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Algebraic Properties of the Invariant Charges of the Nambu-Goto Theory*

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Abstract. We analyse the infinite dimensional algebra of observable non-local integrals of motion of the Nambu-Goto string theory.

I. Introduction

Some time ago one of the present authors suggested a reparametrization invariant approach towards the quantization of the free relativistic closed bosonic string [1, 2]. This approach was modelled after the quantization of the free relativistic particle in terms of irreducible representations of the Poincaré algebra. In the Nambu-Goto theory [3] of the string moving in d-dimensional space-time $\mathbb{M}^d$, the analogue $\mathfrak{g}$ of the Poincaré algebra has been shown [1] to be of the following type

$$\mathfrak{g} = \mathrm{so}(1, d-1) \oplus\!\!\!\!\!\!\rightarrow (\mathbb{M}^d \oplus (\mathfrak{h}_{\mathscr{P}}^+ \oplus \mathfrak{h}_{\mathscr{P}}^-)),$$

where $\mathrm{so}(1,d-1)$ stands for the Lie algebra of the homogeneous Lorentz transformations, $\mathbb{M}^d$ for the Lie algebra of translations, $\mathfrak{h}_{\mathscr{P}}^+$ and $\mathfrak{h}_{\mathscr{P}}^-$ for the infinite-dimensional algebra of infinitesimal generators of certain "internal" symmetry transformations of the string. Explicitly, a basis of $\mathrm{so}(1,d-1)$ is furnished by the infinitesimal generators $M_{\mu\nu}$ of Lorentz transformations in the μ,ν plane, $\mu \neq \nu$, $\mu,\nu = 0,1,\dots,d-1$, $M_{\mu\nu} = -M_{\nu\mu}$, a basis of $\mathbb{M}^d$ by the components $\mathscr{P}_\mu, \mu = 0,1,\dots,d-1$ of the energy-momentum operator, i.e. the infinitesimal generators of translations in the μ direction, and finally a basis of $\mathfrak{h}_{\mathscr{P}}^+$ and $\mathfrak{h}_{\mathscr{P}}^-$ is furnished by certain reparametrization invariant conserved "internal", "non-local" charges $\mathscr{Z}^{\mathrm{red}+}$ and $\mathscr{Z}^{\mathrm{red}-}$ respectively. The charges $\mathscr{Z}^{\mathrm{red}+}$ and $\mathscr{Z}^{\mathrm{red}-}$ commute with the momenta $\mathscr{P}_\mu$ and transform covariantly according to finite dimensional (irreducible) representations of the Lorentz group. The elements of $\mathfrak{h}_{\mathscr{P}}^+$ commute with all the elements of $\mathfrak{h}_{\mathscr{P}}^-$.

The central idea of the new approach consists of viewing the loop equations of the Nambu-Goto theory as an infinite collection of representation conditions for

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the infinite dimensional algebras $\mathfrak{h}_{\mathscr{P}}^{+}$ and $\mathfrak{h}_{\mathscr{P}}^{-}$. In order to convert this qualitative idea into a well-defined strategy, it is necessary to clarify the structure of the algebras $\mathfrak{h}_{\mathscr{P}}^{\pm}$. It would be sufficient to perform the corresponding analysis in the momentum rest frame provided that $\mathscr{P}^2=\mathscr{P}_\mu g^{\mu\nu}\mathscr{P}_\nu=m^2>0$, $g_{\mu\nu}=g^{\mu\nu}=\mathrm{diag}(+1,-1,\ldots,-1)$. Undeniably, the specialization to the momentum rest frame would facilitate the analysis. However, in contradistinction to the intrinsic role which this particular choice of reference frame plays for the construction of the positive energy representations of groups containing the Lorentz group, the specialization to the momentum rest frame does not result in an *essential* simplification of the investigations at hand. Thus only at a late stage the momentum rest frame will be employed.

The present article reports recent progress in a detailed and systematic analysis of the classical algebraic structure of $\mathfrak{h}_{\mathscr{P}}^{+}$ and $\mathfrak{h}_{\mathscr{P}}^{-}$. Our principal motivation for this effort derives from the role which we attribute to the algebras $\mathfrak{h}_{\mathscr{P}}^{+}$ and $\mathfrak{h}_{\mathscr{P}}^{-}$ for the classification of the string states. In this context let us recall that the degeneracy of states is expected to increase exponentially with energy. In addition, we have studied the one loop renormalization of the elements of $\mathfrak{h}_{\mathscr{P}}^{+}$ and $\mathfrak{h}_{\mathscr{P}}^{-}$. There are strong indications that this renormalization involves not more than a single (already familiar) free parameter [10].

As regards Sects. II and III, a still more extensive discussion can be found in [4]. We aimed at setting the present analysis into a larger mathematical context in order to make the methods and results developed in this article applicable to a variety of algebraic and combinatorial aspects of path ordered exponentials.

II. Definition and Properties of the Algebras $\mathfrak{h}_{\mathscr{P}}^{\pm}$ in the Classical Theory

We consider the linear spans of the classically conserved, reparametrization invariant "non-local" charges, in short: invariants $\mathscr{Z}^{+}_{\mu_1\ldots\mu_N}$ ($N=1,2,\ldots$, $\mu_i=0,1,\ldots,d-1$) and $\mathscr{Z}^{-}_{\mu_1\ldots\mu_N}$ separately and denote them by $\mathfrak{h}_{\mathscr{P}}^{+}$ and $\mathfrak{h}_{\mathscr{P}}^{-}$:

$$\mathscr{Z}^{\pm}_{\mu_1\ldots\mu_N}=\mathscr{R}^{\pm}_{\mu_1\ldots\mu_N}(\tau,\sigma)+\mathscr{R}^{\pm}_{\mu_2\ldots\mu_N\mu_1}(\tau,\sigma)+\ldots+\mathscr{R}^{\pm}_{\mu_N\mu_1\ldots\mu_{N-1}}(\tau,\sigma)$$

with

$$\mathscr{R}^{\pm}_{\mu_1\ldots\mu_N}(\tau,\sigma)=\int\limits_{\sigma+2\pi>\sigma_1>\sigma_2>\ldots>\sigma_N>\sigma}\!\!\!\ldots\int d\sigma_1\ldots d\sigma_N\prod_{i=1}^{N}u^{\pm}_{\mu_i}(\tau,\sigma_i).$$

Here we have set

$$u^{\pm}_{\mu}(\tau,\sigma)=p_\mu(\tau,\sigma)\pm M^2\partial_\sigma x_\mu(\tau,\sigma).$$

The mapping $S^1\to\mathbb{M}^d:\sigma\to x_\mu(\tau,\sigma)$ describes the position of the string, the mapping $S^1\to\bar{V}_+\subset\mathbb{M}^d:\sigma\to p_\mu(\tau,\sigma)$ the energy-momentum distribution over the string.

$$u^{\pm}(\tau,\sigma)^2=0.$$

In the following we employ units such that the mass parameter M is equal to one.

The canonical Poisson brackets read

$$\{u_\mu^\pm(\tau,\sigma), u_\nu^\pm(\tau,\sigma')\} = \mp 2g_{\mu\nu}\partial_\sigma \delta_{2\pi}(\sigma-\sigma'),$$
$$\{u_\mu^+(\tau,\sigma), u_\nu^-(\tau,\sigma')\} = 0.$$

It will become clear in the sequel that $\mathfrak{h}_{\mathscr{P}}^+$ and $\mathfrak{h}_{\mathscr{P}}^-$ each form a commutative and associative algebra under tensorial multiplication and a Lie algebra under Poisson bracket operation.

The above invariants are in general not mutually independent. Apart from the linear cyclic symmetry

$$\mathscr{Z}^\pm_{\mu_1\ldots\mu_N} = \mathscr{Z}^\pm_{\mu_2\ldots\mu_N\mu_1},$$

there exist non-linear relations of increasing complexity, e.g.

$$\mathscr{Z}^\pm_{\mu\nu} = \mathscr{Z}^\pm_\mu \cdot \mathscr{Z}^\pm_\nu = \mathscr{P}_\mu \cdot \mathscr{P}_\nu, \qquad \boxed{\begin{array}{c}1\\2\\3\\4\end{array}}\ \mathscr{P}_{\mu_1} \cdot \mathscr{Z}^\pm_{\mu_2\mu_3\mu_4} = 0,$$

or

$$\boxed{\begin{array}{cc}1&4\\2&5\\3&6\end{array}}_A (\mathscr{Z}^\pm_{\mu_1\mu_2\mu_3} \cdot \mathscr{Z}^\pm_{\mu_4\mu_5\mu_6}) = \boxed{\begin{array}{cc}1&4\\2&5\\3&6\end{array}}_A (3\mathscr{P}_{\mu_1}\mathscr{P}_{\mu_4}\mathscr{Z}^\pm_{\mu_2\mu_3\mu_5\mu_6}).$$

Here the tableaux stand for the unnormalized Young operators $Q \cdot P$ in the group algebra $\mathbf{O}_N$ of the symmetric group $\mathbf{S}_N$ with [5, 6]

$$Q = \sum_{\sigma \in \{\text{permutations within the columns}\}} \operatorname{sign}(\sigma)\sigma, \qquad P = \sum_{\pi \in \{\text{permutations within the rows}\}} \pi.$$

For the identification of the algebras $\mathfrak{h}_{\mathscr{P}}^+$ and $\mathfrak{h}_{\mathscr{P}}^-$ as enveloping algebras it would be desirable to isolate a minimal linear subspace of $\mathfrak{h}_{\mathscr{P}}^+$ and $\mathfrak{h}_{\mathscr{P}}^-$ respectively, each one closed under Poisson bracket operation, which by taking tensor products and real linear combinations generates all of $\mathfrak{h}_{\mathscr{P}}^+$ and all of $\mathfrak{h}_{\mathscr{P}}^-$ respectively. With this aim in mind one might try to solve all the relations among the invariants $\mathscr{Z}^\pm_{\mu_1\ldots\mu_N}$ explicitly and systematically in terms of a natural complete set of mutually independent "reduced" invariants $\mathscr{Z}^{\text{red}\,\pm}_{\mu_1\ldots\mu_N}$ such that all invariants $\mathscr{Z}^\pm_{\mu_1\ldots\mu_N}$ are tensorial polynomials in the reduced invariants. This turns out to be a very difficult, as yet unsolved problem. Even the very existence of an unambiguous algorithm for the identification of the reduced invariants has not been derived from general theorems. However, important partial aspects of the solution of the problem in question will be presented in the next two subsections.

A less ambitious goal, the construction of an explicit though somewhat arbitrary basis for the set of all invariants, will be achieved in Sect. IV.

In the following the dimension d of the Minkowski space-time in which the string moves will not be specified unless stated otherwise in order to exhibit the relevant structures most transparently. It is true that in low dimensions tensors of pronounced antisymmetric symmetry types vanish identically. However, this fact is not all that helpful.

Many of the propositions of the subsequent analysis can be traced back to identities in the group algebra $\mathbf{O}_N$ of the symmetric group acting on the indices of the tensors $\mathscr{R}$ and $\mathscr{Z}$. We shall use the following conventions and notations: Occasionally for a Lorentz tensor $\mathscr{T}$ of rank N we shall write $\mathscr{T}_{1\dots N}$ instead of $\mathscr{T}_{\mu_1\dots\mu_N}$. If $X=\sum_{\pi\in S_N}\lambda_\pi\cdot\pi$, $\lambda_\pi\in\mathbb{R}$, is an element of $\mathbf{O}_N$, we shall write $\mathscr{T}_X$ instead of $\sum_{\pi\in S_N}\lambda_\pi\mathscr{T}_{\mu_{\pi(1)}\dots\mu_{\pi(N)}}$.

We shall represent permutations π of the numbers $1,\dots,N$ by their scheme

$$\begin{pmatrix}1 & \dots & N\\ \pi(1) & \dots & \pi(N)\end{pmatrix}\mathrel{\hat{=}}\pi(1)\dots\pi(N),$$

rather than by their cycles.

Some elements of the group algebra which occur frequently are

- the cyclic permutation $\mathfrak{z}_N=2\,3\dots N\,1$;
- the cyclic symmetrizer $Z_N=\mathrm{id}_N+\mathfrak{z}_N+\dots+\mathfrak{z}_N^{N-1}$;
- the inversion $I_N=N(N-1)\dots 2\,1$.

Finally we shall focus our attention on $\mathfrak{h}_{\mathscr{P}}^+$ and suppress the superscript $+$. The algebras $\mathfrak{h}_{\mathscr{P}}^+$ and $\mathfrak{h}_{\mathscr{P}}^-$ with the tensor product as the composition law are isomorphic while as Lie algebras with the Poisson bracket operation as the composition law they differ by a global factor -1 for corresponding structure constants.

1. The Tensors $\mathscr{R}_{\mu_1\dots\mu_N}(\tau,\sigma)$

The building units for the invariants $\mathscr{Z}$, the tensors $\mathscr{R}_{\mu_1\dots\mu_N}(\tau,\sigma)$, arise essentially as entries of the monodromy matrix for a parameter dependent system of linear differential equations associated with the classical equations of motion [1, 2]. Apart from its dependence on the parameters, the monodromy matrix is not only a functional of $u_\mu(\tau,\cdot)$, in particular a function of τ, it also depends on the choice of a reference point σ on the string $x_\mu(\tau,\cdot)$. The monodromy matrix and hence the tensors $\mathscr{R}_{\mu_1\dots\mu_N}(\tau,\sigma)$ are reparametrization *variant*. It is the eigenvalues of the monodromy matrix, in other words [2] the cyclic sums

$$Z_N\mathscr{R}_{\mu_1\dots\mu_N}=\mathscr{Z}_{\mu_1\dots\mu_N},$$

which are reparametrization *invariant*, conserved quantities. Notwithstanding the variance of the tensors $\mathscr{R}_{\mu_1\dots\mu_N}$, it is recommendable to study the properties of these tensors. By solving all the relations which may exist among the components $\mathscr{R}_{\mu_1\dots\mu_N}$, one gets a handle on an important class of relations among the invariant tensors $\mathscr{Z}$.

In this subsection a minimal basis of "truncated" tensors $\mathscr{R}^t$ will be identified with the property that every tensor component $\mathscr{R}_{\mu_1\dots\mu_N}$ systematically and unambiguously can be represented as a polynomial in the components $\mathscr{R}^t_{\mu_1\dots\mu_r}$ of the truncated tensors.

The next two propositions give complete information about all possible linear and non-linear relations among arbitrary components of the tensor-valued functionals $\mathscr{R}_{\mu_1\dots\mu_N}(\tau,\sigma)$ for a fixed value of τ, $N=1,2,\dots$.

Proposition 1. *All the components $\mathscr{R}_{\mu_1 \dots \mu_N}(\tau, \sigma)$ for fixed values of τ and σ, $N=1,2,\dots$ are linearly independent functionals of the string-variables $u_\mu(\tau, \sigma')$.*

Corollary. *The equation $X\mathscr{R}_{\mu_1 \dots \mu_N}(\tau, \sigma)=0, X \in \mathbf{O}_N$, for all $\mu_i=0,1,\dots, d-1$, $i=1,2,\dots,N$ for some $d \in \mathbb{N}$, $d \geqq N$, implies $X=0$.*

Proof. As a preliminary we shall treat the string variables $u_\mu(\tau, \sigma)$ as if they were not subject to the constraints $u^2=0$. Every tensor component $\mathscr{R}_{\mu_1 \dots \mu_N}$ can be uniquely characterized by its "class" $\{i_1, \dots, i_r\}$ (order!) and within the class by the lengths $a_j - a_{j-1}, j=1,\dots,r$, of the sequence $0=a_0<a_1<\dots<a_r=N$, i.e. by the numbers $(a_0), a_1, \dots, a_{r-1}, (a_r)$. The new labels are obtained from the tensor indices according to the following scheme:

$$\mu_1=\dots=\mu_{\alpha_1} \doteqdot i_1 \mp \mu_{a_1+1}=\dots=\mu_{a_2} \doteqdot i_2 \mp \mu_{a_2+1}=\dots=i_{r-1} \\ \mp \mu_{a_{r-1}+1}=\dots=\mu_N \doteqdot i_r\,.$$

The linear independence of all components will be established by induction with respect to r. First we prove that the tensor components of a class with $r=1$ are linearly independent of each other and of the rest of the tensor components. To this end we choose $u_\mu(\tau, \sigma')=\lambda \delta_{\mu i_1} f(\sigma')$, where λ is a real parameter and $f(\sigma')$ is a fixed, sufficiently smooth periodic function. With this choice we have demonstrated that the only non-vanishing components $\mathscr{R}_{i_1 \dots i_1}$ are linearly independent of the rest of the tensor components. Varying the index i_1, the components belonging to different classes $\{i_1\}$ are seen to be linearly independent of each other. Varying the parameter λ, the components of class $\{i_1\}$ corresponding to different sequences $0=a_0<a_1=N$ (i.e. of different ranks) are seen to be linearly independent of each other. Putting the various findings together, we have verified the induction hypothesis for $r=1$.

