C′ are:

Theorem D (Uniqueness). *Under assumptions* (H1)–(H9) *and* (3.1), $\exp(-tQ)$ *is the only contraction semi-group on* E^+ *solving the radiative transfer equations in the classical sense for* $(\mathscr{E}_0, I_0)$ *satisfying the assumption*

$$(\mathscr{E}_0, I_0) \in D(A_1) \cap D(B_1)$$

of Theorem C′.

The continuity of the temperature, in the one-dimensional case results in

Corollary 2. *Under the same assumptions as in Theorem C′, with* $N=0$ *and if* $\mathscr{E}_0 \in C(\bar{X})$, *then,*

$$\mathscr{E}(t) \text{ is continuous on } \bar{X}$$

and

$$\int_{S^N} I(t)\, \delta\Omega \text{ is continuous on } \bar{X} \text{ with values in } L^1(\mathbb{R}_v^{+*})$$

both uniformly on each compact t-interval.

We now turn to prove Theorem C′. Assumptions (H1)–(H9) and (3.1) ensure that Q is a perturbation of a m-accretive linear operator by a Lipschitz continuous operator (see below). According to this remark, Theorem C′ would be obvious if we were dealing with Hilbert spaces. Since we are working in E_1 which is non-reflexive, we do not know whether $(\mathscr{E}, I)(t)$ is a.e. differentiable with respect to the time variable. However, the proof of Theorem C′ is not far from the Hilbert space case.

Proof of Theorem C′. The proof is divided in three steps: Estimates on the nonlinear terms, Discretization in time and estimates on the derivatives, Taking the limits when the time step goes to 0.

Step 1.

Lemma 5. *Under assumptions* (H7)–(H9), *the operator* B_1 *is Lipschitz continuous.*

The proof of this lemma is quite straightforward, and is omitted here.

Now, we set:

$$u_k^n(t) = (\mathscr{E}_k^n(t), I_k^n(t)) = (J_{t/n}^Q)^k \cdot (\mathscr{E}_0, I_0).$$

Lemma 6. *Assume* (H1)–(H6) *and* (3.1). *Then, if* $(\mathscr{E}_0, I_0) \in D(B_1)$, *we have that:* $u_k^n(t) \in D(B_1)$ *for each* t, k, n.

This lemma is merely a restatement of Corollary 1 and Lemma 4.

According to (H1) and Lemma 6, we have that:

$$\begin{cases} \sigma_\nu(\mathscr{E}_k^n(t))\, I_k^n(t) \leqq \sigma_\nu(\mathscr{E}_{\min})\, B_\nu(\mathscr{E}_{\max}), \\ q_\nu(\mathscr{E}_k^n(t)) \leqq q_\nu(\mathscr{E}_{\max}), \end{cases} \tag{3.3}$$

for each t, n, k. We recall here that $\sigma_\nu(\mathscr{E}_{\min})\, B_\nu(\mathscr{E}_{\max}) \in L^1(\mathbb{R}_\nu^{+*})$ [see (H4)].

Step 2. We are now going to use the accretiveness of A_1.

Lemma 7. *Assume* (H1)–(H9) *and* (3.1). *Then we have that, for each* $T>0$*:*

$$\forall t \in [0, T],\ \|u_k^n(t) - u_{k-1}^n(t)\|_{E_1} \leqq (Ce^{CT}) \frac{t}{n},$$

where C *is a positive constant (depending on* $\|A_1 \cdot (\mathscr{E}_0, I_0)\|_{E_1}$, $\mathscr{E}_{\min}$, $\mathscr{E}_{\max}$ *only).*

Proof of Lemma 7. We have that

$$u_k^n(t) + \frac{t}{n} A_1 \cdot u_k^n(t) = -\frac{t}{n} B_1 U_k^n(t) + u_{k-1}^n(t).$$

This equality yields, after a straightforward induction:

$$u_k^n(t) = (J_{t/n}^{A_1})^k \cdot (\mathscr{E}_0, I_0) + \frac{t}{n} \sum_{p=1}^{k} (J_{t/n}^{A_1})^{k+1-p} \cdot (-B_1 u_p^n(t)).$$

Therefore, we have that

$$\begin{aligned} u_k^n(t) - u_{k-1}^n(t) &= [(J_{t/n}^{A_1})^k \cdot (\mathscr{E}_0, I_0) - (J_{t/n}^{A_1})^{k-1} \cdot (\mathscr{E}_0, I_0)] \\ &\quad - \frac{t}{n} (J_{t/n}^{A_1})^k \cdot (B_1 \cdot (\mathscr{E}_0, I_0)) - \sum_{l=1}^{k} (J_{t/n}^{A_1})^{k+l-1} [B_1 \cdot u_l^n(t) - B_1 \cdot u_{l-1}^n(t)]. \end{aligned}$$

Since A_1 is accretive, we have that

$$\|(J_{t/n}^{A_1})^k \cdot (\mathscr{E}_0, I_0) - (J_{t/n}^{A_1})^{k-1} \cdot (\mathscr{E}_0, I_0)\|_{E_1} \leqq \frac{t}{n} \|A_1 \cdot (\mathscr{E}_0, I_0)\|_{E_1},$$

$$\|(J_{t/n}^{A_1})^k \cdot (B_1 \cdot (\mathscr{E}_0, I_0))\|_{E_1} \leqq \|B_1 \cdot (\mathscr{E}_0, I_0)\|_{E_1},$$

and

$$\|(J_{t/n}^{A_1})^{k+1-l} \cdot [B_1 \cdot u_l^n(t) - B_1 \cdot u_{l-1}^n(t)]\|_{E_1} \leqq C_1 \|u_l^n(t) - u_{l-1}^n(t)\|_{E_1}$$

according to Lemma 5 and Lemma 6. Therefore:

$$\|u_k^n(t) - u_{k-1}^n(t)\|_{E_1} \leqq \frac{Ct}{n} + \frac{Ct}{n} \sum_{p=1}^{k} \|u_p^n(t) - u_{p-1}^n(t)\|_{E_1}$$

for some positive constant C, depending on $\|A_1 \cdot (\mathscr{E}_0, I_0)\| E_1$, $\mathscr{E}_{\min}$, $\mathscr{E}_{\max}$ only. Gronwall's lemma, together with this inequality yields the estimate of Lemma 7.

