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Super-Kac-Moody Algebras and Supersymmetric 2d-Free Fermions*

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Abstract. Explicit representations of super-Kac-Moody algebra are constructed in terms of 2d-free fermions which form a non-linear representation of supersymmetry with the fermions grouped with the generators of the algebra into superfields. It is shown how the most general construction of this type corresponds to homogeneous spaces G/H and how supersymmetry alone is responsible for that structure.

It is well known that representations of Kac-Moody algebra [1] can be constructed using two-dimensional free fermions [2]. This construction was crucial in the proof by Witten [3] of the equivalence between non-linear sigma models with a Wess-Zumino term [4] and free fermion systems. This equivalence was later developed in a beautiful paper by Knizhnik and Zamolodchikov [5] using the techniques of conformal field theory [6]. It was then noticed that the supersymmetric extension of the sigma model [7] also had a rich algebraic structure and that it gave a representation of a supersymmetric extension of the Kac-Moody algebra [8]. In the case of SO(N) for example the content of the model in terms of free fermions is the following: there are two types of decoupled fields, one transforming under the adjoint representation of the group while the other (corresponding to the fermionization of the bosonic field of the original model) is in the fundamental representation. These two fields form a nonlinear representation of supersymmetry [8]. A similar property was also observed in Goddard and Olive in [9]. The purpose of this note is to show that a large class of representations of super-Kac-Moody algebra can be constructed in terms of free fermions which realize a non-linear representation of the two dimensional superconformal (Neveu et al. [10]) algebra (for another point of view on this latter construction and related considerations about superstrings see [11]). As we will

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show, supersymmetry alone determines the precise structure of the super-Kac-Moody algebra and the required group theoretical content of the theory (in particular the only allowed representations of the fermion fields). More precisely the requirement of supersymmetry will be equivalent to the existence of an invariant connection on an homogeneous space G/H and the two sets of fields can be regarded respectively as vertical and horizontal vector fields on the principal bundle $G(H, G/H)$. This remarkable fact points at a deeper connection between supersymmetry and the geometry of Lie groups.

We will consider free Weyl-Majorana $2d$-fermion fields which are functions of the coordinates $z=x+iy$ and $\bar{z}=x-iy$. For later use we also introduce at this point the Grassmannian coordinate θ associated with z. The superconformal transformations defined by an infinitesimal displacement vector field $V(z,\theta)=v_0(z)+\theta v_1(z)$ which is any analytic function of z and the Grassmannian coordinate θ (note that v_1 is anti-commuting) will be given explicitly by:

$$\delta z=v_0(z)+\tfrac{1}{2}\theta v_1(z) \quad \text{and} \quad \delta\theta=\tfrac{1}{2}(v_1(z)+\theta\partial_z v_0(z)). \tag{1}$$

The model is described by the Lagrangian

$$L=\tfrac{1}{2}\psi^i\partial_{\bar{z}}\psi^i+\tfrac{1}{2}\chi^a\partial_{\bar{z}}\chi^a, \tag{2}$$

where the $\psi^i(z,\bar{z})$ transform under an as yet unspecified real representation r of dimension $d(r)$ of a compact Lie group H and the $\chi^a(z,\bar{z})$ are in the adjoint representation of H whose dimension we denote by D. By the equations of motion ψ^i and χ^a are functions of z only, and we have

$$\psi^i(z)\psi^j(w)\sim-\frac{\delta^{ij}}{z-w}, \tag{3}$$

and similarly for the χ^a field. One can define two currents:

$$J^a_\psi=-\tfrac{1}{2}\psi^i T^a_{ij}\psi^j \tag{4}$$

and

$$J^a_\chi=\tfrac{1}{2}f_{abc}\chi^b\chi^c, \tag{5}$$

where f_{abc} are the structure constants of the group H and the generators T^a_{ij} in the representation R satisfy

$$[T^a, T^b]_{ij}=f_{abc}T^c_{ij}. \tag{6}$$

$J^a_\chi(z)$ and $J^a_\psi(z)$ will satisfy

$$\partial_{\bar{z}}J^a_\chi=0 \tag{7}$$

and

$$\partial_{\bar{z}}J^a_\psi=0 \tag{8}$$

which reflects the fact that the transformations of H which leave the Lagrangian (2) invariant can be any analytic functions of z. We have then the well known Kac-