Next, let us assume that the induction hypothesis holds true for all classes of tensor components $\{i'_1, \dots, i'_{r'}\}$ with $r'<r$. In order to show that this implies its validity also for the class $\{i_1, \dots, i_r\}$ of tensor components, we subdivide the circle $(\sigma, \sigma+2\pi]$ into smaller intervals $I_j=\left(\sigma+\frac{r-j}{r}\cdot 2\pi,\ \sigma+\frac{r-j+1}{r}2\pi\right], j=1,\dots,r$. We choose the string variables $u_\mu(\tau, \sigma)$ to have the following supports:

$$\operatorname*{supp}_{\sigma'} u_\mu(\tau, \sigma') \subset I_j \quad \text{if} \quad \mu=i_j$$

$$u_\mu(\tau, \sigma') \equiv 0 \quad \text{otherwise}.$$

With this choice the only non-vanishing tensor components either belong to the class $\{i_1, \dots, i_r\}$ or to classes $\{i'_1, \dots, i'_{r'}\}$ with $r'<r$. By induction hypothesis the tensor components of the latter classes are in particular linearly independent of the tensor components of the former class. Thus we have demonstrated that the tensor components of the class $\{i_1, \dots, i_r\}$ are linearly independent of the rest of the tensor components. If the string variables $u_\mu(\tau, \sigma')$ subject to the above mentioned choice are scaled by independent factors,

$$u_\mu(\tau, \sigma') \to \lambda_j u_\mu(\tau, \sigma') \quad \text{for} \quad \mu=i_j,$$

the tensor components $\mathscr{R}_{\mu_1 \dots \mu_N}$ of the class $\{i_1, \dots, i_r\}$ pick up a factor $\prod_1^N \lambda_j^{(a_j-a_{j-1})}$.

Taking into account the qualitatively different types of variation of these factors for different sequences $0=a_0<a_1<\ldots<a_r=N$, the components of the class $\{i_1,\ldots,i_r\}$ are seen to be linearly independent of each other. Putting the various findings together we conclude that the induction hypothesis holds true for the value of r in question, too, and consequently for every value of r.

The proof becomes more complicated when we pay attention to the constraint $u^2(\tau,\sigma)\equiv 0$. From the foregoing argument we know that all of the tensor components not carrying any 0-index are linearly independent functionals on the space of string variables $u_\ell(\tau,\sigma')$, $\ell\neq 0$ with disjoint supports. For simplicity we shall restrict the discussion to those functions $u_\ell(\tau,\sigma')$ which do not change their sign. This will be sufficient for our purposes. Now we solve the constraints $u^2(\tau,\sigma')\equiv 0$ by

$$u_0(\tau,\sigma')=\sum_{\ell=1}^{d-1}|u_\ell(\tau,\sigma')|=\sum_{\ell=1}^{d-1}\operatorname{sign}(u_\ell)\cdot u_\ell(\tau,\sigma').$$

This leads to expressions for tensor components with 0-indices in terms of linear combinations of tensor components without any 0-index:

$$\mathscr{R}_{\ldots 0\ldots}(\tau,\sigma)=\sum_{\ell=1}^{d-1}\operatorname{sign}(u_\ell)\cdot\mathscr{R}_{\ldots\ell\ldots}(\tau,\sigma).$$

Global linear relations on the space of all possible string variables must be compatible with these relations which hold for special classes of the string variables only. However, the possibility to vary the signs of the functions $u_\ell(\tau,\sigma')$ independently rules out the existence of any such global linear relation among the tensor components including those with 0-indices. (Actually, the last argument does not apply for $d=2$, where e.g. $\mathscr{R}_{00}=\operatorname{sign}(u_1)^2\mathscr{R}_{11}=\mathscr{R}_{11}$ for all functions $u_1(\tau,\sigma')$ which do not change their sign on S^1. In this case also functions of varying sign must be considered.)

Proposition 2. *The linear span of all tensors $\mathscr{R}_{\mu_1\ldots\mu_N}$, $N=1,2,\ldots$ is closed under the tensor product operation. More precisely, the following equation holds true*

$$\mathscr{R}_{\mu_1\ldots\mu_M}(\tau,\sigma)\cdot\mathscr{R}_{\mu_{M+1}\ldots\mu_N}(\tau,\sigma)=\boxed{\begin{matrix}1\ldots M\\(M+1)\ldots N\end{matrix}}\,\mathscr{R}_{\mu_1\ldots\mu_N}(\tau,\sigma)=\mathscr{R}_{\boxed{\begin{smallmatrix}1\ldots M\\(M+1)\ldots N\end{smallmatrix}}}(\tau,\sigma).$$

Here the symbol $\boxed{\begin{matrix}1\ldots M\\(M+1)\ldots N\end{matrix}}$ denotes the sum over all permutations π such that the numbers $1,\ldots,M$ and the numbers $M+1,\ldots,N$ appear in the symbol $\pi(1),\ldots,\pi(N)$ in their orginal order.

For instance

$$\boxed{\begin{matrix}1&2\\3&4\end{matrix}}=1234+1324+1342+3124+3142+3412.$$

Proof. The above equation follows directly from the definition of the tensors $\mathscr{R}_{\mu_1\mu_2\ldots}$. The domain of integration of the product $\mathscr{R}_{\mu_1\ldots\mu_M}\cdot\mathscr{R}_{\mu_{M+1}\ldots\mu_N}$ is given by the inequalities $\sigma+2\pi>\sigma_1>\ldots>\sigma_M>\sigma$ and $\sigma+2\pi>\sigma_{M+1}>\ldots>\sigma_N>\sigma$ with no restrictions involving simultaneously variables of integration of both factors.

Obviously, the multiplication rule of Proposition 2 as well as the block-notation introduced in that context can be generalized to more than two factors. The blocks should not be confused with the Young-tableaux.

Proposition 1 (in particular its corollary) and proposition 2 imply that for fixed τ and σ the tensors $\mathscr{R}_{\mu_1 \ldots \mu_N}(\tau, \sigma)$ can be represented unambiguously by elements of the group algebra and, moreover, that the multiplicative structure of the tensor product is characterized by certain permutations of the tensor indices. This observation is the basis of the methodical approach of this section which consists of transferring the analysis of the tensors to an analysis of the group algebra $\mathbf{O}_N$.

First we shall show that no information about the string variables $u_\mu(\tau, \sigma')$ is lost if we eliminate the tensors $\mathscr{R}$ in favour of the so-called *truncated* tensors $\mathscr{R}^t$ (generated by the logarithm of the monodromy matrix). For fixed τ and σ an *algebraically independent* basis for the tensors in terms of the tensors $\mathscr{R}^t$ can be given.

We define tensors $\mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N}$ generated by $(K!)^{-1}$ times the K^{th} power of the logarithm of the monodromy matrix,

$$\mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N}(\tau, \sigma) = P_N^{(K)} \mathscr{R}_{\mu_1 \ldots \mu_N}(\tau, \sigma).$$

Here the symbol $P_N^{(K)}$ denotes the following element of the group algebra $\mathbf{O}_N$:

$$P_N^{(K)} = \sum_{\nu=K}^{N} c_{K\nu} C_N^{(\nu)}$$

with

$$C_N^{(\nu)} = \sum_{0<a_1<\ldots<a_{\nu-1}<N} \boxed{\begin{matrix} 1\ldots a_1 \\ \ldots \\ (a_{\nu-1}+1)\ldots N \end{matrix}}, \quad C_N^{(\nu)} = 0 \quad \text{for} \quad \nu > N,\ C_N^{(0)} = 0,\ C_N^{(1)} = \mathrm{id}_N$$

and $c_{K\nu}$ being a coefficient of the Taylor series

$$\frac{1}{K!}(\ln(1+x))^K = \sum_{\nu=0}^{\infty} c_{K\nu} x^\nu \quad \text{for} \quad |x| < 1,$$

or, equivalently,

$$\frac{\Gamma(y+1)}{\Gamma(\nu+1)\Gamma(y-\nu+1)} = \sum_{K=0}^{\infty} c_{K\nu} y^K \quad \text{for} \quad \mathrm{Re}\, y > -1,$$

$c_{K\nu} = 0$ for $\nu < K$, $c_{1\nu} = (-1)^{\nu-1}/\nu$.

For the special case $K=1$ we obtain the truncated tensor

$$\mathscr{R}^t_{\mu_1 \ldots \mu_N} = \mathscr{R}^{(1)}_{\mu_1 \ldots \mu_N} = P_N^{(1)} \mathscr{R}_{\mu_1 \ldots \mu_N} = \sum_{\nu=1}^{N} (-1)^{\nu-1} \frac{1}{\nu} C_N^{(\nu)} \mathscr{R}_{\mu_1 \ldots \mu_N}.$$

For instance,

$$\mathscr{R}^t_\mu = \mathscr{P}_\mu, \quad \mathscr{R}^t_{\mu\nu} = \tfrac{1}{2}(\mathscr{R}_{\mu\nu} - \mathscr{R}_{\nu\mu}).$$

The following proposition is an easy consequence of the various definitions. It states that every tensor $\mathscr{R}_{\mu_1 \ldots \mu_N}$ can be expressed as a polynomial in the truncated tensors $\mathscr{R}^t$. The homogeneous part of degree K of this polynomial is nothing but the tensor $\mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N}$.

Proposition 3.

$$\mathscr{R}_{\mu_1 \ldots \mu_N} = \sum_{K=1}^{N} \mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N},$$

$$\mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N} = \frac{1}{K!} \sum_{0 < a_1 < \ldots < a_{K-1} < N} \mathscr{R}^t_{1 \ldots a_1} \cdot \mathscr{R}^t_{(a_1+1) \ldots a_2} \cdot \ldots \cdot \mathscr{R}^t_{(a_{K-1}+1) \ldots N}.$$

Conversely every homogeneous polynomial in the tensors $\mathscr{R}^t$ of degree K is obtained by a linear combination of $\pi \mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N}$, $\pi \in \mathbf{S}_N$.

Furthermore, there exist no linear relations among the polynomials of different degrees. Both statements are implied by the following two propositions which form the center of this subsection.

Proposition 4. *The collection $P^{(K)}_N \in \mathbf{O}_N$, $K = 1, \ldots, N$, provides a resolution of the group identity*

$$\mathrm{id}_N = \sum_{K=1}^{N} P^{(K)}_N$$

in terms of orthogonal projectors

$$P^{(L)}_N \circ P^{(K)}_N = \delta_{LK} P^{(K)}_N .$$

Proposition 5.

$$\boxed{\begin{matrix} 1 \ldots M \\ (M+1) \ldots N \end{matrix}} \mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N} = \sum_{K_1 + K_2 = K} \mathscr{R}^{(K_1)}_{\mu_1 \ldots \mu_M} \mathscr{R}^{(K_2)}_{\mu_{M+1} \ldots \mu_N},$$

or, equivalently, formulated as an identity in the group algebra

$$\boxed{\begin{matrix} 1 \ldots M \\ (M+1) \ldots N \end{matrix}} \circ P^{(K)}_N = \sum_{K_1 + K_2 = K} P^{(K_1)}_M \circ (\mathfrak{z}^M_N \circ P^{(K_2)}_{N-M} \circ \mathfrak{z}_N^{-M}) \circ \boxed{\begin{matrix} 1 \ldots M \\ (M+1) \ldots N \end{matrix}} .$$

Notice, that in the group algebra the projectors operate from the right: if $X \mathscr{R}_{1 \ldots N}$ is an arbitrary tensor, then its homogeneous parts are given by

$$X \mathscr{R}^{(K)}_{1 \ldots N} = X \circ P^{(K)}_N \mathscr{R}_{1 \ldots N} .$$

Proof of the Propositions 4 and 5. We define elements $D^{[p]}_N$ of the group algebra by

$$D^{[p]}_N = \sum_{0 \leqq a_1 \leqq \ldots \leqq a_{p-1} \leqq N} \boxed{\begin{matrix} 1 \ldots a_1 \\ \ldots \\ (a_{p-1}+1) \ldots N \end{matrix}} .$$

$\mathscr{R}^{[p]}_{\mu_1\dots\mu_N} \doteq D^{[p]}_N \mathscr{R}_{\mu_1\dots\mu_N}$ is generated by the p^{th} power of the monodromy matrix. Obviously, the elements $D^{[p]}_N$ are related to the elements $C^{(\nu)}_N$ by

$$D^{[p]}_N = \sum_{\nu=1}^{p} \binom{p}{\nu} C^{(\nu)}_N , \qquad C^{(\nu)}_N = \sum_{p=1}^{\nu} (-1)^{\nu-p} \binom{\nu}{p} D^{[p]}_N .$$

The elements $D^{[p]}_N$ are different from zero for arbitrarily large integer values of p. However, for $p > N$ they become linearly dependent of each other, the precise linear dependence being given by the second equation. The projectors $P^{(K)}_N$ can be expressed in terms of $D^{[p]}_N$. The rule of composition of two such elements $D^{[p]}_N$ and $D^{[q]}_N$ reads

$$D^{[p]}_N \circ D^{[q]}_N = D^{[p\cdot q]}_N .$$

The rule for the tensorial multiplication of $\mathscr{R}^{[p]}_{1\dots M} \cdot \mathscr{R}^{[p]}_{(M+1)\dots N}$ is

$$\mathscr{R}^{[p]}_{1\dots M} \cdot \mathscr{R}^{[p]}_{(M+1)\dots N} = \boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}} \, \mathscr{R}^{[p]}_{1\dots N} .$$

We conclude:

$$\boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}} \, \mathscr{R}^{[p]}_{1\dots N} = \boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}} \circ D^{[p]}_N \mathscr{R}_{1\dots N}$$

$$= \mathscr{R}^{[p]}_{1\dots M} \mathscr{R}^{[p]}_{(M+1)\dots N} = D^{[p]}_M \circ (\mathfrak{z}^M_N \circ D^{[p]}_{N-M} \circ \mathfrak{z}^{-M}_N) \circ \boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}} \, \mathscr{R}_{1\dots N} .$$

Hence we obtain the following identity in the group algebra

$$\boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}} \, D^{[p]}_N = D^{[p]}_M \circ (\mathfrak{z}^M_N \circ D^{[p]}_{N-M} \circ \mathfrak{z}^{-M}_N) \circ \boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}} .$$

This identity together with the expression of $D^{[p]}_N$ in terms of $C^{(\nu)}_N$ and vice versa leads to

$$\text{i)} \quad P^{(L)}_N \circ P^{(K)}_N = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} c_{L\nu} c_{K\mu} C^{(\nu)}_N \circ C^{(\mu)}_N = \sum_{\varrho=0}^{\infty} X_{LK,\varrho} C^{(\varrho)}_N$$

with

$$X_{LK,\varrho} = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} c_{L\nu} c_{K\mu} (-1)^{\nu-p} \binom{\nu}{p} (-1)^{\mu-q} \binom{\mu}{q} \binom{pq}{\varrho} ,$$

$$\text{ii)} \quad \boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}} \circ P^{(K)}_N = \sum_{\lambda=0}^{\infty} \sum_{\kappa=0}^{\infty} Y_{\lambda\kappa} C^{(\lambda)}_M \circ (\mathfrak{z}^M_N \circ C^{(\kappa)}_{N-M} \circ \mathfrak{z}^{-M}_N) \circ \boxed{\begin{matrix} 1\dots M \\ (M+1)\dots N \end{matrix}}$$

with

$$Y_{\lambda\kappa} = \sum_{\nu=0}^{\infty} \sum_{p=0}^{\infty} c_{K\nu} (-1)^{\nu-p} \binom{\nu}{p} \binom{p}{\lambda} \binom{p}{\kappa} .$$

For the sake of transparency we have formally extended the sums to run from zero to infinity. From the definitions of $X_{LK,\varrho}$ and $Y_{\lambda\kappa}$ we find for $-1<x<0$, $\operatorname{Re}y>-1$, $\operatorname{Re}z>-1$:

$$\begin{aligned}
&\sum_{\varrho=0}^{\infty}\sum_{L=0}^{\infty}\sum_{K=0}^{\infty} X_{LK,\varrho}x^{\varrho}y^{L}z^{K} \\
&=\sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty}\frac{\Gamma(y+1)}{\Gamma(\nu+1)\Gamma(y-\nu+1)}\cdot\frac{\Gamma(z+1)}{\Gamma(\mu+1)\Gamma(z-\mu+1)}(-1)^{\nu+\mu} \\
&\quad\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(-1)^{p+q}\binom{\nu}{p}\binom{\mu}{q}((1+x)^{p})^{q} \\
&=(1+x)^{yz}=\sum_{\varrho=0}^{\infty}\sum_{L=0}^{\infty}\sum_{K=0}^{\infty}(\delta_{LK}c_{K\varrho})x^{\varrho}y^{L}z^{K}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\lambda=0}^{\infty}\sum_{\kappa=0}^{\infty} Y_{\lambda\kappa}y^{\lambda}z^{\kappa}=\sum_{\nu=0}^{\infty}\sum_{p=0}^{\infty}c_{K\nu}(-1)^{\nu-p}\binom{\nu}{p}(1+y)^{p}(1+z)^{p} \\
&=\frac{1}{K!}(\ln(1+y)+\ln(1+z))^{K}=\sum_{\lambda=0}^{\infty}\sum_{\kappa=0}^{\infty}\left(\sum_{K_1+K_2=K}c_{K_1\lambda}c_{K_2\kappa}\right)y^{\lambda}z^{\kappa}.
\end{aligned}$$

This can be true only if

$$X_{LK,\varrho}=\delta_{LK}c_{K\varrho}\quad\text{and}\quad Y_{\lambda\kappa}=\sum_{K_1+K_2=K}c_{K_1\lambda}c_{K_2\kappa}.$$

Finally, realizing that

$$\sum_{K}P_N^{(K)}=\sum_{\nu}\left(\sum_{K}c_{K\nu}\right)C_N^{(\nu)}=\sum_{\nu}\frac{\Gamma(2)}{\Gamma(1+\nu)\Gamma(2-\nu)}C_N^{(\nu)}=\mathrm{id}_N,$$

the proof of Propositions 4 and 5 is accomplished.