We introduce the discretized version of the transfer equation:

$$\frac{I_k^n(t)-I_{k-1}^n(t)}{\frac{t}{n}}+\Omega\cdot\nabla_x I_k^n(t)=q_\nu(\mathscr{E}_k^n(t))-\sigma_\nu(\mathscr{E}_k^n(t))\,I_k^n(t)+\mathscr{D}I_k^n(t).$$

According to Lemma 7 and the estimates (3.3), we have that

$$\sup_{\substack{t\in[0,T]\\ n,k\geqq 0}}\|A_1\cdot u_k^n(t)\|_{E_1}<+\infty. \tag{3.4}$$

Step 3. Now, we have enough estimates to finish the proof of Theorem C′. We have the following convergences when $n\to+\infty$:

$$u_n^n(t)\underset{S}{\rightarrow}u(t) \text{ in } E, \text{ uniformly on each compact } t\text{-set};$$

(see Crandall and Ligett [9]);

$$u_n^n(t)\underset{W^*}{\longrightarrow}u(t)\qquad A\cdot u_n^n(t)\underset{W^*}{\longrightarrow}A\cdot u(t),\quad\text{and}\quad\frac{u_n^n(t)-u_{n-1}^n(t)}{\frac{t}{n}}\underset{W^*}{\longrightarrow}\frac{\partial u}{\partial t}$$

in $L^\infty([0,T]\times X\times S^N;M^1(\mathbb{R}_\nu^+))$.

These three assertions are easy consequences of Lemma 6, Lemma 7 (3.4) and the fact that $L^\infty([0,T]\times X\times S^N;\ M^1(\mathbb{R}_\nu^+))$ is a dual space (see Treves [16]). Moreover, for the third assertion, we need the following obvious remark:

$$u_{n-1}^n(t)=u_{n-1}^{n-1}\left(t-\frac{t}{n}\right),$$

and, therefore:

$$\frac{u_n^n(t)-u_{n-1}^n(t)}{\frac{t}{n}}\to\frac{\partial u}{\partial t}\quad\text{in}\quad\mathscr{D}'(]0,T[\times X\times S^N\times\mathbb{R}_\nu^{+*}).$$

Now we have that $B_1: D(B_1)\subset E\to E$ is Lipschitz continuous for the E-norm; (see Mercier [13]). Therefore:

$$B\cdot u_n^n(t)\underset{S}{\rightarrow}B\cdot u(t) \text{ in } E, \text{ uniformly on each compact } t\text{-set}.$$

To finish the proof of Theorem C, we notice that

$$I(t)|_{(\partial X\times S^N)_-}=h,$$

since

$$I_n^n(t)\underset{W^*}{\longrightarrow}I(t)\quad\text{in}\quad L^\infty([0,T]\times X\times S^N;M^1(\mathbb{R}_\nu^{+*}))$$

and $\Omega\cdot\nabla_x I_n^n(t)$ is bounded in $L^\infty([0,T]\times X\times S^N;M^1(\mathbb{R}_\nu^{+*}))$. □

We now turn to prove the corollaries of Theorem C′. The proof of Theorem D is rather straightforward (it uses merely the fact that E_1 is dense in E); we do not give it here.

Proof of Corollary 2. In the one dimensional case, estimate (3.4) can be written as:

$$\int_0^\infty \left|\mu \frac{\partial I}{\partial X}\right| d\nu \leqq C$$

for all $t \in [0, T]$, $x \in X$, and $\mu \in [-1, 1]$. Thus, we have that:

$$\int_0^{+\infty} \left| \int_{-1}^{1} I(x) \frac{d\mu}{2} - \int_{-1}^{1} I(y) \frac{d\mu}{2} \right| d\nu \leqq \int_x^y \int_{|\mu|>\varepsilon} \frac{C}{\mu} \frac{d\mu}{2} + C\varepsilon \leqq \frac{C}{\varepsilon} |y-x| + C\varepsilon .$$

Taking $\varepsilon = |y-x|^{1/2}$ when $|y-x| < 1$ yields that $\int_{-1}^{1} I(t) \frac{d\mu}{2}$ is uniformly bounded in $C^{1/2}(X; L^1(\mathbb{R}_\nu^{+*}))$ on each compact t-interval.

Now, we look at the regularity of $\mathscr{E}$. We know that

$$\frac{\partial \mathscr{E}}{\partial t} + \int_{\mathbb{R}_\nu^{+*}} q_\nu(\mathscr{E})\, d\nu = \int_{-1}^{1} \int_{\mathbb{R}_\nu^{+*}} \sigma_\nu(\mathscr{E}) I \, d\nu \, d\mu ; \qquad \mathscr{E}(0) = \mathscr{E}_0 .$$

Thus, we have the following inequality:

$$\begin{aligned} |\mathscr{E}(t,x) - \mathscr{E}(t,y)| &\leqq |\mathscr{E}_0(x) - \mathscr{E}_0(y)| + C \int_0^t |\mathscr{E}(s,x) - \mathscr{E}(s,y)|\, ds \\ &+ \int_0^t \int_{-1}^{1} \int_{\mathbb{R}_\nu^{+*}} |\sigma_\nu(\mathscr{E}(s,y)| - \sigma_\nu(\mathscr{E}(s,x))| I(s,\mu,x) d\nu \frac{d\mu}{2} ds \\ &+ \int_0^t \int_{-1}^{1} \int_{\mathbb{R}_\nu^{+*}} |\sigma_\nu(\mathscr{E}(s,y))| \cdot |I(s,\mu,x) - I(s,\mu,y)|\, d\nu \frac{d\mu}{2} ds \\ &\leqq |\mathscr{E}_0(x) - \mathscr{E}_0(y)| + C_1 \int_0^t |\mathscr{E}(s,x) - \mathscr{E}(s,y)|\, ds + \int_0^t \delta(|x-y|)\, ds , \end{aligned}$$

where C and C_1 are two constants independent of t, x, y, and δ a non-negative function such that $\delta(t) \to 0$ when $t \to 0$. Using Gronwall's lemma now yields the desired result. □

Let us go back to the $N+1$-dimensional case. We want to prove a continuity result for the stationary problem.