Moody algebra in operator product form:

$$J^a_\psi(z)J^b_\psi(w)\sim\frac{k\delta^{ab}}{2(z-w)^2}+\frac{f_{abc}J^c_\psi(w)}{z-w}, \tag{9}$$

where $k=\frac{c(r)d(r)}{D}$, the central extension, is equal to the Dynkin number $k(r)$ and $c(r)\delta_{ij}=(T^aT^a)_{ij}$. As usual we should interpret this equation in the following way [5, 6]. Define the generator J_ω of an H-transformation parametrized by analytic functions $\omega^a(z)$ by:

$$J_\omega=\frac{1}{2\pi i}\oint dz\omega^a(z)J^a_\psi(z). \tag{10}$$

The variation of the field ψ^i will be

$$\delta_\omega\psi^i(w)=[J_\omega,\psi^i(w)]=\frac{1}{2\pi i}\oint dz\omega^a(z)J^a_\psi(z)\psi^i(w)=-\omega^a(w)T^a_{ij}\psi^j(w), \tag{11}$$

and in particular the transformation of the generators themselves will be

$$\delta_\omega J^a_\psi(w)=\frac{1}{2\pi i}\oint dz\omega^b(z)J^b_\psi(z)J^a_\psi(w)=f_{abc}\omega^c(w)J^b_\psi(w)+\frac{k(r)}{2}\partial_w\omega^a(w), \tag{12}$$

where the contour circles around w. From these remarks follows

$$[J^a_m,J^b_n]=f_{abc}J^c_{m+n}+m\frac{k(r)}{2}\delta^{a,b}\delta_{m+n,0}, \tag{13}$$

where $J^a_\psi(z)=\sum_n z^{-n-1}J_n$. We have similar relations for $J^a_\chi(z)$ with $k=-f_{abc}f_{abc}\equiv c_v$.

The only non-vanishing component of the energy momentum tensor, given by

$$T_B(z)=\tfrac{1}{2}\psi^i\partial_z\psi^i+\tfrac{1}{2}\chi^a\partial_z\chi^a \tag{14}$$

is an analytic function of z and the generator of the conformal transformations (1). The Virasoro algebra [12] is given by

$$T_B(z)T_B(w)\sim\frac{c}{2(z-w)^4}+\frac{2T_B(w)}{(z-w)^2}+\frac{\partial_wT_B(w)}{(z-w)}, \tag{15}$$

where $c=\frac{d(r)+D}{2}$ is the central charge for free fermions. The operator product expansion (O.P.E.) of T_B with the fermionic field ψ^i is

$$T_B(z)\psi^i(w)\sim\frac{1}{2}\frac{\psi^i(w)}{(z-w)^2}+\frac{\partial_w\psi^i(w)}{(z-w)}. \tag{16}$$

This equation expresses the fact that the fermionic fields have a conformal weight $\frac{1}{2}$ under a conformal transformation. One checks similarly that the conformal weight of the currents is one.

If one wants to have a nontrivial realization of supersymmetry linking these two sets of fields, one first has to construct a candidate for the generator of superconformal transformations $T_F(z)$ which will be the partner of $T_B(z)$, and satisfy the Neveu-Schwarz-Ramond algebra [10]:

$$T_F(z)T_F(w) \sim \frac{\hat{c}}{4(z-w)^3} + \frac{\frac{1}{2}T_B(w)}{(z-w)}, \tag{17}$$

$$T_B(z)T_F(w) \sim \frac{3T_F(w)}{2(z-w)^2} + \frac{\partial_w T_F(w)}{(z-w)}. \tag{18}$$

The first of these equations is the expression of supersymmetry, while the last one simply shows that the conformal weight of T_F is $\frac{3}{2}$. As in (15) all the coefficients are determined unambiguously by the fact that T_B and T_F must be the generators of superconformal transformations (1) except for the central charge which to be consistent with (15) must be $\hat{c}=2/3c$ (for more details about super conformal field theory see [13]). The most general operator of weight 3/2 that can be built out of the ψ and χ fields is

$$T_F(z) = \tfrac{1}{6} i\alpha f_{abc}\chi^a\chi^b\chi^c + \tfrac{1}{6} i\beta\eta_{ijk}\psi^i\psi^j\psi^k + i\gamma\chi^a J^a_\psi . \tag{19}$$