Proposition 4 rules out any truly inhomogeneous polynomial relations among the truncated tensors $\mathscr{R}^t$, since with the help of the projectors $P_N^{(K)}$ every possible inhomogeneous relation could be reduced to a set of homogeneous relations. Now we shall demonstrate that the decomposition of Proposition 3 is unique, i.e. that apart from the linear symmetries $P_N^{(L)}\mathscr{R}^t_{\mu_1\dots\mu_N}=0$ for $L>1$ there are no further linear or nonlinear homogeneous identities among the truncated tensors $\mathscr{R}^t$.

Lemma. *Exactly $(N-1)!$ elements of the set of tensors $\{\mathscr{R}^t_{\mu_{\pi(1)}\dots\mu_{\pi(N)}}, \pi\in\mathbf{S}_N\}$ are linearly independent. In d dimensions $d\geqq N$, the number of independent components of the tensor $\mathscr{R}^t_{\mu_1\dots\mu_N}$ coincides with the dimension of $\ell^{(N)}$ with $\ell=\bigoplus_{N=1}^{\infty}\ell^{(N)}$ the free Lie algebra generated by d basis elements of $\ell^{(1)}$: $n(d,N)=\frac{1}{N}\sum_{D|N}\mu(D)d^{N/D}$, where the sum extends over all divisors D of N and where $\mu(D)$ denotes the Möbius function.*

Proof. The number of linearly independent tensors $\mathscr{R}^t_{\mu_{\pi(1)}\dots\mu_{\pi(N)}}$ coincides with the dimension of the left-sided ideal in $\mathbf{O}_N$ generated by the idempotent $P_N^{(1)}$. In turn, this coincides with the value of $\operatorname{tr}P_N^{(1)}$ in the regular representation, i.e. with $N!$ times the component of $P_N^{(1)}$ in the direction of the group identity. Now,

$P_N^{(1)} = \sum_{\nu=1}^{N} (-1)^{\nu-1} \nu^{-1} C_N^{(\nu)}$, and the sum $C_N^{(\nu)}$ consists of $\binom{N-1}{\nu-1}$ terms each of which has component one in the direction of the group identity. Hence the number of linearly independent tensors $\mathscr{R}^t_{\mu_{\pi(1)} \dots \mu_{\pi(N)}}$ is given by

$$N! \sum_{\nu=1}^{N} (-1)^{\nu-1} \nu^{-1} \binom{N-1}{\nu-1} = (N-1)!\,.$$

The proof of the second part of the lemma is omitted. It uses projectors $Q_N^{(1)}$ equivalent to $P_N^{(1)}$ [4] and properties of $Q_N^{(1)}$ which otherwise are not essential for the present article (see, however, further below).

Proposition 1 implies that there exist exactly $N!$ *linearly independent* tensors $\mathscr{R}_{\mu_{\pi(1)} \dots \mu_{\pi(N)}}$, $d \geqq N$. Proposition 3 makes sure that each one of these tensors is a polynomial in the tensors $\mathscr{R}^t$ of rank $\leqq N$. Hence there are at least $N!$ linearly independent monomials of rank N in the tensors $\mathscr{R}^t$ of rank $\leqq N$. On the other hand, this is the maximal number of linearly independent monomials of rank N in the tensors $\mathscr{R}^t$ of rank $\leqq N$ allowed by the above lemma. In order to understand this, we keep the ranks of the factors preliminarily fixed up to the order in which the factors appear. The ranks correspond to a partition $(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_r)$, $\lambda_1 \geqq \lambda_2 \geqq \dots \geqq \lambda_r$, of N. We can distribute the indices $\mu_1 \dots \mu_N$ freely over the various factors such that the j^{th} factors carries λ_j indices. For a given distribution of the indices the j^{th} factor contributes $(\lambda_j - 1)!$ linearly independent tensors $\mathscr{R}^t$. We realize that this figure coincides with the number of different cycles of length λ_j, $(i_1, \dots, i_{\lambda_j})$, $i_k \neq i_\ell$, $i_k \in \{1, \dots, \lambda_j\}$. Hence the number of linearly independent monomials corresponding to a fixed partition (λ) is given by the order of the conjugacy class $C_{(\lambda)}$ in $\mathbf{S}_N$. Varying the partitions (λ), these orders sum up to $N!$. Hence the maximal number of linearly independent monomials of rank N is $N!$.

The above argument implies

Proposition 6. *The tensors $\mathscr{R}^t$ do not satisfy linear or non-linear relations other than*

$$P_N^{(L)} \mathscr{R}^t_{\mu_1 \dots \mu_N} = 0 \quad \text{for} \quad L = 2, \dots, N\,.$$

The existing relations among the tensors $\mathscr{R}^t$ can be solved explicitly.

Proposition 7.

$$\mathscr{R}^t_{1\dots N} = (-1)^{i-1} \mathscr{R}^t_{\;i\,\boxed{\begin{smallmatrix}(i-1)\dots 1\\(i+1)\dots N\end{smallmatrix}}} = (-1)^{N-j} \mathscr{R}^t_{\boxed{\begin{smallmatrix}1\dots(j-1)\\N\dots(j+1)\end{smallmatrix}}\,j} = (-1)^{N-1} \mathscr{R}^t_{N\dots 1}\,.$$

Proof. Proposition 5 implies

$$\boxed{\begin{matrix}1\dots M\\(M+1)\dots N\end{matrix}}\; \mathscr{R}^t_{1\dots N} = 0\,, \qquad 1 \leqq M < N\,.$$

We apply to this equation the inversion I_M and obtain

$$\mathscr{R}^t_{\boxed{\begin{smallmatrix}M\dots 1\\(M+1)\dots N\end{smallmatrix}}} = 0\,.$$

Starting from

$$\mathscr{R}^t_{1\ldots N} = -\mathscr{R}^t_{2\,\boxed{\begin{smallmatrix}1\\3\ldots N\end{smallmatrix}}} = -\mathscr{R}^t_{\boxed{\begin{smallmatrix}1\ldots(N-2)\\N\end{smallmatrix}}\,(N-1)},$$

we arrive at the desired equations if we make repeatedly use of the group algebra identities

$$\boxed{\begin{matrix}M\ldots 1\\(M+1)\ldots N\end{matrix}} = M\,\boxed{\begin{matrix}(M-1)\ldots 1\\(M+1)\ldots N\end{matrix}} + (M+1)\,\boxed{\begin{matrix}M\ldots 1\\(M+2)\ldots N\end{matrix}}\,.$$

In these identities the sum of permutations on the left-hand side has been divided up into a sum of permutations π with $\pi(1)=M$ and into a sum of permutations π with $\pi(1)=M+1$.

According to Proposition 7 the symmetries of $\mathscr{R}^t$ can be exploited such that an arbitrary index μ_j stands by choice at the extreme left or at the extreme right. Then the tensors obtained by permutations of the remaining $(N-1)$ indices are linearly independent as can be seen by counting.

From Proposition 7 we read off the "*parity*" of the homogeneous tensors $\mathscr{R}^{(K)}$ under the transformation $u_\mu(\tau,\sigma)\to u_\mu(\tau,-\sigma)$ which effects an inversion of the order of the indices:

$$\mathscr{R}^{(K)}_{\mu_N\ldots\mu_1} = (-1)^{N-K}\mathscr{R}^{(K)}_{\mu_1\ldots\mu_N},$$
$$\sum_{N-K=\text{even}} P^{(K)}_N = \tfrac{1}{2}(\mathrm{id}_N + I_N), \quad \sum_{N-K=\text{odd}} P^{(K)}_N = \tfrac{1}{2}(\mathrm{id}_N - I_N).$$

The coordinate dependence of the homogeneous tensors $\mathscr{R}^{(K)}$ with $K \leqq N$ is specified by

$$\partial_\sigma \mathscr{R}^{(K)}_{\mu_1\ldots\mu_N}(\tau,\sigma) = u_{\mu_1}(\tau,\sigma)\mathscr{R}^{(K)}_{\mu_2\ldots\mu_N}(\tau,\sigma) - \mathscr{R}^{(K)}_{\mu_1\ldots\mu_{N-1}}(\tau,\sigma)u_{\mu_N}(\tau,\sigma),$$
$$\partial_\tau \mathscr{R}^{(K)}_{\mu_1\ldots\mu_N}(\tau,\sigma) \propto \partial_\sigma \mathscr{R}^{(K)}_{\mu_1\ldots\mu_N}(\tau,\sigma).$$

We conclude this subsection by pointing out that the decomposition of $\mathscr{R}$ into a sum of homogeneous parts defined by the action of some projectors with the properties of Propositions 4 and 5 is not unique.

For instance

$$\tilde{\mathscr{R}}^t_{1\ldots N} = Q^{(1)}_N \mathscr{R}_{1\ldots N},$$

with

$$Q^{(1)}_N = \frac{1}{N}\sum_{i=1}^{N}(-1)^{i-1}\left(i\,\boxed{\begin{matrix}(i-1)\ldots 1\\(i+1)\ldots N\end{matrix}}\right)$$

defines an equivalent truncation as seen from the algebra

$$O_i \circ O_j = O_j, \quad i,j=1,2, \quad O_1 = P^{(1)}_N, \quad O_2 = Q^{(1)}_N.$$

The coefficients of the polynomial expressing $\mathscr{R}_{1\ldots N}$ in terms of $\tilde{\mathscr{R}}^t$ would be different from the ones given in Proposition 3, of course.

However, the projectors $P_N^{(K)}$ have the extremely important property distinguishing them from all other candidates that they decompose the invariant tensors $\mathscr{Z}$ into homogeneous constituents which are separately invariant. This will be considered in the following subsection.

2. The Invariant Tensors $\mathscr{Z}_{\mu_1 \dots \mu_N}$.

There are three equivalent ways to define the invariant tensors $\mathscr{Z}_{\mu_1 \dots \mu_N}$:

$$\mathscr{Z}_{\mu_1 \dots \mu_N} = Z_N \mathscr{R}_{\mu_1 \dots \mu_N},$$

$$\mathscr{Z}_{\mu_1 \dots \mu_N} = \oint d\sigma \mathscr{R}_{\mu_1 \dots \mu_{N-1}}(\tau, \sigma) u_{\mu_N}(\tau, \sigma)$$

and

$$\mathscr{Z}_{\mu_1 \dots \mu_N} = \oint d\sigma u_{\mu_1}(\tau, \sigma) \mathscr{R}_{\mu_2 \dots \mu_N}(\tau, \sigma).$$

The initial and final points of the loop integrations on the right-hand side of the two last equations need not be specified since both constituents of the integrands, u and $\mathscr{R}$, are periodic functions of σ. The equivalence of the definitions can be perceived by comparing the domains of integration on the torus $(S^1)^N$.

According to Proposition 1, any invariant tensor of rank N

$$X\mathscr{Z}_{\mu_1 \dots \mu_N} = X \circ Z_N \mathscr{R}_{\mu_1 \dots \mu_N}, \qquad X \in \mathbf{O}_N$$

is characterized by the element $X \circ Z_N$ of the left-sided ideal $\mathbf{O}_N Z_N$ generated by the idempotent $\frac{1}{N} Z_N$. This element can be represented unambiguously by an element $\tilde{X} \in \mathbf{O}_{N-1}$ such that

$$X \circ Z_N = \tilde{X} \circ Z_N.$$

For instance

$$X\mathscr{Z}_{\mu_1 \dots \mu_N} = \oint d\sigma (\tilde{X} \mathscr{R}_{\mu_1 \dots \mu_{N-1}}(\tau, \sigma)) u_{\mu_N}(\tau, \sigma).$$

Of course, the tensorial product of two invariants is again an invariant.

Proposition 8. *The linear span* $\mathfrak{h}_{\mathscr{P}}$ *of all invariant tensors* $\mathscr{Z}_{\mu_{\pi(1)} \dots \mu_{\pi(N)}}$, $\pi \in \mathbf{S}_N$, $N = 1, 2, \dots$ *is closed under the tensorial multiplication.*

More precisely

$$\mathscr{Z}_{\mu_1 \dots \mu_M} \cdot \mathscr{Z}_{\mu_{M+1} \dots \mu_N} = Z_M \circ \boxed{\begin{matrix} 1 \dots M \\ (M+1) \dots (N-1) \end{matrix}} \, \mathscr{Z}_{\mu_1 \dots \mu_N}.$$

Proof.

$$\begin{aligned} \mathscr{Z}_{1 \dots M} \mathscr{Z}_{(M+1) \dots N} &= \mathscr{Z}_{1 \dots M} \cdot \oint d\sigma \mathscr{R}_{(M+1) \dots (N-1)}(\tau, \sigma) u_N(\tau, \sigma) \\ &= Z_M \oint d\sigma \mathscr{R}_{1 \dots M}(\tau, \sigma) \mathscr{R}_{(M+1) \dots (N-1)}(\tau, \sigma) u_N(\tau, \sigma) \\ &= Z_M \circ \boxed{\begin{matrix} 1 \dots M \\ (M+1) \dots (N-1) \end{matrix}} \oint d\sigma \mathscr{R}_{1 \dots (N-1)}(\tau, \sigma) u_N(\tau, \sigma) \\ &= Z_M \circ \boxed{\begin{matrix} 1 \dots M \\ (M+1) \dots (N-1) \end{matrix}} \, \mathscr{Z}_{1 \dots N}. \end{aligned}$$

Now, we introduce the concept of reduced invariants: we propose to define the reduced invariants $\mathscr{Z}^{\text{red}}$ of rank N to be those invariants forming complete Lorentz multiplets which cannot be represented as polynomials in invariants of rank $<N$. Reduced invariants $\mathscr{Z}^{\text{red}\,1}, \ldots, \mathscr{Z}^{\text{red}\,r}$ are said to be linearly independent if the statement

$$\alpha_1 \mathscr{Z}^{\text{red}\,1} + \ldots + \alpha_r \mathscr{Z}^{\text{red}\,r} = \text{sum of products of invariants}$$

$$-\alpha_1, \ldots, \alpha_r = \text{scalar} - \text{implies: } \alpha_1 = \ldots = \alpha_r = 0\,.$$

The linear span $\mathfrak{h}_{\mathscr{P}}^{\text{red}}$ of a complete set of linearly independent reduced invariants on the one hand and the linear span $\mathfrak{h}_{\mathscr{P}}^{\text{prod}}$ of products of invariants on the other hand defines a decomposition of $\mathfrak{h}_{\mathscr{P}}$ into two disjoint subspaces. In fact, algebraically any linear basis of the subspace $\mathfrak{h}_{\mathscr{P}}^{\text{red}}$ provides a generating basis for all of $\mathfrak{h}_{\mathscr{P}}$. It must be pointed out, however, that the subspace $\mathfrak{h}_{\mathscr{P}}^{\text{red}}$ has not been unambiguously defined, yet.

An important step towards the solution of this uniqueness problem is achieved by the following Proposition 9. It makes sure that the homogeneous portions of the invariants $\mathscr{Z}$ considered as polynomials in $\mathscr{R}^t$ are separately invariant.

Proposition 9.

$$Z_N \circ P_N^{(K)} = P_{N-1}^{(K-1)} \circ Z_N = Z_N \circ P_{N-1}^{(K-1)}\,.$$

Corollary.

i) $\mathscr{Z}_{\mu_1 \ldots \mu_N} = \sum_{K=1}^{N} \mathscr{Z}^{(K)}_{\mu_1 \ldots \mu_N}$,

where the homogeneous terms

$$\mathscr{Z}^{(K)}_{\mu_1 \ldots \mu_N} \doteq Z_N \mathscr{R}^{(K)}_{\mu_1 \ldots \mu_N} = P_{N-1}^{(K-1)} \mathscr{Z}_{\mu_1 \ldots \mu_N}$$
$$= \oint d\sigma\, \mathscr{R}^{(K-1)}_{\mu_1 \ldots \mu_{N-1}}(\tau, \sigma)\, u_{\mu_N}(\tau, \sigma)$$

are separately invariant: homogeneous invariants.

ii) *Apart from the case* $N=1$: $\mathscr{Z}^{(1)}_\mu = \mathscr{Z}_\mu = \mathscr{R}^t_\mu = \mathscr{R}_\mu = \mathscr{P}_\mu$, *there are no homogeneous invariants of degree one:*

$$Z_N \mathscr{R}^t_{\mu_1 \ldots \mu_N} = 0\,.$$

Proof. We start from the observation that the cyclic sum of the tensors $\mathscr{R}^{[p]}_{\mu_1 \ldots \mu_N}(\tau, \sigma)$ (the latter ones correspond to the p^{th} powers of the monodromy matrix or equivalently the monodromy matrix for the interval $(\sigma, \sigma + p \cdot 2\pi)$) can be represented by the integral of $\mathscr{R}^{[p]}_{\mu_1 \ldots \mu_{N-1}}(\tau, \sigma)\, u_{\mu_N}(\tau, \sigma)$ over p periods. However, the periodicity of the integrand simplifies the representation to

$$Z_N \mathscr{R}^{[p]}_{1 \ldots N} = p \cdot \oint d\sigma\, \mathscr{R}^{[p]}_{1 \ldots (N-1)}(\tau, \sigma)\, u_N(\tau, \sigma)\,.$$

This equation implies the group algebra identities

$$Z_N \circ D_N^{[p]} = p \cdot D_{N-1}^{[p]} \circ Z_N\,.$$

Linear manipulations of these identities lead to

$$Z_N \circ P_N^{(K)} = \sum_{\varrho=0}^{N-1} X_\varrho C_{N-1}^{(\varrho)} \circ Z_N\,,$$

with

$$X_\varrho = \sum_{\nu=0}^{\infty} \sum_{p=0}^{\infty} c_{K\nu}(-1)^{\nu-p} \binom{\nu}{p} p \binom{p}{\varrho}.$$

We compute $\sum_{\varrho=0}^{\infty} X_\varrho x^\varrho$:

$$\begin{aligned}\sum_{\varrho=0}^{\infty} X_\varrho x^\varrho &= \sum_{\nu=0}^{\infty} \sum_{p=0}^{\infty} c_{K\nu}(-1)^{\nu-p} \binom{\nu}{p} (1+x)\frac{d}{dx}(1+x)^p \\ &= (1+x)\frac{d}{dx} \sum_{\nu=0}^{\infty} c_{K\nu} x^\nu = (1+x)\frac{d}{dx}\frac{1}{K!}[\ln(1+x)]^K \\ &= \frac{1}{(K-1)!}[\ln(1+x)]^{K-1} = \sum_{\varrho=0}^{\infty} c_{K-1,\varrho} x^\varrho,\end{aligned}$$

whence we conclude

$$X_\varrho = c_{K-1,\varrho}.$$

Insertion of this result into the above equation for $Z_N \circ P_N^{(K)}$ completes the proof of the first equation of Proposition 9. The proof of the second equation is omitted since it does not enter the subsequent analysis in an essential way. The proof can be found, however, in Appendix B of [4].