Proposition 2. *Under assumptions* (H1)–(H9) *and* (3.1) *if* $(\mathscr{E}_0, I_0) \in D(B)$ *and*

$$(\mathscr{E}_0, I_0) \in C(\bar{X}) \times C(\bar{X}; L^\infty(S^N \times L^1(\mathbb{R}_\nu^{+*})))$$

then, we have, for any $\lambda \in]0, \lambda_0]$

$$(\mathscr{E}^\lambda, I^\lambda) \in C(\bar{X}) \times C(\bar{X}; L^\infty(S^N \times L^1(\mathbb{R}_\nu^{+*}))) ,$$

where:

$$\lambda_0 = \lambda_0(\mathscr{E}_{\min}, \mathscr{E}_{\max}) \textit{ is a positive constant:}$$

$$(\mathscr{E}^\lambda, I^\lambda) = J_\lambda^Q \cdot (\mathscr{E}_0, I_0) .$$

To obtain this result, we perform a truncation of the operator B, as in [13], and we use a standard fixed point theorem.

IV. Bounded Variation Regularity of the Generalized Solutions (in the Degenerate Case)

The present section is devoted to the proof of Theorem E. Here again, for the sake of simplicity, we assume that the incoming density h is the boundary value on $(\partial X\times S^N)_-\times\mathbb{R}^{+*}$ of a function, still denoted by h, such that:

$$h\in C^1(\bar{X};L^\infty(S^N;L^1(\mathbb{R}_\nu^{+*}))). \tag{4.1}$$

Moreover, we assume that there exists a positive constant $\mathscr{E}_{\max}$ such that:

$$0\leqq h\leqq B_\nu(\mathscr{E}_{\max}). \tag{4.2}$$

We recall the definition of the space BV: let V be a Banach space; then $BV(X;V)=\{u\in L^1(X;V)$ such that $\nabla_x u$ is a bounded (V-valued) Radon measure on $X\}$; [the notation $BV(X)$ means $BV(X;\mathbb{R})$].

Now, we recall the statement of Theorem E:

Theorem E. *Assume* (H1)–(H9), (4.1), *and* (4.2). *Let* $(\mathscr{E}_0,I_0)\in BV(X)\times BV(X,L^1(S^N\times\mathbb{R}_\nu^{+*}))$ *such that:*

$$0\leqq(\mathscr{E}_0,I_0)\leqq(\mathscr{E}_{\max},B_\nu(\mathscr{E}_{\max})),$$

and let $(\mathscr{E}(t);I(t))=\exp(-tQ)\cdot(\mathscr{E}_0,I_0)$, *for each non-negative* t.

Then, for each non-negative t,

$$(\mathscr{E}(t),I(t))\in BV(X)\times BV(X;L^1(S^N\times\mathbb{R}_\nu^{+*})).$$

Moreover, $(\mathscr{E}(t),I(t))$ *is bounded in* $BV(X)\times BV(X,L^1(S^N\times\mathbb{R}_\nu^{+*}))$ *uniformly on each compact* t*-interval.*

The proof of Theorem E relies mainly on the following remark. In the non-linear terms, as well as in the scattering term, the space variable can be regarded merely as a parameter. Thus, using accretiveness for these terms yields an estimate for:

$$\left|\frac{\mathscr{E}(x)-\mathscr{E}(x')}{x-x'}\right|+\iint_{S^N\times\mathbb{R}_\nu^{+*}}\left|\frac{I(x)-I(x')}{x-x'}\right|\delta\Omega\,d\nu.$$

Proof of Theorem E. This proof is divided in three steps: BV estimate for the stationary problem with regular data, BV estimate for the evolution problem with regular data, BV estimate for the evolution problem with general data.

Step 1. We assume that there exists a positive constant $\mathscr{E}_{\min}<\mathscr{E}_{\max}$, such that:

$$B_\nu(\mathscr{E}_{\min})\leqq h\leqq B_\nu(\mathscr{E}_{\max}), \tag{4.3}$$

$$(\mathscr{E}_{\min},B_\nu(\mathscr{E}_{\min}))\leqq(\mathscr{E}_0,I_0)\leqq(\mathscr{E}_{\max},B_\nu(\mathscr{E}_{\max})), \tag{4.4}$$

$$(\mathscr{E}_0,I_0)\in C^0(\bar{X})\times C^0(\bar{X};L^\infty(S^N;L^1(\mathbb{R}_\nu^{+*}))). \tag{4.5}$$

We define

$$(\mathscr{E}^\lambda,I^\lambda)=J_\lambda^Q\cdot(\mathscr{E}_0,I_0)\quad\text{for}\quad\lambda>0.$$

We recall that there exists $\lambda(\mathscr{E}_{\min},\mathscr{E}_{\max})>0$ such that $\forall\lambda\in]0,\lambda(\mathscr{E}_{\min},\mathscr{E}_{\max})]$:

$$(\mathscr{E}^\lambda,I^\lambda)\in C^0(\bar{X})\times C^0(\bar{X};L^\infty(S^N;L^1(\mathbb{R}_\nu^{+*}))),$$

[according to (4.1)–(4.5) and Proposition 2]. In this step, we assume that $0<\lambda\leqq\lambda(\mathscr{E}_{\min},\mathscr{E}_{\max})$. Our goal is to prove the following result:

Proposition 3. *Under assumptions* (H1)–(H9) *and* (4.1)–(4.5), $(\mathscr{E}^\lambda, I^\lambda)$ *satisfies the following BV estimate:*

$$\overline{\lim_{\varepsilon\to 0}}\left\{\int_{X_\varepsilon}\left|\frac{\mathscr{E}^\lambda(x+\varepsilon e_i)-\mathscr{E}^\lambda(x)}{\varepsilon}\right|dx+\iiint_{X_\varepsilon\times S^N\times\mathbb{R}_\nu^{+*}}\left|\frac{I^\lambda(x+\varepsilon e_i)-I^\lambda(x)}{\varepsilon}\right|dx\,\delta\Omega\,d\nu\right\}$$

$$\leqq\lim_{\varepsilon\to 0}\left\{\int_{X_\varepsilon}\left|\frac{\mathscr{E}_0(x+\varepsilon e_i)-\mathscr{E}_0(x)}{\varepsilon}\right|dx+\iiint_{X_\varepsilon\times S^N\times\mathbb{R}_\nu^{+*}}\left|\frac{I_0(x+\varepsilon e_i)-I_0(x)}{\varepsilon}\right|dx\,\delta\Omega\,d\nu\right\}$$

$$+C\lambda+\iiint_{\partial X\times S^N\times\mathbb{R}^{+*}}(\mathscr{E}^\lambda-\mathscr{E}_0)\,d\Gamma\,\delta\Omega\,d\nu+\iiint_{(\partial X\times S^N)_-\times\mathbb{R}_\nu^{+*}}|I^\lambda-I_0|\,d\Gamma\,\delta\Omega\,d\nu\,; \tag{4.6}$$