It is understood that the product of several operators at the same point is normal ordered with respect to the modes of these operators. Here α, β, γ are unknown constants. Notice the presence of a term tri-linear in ψ^i. Since T_F must be a scalar under a transformation of H, and since the ψ^i anti-commute η_{ijk} must be a totally antisymmetric tensor satisfying

$$T^a_{ir}\eta_{jkr} + T^a_{kr}\eta_{ijr} + T^a_{jr}\eta_{kir} = 0 \tag{20}$$

and

$$T^a_{ij}\eta_{ijk} = 0 . \tag{21}$$

We want to show now that it is possible to fix the constants α, β, and γ in such a way that T_B and T_F satisfy (15) and (17) and that the central charge is exactly the one corresponding to free fermions. In taking the O.P.E. of T_F with itself one sees that there are two types of dimension two operators which can appear: either the usual current-current terms $J^a_\psi J^a_\psi$ or $J^a_\chi J^a_\chi$ which come in the Sugawara construction of the energy-momentum tensor given any Kac-Moody algebra (this was extensively studied by Goddard and Olive in [9], see also Zamolodchikov and Knizhnik [5]), or terms in four fermions like $\eta_{ijk}\eta_{rsk}\psi^i\psi^j\psi^r\psi^s$ and $T^a_{ij}T^a_{kl}\psi^i\psi^j\psi^k\psi^l$. Since these terms are not present in (14) they must appear either in combinations such that they cancel each other or be simply absent. One sees immediately that two cases are possible depending on whether the tensor η_{ijk} exists or not in the particular representation r we are considering. However, we will see that it is not necessary to make an exhaustive study of the representations which do admit such a tensor for different groups since the possible solutions will have a simple geometrical interpretation.

a) In the case where η_{ijk} is absent, the above condition imposes the following constraints on the generator T^a_{ij}:

$$T^a_{ij}T^a_{kl} + T^a_{ki}T^a_{jl} + T^a_{jk}T^a_{il} = 0 . \tag{22}$$

Then from (17) and (14) one finds

$$\alpha^2=\gamma^2=\frac{1}{2(c_v+k(r))}. \tag{23}$$

Note finally that in this case the energy momentum tensor is

$$T_B(z)=\frac{1}{\alpha}(J^a_\chi J^a_\chi+J^a_\psi J^a_\psi), \tag{24}$$

and that the normalization can be easily checked using the null vector condition of [5] by applying the two sides of this equation on some highest weight vector of the Virasoro and Kac-Moody algebra. Let's stress that this equality is a purely quantum mechanical effect. Also from the equality $\hat{c}=\frac{2}{3}c=\frac{1}{3}(d(r)+D)$ we have the constraint

$$\frac{2c(r)}{c_v+k(r)}=1, \tag{25}$$

which severely restrict the possible representations the ψ^i can be in; but we will return to this later on. The very same constraint (25) comes in the study of Goddard and Olive [9] through the requirement that the Virasoro algebra associated with the ordinary Kac-Moody algebra (9) be precisely (24).

b) In the second case the constraint (22) is two restrictive. We define:

$$\eta_{ijk}\eta_{ijr}=-\bar{c}(r)\delta_{kr}, \tag{26}$$

$$\mathbf{T}_{ijkl}=T^a_{ij}T^a_{kl}+T^a_{ki}T^a_{jl}+T^a_{jk}T^a_{il}, \tag{27}$$

and

$$\mathbf{N}_{ijkl}=\eta_{ijr}\eta_{klr}+\eta_{kjr}\eta_{jlr}+\eta_{jkr}\eta_{ilr}, \tag{28}$$

from which follows

$$\tfrac{1}{2}\mathbf{N}\cdot\mathbf{T}=-c(r)d(r)\bar{c}(r), \tag{29}$$

and

$$\tfrac{1}{3}\mathbf{T}\mathbf{T}=c(r)d(r)[k(r)+c_v-2c(r)]. \tag{30}$$

We will then replace the constraint (22) by

$$\mathbf{N}_{ijkl}=-\lambda\mathbf{T}_{ijkl}, \tag{31}$$

and one finds $\lambda=\frac{\bar{c}(r)}{k(r)+c_v-2c(r)}$. Again we determine the constants from (17)

$$\alpha^2=\lambda\beta^2=\gamma^2; \qquad \alpha^2=\frac{1}{2(c_v+k(r))}; \qquad \lambda=\frac{\bar{c}(r)}{c_v+k(r)-2c(r)}. \tag{32}$$