The following two propositions contain information about the invariant "factorability" of invariant tensors.

Proposition 10. *Apart from the case* $N=2$: $\mathscr{Z}^{(2)}_{\mu_1\mu_2} = \mathscr{P}_{\mu_1}\mathscr{P}_{\mu_2}$, *there are no invariant tensors homogeneous of degree 2:* $X\mathscr{Z}^{(2)}_{\mu_1\ldots\mu_N}$, $X \in \mathbf{O}_N$, *which can be written as a sum of products of other invariants* $\mathscr{Z}_{\nu_1\ldots\nu_r}$.

For $N>3$, the only "factorable" invariant tensors homogeneous of degree 3: $X\mathscr{Z}^{(3)}_{\mu_1\ldots\mu_N}$, $X \in \mathbf{O}_N$, are of the type $Y(\mathscr{P}_{\mu_1}\cdot\mathscr{Z}^{(2)}_{\mu_2\ldots\mu_N})$, $Y \in \mathbf{O}_N$.

Proposition 11. *For* $K > \frac{N+1}{2}$, *every invariant tensor* $\mathscr{Z}^{(K)}_{\mu_1\ldots\mu_N}$ *is factorable at least like* $Y(\mathscr{P}_{\mu_1}\mathscr{Z}^{(K-1)}_{\mu_2\ldots\mu_N})$, $Y \in \mathbf{O}_N$. *For* $K = \frac{N+1}{2}$, *only the totally antisymmetric part of a tensor* $\mathscr{Z}^{(K)}_{\mu_1\ldots\mu_N}$ *is non-factorable.*

Proof. Proposition 10 is a simple consequence of the corollary of Proposition 9. In particular, for each $N>2$ the invariants of rank N and degree (of homogeneity) 2 are algebraically independent of all invariant tensors of rank $<N$. The first statement of Proposition 11 follows from the representation

$$\mathscr{Z}^{(K)}_{1\ldots N} = \oint d\sigma \mathscr{R}^{(K-1)}_{1\ldots(N-1)}(\tau,\sigma) u_N(\tau,\sigma)$$

together with the observation that for $(K-1)>(N-1)/2$ each term in the defining sum of

$$\mathscr{R}^{(K-1)}_{1\ldots(N-1)} = \frac{1}{(K-1)!} \sum_{0<a_1<\ldots<a_{K-2}<N} \mathscr{R}^t_{1\ldots a_1}\mathscr{R}^t_{(a_1+1)\ldots a_2}\cdot\ldots\cdot\mathscr{R}^t_{(a_{K-2}+1)\ldots(N-1)}:$$

contains at least one factor $\mathscr{P}_i$. This factor is independent of σ and can be pulled out of the integral, whereas the remaining integral yields an invariant tensor homogeneous of degree $(K-1)$.

If $K=\frac{N+1}{2}$, then $\mathscr{R}^{(K-1)}=\frac{1}{(K-1)!}\mathscr{R}^t_{12}\mathscr{R}^t_{34}\cdot\ldots\cdot\mathscr{R}^t_{(N-2)(N-1)}+\sum_\alpha \mathscr{P}_\alpha(\ldots)$.

We notice the relation

$$\mathscr{R}^t_{12}\mathscr{R}^t_{34}+\mathscr{R}^t_{32}\mathscr{R}^t_{14}=\mathscr{Z}^{(2)}_{1234}-Z_4(\mathscr{P}_1\mathscr{R}^t_{234}).$$

Thus, if $\mathscr{Z}^{(K)}_{\mu_1\ldots\mu_N}$ is symmetrized in any two indices, the above argument applies ensuring factorability of that part of $\mathscr{Z}^{(K)}_{\mu_1\ldots\mu_N}$. On the other hand, the totally antisymmetric part of $\mathscr{Z}^{(K)}_{\mu_1\ldots\mu_N}$ cannot be written as a polynomial in invariant tensors of lower rank since there are no totally antisymmetric cyclic tensors of even rank. This completes the proof of Proposition 11.

We return to the problem of singling out reduced invariants. Group theoretically, a systematic definition of them is equivalent to a decomposition of $Z_N: Z_N=E_N^{\text{red}}+E_N^{\text{prod}}$ with the following properties:

i) the elements of the group algebra $\mathbf{O}_N$: E_N^{red} and E_N^{prod}, are (up to normalization) orthogonal projectors: $E^i\circ E^j=N\delta_{ij}E^j$.

ii) $E_N^{\text{prod}}\mathscr{R}_{\mu_1\ldots\mu_N}$ is a sum of products each of which involves at least two invariant factors.

iii) Every product of invariant tensors can be written in the form $X\circ E_N^{\text{prod}}\mathscr{R}_{\mu_1\ldots\mu_N}$, $X\in\mathbf{O}_N$.

iv) E_N^{red} and E_N^{prod} commute with $Z_N\circ P_N^{(K)}$.

Property i) implies that application of E_N^{red} or E_N^{prod} to the tensors $\mathscr{R}_{\mu_1\ldots\mu_N}$ yields invariant tensors:

$$E^i\mathscr{R}_{\mu_1\ldots\mu_N}=\frac{1}{N}E^i\mathscr{Z}_{\mu_1\ldots\mu_N}.$$

Moreover, property i) ensures that every invariant $X\circ Z_N\mathscr{R}_{\mu_1\ldots\mu_N}$, $X\in\mathbf{O}_N$, can be decomposed uniquely into two invariant terms: its reduced part $X\circ E^{\text{red}}\mathscr{R}_{\mu_1\ldots\mu_N}$ and its factorable part $X\circ E^{\text{prod}}\mathscr{R}_{\mu_1\ldots\mu_N}$. According to property ii), the second term is a sum of products of invariants of rank $<N$.

Property iii) entails that the reduced part of an arbitrary sum of products of invariants – each of the products involving at least two factors – vanishes.

Properties i)–iii) are necessary and sufficient to define $\mathscr{Z}^{\text{red}}_{\mu_1\ldots\mu_N}\doteqdot E_N^{\text{red}}\mathscr{R}_{\mu_1\ldots\mu_N}$ consistently as invariant tensors. Finally, property iv) formulates the additional requirement that for a homogeneous invariant both its reduced and its factorable part separately are homogeneous (of the same degree).

A priori, only the left-sided ideal $\mathbf{O}_N E_N^{\text{prod}}$ generated by E_N^{prod} is known. It corresponds to the linear span $\mathfrak{h}_{\mathscr{P}}^{\text{prod}}$ of products of invariant tensors with rank $<N$. In general E_N^{prod} itself is not uniquely determined, hence also E_N^{red} is not unambiguously defined by the properties i)–iv). Actually an ambiguity arises whenever there exist two equivalent representations of the symmetric group $\mathbf{S}_N$ in $\mathbf{O}_N Z_N\circ P_N^{(K)}$, only one of which being furnished by factorable invariants. According to [6], (Theorem III, 3.8) in that case there exist nontrivial nilpotent elements $W=E_N^{\text{red}}\circ X\circ E_N^{\text{prod}}$ such that $\tilde{E}_N^{\text{red}}=E_N^{\text{red}}+\lambda W$ and $\tilde{E}_N^{\text{prod}}=E_N^{\text{prod}}-\lambda W$ for an arbitrary scalar λ possess properties i)–iv), too. The multiplicity Tables 1 to 8 below

Tables 1–8. Symmetry types of homogeneous invariant tensors of rank 1–8

$N=1$

	$K=1$
X1 = (1)	1

N = 2

	$K=2$
X1 = (2)	1
$X2=(1^2)$	.

N = 3

	$K=2$	$K=3$
X1 = (3)	.	1
$X2=(2,1)$	.	.
$X3=(1^3)$	1(1)	.

N = 4

	$K=2$	$K=3$	$K=4$
X1 = (4)	.	.	1
$X2=(3,1)$	.	.	.
$X3=(2^2)$	1(1)	.	.
$X4=(2,1^2)$	.	1	.
$X5=(1^4)$	.	.	.

N = 5

	$K=2$	$K=3$	$K=4$	$K=5$
X1 = (5)	.	.	.	1
$X2=(4,1)$	.	.	.	.
$X3=(3,2)$	.	1	.	.
$X4=(3,1^2)$	1(1)	.	1(1)	.
$X5=(2^2,1)$	.	1	.	.
$X6=(2,1^3)$	.	.	.	.
$X7=(1^5)$	.	1(1)	.	.

N = 6

	$K=2$	$K=3$	$K=4$	$K=5$	$K=6$
X1 = (6)	.	.	.	.	1
$X2=(5,1)$	.	.	.	.	.
$X3=(4,2)$	1(1)	.	1	.	.
$X4=(4,1^2)$	.	1	.	1	.
$X5=(3^2)$	.	1(1)	.	.	.
$X6=(3,2,1)$	.	1	1	.	.
$X7=(3,1^3)$	1(1)	1	.	.	.
$X8=(2^3)$	1(1)	.	1	.	.
$X9=(2^2,1^2)$	.	1(1)	.	.	.
$X10=(2,1^4)$	.	.	1	.	.
$X11=(1^6)$	.	.	.	.	.

N=7

	$K=2$	$K=3$	$K=4$	$K=5$	$K=6$	$K=7$
$X1=(7)$	.	.	.	.	.	1
$X2=(6,1)$	.	.	.	.	.	.
$X3=(5,2)$	.	1	.	1	.	.
$X4=(5,1^2)$	1(1)	.	1	.	1	.
$X5=(4,3)$	.	1	1	.	.	.
$X6=(4,2,1)$	1(1)	2(1)	1	1	.	.
$X7=(4,1^3)$	.	1	1	.	.	.
$X8=(3^2,1)$	1(1)	.	2	.	.	.
$X9=(3,2^2)$	.	2(1)	.	1	.	.
$X10=(3,2,1^2)$	1(1)	2(1)	2	.	.	.
$X11=(3,1^4)$	.	2(1)	.	1	.	.
$X12=(2^3,1)$	.	1	1	.	.	.
$X13=(2^2,1^3)$	1(1)	.	1	.	.	.
$X14=(2,1^5)$	.	.	.	.	.	.
$X15=(1^7)$	.	.	1(1)	.	.	.

N=8

	$K=2$	$K=3$	$K=4$	$K=5$	$K=6$	$K=7$	$K=8$
$X1=(8)$	.	.	.	.	.	.	1
$X2=(7,1)$	.	.	.	.	.	.	.
$X3=(6,2)$	1(1)	.	1	.	1	.	.
$X4=(6,1^2)$	.	1	.	1	.	1	.
$X5=(5,3)$	.	1(1)	1	1	.	.	.
$X6=(5,2,1)$	1(1)	3(1)	2	1	1	.	.
$X7=(5,1^3)$	1(1)	1	1	1	.	.	.
$X8=(4^2)$	1(1)	.	2	.	.	.	.
$X9=(4,3,1)$	1(1)	3(1)	2	2	.	.	.
$X10=(4,2^2)$	2(2)	1	4	.	1	.	.
$X11=(4,2,1^2)$	1(1)	5(3)	3	2	.	.	.
$X12=(4,1^4)$	1(1)	1(1)	2	.	1	.	.
$X13=(3^2,2)$	.	3(2)	1	1	.	.	.
$X14=(3^2,1^2)$	2(2)	2	3	1	.	.	.
$X15=(3,2^2,1)$	1(1)	3(2)	3	1	.	.	.
$X16=(3,2,1^3)$	1(1)	3(1)	3	1	.	.	.
$X17=(3,1^5)$	.	1(1)	1	.	.	.	.
$X18=(2^4)$	1(1)	.	2	.	.	.	.
$X19=(2^3,1^2)$	.	2(1)	.	1	.	.	.
$X20=(2^2,1^4)$	1(1)	1	1(1)	.	.	.	.
$X21=(2,1^6)$	.	.	.	1	.	.	.
$X22=(1^8)$	.	.	.	.	.	.	.

The tables give the numbers $m^{(\lambda)K}$ of linearly independent tensors of homogeneous invariants for every symmetry type (in brackets the numbers of linearly independent non-factorable homogeneous charge tensors)

show that such a situation occurs for the first time for tensors of rank 7, degree $K=3$, and symmetry type $X6$, $X9$, $X10$, $X11$.

In spite of this non-uniqueness, the number of independent invariants of degree K and given symmetry type is determined unambiguously. In the group theoretical language Propositions 10 and 11 can be formulated as follows:

Proposition 10′.

$$E_N^{\mathrm{red}} \circ P_N^{(2)} = Z_N \circ P_N^{(2)} \quad \textit{for} \quad N \geqq 3\,.$$

Proposition 11′.

$$E_N^{\mathrm{red}} \circ P_N^{(K)} = 0 \qquad \textit{for} \quad K > \frac{N+1}{2}\,,$$

$$E_N^{\mathrm{red}} \circ P_N^{(K)} = \frac{1}{N!} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline N \\ \hline \end{array} \circ Z_N \qquad \textit{for} \quad N \textit{ odd}, \; K = \frac{N+1}{2}\,.$$

The clearance-space between the assertions of Propositions 10 and 11 admits invariant charges of degree 3 for ranks $\geqq 5$ only. The complete list displayed below of decompositions of invariants with rank $\leqq 6$ into their reduced part and their factorable part shows that these invariant charges do appear for $N = 5, 6, \ldots$ (compare also Tables 1–8):

$N=1$: $\quad \mathscr{L}_1^{(1)} = \mathscr{P}_1\,.$

$N=2$: $\quad \mathscr{L}_{12}^{(2)} = \mathscr{P}_1 \cdot \mathscr{P}_2\,.$

$N=3$: $\quad \mathscr{L}_{123}^{(2)} = \mathscr{L}_{123}^{\mathrm{red}}\,,$

$\qquad \mathscr{L}_{123}^{(3)} = \frac{1}{2}\mathscr{P}_1\mathscr{P}_2\mathscr{P}_3\,.$

$N=4$: $\quad \mathscr{L}_{1234}^{(2)} = \mathscr{L}_{1234}^{\mathrm{red}}\,,$

$\qquad \mathscr{L}_{1234}^{(3)} = \frac{1}{4}Z_4(\mathscr{L}_{123}^{(2)}\mathscr{P}_4)\,,$

$\qquad \mathscr{L}_{1234}^{(4)} = \frac{1}{6}\mathscr{P}_1\mathscr{P}_2\mathscr{P}_3\mathscr{P}_4\,.$

$N=5$: $\quad \mathscr{L}_{12345}^{(2)} = \mathscr{L}_{12345}^{\mathrm{red}\,(2)}\,,$

$\qquad \mathscr{L}_{12345}^{(3)} = \mathscr{L}_{12345}^{\mathrm{red}\,(3)} + \frac{1}{12}Z_5(\mathscr{L}_{[12][34]}^{(2)}\mathscr{P}_5)\,,$

$\qquad \mathscr{L}_{12345}^{(4)} = \frac{1}{30}Z_5(2\mathscr{L}_{123}^{(2)}\mathscr{P}_4\mathscr{P}_5 + \mathscr{L}_{124}^{(2)}\mathscr{P}_3\mathscr{P}_5)\,,$

$\qquad \mathscr{L}_{12345}^{(5)} = \frac{1}{24}\mathscr{P}_1\mathscr{P}_2\mathscr{P}_3\mathscr{P}_4\mathscr{P}_5\,.$