$\Big($*where* $X_\varepsilon=\{x\in X|\operatorname{dist}(x,\partial X)>\varepsilon\}$, $(e_i)_{1\leqq i\leqq N+1}$ *is an orthonormal basis of* $\mathbb{R}^{N+1}$, c *is a positive constant which depends only on* $\mathscr{E}_{\max}$ *and* $\sup\limits_{(\partial X\times S^N)_-}\int_\nu\|T_x h\|\,d\nu\Big)$.

We begin the proof of Proposition 3 with

Lemma 8. *Under assumptions* (H1)–(H9) *and* (4.1)–(4.5) *we have that:*

$$\int_{X_\varepsilon}\left|\frac{\mathscr{E}^\lambda(x+\varepsilon e_i)-\mathscr{E}^\lambda(x)}{\varepsilon}\right|dx+\iiint_{X_\varepsilon\times S^N\times\mathbb{R}^{+*}}\left|\frac{I^\lambda(x+\varepsilon e_i)-I^\lambda(x)}{\varepsilon}\right|dx\,\delta\Omega\,d\nu$$

$$\leqq\int_{X_\varepsilon}\left|\frac{\mathscr{E}_0(x+\varepsilon e_i)-\mathscr{E}_0(x)}{\varepsilon}\right|dx+\iiint_{X_\varepsilon\times S^N\times\mathbb{R}^{+*}}\left|\frac{I_0(x+\varepsilon e_i)-I_0(x)}{\varepsilon}\right|dx\,\delta\Omega\,d\nu$$

$$+\lambda\iiint_{(\partial X_\varepsilon\times S^N)_-\times\mathbb{R}_\nu^{+*}}|\Omega\cdot n^\varepsilon|\left|\frac{I^\lambda(x+\varepsilon e_i)-I^\lambda(x)}{\varepsilon}\right|d\Gamma^\varepsilon\,\delta\Omega\,d\nu\,. \tag{4.7}$$

Proof of Lemma 8. Let us introduce the following operator on the Banach space $\mathbb{R}\times L^1(S^N\times\mathbb{R}_\nu^{+*})$:

$$C\cdot(\mathscr{E},I)=\left(\iint_{S^N\times\mathbb{R}^{+*}}[q_\nu(\mathscr{E})-\sigma_\nu(\mathscr{E})I]\,\delta\Omega\,d\nu;\sigma_\nu(\mathscr{E})I-q_\nu(\mathscr{E})-\mathscr{D}I\right)$$

with domain

$$D(C)=\{(\mathscr{E},I)\in\mathbb{R}^+\times L^1(S^N\times\mathbb{R}_\nu^{+*})^+\,|\,\sigma_\nu(\mathscr{E})I\in L^1(S^N\times\mathbb{R}_\nu^{+*})\}$$

[with the convention that $0\cdot\sigma_\nu(0)=0$]. We can easily prove that C is T-accretive, by using the same kind of proof as in Lemma 1.

Now, for a.e. $x\in X_\varepsilon$, we have that

$$(\mathscr{E}^\lambda,I^\lambda)(x)+\lambda A\cdot(\mathscr{E}^\lambda,I^\lambda)(x)+\lambda C\cdot(\mathscr{E}^\lambda(x),I^\lambda(x))=(\mathscr{E}_0(x),I_0(x))$$

and

$$(\mathscr{E}^\lambda,I^\lambda)(x+\varepsilon e_i)+\lambda A\cdot(\mathscr{E}^\lambda,I^\lambda)(x+\varepsilon e_i)+\lambda C\cdot(\mathscr{E}^\lambda(x+\varepsilon e_i),I^\lambda(x+\varepsilon e_i))$$
$$=(\mathscr{E}_0(x+\varepsilon e_i),I_0(x+\varepsilon e_i))\,.$$

Subtracting the first equality from the second one, and using the T-accretiveness of C yields exactly estimate (4.7). □

Now, we must study carefully the "boundary term":

$$\iiint_{(\partial X_\varepsilon \times S^N)_- \times \mathbb{R}_v^{+*}} |\Omega \cdot n^\varepsilon| \left| \frac{I^\lambda(x+\varepsilon e_i) - I^\lambda(x)}{\varepsilon} \right| d\Gamma^\varepsilon \, \delta\Omega \, dv .$$

We begin with the following elementary lemma. Since it is not the heart of the matter, we shall only give the main tricks its proof.

Lemma 9. *Under assumptions* (H1)–(H9) *and* (4.1)–(4.5), *we have that:*

$$\overline{\lim_{\varepsilon \to 0}} \iiint_{(\partial X_\varepsilon \times S^N)_- \times \mathbb{R}_v^{+*}} |\Omega \cdot n^\varepsilon| \left| \frac{I^\lambda(x+\varepsilon e_i) - I^\lambda(x)}{\varepsilon} \right| d\Gamma^\varepsilon \, \delta\Omega \, dv$$
$$\leqq \iiint_{(\partial X \times S^N)_- \times \mathbb{R}_v^{+*}} \|T_x h\| \, d\Gamma \, \delta\Omega \, dv + \iiint_{(\partial X \times S^N)_- \times \mathbb{R}_v^{+*}} |\Omega \cdot \nabla I^\lambda| \, d\Gamma \, \delta\Omega \, dv .$$

Proof of Lemma 9. We choose $\varepsilon_0 > 0$ such that $X_{\varepsilon_0} \neq \emptyset$; and in the following ε will be such that $0 < \varepsilon < \varepsilon_0$. We know that ∂X and ∂X_ε are parallel hypersurfaces in $\mathbb{R}^{N+1}$. Using a C^∞ partition of unity on the set $X - X_{\varepsilon_0}$ and straightening locally the boundaries ∂X and ∂X_ε allows us to consider the simplest situation, where ∂X and ∂X_ε are parallel hyperplane sets.