So we see that when the constraint (22) or (31) are satisfied, the Lagrangian (2) is invariant under the superconformal transformation (1) whose generator is given by (19). The transformations of the fermionic fields are

$$\delta_{v_1}\chi^a(w)=\frac{1}{2\pi i}\oint dz v_1(z)T_F(z)\chi^a(w)=-i2\alpha[J^a_\chi(w)+J^a_\psi(w)] \tag{33}$$

and

$$\delta_{v_1}\psi^i(w)=\frac{1}{2\pi i}\oint dz v_1(z)T_F(z)\psi^i(w)$$
$$=-i2\alpha\left[T^a_{ij}\chi^a(w)\psi^j(w)+\frac{1}{2\sqrt{\lambda}}\eta_{ijk}\psi^j(w)\psi^k(w)\right]. \tag{34}$$

We define the two superfields

$$S^a(z,\theta)=\frac{i}{2\alpha}\chi^a(z)+\theta[J^a_\psi(z)+J^a_\chi(z)]\equiv\frac{i}{2\alpha}\chi^a(z)+\theta J^a(z), \tag{35}$$

$$\Psi^i(z,\theta)=\frac{i}{2\alpha}\psi^i(z)+\theta\left[T^a_{ij}\chi^a(z)\psi^j(z)+\frac{1}{2\sqrt{\lambda}}\eta_{ijk}\psi^j(z)\psi^k(z)\right]$$
$$\equiv\frac{i}{2\alpha}\psi^i(z)+\theta\phi^i(z). \tag{36}$$

We are now in the position to see the full structure of the super-Kac-Moody algebra by simply taking the operator product of these superfields:

$$S^a(z_1,\theta_1)S^b(z_2,\theta_2)=\frac{k/2}{z_{12}}\delta^{ab}+\frac{\theta_{12}}{z_{12}}f_{abc}S^c(z_2,\theta_2), \tag{37}$$

$$S^a(z_1,\theta_1)\Psi^i(z_2,\theta_2)=\frac{\theta_{12}}{z_{12}}[-T^a_{ik}\Psi^k(z_2,\theta_2)], \tag{38}$$

$$\Psi^i(z_1,\theta_1)\Psi^j(z_2,\theta_2)=\frac{k/2}{z_{12}}\delta^{ij}+\frac{\theta_{12}}{z_{12}}\left[\frac{\eta_{ijk}}{\sqrt{\lambda}}\Psi^k(z_2,\theta_2)-T^a_{ij}S^a(z_2,\theta_2)\right], \tag{39}$$

with $k=\frac{1}{2\alpha^2}=c_v+k(r)$, and where we have used the notation $z_{12}=z_1-z_2-\theta_1\theta_2$ and $\theta_{12}=\theta_1-\theta_2$. The first of these equations is nothing but the supersymmetric extension of the Kac-Moody algebra (9), the central charge k being as it should be the sum of the one corresponding to J^a_ψ and J^a_χ since they commute with each other. The second expresses the transformations properties of the Ψ^i field under H. More surprising is the structure of the last equation. Notice that it exhibits the same central charge k and that the Ψ^i play a role very similar to the supercurrents S^a. All this is best understood by looking at the bosonic part of the equations above. If we define $J^a(z)=\sum_n z^{-n-1}J^a_n$ and $\phi^i(z)=\sum_n z^{-n-1}\phi^i_n$, we have for the modes zero denoted by j^a and φ^i, respectively:

$$[j^a,j^b]=f_{abc}j^c, \tag{40}$$

$$[\varphi^i,\varphi^j]=-T^a_{ij}j^a+\frac{1}{\sqrt{\lambda}}\eta_{ijk}\varphi^k, \tag{41}$$

and

$$[j^a, \varphi^i] = -T^a_{ij}\varphi^j. \tag{42}$$

Since for all possible combinations of φ^i and j^a the Jacobi identity easily obtains, this is the Lie algebra $g = h + m$ of a group $G \supset H$ with $\mathrm{ad}(H)m = m$; j^a and φ^i span h and m respectively and G/H is an homogeneous space. This is well known to correspond to the decomposition of the tangent fields of the principle bundle $G(H, G/H)$ at the identity into its vertical and horizontal components, and given the corresponding invariant connection we recognize $-\frac{1}{\sqrt{\lambda}}\eta_{ijk}$ as the torsion and $-T^a_{ij}T^a_{kl}$ as the Riemann tensor [14]. The constraint (31) is the Bianchi identity. These identifications provide the geometrical interpretation we were looking for. Specializing to the case were the tensor η_{ijk} is absent is then equivalent to imposing that G/H is a symmetric space [15]. One also recovers the result of [9] that free fermions in the adjoint representation of a compact Lie group are supersymmetric by themselves. As a last remark we would like to stress once more that all this rich structure is the result of imposing supersymmetry.

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