$N=6$: $\quad \mathscr{L}_{123456}^{(2)} = \mathscr{L}_{123456}^{\mathrm{red}\,(2)}\,,$

$$\begin{aligned} \mathscr{L}_{123456}^{(3)} = {} & \mathscr{L}_{123456}^{\mathrm{red}\,(3)} + \tfrac{1}{40}Z_6([12\mathscr{L}_{12345}^{(2)} + 3\mathscr{L}_{12435}^{(2)} \\ & + 3\mathscr{L}_{13245}^{(2)} + 2\mathscr{L}_{13425}^{(2)} + 2\mathscr{L}_{14235}^{(2)} - 2\mathscr{L}_{14325}^{(2)}]\mathscr{P}_6)\,, \end{aligned}$$

$$\begin{aligned} \mathscr{L}_{123456}^{(4)} = {} & \tfrac{1}{6}Z_6(\mathscr{L}_{12345}^{\mathrm{red}\,(3)}\mathscr{P}_6) \\ & + \tfrac{1}{72}Z_6(2\mathscr{L}_{[12][34]}^{(2)}\mathscr{P}_5\mathscr{P}_6 + \mathscr{L}_{[12][45]}^{(2)}\mathscr{P}_3\mathscr{P}_6)\,, \end{aligned}$$

$$\mathscr{L}_{123456}^{(5)} = \tfrac{1}{72}Z_6(\mathscr{L}_{123}^{(2)}\mathscr{P}_4\mathscr{P}_5\mathscr{P}_6 + \mathscr{L}_{124}^{(2)}\mathscr{P}_3\mathscr{P}_5\mathscr{P}_6)\,,$$

$$\mathscr{L}_{123456}^{(6)} = \tfrac{1}{120}\mathscr{P}_1\mathscr{P}_2\mathscr{P}_3\mathscr{P}_4\mathscr{P}_5\mathscr{P}_6\,.$$

$[ij] \equiv ij - ji\,.$

In $\mathscr{L}^{(4)}_{123456}$ no terms of the kind $\mathscr{L}^{(2)}_{123}\mathscr{L}^{(2)}_{456}$ occur. The reason for this lies in the fact that every product of an arbitrary invariant tensor with $\mathscr{L}^{(2)}_{123}$ can be remodelled into a sum of terms each containing a factor $\mathscr{P}_i$. Explicitly

$$\mathscr{L}^{(2)}_{123}=\tfrac{1}{2}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \mathscr{R}^{(2)}_{123}=\tfrac{1}{2}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ (\mathscr{P}_1\mathscr{R}^t_{23}),$$

$$\begin{aligned}\mathscr{L}^{(2)}_{123}\cdot\mathscr{L}_{4\dots N}&=\oint d\sigma\,\mathscr{L}^{(2)}_{123}\mathscr{R}_{4\dots(N-1)}(\tau,\sigma)u_N(\tau,\sigma)\\ &=\tfrac{1}{2}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\ \mathscr{P}_1\oint d\sigma\,\mathscr{R}^t_{23}(\tau,\sigma)\mathscr{R}_{4\dots(N-1)}(\tau,\sigma)u_N(\tau,\sigma)\\ &=X(\mathscr{P}_1\mathscr{L}_{2\dots N})\ \text{with an appropriately defined } X\in\mathbf{O}_N.\end{aligned}$$

An example of this rule is the following identity:

$$\begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}_{\,A}\ (\mathscr{L}^{\text{red}}_{123}\cdot\mathscr{L}^{\text{red}}_{456})=3\cdot\ \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline\end{array}_{\,A}\ (\mathscr{P}_1\mathscr{P}_4\mathscr{L}^{\text{red}}_{2356}),$$

which illustrates that an assertion like Proposition 6 cannot hold true for the hitherto only provisionally defined reduced invariant tensors.

We turn now to the classification of invariant tensors according to their symmetry types and their behaviour under Lorentz transformations. (More detailed classification schemes are available and will be discussed towards the end of this section.)

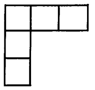
The rule for the evaluation of tensor products of invariants (compare Proposition 8) is homogeneous in the string variables $u_\mu(\tau,\sigma)$ as a relation among functionals of u, homogeneous as a polynomial in $\mathscr{R}^t$, and Lorentz covariant. Thus we are justified to investigate the factorability of the invariant tensors with different ranks, degrees of homogeneity and different behaviour under Lorentz transformations separately.

The inequivalent tensor representations of the Lorentz group are traceless tensors of definite symmetry type. According to Proposition 1 contraction of the invariant tensors with the metric tensor $g^{\mu_i\mu_j}$ does not introduce other dependences than those generated by symmetrization in μ_i and μ_j. Hence, a contracted invariant tensor is factorable if and only if the appropriately symmetrized tensor is factorable. Therefore as far as the factorability is concerned it suffices to decompose the uncontracted invariant tensors $\mathscr{L}_{\mu_1\dots\mu_N}$ into their homogeneous parts and to study their factorability.

From the theory of the regular representations of the symmetric groups $\mathbf{S}_N$ [5, 6] we know projectors $e^{(\lambda)}\in\mathbf{O}_N$ which project onto tensors of the symmetry type characterized by that Young frame which corresponds to the partition $(\lambda)=(\lambda_1,\dots,\lambda_r)$, $\lambda_1\geqq\lambda_2\geqq\dots\geqq\lambda_r>0$, $\sum\lambda_i=N$. These projectors define an orthogonal decomposition of the group identity:

$$\mathrm{id}_N=\sum_{(\lambda)}e^{(\lambda)}.$$

The projectors $e^{(\lambda)}$ form a basis of the center of the group algebra $\mathbf{O}_N$, i.e. they commute with all elements of $\mathbf{O}_N$. They decompose the group algebra into a direct sum of two-sided ideals $\mathbf{O}_N e^{(\lambda)} = e^{(\lambda)}\mathbf{O}_N$. Each one of these ideals is a representation space for the group $\mathbf{S}_N$ carrying equivalent irreducible representations $D^{(\lambda)}$ only, in fact as many linearly independent equivalent representations $D^{(\lambda)}$ as the dimension $f^{(\lambda)}$ of $D^{(\lambda)}$ amounts to.

When the projector $e^{(\lambda)}$ is applied to the invariant tensors, one is essentially left with the representation space $\mathbf{O}_N e^{(\lambda)} \circ Z_N$ which carries only $m^{(\lambda)} \leqq f^{(\lambda)}$ copies of $D^{(\lambda)}$. This implies that with $\mathbf{S}_N$ acting from the left there exist exactly $m^{(\lambda)}$ linearly independent invariant tensors of fixed symmetry type (λ). For instance, for $N=5$, $(\lambda)=(3,1^2)=$  , $f^{(\lambda)}=6$, $m^{(\lambda)}=2$, the two tensors in question are $\mathscr{X}^{(2)}_{12345}$ and $\mathscr{X}^{(4)}_{12345}$.

The degree of homogeneity helps to discriminate between the invariant tensors of given symmetry type. Thus we reconsider and apply the projector $e^{(\lambda)}$ to the homogeneous invariant tensors of degree K. Essentially this yields the representation space $\mathbf{O}_N e^{(\lambda)} \circ Z_N \circ P_N^{(K)}$. Let it contain $m^{(\lambda),K}$ times the irreducible representation $D^{(\lambda)}$. With the help of the tabulated characters $\zeta^{(\lambda)}$ [7] these multiplicities can be computed according to the formula

$$m^{(\lambda),K} = \zeta^{(\lambda)}\left(\frac{1}{N} Z_N \circ P_N^{(K)}\right), \qquad m^{(\lambda)} = \sum_{K=1}^{N} m^{(\lambda),K}, \qquad N = \sum_{i=1}^{r} \lambda_i .$$

There are $m^{(\lambda),K}$ linearly independent homogeneous invariant tensors of degree K and fixed symmetry type (λ). The situation for $N \leqq 8$ is illustrated by Tables 1 to 8. The figures give the number $m^{(\lambda),K}$ of linearly independent homogeneous invariant tensors for each pair $(\lambda), K$ and the figures in brackets, the number of the *non-factorable* ones. The latter figures come about by subtracting from $m^{(\lambda),K}$ the number of independent factorable invariant tensors compatible with the following two independent necessary conditions: A homogeneous invariant tensor of rank N, degree K and given symmetry type (λ) can be written as a sum of products of homogeneous invariants of rank $N_i < N$, degree K_i and symmetry type (λ_i) only if

i) $\sum N_i = N, \quad \sum K_i = K$,

ii) the Littlewood-Richardson rule [5] for the outer product of irreducible representations is satisfied.

We turn now to a second important composition law for the invariant tensors: the Poisson bracket operation. Taking Poisson brackets of integrals involving the string variables u_μ multiplied by non-periodic functions – which is the case for the tensors $\mathscr{R}_{\mu_1 \ldots \mu_N}(\tau, \sigma)$ – leads to ambiguities. These can be traced back to the appearance of the derivative of the periodic δ-function in the canonical Poisson bracket relation for the variables $u_\mu(\tau, \sigma)$. We define the Poisson brackets between the components of the various tensors $\mathscr{R}$ by interpreting $\delta_{2\pi}(\sigma)$ as the limit of the Gaussian regularization

$$\delta_{2\pi}(\sigma) = \frac{1}{2\pi} \lim_{\varepsilon \downarrow 0} \sum_{k=-\infty}^{+\infty} e^{-\varepsilon k^2} e^{ik\sigma}$$

in the sense of distributions. This guarantees the antisymmetry of

$$\{\mathscr{R}_{\mu_1 \ldots \mu_M}, \mathscr{R}_{\nu_1 \ldots \nu_N}\}.$$

Moreover, with this definition of the Poisson brackets the components of the tensors $\mathscr{R}_{\mu_1\ldots\mu_M}$ form a closed Lorentz covariant algebra whose structure constants coincide with the components of the metric tensor $g_{\mu\nu}$. However, this algebra does not satisfy the Jacobi identity, a fact which can be realized most easily by considering the appropriate Poisson brackets for the tensors $\mathscr{R}_{\mu_1\mu_2}$, $\mathscr{R}_{\nu_1\nu_2}$ and $\mathscr{R}_{\kappa_1\kappa_2\kappa_3}$.

It can be shown by cyclic symmetrization that also the invariant tensors $\mathscr{Z}_{\mu_1\ldots\mu_M}$ form a closed Lorentz covariant algebra with structure constants $g_{\mu\nu}$. This time the Jacobi identity is satisfied [4], a fact which reassures us that the invariant tensors are physically meaningful functionals of the string variables. The Poisson bracket of an invariant tensor of rank N and another invariant tensor of rank N' can be expressed by a linear combination of invariant tensors of rank $N+N'-2$. The integer $n=$(rank N minus two) defines a gradation. Explicitly

$$\{\mathscr{Z}_{\mu_1\ldots\mu_N}, \mathscr{P}_\mu\}=0,$$

$$\{\mathscr{Z}_{\mu_1\ldots\mu_N}, \mathscr{Z}_{\nu_1\ldots\nu_{N'}}\}=\sum_{i=1}^{N}\sum_{j=1}^{N'} 2g_{\mu_i\nu_j}$$

$$\times\left(\mathscr{Z}_{\mu_{i+1}\boxed{\begin{smallmatrix}\mu_{i+2}\ldots\mu_{i-1}\\ \nu_{j+1}\ldots\nu_{j-2}\end{smallmatrix}}\nu_{j-1}}-\mathscr{Z}_{\nu_{j+1}\boxed{\begin{smallmatrix}\mu_{i+1}\ldots\mu_{i-2}\\ \nu_{j+2}\ldots\nu_{j-1}\end{smallmatrix}}\mu_{i-1}}\right)\quad \text{for}\quad N,N'\geqq 2.$$

The invariant which appears on the right-hand side with the coefficient $2g_{\mu_i\nu_j}$ will be denoted by $\{\mathscr{Z}_{\mu_1\ldots\mu_N}, \mathscr{Z}_{\nu_1\ldots\nu_{N'}}\}/(2g_{\mu_i\nu_j})$,

$$\{\mathscr{Z}_{\mu\mu_1\ldots\mu_N}, \mathscr{Z}_{\nu\mu_{N+1}\ldots\mu_{N'}}\}/(2g_{\mu\nu})$$

$$=\left(1\boxed{\begin{matrix}2\ldots N\\ (N+1)\ldots(N'-1)\end{matrix}}N'-(N+1)\boxed{\begin{matrix}1\ldots(N-1)\\ (N+2)\ldots N'\end{matrix}}N\right)\mathscr{Z}_{\mu_1\ldots\mu_{N'}},\; N, N'-N\geqq 1.$$

The Poisson bracket operation defines a skew-symmetric product. It has all the properties of a derivation.

The degree of homogeneity leads to yet another gradation of $\mathfrak{h}_{\mathscr{P}}$:

Proposition 12. *The Poisson bracket of two homogeneous invariants of degree K and K' respectively yields a homogeneous invariant of degree $(K+K'-1)$. Hence, if $V_k(\mathfrak{h}_{\mathscr{P}})$ denotes the linear span of all tensor components $\mathscr{Z}^{(k+1)}_{\mu_1\ldots\mu_N}$, then the inclusion relation $\{V_k(\mathfrak{h}_{\mathscr{P}}), V_{k'}(\mathfrak{h}_{\mathscr{P}})\}\subset V_{k+k'}(\mathfrak{h}_{\mathscr{P}})$ is valid.*

Proof. $V_0(\mathfrak{h}_{\mathscr{P}})=$ linear span of $\mathscr{P}_\mu$, $\mu=0,\ldots,d-1$.

$$\begin{aligned}&\{\mathscr{Z}^{(K+1)}_{\mu_1\ldots\mu_N\mu}, \mathscr{Z}^{(K'+1)}_{\mu_{N+1}\ldots\mu_{N'}\nu}\}/(2g_{\mu\nu})=\oint d\sigma\oint d\sigma'\mathscr{R}^{(K)}_{\mu_1\ldots\mu_N}(\tau,\sigma)\,\{u_\mu(\tau,\sigma), u_\nu(\tau,\sigma')\}/(2g_{\mu\nu})\\ &\quad\times\mathscr{R}^{(K')}_{\mu_{N+1}\ldots\mu_{N'}}(\tau,\sigma')=-\oint d\sigma\oint d\sigma'\mathscr{R}^{(K)}_{\mu_1\ldots\mu_N}(\tau,\sigma)\,\delta'(\sigma-\sigma')\mathscr{R}^{(K')}_{\mu_{N+1}\ldots\mu_{N'}}(\tau,\sigma')\\ &\quad=\oint d\sigma(u_{\mu_1}(\tau,\sigma)\mathscr{R}^{(K)}_{\mu_2\ldots\mu_N}(\tau,\sigma)-\mathscr{R}^{(K)}_{\mu_1\ldots\mu_{N-1}}(\tau,\sigma)u_{\mu_N}(\tau,\sigma))\mathscr{R}^{(K')}_{\mu_{N+1}\ldots\mu_{N'}}(\tau,\sigma)\\ &\quad=X\oint d\sigma\mathscr{R}^{(K+K')}_{\mu_1\ldots\mu_{N'-1}}(\tau,\sigma)u_{\mu_{N'}}(\tau,\sigma)\quad\text{with}\quad X\in\mathbf{O}_{N'}\\ &\quad=X\mathscr{Z}^{(K+K'+1)}_{\mu_1\ldots\mu_{N'}}\in V_{K+K'}(\mathfrak{h}_{\mathscr{P}}).\quad\text{q.e.d.}\end{aligned}$$

If we write $\mathscr{Z}^{(K+1)}_{\mu_1\ldots\mu_N\mu}$ and $\mathscr{Z}^{(K'+1)}_{\mu_{N+1}\ldots\mu_{N'}\nu}$ as homogeneous polynomials in $\mathscr{R}^t$, their Poisson bracket is given by a sum of terms of the form

$$(\mathscr{R}^t)^K\{\mathscr{R}^t,\mathscr{R}^t\}\,(\mathscr{R}^t)^{K'}.$$

According to the preceding proposition, the sum must be homogeneous of degree $(K+K'+1)$. Thus all contributions from $\{\mathscr{R}^t,\mathscr{R}^t\}$ which are non-linear in $\mathscr{R}^t$ ultimately must add up to zero. This suggests the definition and use of a *modified Poisson bracket* $\{\,,\}^*$ of the tensors $\mathscr{R}^t_{\mu_1\ldots\mu_N}$ as the linear contribution of the canonical Poisson-brackets, i.e.

$$\{\mathscr{R}^t_{\mu_1\ldots\mu_N},\mathscr{P}_\nu\}^*=0\,,$$

$$\{\mathscr{R}^t_{\mu_1\ldots\mu_N},\mathscr{R}^t_{\mu_{N+1}\ldots\mu_{N'}}\}^*/(2g_{\mu_N\mu_{N+1}})=-\mathscr{R}^t_{\mu_1\ldots\mu_{N-1}\hat{\mu}_N\hat{\mu}_{N+1}\mu_{N+2}\ldots\mu_{N'}},$$

plus extension by linearity to the general term $\{\mathscr{R}^t_{\mu_1\ldots\mu_N},\mathscr{R}^t_{\mu_{N+1}\ldots\mu_{N'}}\}^*/(2g_{\mu_i\mu_j})$, $1\leqq i\leqq N$, $N+1\leqq j\leqq N'$ with the help of Proposition 7.

Finally, by extension as a derivation this modified Poisson bracket can be defined for all tensors $\mathscr{R}$. With this modified composition law the components of the tensors $\mathscr{R}(\mathscr{R}^t)$ form a closed Lorentz covariant algebra satisfying the Jacobi identity.