We pick some $x_0 \in \partial X$ and we consider the cylinder $C(\partial X \cap B(x_0, 4\alpha), -\varepsilon_0 n(x_0))$[4] to which the restrictions of ∂X and ∂X_ε are parallel hyperplane sets. We pick some $\Omega \in S^N$ such that $\Omega \cdot n(x_n) < 0$, and for $x \in C(\partial X \cap B(x_0, 4\alpha), -\varepsilon_0 n(x_0))$ we consider the following points:

$$x^\varepsilon = x + \frac{\varepsilon \Omega}{|\Omega \cdot n|};$$

$y_i(x, \Omega, h) =$ projection of $x + h e_i$ on the parallel hyperplane to ∂X containing x, following the direction Ω;

$$y_i^\varepsilon(x, \Omega, h) = y_i(x, \Omega, h) + \frac{\varepsilon \Omega}{|\Omega \cdot n|}.$$

We notice that:

$$\operatorname{dist}(x^\varepsilon + h e_i; y_i^\varepsilon(x, \Omega, h)) = \frac{|e_i \cdot n|}{|\Omega \cdot n|} \varepsilon .$$

Then, using the dominated convergence theorem, and the fact that $\Omega \cdot \nabla_x I \in C(\bar{X}; L^\infty(S^N; {}^{v\,1}(\mathbb{R}^{+*})))$, we get that, for a.e. x, Ω such that $\Omega \cdot n(x_0) < 0$ and $x \in \partial X \cap B(x_0, \alpha)$:

$$\int_{\mathbb{R}_v^{+*}} |\Omega \cdot n| \left| \frac{I^\lambda(x^\varepsilon + \varepsilon e_i) - I^\lambda(y_i^\varepsilon)}{\varepsilon} \right| dv = \int_{\mathbb{R}_v^{+*}} |e_i \cdot n| \left| \frac{I^\lambda(x + \varepsilon e_i) - I^\lambda(y_i^\varepsilon)}{\frac{|e_i \cdot n|}{|\Omega \cdot n|} \varepsilon} \right| dv$$
$$\xrightarrow[\varepsilon \to 0]{} \int_{\mathbb{R}_v^{+*}} |e_i \cdot n| \, |\Omega \cdot \nabla_x I(x)| \, dv .$$

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[4] With the notation: $C(A, \mathrm{V}) = \{X + \lambda \mathrm{V}$, where $X \in A$ and $\lambda \in [0,1]\}$

Now, let us call:

$$\Sigma = \partial X \cap B(x,\alpha), \qquad \Sigma_\varepsilon = \Sigma + \frac{\varepsilon\Omega}{|\Omega\cdot n|}.$$

$C_\varepsilon(\Sigma,\Sigma_\varepsilon)$ = the open cylinder of basis Σ and Σ_ε. Then, we choose a test function φ such that:

$$\begin{cases} \varphi \sim 1 & \text{on} \quad C_\varepsilon \cup \Sigma \cup \Sigma_\varepsilon \\ \varphi = 0 & \text{on} \quad \bar{C}_\varepsilon - (C_\varepsilon \cup \Sigma \cup \Sigma_\varepsilon) \end{cases}.$$

Using Stokes formula on C_ε yields:

$$\left| \frac{1}{\varepsilon} \iint_{\mathbb{R}_v^{+*}\times\Sigma_\varepsilon} |\Omega\cdot n|\, |I^\lambda(x^\varepsilon) - I^\lambda(y_i^\varepsilon)|\, d\Gamma^\varepsilon\, dv - \frac{1}{\varepsilon} \iint_{\mathbb{R}_v^{+*}\times\Sigma} |\Omega\cdot n|\, |I^\lambda(x) - I^\lambda(y_i^\varepsilon)|\, d\Gamma^\varepsilon\, dv \right|$$

$$\leqq \frac{1}{\varepsilon} \iint_{\mathbb{R}_v^{+*}\times C_\varepsilon} |\Omega\cdot \nabla_x [I^\lambda(y_i) - I^\lambda(x)]|\, \varphi(x)\, dx\, dv + \frac{1}{\varepsilon} \iint_{\mathbb{R}_v^{+*}\times C_\varepsilon} |\Omega\cdot \nabla_x \varphi|\, |I^\lambda(y_i) - I^\lambda(x)|\, dx\, dv\,. \tag{4.8}$$

Since I and $\Omega\cdot\nabla_x I \in C(\bar{X}; L^\infty(S^N; L^1(\mathbb{R}_v^{+}*)))$, and since $\operatorname{mes} C_\varepsilon = 0(\varepsilon)$, we easily get from (4.8) that:

$$\lim_{\varepsilon\to 0} \left| \frac{1}{\varepsilon} \iint_{\mathbb{R}_v^{+*}\times\Sigma_\varepsilon} |\Omega\cdot n|\, |I^\lambda(x^\varepsilon) - I^\lambda(y_i^\varepsilon)|\, d\Gamma^\varepsilon\, dv - \frac{1}{\varepsilon} \iint_{\mathbb{R}_v^{+*}\times\Sigma} |\Omega\cdot n|\, |I^\lambda(x) - I^\lambda(y_i)|\, d\Gamma\, dv \right| = 0\,.$$

Then, we notice that

$$\overline{\lim_{\varepsilon\to 0}} \iint_{\mathbb{R}_v^{+*}\times\Sigma} |\Omega\cdot n| \left| \frac{I^\lambda(x) - I(y_i)}{\varepsilon} \right| d\Gamma\, dv$$

$$= \overline{\lim_{\varepsilon\to 0}} \iint_{\mathbb{R}_v^{+*}\times\Sigma} |\sin[(e_i,n) - (\Omega,n)]| \cdot \left| \frac{I^\lambda(x) - I^\lambda(y_i)}{|x-y_i|} \right| d\Gamma\, dv$$

$$\leqq \iint_{\mathbb{R}_v^{+*}\times\Sigma} \|T_x h\|\, |\sin[(e_i,n) - (\Omega,n)]|\, d\Gamma\, dv\,.$$

The remaining details are sheer routine. □

Now, using the equation $(\mathscr{E}^\lambda, I^\lambda) = J_\lambda^Q \cdot (\mathscr{E}^0, I^0)$ yields:

$$\lambda|\Omega\cdot\nabla_x I^\lambda| \leqq \lambda\sigma_v(\mathscr{E}^\lambda) I^\lambda + \lambda q_v(\mathscr{E}^\lambda) + \lambda|I^\lambda - I_0|\,,$$

whence we get

$$\lambda \iiint_{(\partial X\times S^N)_-\times\mathbb{R}^{+*}} |\Omega\cdot\nabla_x I^\lambda|\, d\Gamma\, \delta\Omega\, dv \leqq \lambda \iiint_{\partial X\times S^N\times\mathbb{R}^{+*}} [\sigma_v(\mathscr{E}_\lambda) I^\lambda - q_v(\mathscr{E}^\lambda)]\, d\Gamma\, \delta\Omega\, dv$$

$$+ 2\lambda \iiint_{\partial X\times S^N\times\mathbb{R}^{+*}} q_v(\mathscr{E}_\lambda)\, d\Gamma\, \delta\Omega\, dv + \lambda \iiint_{(\partial X\times S^N)_-\times\mathbb{R}^{+*}} |I^\lambda - I_0|\, d\Gamma\, \delta\Omega\, dv\,.$$

Thus, we have that:

$$\lambda \iiint_{(\partial X\times S^N)_-\mathbb{R}_v^{+*}} |\Omega\cdot\nabla_x I^\lambda|\, d\Gamma\,\delta\Omega\, dv \leqq C\lambda + \iiint_{(\partial X\times S^N)_-\times\mathbb{R}_v^{+*}} |I^\lambda - I_0|\, d\Gamma\,\delta\Omega\, dv + \iiint_{(\partial X\times S^N)_-\times\mathbb{R}_v^{+*}} (\mathscr{E}^\lambda-\mathscr{E}_0)\, d\Gamma\,\delta\Omega\, dv\,, \tag{4.9}$$

since we know that

$$\lambda \iint_{S^N\times\mathbb{R}_v^{+*}} [q_v(\mathscr{E}^\lambda)-\sigma_v(\mathscr{E}^\lambda)I^\lambda]\,\delta\Omega\, dv = \mathscr{E}_0 - \mathscr{E}_\lambda\,.$$

Equation (4.9) is the crucial point of the present section. Now, Proposition 3 is clearly a consequence of Lemma 8, Lemma 9, together with (4.9). □

Step 2. We keep the same regularity assumptions as in Step 1. We are going to prove

Proposition 4. *Under assumptions* (H1)–(H9) *and* (4.1)–(4.5) *we have that* $\forall t>0$, $e^{-tQ}\cdot(\mathscr{E}_0, I_0)\in BV(X)\times BV(X; L^1(S^N\times\mathbb{R}_v^{+*}))$ *with the estimate*

$$\|e^{-tQ}\cdot(\mathscr{E}_0, I_0)\|_{BV} \leqq \|(\mathscr{E}_0, I_0)\|_{BV} + C(t+1)\,. \tag{4.10}$$

Proof of Proposition 4. We define

$$(\mathscr{E}_n^k, I_n^k) = (J_{t/n}^Q)^k\cdot(\mathscr{E}_0, I_0)\,.$$

An easy induction yields, with estimate (4.6),

$$\begin{aligned}
&\overline{\lim_{\varepsilon\to 0}}\left\{\int_{X_\varepsilon}\left|\frac{\mathscr{E}_n^k(x+\varepsilon e_i)-\mathscr{E}_n^k(x)}{\varepsilon}\right| dx + \iiint_{X_\varepsilon\times S^N\times\mathbb{R}_v^{+*}}\left|\frac{I_n^k(x+\varepsilon e_i)-I_n^k(x)}{\varepsilon}\right| dx\,\delta\Omega\, dv\right\}\\
&\leqq \overline{\lim_{\varepsilon\to 0}}\left\{\int_{X_\varepsilon}\left|\frac{\mathscr{E}_0(x+\varepsilon e_i)-\mathscr{E}_0(x)}{\varepsilon}\right| dx + \iiint_{X_\varepsilon\times S^N\times\mathbb{R}^{+*}}\left|\frac{I_0(x+\varepsilon e_i)-I_0(x)}{\varepsilon}\right| dx\,\delta\Omega\, dv\right\}\\
&\quad + Ct + \iiint_{\partial X\times S^N\times\mathbb{R}^{+*}} (\mathscr{E}_n^k-\mathscr{E}_0)\, d\Gamma\,\delta\Omega\, dv + \iiint_{(\partial X\times S^N)_-\times\mathbb{R}^{+*}} |I_n^1 - I_0|\, d\Gamma\,\delta\Omega\, dv
\end{aligned} \tag{4.11}$$

for $k\geqq 1$, and $n>\dfrac{t}{\lambda(\mathscr{E}_{\min}, \mathscr{E}_{\max})}$, since, for $k\geqq 1$, $I_n^{k+1}|_{(\partial X\times S^N)_-} = I_n^k|_{(\partial X\times S^N)_-}$.

Now, taking the limits when $n\to+\infty$ yields estimate (4.10), without any supplementary difficulty. □

Step 3. To finish the proof of Theorem E it is enough to prove that estimate (4.10) is still valid under assumptions (H1)–(H9) and (4.1)–(4.2) with

$$0\leqq(\mathscr{E}_0, I_0)\leqq(\mathscr{E}_{\max}, B_v(\mathscr{E}_{\max}))$$

and

$$(\mathscr{E}_0, I_0)\in BV(X)\times BV(X; L^1(S^N\times\mathbb{R}^{+*}))\,.$$

We take $(\mathscr{E}_0^\varepsilon, I_0^\varepsilon)\in C(\bar{X})\times C(\bar{X}; L^\infty(S^N; L^1(\mathbb{R}_v^{+*}))$ such that

$$(\mathscr{E}_0^\varepsilon, I_0^\varepsilon)\to(\mathscr{E}_0, I_0),\ \text{when } \varepsilon\to 0,\ \text{in } BV(X)\times BV(X; L^1(S^N\times\mathbb{R}_v^{+*}))\,.$$

We apply Proposition 4 of Step 2 with

$$(\varepsilon+\sup(0,\mathscr{E}_0^\varepsilon); B_\nu(\varepsilon)+\sup(0,I_0^\varepsilon)) \text{ instead of } (\mathscr{E}_0, I_0)$$

and $B_\nu(\varepsilon)+h$ instead of h. Taking $\varepsilon\to 0$, and using Theorem B′ yields exactly estimate (4.10).