By definition the modified Poisson bracket and the canonical Poisson bracket give identical composition laws for invariant tensors. Either definition may be used for the evaluation of Poisson brackets of homogeneous invariants

$$\begin{aligned}&\{\mathscr{Z}^{(K)}_{1\ldots N_\mu},\mathscr{Z}^{(K')}_{(N+1)\ldots N'\nu}\}/(2g_{\mu\nu})\\&=\frac{1}{(K-1)!}\,\frac{1}{(K'-1)!}\sum_{\substack{0\leqq a_1<\ldots<a_K\leqq N\\ N\leqq b_1<\ldots<b_{K'}\leqq N'}}\mathscr{R}^t_{(a_1+1)\ldots a_2}\cdot\ldots\cdot\mathscr{R}^t_{(a_{K-1}+1)\ldots a_K}\\&\quad\times(-1)^{N'+1-a_1-b_{K'}}\mathscr{R}^t_{\boxed{\substack{(a_K+1)\ldots N\\ a_1\ldots 1}}\ \boxed{\substack{(N+1)\ldots b_1\\ N'\ldots(b_{K'}+1)}}}\cdot\mathscr{R}^t_{(b_1+1)\ldots b_2}\cdot\ldots\cdot\mathscr{R}^t_{(b_{K'-1}+1)\ldots b_{K'}}\cdot\end{aligned}$$

Already simple examples show that the Poisson bracket in general takes reduced invariant charges $\mathscr{Z}^{\text{red}}$ out of their linear span if the components $\mathscr{P}_\mu$ of the energy momentum vector are counted as reduced charges. In particular, under Poisson bracket operation the invariant charge $\mathscr{Z}^{(2)}_{\mu\nu\kappa}=\mathscr{Z}^{\text{red}}_{\mu\nu\kappa}$ acts like an infinitesimal generator of Lorentz transformations combined with a multiplication by some component of the energy momentum vector. The fact that the momenta $\mathscr{P}_\mu$ have vanishing Poisson brackets with the entire algebra $\mathfrak{h}_{\mathscr{P}}$ suggests that we treat the $\mathscr{P}_\mu$'s as scalars which enter the structure constants and not as elements of the algebra. Accordingly, from the outset we have indicated the dependence of the algebra on $\mathscr{P}:\mathfrak{h}_{\mathscr{P}}$. A similar situation arises in the $O(4)$-symmetric treatment of the hydrogen atom [8].

Linear relations involving $\mathscr{P}$-dependent coefficients will be called *$\mathscr{P}$-linear*. Reduced invariants $\mathscr{Z}^{\text{red}\,1},\ldots,\mathscr{Z}^{\text{red}\,r}$ are said to be $\mathscr{P}$-linearly independent if the equation

$$\alpha_1\mathscr{Z}^{\text{red}\,1}+\ldots+\alpha_r\mathscr{Z}^{\text{red}\,r}=\text{sum of products of invariants}$$

$$\text{other than momenta with }\mathscr{P}\text{-dependent coefficients}$$

$$-\alpha_1,\ldots,\alpha_r=\mathscr{P}\text{-dependent “scalars”}-\text{implies: }\alpha_1=\ldots=\alpha_r=0\,.$$

The examples for non-linear relations in the beginning of this section are in fact examples of $\mathscr{P}$-linear dependences.

We denote the $\mathscr{P}$-linear span of a complete set of $\mathscr{P}$-linearly independent ($\mathscr{P}$-) reduced invariants $\mathscr{Z}^{\mathscr{P}\text{-red}}$ by the symbol $\mathfrak{h}^{\mathscr{P}\text{-red}}$. The homogeneous invariants $\mathscr{Z}^{\mathrm{red}(K)}_{\mu_1\dots\mu_{2K-1}}$ for $K>2$ do not belong to $\mathfrak{h}^{\mathscr{P}\text{-red}}$, provided that $\mathscr{P}_\mu\mathscr{P}^\mu=m^2>0$. This is true since there exist identities, as for example

$$\mathscr{P}_\mu\mathscr{P}^\mu\mathscr{Z}^{\mathrm{red}(3)}_{\mu_1\dots\mu_5}=\tfrac{1}{48}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array}\ (\mathscr{P}_{\mu_1}\mathscr{Z}^{(2)}_{\mu_2\mu_3\mu}\mathscr{Z}^{(2)}_{\mu_4\mu_5\nu})g^{\mu\nu}\,.$$

Hence, we may assume for the $\mathscr{P}$-reduced invariants of rank $N>3$ that the degree of homogeneity K satisfies the bound $K\leqq N/2$, a stronger bound than the one established by Proposition 11.

If we consider the momenta $\mathscr{P}_\mu$ as scalars and not as elements of the algebra – as we shall do throughout the following – only a single gradation of $\mathfrak{h}_{\mathscr{P}}$ with respect to the degree $\ell=$(rank N minus degree of homogeneity K minus one) remains:

$$\mathfrak{h}_{\mathscr{P}}=\bigoplus_{\ell=0}^{\infty}V^{(\ell)}(\mathfrak{h}_{\mathscr{P}})\,,$$
$$\{V^{(\ell)}(\mathfrak{h}_{\mathscr{P}}),V^{(\ell')}(\mathfrak{h}_{\mathscr{P}})\}\subset V^{(\ell+\ell')}(\mathfrak{h}_{\mathscr{P}})\,,$$

where $V^{(\ell)}(\mathfrak{h}_{\mathscr{P}})$ denotes the $\mathscr{P}$-linear span of all invariant tensors $\mathscr{Z}^{(K)}_{\mu_1\dots\mu_N}$ with $N>2$, $K\leqq N-1$ and degree ℓ. $V^{(0)}(\mathfrak{h}_{\mathscr{P}})=\mathscr{P}$-linear span of $\mathscr{Z}^{(2)}_{\mu\nu\kappa}$ is a Lie algebra. It is isomorphic to $\mathrm{so}(d-1)$, the Lie algebra of the little group of the Lorentz group, for $\mathscr{P}_\mu\mathscr{P}^\mu>0$.

All spaces $V^{(\ell)}(\mathfrak{h}_{\mathscr{P}})$, and in particular the linear spans of the tensor components $\mathscr{Z}^{\mathrm{red}(K)}_{\mu_1\dots\mu_N}$ are representation spaces of $V^{(0)}(\mathfrak{h}_{\mathscr{P}})$. Hence the homogeneous invariant tensors of given symmetry type may be decomposed further according to inequivalent irreducible representations of $V^{(0)}(\mathfrak{h}_{\mathscr{P}})$. This provides an additional, though space time dimension dependent criterion, for singling out reduced invariants. (This is the point where it will be advantageous to pass to the momentum rest frame. See Sect. IV below.) There are no other simple Lie subalgebras with non-trivial representation spaces contained in some single $V^{(\ell)}(\mathfrak{h}_{\mathscr{P}})$.

III. Subalgebras and Ideals of $\mathfrak{h}_{\mathscr{P}}=\mathfrak{h}^{\pm}_{\mathscr{P}}$

The set of all invariants from $\mathfrak{h}_{\mathscr{P}}(=\mathfrak{h}^{\pm}_{\mathscr{P}})$, the degrees and ranks of which satisfy the following inequality

$$(K-1)\leqq\beta'(N-2)+\beta_0,\quad(0<\beta'\leqq1,\ \beta_0\leqq0)\,,$$

forms an infinite dimensional Lie subalgebra. The relations

$$\ell=N-K-1\geqq\beta$$

define infinite dimensional ideals.

The set of all factorable invariants is an ideal. Another infinite dimensional ideal $\mathfrak{i}$ of $\mathfrak{h}_{\mathscr{P}}$ is given by the $\mathscr{P}$-linear span of all invariants which *do not* contain dominant monomials. Here a monomial is called dominant if it is of the form $X(\mathscr{P}_1 \dots \mathscr{P}_{K-1} \mathscr{R}^t_{K\dots N}) \neq 0$, $X \in \mathbf{O}_N$, $1 < K < N$. In particular, $\mathfrak{i}$ comprises all products of invariants other than momenta. Moreover, $\mathfrak{i}$ contains all those homogeneous invariants of degree K which belong to a symmetry type characterized by a Young frame with less than $(K-1)$ columns.

The fact that $\mathfrak{i}$ is an ideal of $\mathfrak{h}_{\mathscr{P}}$ immediately implies the first part of

Proposition 13. *The quotient* $\hat{\mathfrak{h}} = \mathfrak{h}_{\mathscr{P}}/\mathfrak{i}$ *is a Lie algebra with a gradation. Its structure constants are determined by the modified Poisson brackets for the dominant monomials.*

In order to prove the second part of the proposition, it suffices to remark that two invariants whose dominant monomials coincide differ by an element of $\mathfrak{i}$, at most.

The dominant polynomials $X \circ Z_N(\mathscr{P}_1 \dots \mathscr{P}_{K-1} \mathscr{R}^t_{K\dots N})$ (which by themselves are not invariant!) are suited to serve as representatives for the equivalence classes in $\mathfrak{h}_{\mathscr{P}}/\mathfrak{i}$.

In view of the ultimate goal to pass to the quantum theory by constructing positive energy representations for the algebra $\mathfrak{h}_{\mathscr{P}}$, we are particularly interested in abelian Lie subalgebras with dimension as large as possible. We have found several infinite dimensional abelian Lie subalgebras of $\mathfrak{h}_{\mathscr{P}}$, whose elements in addition are in involution with the Casimir operators and the elements of the usual Cartan algebra of the Poincaré group (compare [4]). Here we discuss only one of them.

Proposition 14. *All Lorentz-scalar invariants of the form* $g^{\mu_1\mu_2} \dots g^{\mu_{N-1}\mu_N} \mathscr{Z}^{(K)}_{\mu_1 \dots \mu_N}$, *N and K being even integers, are in involution, i.e. commute among each other with respect to the Poisson bracket operation.*

Proof. For the sake of transparency we use Euclidean notation. The parity of the invariants $\mathscr{Z}^{(2k)}_{\lambda\lambda\dots\nu\nu}$ is even. Hence, when we take the Poisson bracket of two such invariants, each pair of indices contributes the same amount:

$$\{\mathscr{Z}^{(2k)}_{\underbrace{\lambda\lambda\dots\nu\nu}_{2m}}, \mathscr{Z}^{(2\ell)}_{\underbrace{\varrho\varrho\dots\tau\tau}_{2n}}\} \doteq 4mn\, X^{(k,\ell)}_{m,n}$$

$$= 8mn \oint d\sigma (u_\lambda(\tau,\sigma) \mathscr{R}^{(2k-1)}_{\lambda\dots\nu\nu\varrho}(\tau,\sigma) - \mathscr{R}^{(2k-1)}_{\lambda\lambda\dots\nu\nu}(\tau,\sigma) u_\varrho(\tau,\sigma)) \mathscr{R}^{(2\ell-1)}_{\varrho\sigma\sigma\dots\tau\tau}(\tau,\sigma).$$

From the previously established identities

$$\sum_{N-K=\text{even}} P^{(K)}_N = \tfrac{1}{2}(\mathrm{id}_N + I_N), \qquad \sum_{N-K=\text{odd}} P^{(K)}_N = \tfrac{1}{2}(\mathrm{id}_N - I_N),$$

we infer:

$$\mathscr{R}^{(2k-1)}_{\lambda\lambda\dots\nu\nu} = 0, \qquad \mathscr{R}^{(2k-1)}_{\lambda\mu\mu\dots\nu\nu\varrho} = -\mathscr{R}^{(2k-1)}_{\varrho\nu\nu\dots\mu\mu\lambda}, \qquad \mathscr{R}^{(2\ell-1)}_{\varrho\sigma\sigma\dots\tau\tau} = \mathscr{R}^{(2\ell-1)}_{\tau\tau\dots\sigma\sigma\varrho}.$$

We thus obtain

$$X^{(k,\ell)}_{m,n} = 2 \oint d\sigma\, \mathscr{R}^{(2k-1)}_{\lambda\mu\mu\dots\nu\nu\varrho} \cdot \tfrac{1}{2}(\mathscr{R}^{(2\ell-1)}_{\varrho\sigma\sigma\dots\tau\tau} u_\lambda - u_\varrho \mathscr{R}^{(2\ell-1)}_{\sigma\sigma\dots\tau\tau\lambda})$$

$$= -\oint d\sigma\, \mathscr{R}^{(2k-1)}_{\underbrace{\lambda\mu\mu\dots\nu\nu\varrho}_{2(m-1)}} \cdot \partial_\sigma \mathscr{R}^{(2\ell-1)}_{\underbrace{\varrho\sigma\sigma\dots\tau\tau\lambda}_{2n}} \doteq Y^{(k,\ell)}_{m-1,n}.$$

Both, $X^{(k,\ell)}_{m,n}$ and $Y^{(k,\ell)}_{m,n}$ are antisymmetric under the replacement $m,k\leftrightarrow n,\ell$. Thus we deduce

$$X^{(k,\ell)}_{m,n}=Y^{(k,\ell)}_{m-1,n}=-Y^{(\ell,k)}_{n,m-1}=-X^{(\ell,k)}_{n+1,m-1}=X^{(k,\ell)}_{m-1,n+1}\,.$$

Applying this relation repeatedly, we arrive at the final conclusion

$$X^{(k,\ell)}_{m,n}=X^{(k,\ell)}_{k,n+m-k}=\frac{1}{4k(n+m-k)}\{\mathscr{Z}^{(2k)}_{\underbrace{\mu\mu\ldots\nu\nu}_{2k}},\mathscr{Z}^{(2\ell)}_{\underbrace{\varrho\varrho\ldots\tau\tau}_{2(n+m-k)}}\}=0\,,$$

because

$$\mathscr{Z}^{(2k)}_{\underbrace{\mu\mu\ldots\nu\nu}_{2k}}=\frac{1}{(2k-1)!}(\mathscr{P}^2)^k\,.$$

Explicit computation shows that for $N\leqq 10$ and arbitrary space time dimension there are no other independent Lorentz-scalar invariants in involution than those given by Proposition 14. We suppose that this is true also for general values of N. The number of independent invariants $\mathscr{Z}^{(2k)}_{\underbrace{\lambda\lambda\ldots\nu\nu}_{2m}}$ in involution increases at least as $m/2$ (since $g^{\mu_1\mu_2}\ldots g^{\mu_{2m-1}\mu_{2m}}\mathscr{Z}^{(2k)}_{\mu_1\ldots\mu_{2m}}$ contains for $k\leqq\frac{m+1}{2}$ non-vanishing terms

$$g^{\mu_1\mu_2}\ldots g^{\mu_{2m-1}\mu_{2m}}\mathscr{P}_{\mu_1}\mathscr{R}^t_{\mu_2\mu_3}\mathscr{R}^t_{\mu_4\mu_5}\ldots\mathscr{R}^t_{\mu_{4k-4}\ldots\mu_{2m-1}}\mathscr{P}_{\mu_{2m}}\,,$$

which cannot be decomposed into scalar factors. On the other hand, the number of independent invariants $\mathscr{Z}^{(2k)}_{\underbrace{\lambda\lambda\ldots\nu\nu}_{2m}}$ with $m\geqq 2$ is less than $3m/4$. This bound is a consequence of Proposition 11.

IV. The Invariant Charges in the Momentum Rest Frame

We are interested mainly in the positive energy representations with non-vanishing mass of the algebra of string invariants, i.e. we assume that the energy-momentum vector $\mathscr{P}_\mu$ satisfies the inequality

$$\mathscr{P}_\mu\mathscr{P}^\mu=m^2>0\,.$$

Since the quantities $\mathscr{P}_\mu$ have vanishing Poisson-brackets with all invariant charges of $\mathfrak{h}_{\mathscr{P}}$, we may without further loss of generality perform the entire analysis in the momentum rest frame

$$\mathscr{P}_\mu=m\delta_{\mu 0}\,.$$

In this frame of reference the invariant charges are arranged according to irreducible representations of the little group $SO(d-1)$. The infinitesimal generators of the little group are

$$M_{\mu\nu}=-\frac{1}{2m}(\mathscr{Z}^{(2)+}_{0\mu\nu}-\mathscr{Z}^{(2)-}_{0\mu\nu})\,,\qquad 0\neq\mu\neq\nu\neq 0\,.$$

Thus on the algebra $\mathfrak{h}^{+}_{(m,0,\ldots,0)}$, they can be identified with the invariant charges $-\dfrac{1}{2m}\mathscr{Z}^{(2)+}_{0\mu\nu}$, $0\neq\mu\neq\nu\neq 0$. The latter ones form a basis of the stratum of degree zero: $V^{(0)}(\mathfrak{h}_{(m,0,\ldots,0)})$. The previously discussed irreducible Lorentz multiplets of invariant charges in general decompose into several irreducible representations of the little group. It may happen that certain Lorentz multiplets of invariant charges which were not polynomially factorable, now that we admit division by m are recognized to consist entirely or partly of products of other invariants. For instance:

$$\mathscr{Z}^{\mathrm{red}}_{\mu\nu\kappa\lambda}=\frac{1}{m^2}\left(\mathscr{Z}^{\mathrm{red}}_{0\mu\nu}\mathscr{Z}^{\mathrm{red}}_{0\kappa\lambda}+\mathscr{Z}^{\mathrm{red}}_{0\nu\kappa}\mathscr{Z}^{\mathrm{red}}_{0\lambda\mu}\right)\quad\text{if}\quad \lambda,\mu,\nu,\kappa\neq 0\,.$$

The use of the momentum rest frame helps to gain control over the number and the structure of the independent invariant charges. We shall succeed in obtaining a (minimal and complete) algebraic basis for the set of all conserved charges under consideration.

In the course of the construction those components of the tensors $\mathscr{R}^{t}_{\mu_1\ldots\mu_{K+1}}$, $K>0$, for which the indices in the extremal positions – μ_1 and μ_{K+1} – are different from zero ("space-like"), play a particularly important role.