Now, the proof of Theorem E is complete. □

V. Splitting Formulas for the Radiative Transfer Equations

We know that A, B, and Q are T-accretive, and that Q satisfies the range condition $(\mathscr{R})$. Obviously B satisfies the range condition $(\mathscr{R})$ (by using the same techniques as in the proof of Proposition 1); and it is well-known that A is m-accretive.

Our main goal in this section is to prove a Cranck-Nicholson like representation formula for the semigroup $\exp(-tQ)$, see below.

Let us now state the supplementary assumptions we need for our purpose.

(H10) $\exists\gamma>0$ such that, $\forall\mathscr{E}>0$ $\int_{\mathbb{R}^+} q_\nu(\mathscr{E})d\nu\leqq\gamma\mathscr{E}$,

(H11) $\forall(f,g)\in D(B)$, $\exists 0<\alpha<1$ such that $\forall\lambda\in]\alpha,1[, (\lambda f,\lambda g)\in D(B)$;

(H12) $\forall\nu>0$, $\mathscr{E}\mapsto\sigma_\nu(\mathscr{E})\,q_\nu(\mathscr{E})$ is a nondecreasing function;

(H13) the operator $\mathscr{E}\mapsto\sigma_\nu(\mathscr{E})\,q_\nu(\mathscr{E})$ is continuous from $L^1(X)^+$ to $L^1(X\times S^N\times\mathbb{R}_\nu^{+*})$;

[we recall the convention $0\,\sigma_\nu(0)=0$]. We now state the main result of this section.

Theorem F. *Under assumptions* (H1)–(H6) *and* (H10)–(H13) *we have that, for each* $x\in D(Q)$, *when* $n\to\infty$,

$$(J_{t/n}^A\circ J_{t/n}^B)^n\cdot x\to\exp(-tQ)\cdot x$$

uniformly on each compact t-interval.

This section is organized as follows:

– first, we prove an elementary abstract lemma from functional analysis necessary for the proof of the theorem;

– then, the proof of theorem reduces to the study of $\frac{1}{\lambda}(I-J_\lambda^B)$ when $\lambda\to 0_+$.

We shall end this section with some remarks about the case of a uniformly positive temperature.

1. A Splitting Formula for Single-Valued Accretive Operators in General Banach Spaces

On a Banach space $\mathscr{B}$, we consider the following evolution equation:

$$\begin{cases} \dfrac{du}{dt}+Ru+Su=0 \\ u(0)=u_0, \end{cases} \tag{5.1}$$

where R, S, and $R+S$ are accretive operators on $\mathscr{B}$ with domains $D(R)$, $D(S)$, $D(R+S)=D(R)\cap D(S)$ satisfying the range condition ($\mathscr{R}$). We shall study the Trotter like representation formulas for the generalized solutions of (5.1). We need the following assumptions on R and S:

(i) R is affine with dense domain

(ii) R and S are single-valued

and

(i)′ $\overline{D(S)}\subset D(R)$ and

$$\frac{1}{\lambda}(x-J_\lambda^R x)\to Rx \quad \textit{when} \quad \lambda\to 0_+$$

uniformly on each compact subset of $\overline{D(S)}$.

Lemma 10 (Consistency). *We assume* (i) *and* (ii). *Let $G(\lambda)$ be a family of mappings from $D(S)$ into $\mathscr{B}$ such that:*

$$\frac{1}{\lambda}(x-G(\lambda)\cdot x)\to Sx \quad \textit{when} \quad \lambda\to 0_+\,,$$

for each $x\in D(S)$. Then:

$$\frac{1}{\lambda}(x-J_\lambda^R\circ G(\lambda)\cdot x)\to Rx+Sx$$

when $\lambda\to 0_+$, for each $x\in D(R+S)$. The same conclusion holds if we substitute assumption (i)′ *to assumption* (i).

The proof of this lemma is an adaptation of the Hilbert space case (see [12]). We give the proof here for the sake of completeness.

Proof of Lemma 10. Let $x\in D(R+S)$; and let $x_\lambda=G(\lambda)\cdot x$; thus our assumption is that

$$\frac{1}{\lambda}(x-x_\lambda)\to Sx \quad \text{when} \quad \lambda\to 0_+\,.$$

Therefore, since J_λ^R is a contraction:

$$\frac{1}{\lambda}(J_\lambda^R(x-\lambda Sx)-J_\lambda^R\cdot x_\lambda)\to 0$$

when $\lambda\to 0_+$.

If we assume (i), without restricting the generality of the proof, we can assume that R is linear. Thus, we may write:

$$\frac{1}{\lambda}(x-J_\lambda^R\cdot(x-\lambda S\cdot x))=\frac{1}{\lambda}(x-J_\lambda^R\cdot x)+J_\lambda^R(Sx);$$

and since R is linear and m-accretive, we know that

$$\frac{1}{\lambda}(x-J_\lambda^R\cdot x)\to R\cdot x \quad \text{when} \quad \lambda\to 0_+ \quad \text{and} \quad J_\lambda^R(Sx)\to Sx \quad \text{when} \quad \lambda\to 0_+\,.$$