We start by choosing among the tensor components $\mathscr{R}^{t}_{\mu_1\ldots\mu_{K+1}}$ with space-like indices at the extremal positions a maximal subset $\mathscr{R}^{t}_{K+1,i}=\mathscr{R}^{t}_{\mu_1^i\ldots\mu_{K+1}^i}$, $i=1,2,\ldots$ of linearly independent ones. To each such tensor-component $\mathscr{R}^{t}_{K+1,i}$ we assign a dominant *standard* invariant, namely

$$(K-1)!\,\mathscr{Z}^{(K)}_{\underbrace{0\ldots0}_{K-1}\mu_1^i\ldots\mu_{K+1}^i}=\mathscr{Z}_{K+1,i}$$

unless $K=1$ or $\mu_2=\ldots=\mu_K=0$. In these special cases the assigned dominant standard invariant is

$$K^{-1}\mathscr{Z}^{(2)}_{0\mu_1^i\ldots\mu_{K+1}^i}=\mathscr{Z}_{K+1,i}\,.$$

In any case, the dominant polynomial of the standard invariant consists of one monomial only. In the first case the monomial in question is

$$m^{K-1}\mathscr{R}^{t}_{\mu_1^i\ldots\mu_{K+1}^i}\,,$$

in the second case

$$m\mathscr{R}^{t}_{\mu_1^i0\ldots0\mu_{K+1}^i}\,.$$

The standard invariants are algebraically independent of each other. The standard invariant $\mathscr{Z}_{K+1,i}$ is contained in the stratum $V^{(K-1)}(\mathfrak{h}_{(m,0,\ldots,0)})$ of degree $\ell=K-1$.

Proposition 15. *The identity $\mathscr{R}^{t}_{\mu_1\ldots\mu_{K+1}}\equiv 0$ for a certain specification of the indices $\mu_1,\ldots,\mu_{K+1}\in\{0,1,\ldots,d-1\}$, $K>0$, $d>2$ – $\mathscr{R}^{t}_{\mu_1\ldots\mu_{K+1}}$ viewed as a function of τ,σ and a functional of $u_\mu(\tau,\sigma)$, where $u_\mu(\tau,\sigma)$ is subject to the constraints $u^2(\tau,\sigma)=0$ and $\oint d\sigma\, u_\mu(\tau,\sigma)=m\delta_{\mu0}$ – implies*

$$\mu_1=\mu_2=\ldots=\mu_{K+1}\qquad\qquad \textit{for even values of } K\,,$$

$$\mu_1=\mu_{K+1},\mu_2=\mu_K,\ldots,\mu_{\frac{K+1}{2}}=\mu_{\frac{K+1}{2}+1}\ \ \textit{for odd values of } K\,.$$

The proof is achieved by induction on K using the differential equation for the spatial dependence of $\mathscr{R}^t_{\mu_1\dots\mu_{K+1}}$,

$$\partial_\sigma \mathscr{R}^t_{\mu_1\dots\mu_{K+1}}(\tau,\sigma) = u_{\mu_1}(\tau,\sigma)\mathscr{R}^t_{\mu_2\dots\mu_{K+1}}(\tau,\sigma) - \mathscr{R}^t_{\mu_1\dots\mu_K}(\tau,\sigma)u_{\mu_{K+1}}(\tau,\sigma),$$

and arguing as in the proof of Proposition 1.

Proposition 16. *A standard invariant $\mathscr{Z}$ vanishes identically as a functional of $u_\mu(\tau,\sigma)$ if and only if its dominant monomial $m^x\mathscr{R}^t_{\mu_1\dots\mu_{K+1}}$ vanishes identically.*

Proof. All we have to verify is the claim that the identical vanishing of $\mathscr{R}^t_{\mu_1\dots\mu_{K+1}}$ implies the vanishing of $\mathscr{Z}$. We treat the two cases 1) K is even and 2) K is odd in turn:

1) K is even: According to the previous proposition $\mathscr{R}^t_{\mu_1\dots\mu_{K+1}} \equiv 0$ implies $\mu_1 = \dots = \mu_{K+1} = \mu$. All monomials of $\mathscr{Z}$ vanish since every partition of the cyclic sequence $\underbrace{0\dots0}_{K-1}\ \underbrace{\mu\dots\mu}_{K+1}$ involves at least one factor $\mathscr{R}^t$ of rank two or more with no other but identical indices.

2) K is odd: The inversion $0\dots0\mu_1\mu_2\dots\mu_2\mu_1 \to \mu_1\mu_2\dots\mu_2\mu_1 0\dots0$ takes the original sequence of tensor indices into a cyclically equivalent one. That means

$$\mathscr{Z} = \mathscr{Z}^{(K)}_{0\dots,0\mu_1\mu_2\dots\mu_2\mu_1} = -\mathscr{Z}^{(K)}_{\mu_1\mu_2\dots\mu_2\mu_1 0\dots0} = -\mathscr{Z}.$$

Hence $\mathscr{Z} = 0$.

This last discussion also covers the special cases of the standard invariants.

Proposition 17. *If division by m is admitted, all invariant charges are polynomials in the standard invariants.*

Indication of the Proof. We want to express a given invariant charge $\mathscr{Z}^{(K)}_{\mu_1\dots\mu_N}$ in the momentum rest frame in terms of standard charges. We consider $\mathscr{Z}^{(K)}_{\mu_1\dots\mu_N}$ as a polynomial in the tensor components $\mathscr{R}^t$ of rank two or more. From this polynomial we select the "leading" part of lowest degree q. We notice that we can rewrite this part as a polynomial of the same degree q involving only factors $\mathscr{R}^t$ with space-like indices at the extremal positions. These factors are expressed as linear combinations of the basis elements $\mathscr{R}^t_{K+1,i}$, which in turn are replaced by the corresponding standard invariants $\mathscr{Z}_{K+1,i}$. This replacement involves the appearance of additional (negative) powers of m and of additional monomials in $\mathscr{R}^t$ of higher degree. We subtract the homogeneous polynomial in the standard invariants of degree q which we have constructed just now from the original invariant charge. The remainder is considered again as a polynomial in the tensor components $\mathscr{R}^t$ of rank two or more. Its leading part has degree $q+1$. Taking into account the cancellations of the additional monomials in $\mathscr{R}^t$ mentioned above among themselves as well as against some of the original non-leading terms, the leading part of the remainder can be expressed again as a polynomial (of degree $q+1$) in those $\mathscr{R}^t$'s which have space-like indices at the extremal positions. The previous procedure is repeated again and again at worst until we obtain a remainder with a leading part of degree $[N/2]$. At this stage the replacement of the leading part by a polynomial in the standard invariants leaves no longer a remainder because there are no (standard) invariants with less than two space-like indices.

Corollary. *The standard invariants* $\mathscr{Z}_{K+1,i}$ *provide a (minimal and complete) algebraic basis for* $\mathfrak{h}_{(m,0,\ldots,0)}$. *By evaluating these same invariants in an arbitrary frame of reference, they provide an algebraic basis for* $\mathfrak{h}_{\mathscr{P}}$.

Thus the problem of counting the number n_ℓ of independent invariants in the stratum $V^{(\ell)}(\mathfrak{h}_{\mathscr{P}})$ of degree ℓ (algebraically) independent in particular of the invariant charges in the strata $V^{(\ell')}(\mathfrak{h}_{\mathscr{P}})$, $\ell'<\ell$, amounts to counting the number $q(\ell+2)$ of linearly independent tensor components $\mathscr{R}^t_{\mu_1\ldots\mu_{\ell+2}}$ with space-like indices at the extremal positions.

It is possible to choose, recursively in $N>2$, maximal sets of linearly independent components of the truncated tensors of rank N such that the first $q(N)$ elements are a basis of the space spanned by components $\mathscr{R}^t_{\mu_1\ldots\mu_N}$ with space-like indices at the extremal positions while the others are of the form $\mathscr{R}^t_{0\mu_1^i\ldots\mu_{N-1}^i}$, where $\mathscr{R}^t_{\mu_1\ldots\mu_{N-1}}$ are the previously chosen basis components belonging to rank $N-1$. Now the total number of linearly independent components of the tensor $\mathscr{R}^t_{\mu_1\ldots\mu_N}$ is

$$n(d,N)=\frac{1}{N}\sum_{D|N}\mu(D)\,d^{N/D}.$$

Here the sum extends over all divisors D of N. The symbol $\mu(D)$ denotes the Möbius function:

$$\mu(D)=\begin{cases}1 & \text{if } D=1,\\ (-1)^p & \text{if } D \text{ can be decomposed into exactly } p \text{ different prime factors},\\ 0 & \text{if some prime factors of } D \text{ are equal}.\end{cases}$$

Hence $q(N)=n(d,N)-n(d,N-1)$ and the number n_ℓ of independent charges in the ℓ'th stratum equals $n(d,\ell+2)-n(d,\ell+1)$ behaving asymptotically like $\dfrac{d-1}{\ell+1}\,d^{\ell+1}$ as $\ell\to\infty$.

Next, we want to analyse the structure of the Poisson algebra of the invariant charges in the momentum rest frame. Of particular interest is the question whether $\mathfrak{h}_{(m,0,\ldots,0)}$ is a finitely generated algebra. If so, the relevant information would be already contained in a few elements of $\mathfrak{h}_{(m,0,\ldots,0)}$ and the transition to the quantum theory would be achieved by constructing the corresponding few charge operators [9].

The gradation of $\mathfrak{h}_{\mathscr{P}}$ is one-sided: the degree takes non-negative values only. The dimension of each stratum $V^{(\ell)}(\mathfrak{h}_{\mathscr{P}})$ corresponding to a fixed finite degree ℓ is finite. The Poisson bracket operation never decreases the degree ℓ. Hence, if a finite-dimensional subset of $\mathfrak{h}_{\mathscr{P}}$ were to generate an algebraic basis of $\mathfrak{h}_{\mathscr{P}}$ at least it would have to contain a basis of $V^{(0)}(\mathfrak{h}_{\mathscr{P}})$ and of those elements of $V^{(1)}(\mathfrak{h}_{\mathscr{P}})$ which are independent of the elements of $V^{(0)}(\mathfrak{h}_{\mathscr{P}})$. In fact, it would be appealing if $\mathfrak{h}_{\mathscr{P}}$ would turn out to be a *minimal* extension of the Poincaré algebra in the sense that the closure under forming multiple Poisson brackets of the Lorentz group representation spaces $V^{(0)}(\mathfrak{h}_{\mathscr{P}})$ and $V^{(1)}(\mathfrak{h}_{\mathscr{P}})$ (together with the energy momentum vector $\mathscr{P}$) would supply an algebraic basis of the entire set $\mathfrak{h}_{\mathscr{P}}$.

However, reality looks different. In every stratum $V^{(\ell)}(\mathfrak{h}_{(m,0,\ldots,0)})$ of odd degree ℓ there exists at least one "exceptional" element: the linear combination of the standard invariants

$$L_{0\ell} \doteqdot -[(\ell+1)(d-1)]^{-1} g^{\mu\nu} \mathscr{Z}^{(2)}_{0\mu\underbrace{0\ldots0}_{\ell}\nu},$$

which – even if suitably modified by sums of products of invariant charges of lower degrees – cannot be produced from other invariant charges by the Poisson bracket operation.

To prove this statement, it suffices to show that the dominant part of $L_{0\ell}$

$$+\frac{m}{d-1}(\mathscr{R}^t_{1\underbrace{0\ldots0}_{\ell}1}+\ldots+\mathscr{R}^t_{(d-1)\underbrace{0\ldots0}_{\ell}(d-1)}), \qquad \ell=\text{odd},$$

cannot be produced by the Poisson bracket operation.

Suppose on the contrary that it could be produced. In this case it could arise only in the form of a sum of Poisson brackets of the dominant parts of some standard invariants of the special type

$$\text{const}\cdot\{m\mathscr{R}^t_{i_1\underbrace{0\ldots0}_{K_1-1}j_1}, m\mathscr{R}^t_{i_2\underbrace{0\ldots0}_{K_2-1}j_2}\}^*, \qquad K_1-1+K_2-1=\ell; \qquad i_1,j_1,i_2,j_2\neq 0,$$

and, in addition, of Poisson brackets of the dominant parts of standard invariants one of which carrying more than two space-like indices. Without loss of generality we may assume that K_1-1 is even. Then parity implies: $i_1 \neq j_1$. Thus the only remaining chance to produce the dominant part $-\frac{m}{d-1}g^{\mu\nu}\mathscr{R}^t_{\mu\underbrace{0\ldots0}\nu}$ is to set $\{i_2,j_2\}=\{i_1,j_1\}$, say $i_2=j_1$, $i_1=j_2$. However, in this case the individual Poisson brackets yield dominant parts which transform non-trivially under $\mathrm{SO}(d-1)$:

$$-2\cdot\text{const}\, m^2(\mathscr{R}^t_{i_1 0\ldots0 i_1}-\mathscr{R}^t_{j_1 0\ldots0 j_1})$$

and, in addition, dominant parts of standard invariants with more than two space-like indices. Thus, none of the dominant parts is of the desired form.

Although our experience suggests that all linear combinations of the standard invariants with vanishing components in the direction of the exceptional elements – when suitably modified by sums of products of invariant charges carrying lower degrees – do in fact appear as Poisson brackets, the very existence of those exceptional linear combinations seems to rule out any minimality property of the algebra $\mathfrak{h}_{\mathscr{P}}$. However, as we shall show below, the algebra $\mathfrak{h}_{\mathscr{P}}$ is not considerably reduced by passing to the algebra generated by those elements of $V^{(1)}(\mathfrak{h}_{\mathscr{P}})$ only, which have no components in the exceptional direction. The Poisson bracket operation reintroduces the exceptional elements in the form of "nonlinearities" containing their Poisson brackets with other elements.

Let us examine is some detail the situation in three dimensional space-time. Here the little group is the one parameter group SO(2). We identify its infinitesimal generator of rotations in the 1, 2-plane with the element

$$Q=\frac{-1}{2m}\mathscr{L}^{(2)}_{012}=-\tfrac{1}{2}\mathscr{R}^{t}_{12}$$

of the one-dimensional stratum $V^{(0)}(\mathfrak{h}_{(m,0,0)})$. The six-dimensional stratum $V^{(1)}(\mathfrak{h}_{(m,0,0)})$ is spanned by the elements

$$Q^2=\tfrac{1}{8}Q^{(4)00}, \quad L_0=L_{01}=\tfrac{1}{4}(Q^{(4)11}+Q^{(4)22}),$$
$$L_{\pm 1}=\tfrac{1}{2}(Q^{(4)01}\mp iQ^{(4)02}), \quad L_{\pm 2}=\tfrac{1}{4}(Q^{(4)11}-Q^{(4)22}\mp i\cdot 2Q^{(4)12}),$$

where $Q^{(4)\mu\nu}$ is defined as in [1] by

$$Q^{(4)\mu\nu}=\varepsilon^{\mu\alpha\beta}\varepsilon^{\nu AB}\{\tfrac{2}{3}(\mathscr{P}_\alpha\mathscr{R}^t_{\beta AB}+\mathscr{P}_A\mathscr{R}^t_{B\alpha\beta})+\tfrac{1}{2}\mathscr{R}^t_{\alpha\beta}\mathscr{R}^t_{AB}\}.$$

The six elements are arranged according to their behaviour under rotations in the 1, 2-plane

$$\{Q, L_s\}=isL_s.$$

Arbitrary multiple Poisson brackets of L_s, $s=0, \pm 1, \pm 2$, will be denoted by

$$\mathbb{L}_{(s)}=\{\ldots\{L_{s_1}, L_{s_2}\},\ldots, L_{s_n}\}\doteqdot\{L_{s_1}, L_{s_2},\ldots, L_{s_n}\}.$$

Their "spin" is $s(\mathbb{L}_{(s)})=\sum_1^n s_i$.

There are 10 standard invariants in the stratum $V^{(2)}(\mathfrak{h}_{(m,0,0)})$ and there are also 10 independent Poisson brackets $\mathbb{L}_{(s)}$ which may replace the standard invariants in question. This agrees with the "growth" of a free Lie algebra generated by five elements taking the place of L_0, $L_{\pm 1}$, $L_{\pm 2}$. However, the situation in the stratum $V^{(3)}(\mathfrak{h}_{(m,0,0)})$ is quite different: there are 30 standard invariants but only 29 independent Poisson brackets, whereas in the free Lie algebra generated by five elements the number of independent triple Poisson brackets would amount to 40. In fact beyond the Jacobi identities the following relations are valid:

$$|s|=0: N^{(3)}_0=4\mathbb{L}_{(1,-1,0)}-\mathbb{L}_{(2,-2,0)}=0,$$
$$|s|=1: N^{(3)}_1=12\mathbb{L}_{(1,0,0)}+6\mathbb{L}_{(1,-1,1)}-(\mathbb{L}_{(1,2,-2)}+\mathbb{L}_{(1,-2,2)})$$
$$-2\mathbb{L}_{(2,-2,1)}-3\mathbb{L}_{(0,2,-1)}$$
$$=24\{iQ\cdot\mathbb{L}_{(-1,2)}-3L_{-1}\cdot L_2+24Q^2\cdot L_1\},$$
$$|s|=2: N^{(3)}_2=6\mathbb{L}_{(0,1,1)}-(\mathbb{L}_{(2,1,-1)}+\mathbb{L}_{(2,-1,1)})-2\mathbb{L}_{(1,-1,2)}$$
$$=24\{iQ\cdot\mathbb{L}_{(0,2)}-2L_1^2-8Q^2\cdot L_2\},$$
$$|s|=3: N^{(3)}_3=3\mathbb{L}_{(-1,2,2)}+15\mathbb{L}_{(0,2,1)}+8\mathbb{L}_{(2,1,0)}=72\{iQ\cdot\mathbb{L}_{(1,2)}-L_1\cdot L_2\},$$
$$|s|=4: N^{(3)}_4=9\mathbb{L}_{(0,2,2)}-4\mathbb{L}_{(2,1,1)}=48L_2^2,$$
$$|s|=5: N^{(3)}_5=\mathbb{L}_{(1,2,2)}=0.$$

We have carried out explicit computations still for quartic Poisson brackets. We found the following five pseudoscalar non-linear relations:

$$N^{(4)}_{0,0}=\{N^{(3)}_0, L_0\}=4\mathbb{L}_{(1,-1,0,0)}-\mathbb{L}_{(2,-2,0,0)}=0,$$

$$\begin{aligned}
N^{(4)}_{0,1} &= \{N^{(3)}_1, L_{-1}\} - \{N^{(3)}_{-1}, L_{+1}\} = 12(\mathbb{L}_{(1,0,0,-1)} - \mathbb{L}_{(-1,0,0,1)}) \\
&\quad + 6(\mathbb{L}_{(1,-1,1,-1)} - \mathbb{L}_{(-1,1,-1,1)}) - 2(\mathbb{L}_{(2,-2,1,-1)} - \mathbb{L}_{(-2,2,-1,1)}) \\
&\quad - (\mathbb{L}_{(1,2,-2,-1)} + \mathbb{L}_{(1,-2,2,-1)} - \mathbb{L}_{(-1,-2,2,1)} - \mathbb{L}_{(-1,2,-2,1)}) \\
&\quad - 3(\mathbb{L}_{(0,2,-1,-1)} - \mathbb{L}_{(0,-2,1,1)}) \\
&= 24\{-iQ\cdot(\mathbb{L}_{(2,-1,-1)} + \mathbb{L}_{(-2,1,1)}) - 4(L_1\cdot\mathbb{L}_{(1,-2)} - L_{-1}\cdot\mathbb{L}_{(-1,2)}) \\
&\quad + 48Q^2\cdot\mathbb{L}_{(1,-1)} - 96iQ\cdot L_1\cdot L_{-1}\}, \\
N^{(4)}_{0,2} &= \{N^{(3)}_2, L_{-2}\} - \{N^{(3)}_{-2}, L_2\} = 6(\mathbb{L}_{(0,1,1,-2)} - \mathbb{L}_{(0,-1,-1,2)}) \\
&\quad - (\mathbb{L}_{(2,1,-1,-2)} + \mathbb{L}_{(2,-1,1,-2)} - \mathbb{L}_{(-2,-1,1,2)} - \mathbb{L}_{(-2,1,-1,2)}) \\
&\quad - 2(\mathbb{L}_{(1,-1,2,-2)} - \mathbb{L}_{(-1,1,-2,2)}) \\
&= 24\{iQ\cdot(\mathbb{L}_{(0,2,-2)} + \mathbb{L}_{(0,-2,2)}) + 2(L_2\cdot\mathbb{L}_{(-2,0)} - L_{-2}\cdot\mathbb{L}_{(2,0)}) \\
&\quad - 4(L_1\cdot\mathbb{L}_{(1,-2)} - L_{-1}\cdot\mathbb{L}_{(-1,2)}) - 16Q^2\mathbb{L}_{(2,-2)} + 64iQ\cdot L_2L_{-2}\}, \\
N^{(4)}_{0,3} &= 12\mathbb{L}_{(1,-1,0,0)} - 4(\mathbb{L}_{(1,0,0,-1)} - \mathbb{L}_{(-1,0,0,1)}) + 4(\mathbb{L}_{(2,-1,-1,0)} - \mathbb{L}_{(-2,1,1,0)}) \\
&\quad - 4(\mathbb{L}_{(0,1,-2,1)} - \mathbb{L}_{(0,-1,2,-1)}) + (\mathbb{L}_{(1,-1,2,-2)} - \mathbb{L}_{(-1,1,-2,2)}) \\
&\quad + (\mathbb{L}_{(1,2,-2,-1)} + \mathbb{L}_{(1,-2,2,-1)} - \mathbb{L}_{(-1,-2,2,1)} - \mathbb{L}_{(-1,2,-2,1)}) \\
&= 16\{-6iQ(\mathbb{L}_{(0,1,-1)} + \mathbb{L}_{(0,-1,1)}) - 2iQ\cdot(\mathbb{L}_{(0,2,-2)} + \mathbb{L}_{(0,-2,2)}) \\
&\quad + 3iQ\cdot(\mathbb{L}_{(2,-1,-1)} + \mathbb{L}_{(-2,1,1)}) + 10(L_1\cdot\mathbb{L}_{(1,-2)} - L_{-1}\cdot\mathbb{L}_{(-1,2)}) \\
&\quad - 4(L_1\cdot\mathbb{L}_{(-1,0)} - L_{-1}\cdot\mathbb{L}_{(1,0)}) - 2(L_2\cdot\mathbb{L}_{(-2,0)} - L_{-2}\cdot\mathbb{L}_{(2,0)}) \\
&\quad - 80Q^2\cdot\mathbb{L}_{(1,-1)} + 32Q^2\cdot\mathbb{L}_{(2,-2)} \\
&\quad + 160iQ\cdot L_1\cdot L_{-1} - 96iQ\cdot L_2\cdot L_{-2} - 512iQ^5\}, \\
N^{(4)}_{0,4} &= 12\mathbb{L}_{(1,-1,0,0)} - 3(\mathbb{L}_{(2,0,0,-2)} - \mathbb{L}_{(-2,0,0,2)}) - 4(\mathbb{L}_{(0,2,-1,-1)} - \mathbb{L}_{(0,-2,1,1)}) \\
&\quad + 4(\mathbb{L}_{(1,-1,1,-1)} - \mathbb{L}_{(-1,1,-1,1)}) \\
&= 16\{3iQ(\mathbb{L}_{(0,2,-2)} + \mathbb{L}_{(0,-2,2)}) + 2iQ(\mathbb{L}_{(2,-1,-1)} + \mathbb{L}_{(-2,1,1)}) \\
&\quad - 8(L_1\cdot\mathbb{L}_{(-1,0)} - L_{-1}\cdot\mathbb{L}_{(1,0)}) - 2(L_2\cdot\mathbb{L}_{(-2,0)} - L_{-2}\cdot\mathbb{L}_{(2,0)}) \\
&\quad + 32Q^2\cdot\mathbb{L}_{(1,-1)} - 24Q^2\cdot\mathbb{L}_{(2,-2)} - 64iQ\cdot L_1\cdot L_{-1} - 32iQ\cdot L_2\cdot L_{-2} \\
&\quad + 512iQ^5\}.
\end{aligned}$$

Also, we computed the following spin-5 and spin-3 relations:

$$\begin{aligned}
N^{(4)}_{5,1} &= \mathbb{L}_{(2,1,1,1)} - 3\mathbb{L}_{(-1,2,2,2)} = 240L_2\cdot\mathbb{L}_{(1,2)}, \\
N^{(4)}_{3,1} &= \mathbb{L}_{(-1,2,1,1)} + 2\mathbb{L}_{(-1,1,2,1)} + 3\mathbb{L}_{(-1,1,1,2)} \\
&= 4\{3iQ\cdot\mathbb{L}_{(-1,2,2)} + 10iQ\cdot\mathbb{L}_{(1,2,0)} + 18iQ\cdot\mathbb{L}_{(0,1,2)} \\
&\quad - 18L_1\cdot\mathbb{L}_{(0,2)} - 24L_2\cdot\mathbb{L}_{(0,1)} - 624iQ\cdot L_1\cdot L_2 + 96Q^2\cdot\mathbb{L}_{(1,2)}\}.
\end{aligned}$$

The fact that certain linear combinations of k-fold Poisson-brackets can be expressed as polynomials in Q, $\mathbb{L}_{(s)} = \mathbb{L}_{(s_1,\ldots,s_{k'})}$, and k''-fold Poisson-brackets involving $L_{\pm 1}$, $L_{\pm 2}$ and the exceptional elements $L_{0\ell}$, $\ell = 1, 3, \ldots$ seems very alarming at first sight. However, a closer inspection of the "non-linearities" encountered so far reveals the remarkable feature that the exceptional elements do not occur as factors in the polynomials. If this phenomenon holds true in general –

as we conjecture and as we shall assume in the sequel – then the situation can possibly be cured simply by including the element $Q^2 \in V^{(1)}(\mathfrak{h}_{(m,0,0)})$ and the exceptional elements $L_{0\ell} \in V^{(\ell)}(\mathfrak{h}_{(m,0,0)})$, $\ell = 3, 5, \ldots$ $(L_{01} \equiv L_0)$ into the set of generating elements. Equivalently, as generators we shall use the elements Q^2, $L_s^{(1)} \doteq L_s$, $s = \pm 1, \pm 2$, and the members of the abelian subalgebra (see Sect. III) $L_0^{(\ell)} = \mathscr{L}^{(2)}_{\underbrace{\kappa\lambda\ldots\mu\nu}_{\ell+3}} g^{\kappa\lambda} \ldots g^{\mu\nu}$, $\ell = 1, 3, 5, \ldots$.

Definition. $\mathbb{L}^{(\mathrm{l})}_{(\mathrm{s})} = \mathbb{L}^{(\ell_1,\ldots,\ell_n)}_{(s_1,\ldots,s_n)} = \{\ldots\{L^{(\ell_1)}_{s_1}, L^{(\ell_2)}_{s_2}\}, \ldots, L^{(\ell_n)}_{s_n}\}$ with $s_j = 0$ whenever $\ell_j \neq 1$;

$$\mathbb{L}^{(\mathrm{l})}_{(\mathrm{s})} \in V^{(\Sigma \ell_j)}(\mathfrak{h}_{(m,0,0)}), \quad \{Q, \mathbb{L}^{(\mathrm{l})}_{(\mathrm{s})}\} = i\left(\sum s_j\right) \mathbb{L}^{(\mathrm{l})}_{(\mathrm{s})}.$$

Proposition 18. *Suppose that tensorial multiplication is only explained for products of the form Q times a spin zero invariant. Then all other monomials in Q and in* $\mathbb{L}^{(\mathrm{l})}_{(\mathrm{s})} \neq L_0^{(1)}, L_0^{(3)}, \ldots$ *with more than two factors can be defined with the help of multiple Poisson-brackets involving the element* Q^2 *in an essential way.*

Proof. By induction on the degree of the stratum $V^{(\ell)}(\mathfrak{h}_{(m,0,0)})$ containing a particular "admissible" monomial: The claim is trivially true for $\ell = 1$. Suppose that the claim is true for all admissible monomials contained in the strata $V^{(\ell')}(\mathfrak{h}_{(m,0,0)})$ with $\ell' \leqq \ell$. Then – as we shall show – it is also true for all admissible monomials contained in the stratum $V^{(\ell+1)}(\mathfrak{h}_{(m,0,0)})$.

We observe that we need only consider factors $\mathbb{L}^{(\mathrm{l})}_{(\mathrm{s})}$ involving at least one non-exceptional generator L_s, $s = \pm 1, \pm 2$ because otherwise $\mathbb{L}^{(\mathrm{l})}_{(\mathrm{s})}$ would vanish as a consequence of the "commutativity" of $L_0^{(\ell_1)}$ and $L_0^{(\ell_2)}: \{L_0^{(\ell_1)}, L_0^{(\ell_2)}\} = 0$.

Now, consider an arbitrary admissible monomial M in the stratum $V^{(\ell+1)}(\mathfrak{h}_{(m,0,0)})$. Choose one of the factors of M which involves the smallest number of generators. Call it $\mathbb{L}$. Let ν be the number of generators involved. The factor Q corresponds to the value $\nu = 0$. By induction on ν we shall show that $M = \tilde{M} \cdot \mathbb{L}$, $\tilde{M} \in V^{\left(\ell - \sum_{j=1}^{\nu} \ell_j\right)}(\mathfrak{h}_{(m,0,0)})$ can be defined as explained in the proposition:

$\nu = 0: \mathbb{L} = Q: \tilde{M} \cdot Q$ is defined by tensorial multiplication if $s(\tilde{M}) = 0$,

$$\tilde{M} \cdot Q = \frac{1}{2is(\tilde{M})} \{Q^2, \tilde{M}\} \quad \text{if} \quad s(\tilde{M}) \neq 0.$$

$$\nu = 1: \mathbb{L} = L_s, \quad s = \pm 1, \pm 2: \quad \tilde{M} \cdot L_s = (\{\tilde{M} \cdot Q, L_s\} - \{\tilde{M}, L_s\} \cdot Q)/(is),$$

where multiplication by Q is defined as before. Let $M = \tilde{M} \cdot \mathbb{L} \in V^{(\ell+1)}(\mathfrak{h}_{(m,0,0)})$ be defined for all $\mathbb{L}$ with $\nu = 0, 1, \ldots, n$ and all $\tilde{M} \in \bigoplus_{\ell'=1}^{\ell - \Sigma \ell_j} V^{(\ell')}(\mathfrak{h}_{(m,0,0)})$.

Consider $\mathbb{L}$ with $\nu = n+1 \geqq 2$. $\mathbb{L}$ contains at least one non-exceptional generator. Without loss of generality $\mathbb{L} = \{\mathbb{L}', L_{n+1}\}$, where $\mathbb{L}'$ is an admissible factor involving n generators. Hence $\tilde{M} \cdot \mathbb{L}' \in \bigoplus_{\ell'=1}^{\ell} V^{(\ell')}(\mathfrak{h}_{(m,0,0)})$ is defined as well as $\{\tilde{M}, L_{n+1}\} \cdot \mathbb{L}'$. The product $\tilde{M} \cdot \mathbb{L}$ is finally obtained from the relation

$$\tilde{M} \cdot \mathbb{L} = \{\tilde{M} \cdot \mathbb{L}', L_{n+1}\} - \{\tilde{M}, L_{n+1}\} \cdot \mathbb{L}'.$$

This completes the proof of the proposition.

The non-linear relations $N_{5,1}^{(4)}$ and $N_{3,1}^{(4)}$ exemplify that by passing to the Poisson algebra generated by the elements $L_{\pm 1}$, $L_{\pm 2}$ one cannot get rid of the nonlinear constraints among the Poisson brackets, nor can one avoid the appearance of the exceptional invariants alltogether. For instance, the right hand side of $N_{3,1}^{(4)}$ involves the exceptional invariant L_{01} in form of the Poisson brackets $\{L_{01}, L_s\}$, $s = \pm 1, \pm 2$, which are linearly independent of the brackets $\{L_s, L_{s'}\}$, $s, s' = \pm 1, \pm 2$.

V. Summary and Conclusions

The aim of the research reported in this article is a detailed analysis of the algebraic structures of the "internal" observable conserved charges of the classical closed Nambu-Goto string. These invariant non-local charges originally came about as eigenvalues of monodromy matrices associated to certain Lax pairs of systems of linear differential equations depending on infinitely many parameters.

An important first step towards the aforesaid aim is taken by mapping the relevant algebraic structures of the matrix elements of the monodromy matrices – they are the building blocks of the invariant charges – to natural structures of the group algebra $\mathbf{O}_N$ of the symmetric group $\mathbf{S}_N$. Thereby, both object and method of the investigation are put into a general mathematical context. In particular, well-known theorems from the theory of the regular representation of the symmetric group can be applied.

In principle, for the analysis of the algebraic structures of the invariant charges themselves the situation is the same. However, unfortunately very little is known about the structure of the algebra $Z_N \mathbf{O}_N Z_N$ decisive in this context (Z_N = cyclic symmetrizer). Thus no ready-made mathematical theorems pertinent to the algebraic aspects of the invariant charges are available.

To be specific, we have shown that the set $\mathfrak{h}_{\mathscr{P}}$ of invariant charges for arbitrary space time dimension forms an associative algebra under tensor multiplication and a Lie algebra under the Poisson bracket operation. The set $\mathfrak{h}_{\mathscr{P}}$ can be decomposed in a natural way. This leads to gradations and selection rules for the Lie algebra.

The Lie algebra of the invariant charges does not depend for better or for worse on the canonical Poisson brackets of the (unobservable) string variables. There exists at least one inequivalent albeit non-local modification of the Poisson brackets which yields the same Poisson Lie algebra for the invariant charges.

We have revealed Lie subalgebras of $\mathfrak{h}_{\mathscr{P}}$. In view of the representation problem, special interest was paid to abelian subalgebras. Questions relating to the completeness and maximality of these subalgebras still demand an answer. Finally, we have analysed the invariant charges in the momentum rest frame. We were able to count the independent invariants and to construct an explicit though somewhat arbitrary algebraic basis for them. We established that $\mathfrak{h}_{\mathscr{P}}$ is not a finitely generated algebra. Nevertheless, in a certain sense $\mathfrak{h}_{\mathscr{P}}$ seems to be a minimal extension of the Poincaré algebra.

For the special case of three dimensional space-time we computed examples of non-linear relations among multiple Poisson brackets. We argued that by a slight extension of the set of generating elements ordering problems associated with these nonlinearities can be reduced when representing $\mathfrak{h}_{\mathscr{P}}$ as a commutator algebra.

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