If we assume (i)′ we may write

$$\frac{1}{\lambda}(x-J^R_\lambda\cdot(x-\lambda S\cdot x))=S\cdot x+\frac{1}{\lambda}((x-\lambda S\cdot x)-J^R_\lambda\cdot(x-\lambda S\cdot x)),$$

and assumption (i)′ yields that

$$\frac{1}{\lambda}((x-\lambda S\cdot x)-J^R_\lambda\cdot(x-\lambda S\cdot x))\to R\cdot x$$

when $\lambda\to 0_+$. Thus, in both cases:

$$\frac{1}{\lambda}(x-J^R_\lambda\cdot(x-\lambda S\cdot x))\to R\cdot x+S\cdot x$$

and

$$\frac{1}{\lambda}(x-J^R_\lambda\cdot x_\lambda)=\frac{1}{\lambda}(x-J^R_\lambda\cdot(x-\lambda S\cdot x))+\frac{1}{\lambda}(J^R_\lambda(x-\lambda S\cdot x)-J^R_\lambda\cdot x_\lambda).$$

Thus, the proof is complete. □

Together with Chernoff's formula, for which we refer to Brézis and Pazy [5], Lemma 10 yields the following:

Proposition 5. *Assume* (i) *or* (i)′ *and* (ii) *from Lemma 10. Let $D(R+S)$ be convex, and let $G(\lambda)$ be a family of contractions from $D(S)$ to $\mathscr{B}$ such that, for each $x\in D(S)$:*

$$\frac{1}{\lambda}(x-G(\lambda)\cdot x)\to S\cdot x$$

when $\lambda\to 0_+$. Then, for each $x\in D(R+S)$, we have that

$$\left(J^R_{t/n}G\left(\frac{t}{n}\right)\right)^n\cdot x\to\exp(-t(R+S))\cdot x$$

when $n\to+\infty$, and this limit is uniform on each compact t-interval.

Essentially, Proposition 5 has reduced the proof of Theorem F to the study of $\frac{1}{\lambda}(x-J^B_\lambda\cdot x)$ for $x\in D(B)$, when $\lambda\to 0_+$.

2. Consistency of J^B_λ with B

We now turn to prove

Lemma 11. *Under assumptions* (H1)–(H6) *and* (H10)–(H13) *for each $(f,g)\in D(B)$, when $\lambda\to 0_+$*

$$\frac{1}{\lambda}((f,g)-J^B_\lambda\cdot(f,g))\to B\cdot(f,g).$$

Proof of Lemma 11. Let $(\mathscr{E}_\lambda, I_\lambda)=J_\lambda^B\cdot(f,g)$ for $\lambda>0$. We have the following obvious estimates:

$$\begin{cases} \text{for a.e. } x\in X: \\ \mathscr{E}_\lambda(x)+\iint_{S^N\times\mathbb{R}^{+*}} I_\lambda(x)\,\delta\Omega\,dv=f(x)+\iint_{S^N\times\mathbb{R}^{+*}} g(x)\,\delta\Omega\,dv; \\ \mathscr{E}_\lambda(x)\geqq \dfrac{f(x)}{1+\gamma\lambda}. \end{cases} \tag{5.2}$$

Now, one immediately sees that

$$\frac{\mathscr{E}_\lambda-f}{\lambda}=\iint_{S^N\times\mathbb{R}^{+*}}\sigma_v(\mathscr{E}_v)I_\lambda\,\delta\Omega\,dv-\int_{\mathbb{R}^{+*}} q_v(\mathscr{E}_\lambda)\,dv, \tag{5.3}$$

and

$$\frac{I_\lambda-g}{\lambda}=q_v(\mathscr{E}_\lambda)-\sigma_v(\mathscr{E}_\lambda)I_\lambda. \tag{5.4}$$

This yields the following domination relations:

$$0\leqq q_v(\mathscr{E}_\lambda)\leqq q_v\left(f+\iint_{S^N\times\mathbb{R}^{+*}} g\,\delta\Omega\,dv\right) \tag{5.5}$$

according to (H2), and

$$0\leqq\sigma_v(\mathscr{E}_\lambda)I_\lambda\leqq\sigma_v\left(\frac{f}{1+\gamma\lambda_0}\right)g+\lambda_0\sigma_v\left(f+\iint_{S^N\times\mathbb{R}^{+}} g\,\delta\Omega\,dv\right)q_v\left(f+\iint_{S^N\times\mathbb{R}^{+}} g\,\delta\Omega\,dv\right) \tag{5.6}$$

according to (H2), (5.4), and (H12) for $0<\lambda<\lambda_0$.

Using assumption (H5),

$$q_v\left(f+\iint_{S^N\times\mathbb{R}^{+}} g\,\delta\Omega\,dv\right)\in L^1(X\times S^N\times\mathbb{R}^{+*}),$$

and, according (H11) and (H13) by choosing λ_0 sufficiently small:

$$\sigma_v\left(\frac{f}{1+\gamma\lambda_0}\right)g+\lambda_0\sigma_v\left(f+\iint_{S^N\times\mathbb{R}^{+*}} g\,\delta\Omega\,dv\right)\cdot q_v\left(f+\iint_{S^N\times\mathbb{R}^{+*}} g\,\delta\Omega\,dv\right) \in L^1(X\times S^N\times\mathbb{R}^{+*}).$$

Thus, (5.5) and (5.6), together with (5.3) and (5.4) ensure that

$$(\mathscr{E}_\lambda, I_\lambda)\to(f,g) \text{ in } E \text{ and a.e. when } \lambda\to 0_+. \tag{5.7}$$

Furthermore, putting (5.7) again into (5.3) and (5.4) yields, with (5.5) and (5.6):

$$\frac{\mathscr{E}_\lambda-f}{\lambda}\to\iint_{S^N\times\mathbb{R}^{+*}}[\sigma_v(f)g-q_v(f)]\,\delta\Omega\,dv \quad\text{and}\quad \frac{I_\lambda-g}{\lambda}\to q_v(f)-\sigma_v(f)g$$

in $L^1(X)$ and $L^1(X\times S^N\times\mathbb{R}^{+*})$ respectively, according to the dominated convergence theorem. □

Lemma 11, together with Proposition 5, gives the proof of Theorem F.

The same method can be used to prove the convergence of another splitting algorithm for (TR), introduced by Mercier. But this splitting algorithm is valid for uniformly positive temperatures only (see [13]).

Acknowledgements. We wish to thank Prof. C. Bardas for many helpful discussions on this topic.

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Communicated by J. L. Lebowitz

Received October 1, 1985; in revised form March 10, 1986