Commun. Math. Phys. 105, 281–290 (1986)

Communications in
Mathematical
Physics

Translation Invariant Gibbs States in the q-State Potts Model

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Abstract. We describe the set of all translation invariant Gibbs states in the q-state Potts model for the case of q large enough and the other parameters to be arbitrary.

Introduction

The aim of this note is to describe the set of all translation invariant Gibbs states in the q-state Potts model. We consider only the case of q large enough, assuming the other parameters of the model, i.e. the temperature and the space dimension $\nu \geqq 2$, to be arbitrary. Let $\mathbb{Z}^\nu$ be ν-dimentional lattice, $\nu \geqq 2$. The distance between any two points $x, y \in \mathbb{Z}^\nu$, $x=(x_1, \ldots, x_\nu)$, $y=(y_1, \ldots, y_\nu)$, is defined as $d(x, y)=\sum_{i=1}^{\nu} |x_i - y_i|$. We assume that the spin $\varphi(x)$, $x \in \mathbb{Z}^\nu$, in the model under consideration takes values in the finite set $Q=\{1, \ldots, q\}$, and the formal Hamiltonian is written as follows:

$$H=-\sum_{\langle x, y\rangle} \delta_{\varphi(x), \varphi(y)}, \qquad \varphi(x), \varphi(y) \in Q, \tag{1}$$

where the sum is taken over all the pairs of nearest neighbors x, y on the lattice and δ is the Kronecker symbol. By $\mathrm{g}(\beta, q)$ [respectively by $\mathrm{g}^{(\mathrm{inv})}(\beta, q)$] is denoted the class of all (respectively of all translation invariant) Gibbs states with β parameter and the Hamiltonian (1).

By using reflection positivity Kotecky and Shlosman [1] have proved the coexistence of $q+1$ phases at some $\beta_c(q)$ (critical inverse temperature) for q large enough. Another approach to the solution of this problem, based on the contour technique, was offered by E. Dinaburg and Ya. Sinai [2] and independently by Bricmont et al. [3]. Everywhere below we mean that the value of $\beta_c(q)$ is defined namely as in [2], although the next theorem shows that $\beta_c(q)$ is to be unique.

Now we formulate the main result of this paper.

Theorem. *For any $\nu \geqq 2$, $q_0(\nu)$ may be found so that for all $q > q_0(\nu)$ the following statement is true. There exists such a value $\beta=\beta_c(q)$ of inverse temperature, that*

i) *when* $\beta=\beta_c(q)$ *the class* $\mathfrak{g}^{(\mathrm{inv})}(\beta, q)$ *contains exactly* $q+1$ *extreme points* $P^{(0)}$, $P^{(1)}, \ldots, P^{(q)}$, *i.e. any translation invariant Gibbs state* $P \in \mathfrak{g}^{(\mathrm{inv})}(\beta, q)$ *is expressible as*

$$P=\alpha_0 P^{(0)}+\alpha_1 P^{(1)}+\ldots+\alpha_q P^{(q)}, \quad \alpha_i \geqq 0 \forall i, \quad \sum \alpha_i=1,$$

ii) *for each* $\beta>\beta_c(q)$ *one may constructed* q *Gibbs states* $P_\beta^{(1)}, P_\beta^{(2)}, \ldots, P_\beta^{(q)}$ *so that any translation invariant Gibbs state* $P \in \mathfrak{g}^{(\mathrm{inv})}(\beta, q)$ *is expressible as*

$$P=\alpha_1 P_\beta^{(1)}+\ldots+\alpha_q P_\beta^{(q)}, \quad \alpha_i \geqq 0 \,\forall i, \quad \sum \alpha_i=1,$$

iii) *when* $\beta<\beta_c(q)$ *a Gibbs state is unique in the class* $\mathfrak{g}^{(\mathrm{inv})}(\beta, q)$ *of all translation invariant Gibbs states.*

Remarks. i) when $\beta<\beta_c(q)$ one can prove the uniqueness of the Gibbs state in the class of all Gibbs states, but we omit the proof of this fact, ii) by a quite different method Laanait et al. [4] have received the similar result in the case $v=2$.

1. The Basic Definitions and Notations

In this section we make use of the definitions and notations of [2]. Given any set C, denote by $|C|$ the number of points in C. Let $V \subset \mathbb{Z}^v$. Let $\partial V=\{x \in V \mid$ there exists $y \notin V$ such that $d(x, y)=1\}$, and $\partial_1 V=\{x \notin V \mid$ there exists $y \in V$ such that $d(x, y)=1\}$. The mapping $\varphi: \mathbb{Z}^v \rightarrow Q$ will be called a configuration. The restriction of the configuration φ to the set $V \subset \mathbb{Z}^v$ is denoted by $\varphi(V)$. This $\varphi(V)$ is sometimes called a configuration on V. If φ is a configuration and $x \in \mathbb{Z}^v$, we put $\alpha(x, \varphi)=\{$the number of y, for which $\varphi(y) \neq \varphi(x)$, $d(y, x)=1\}$.

Definition 1.1. Let φ be an arbitrary configuration. We shall say that φ is in the phase 0 at the point $x \in \mathbb{Z}^v$, if $\varphi(y) \neq \varphi(x)$ for all y such that $d(x, y)=1$. If $\varphi(y)=\varphi(x)=p\ (1 \leqq p \leqq q)$ for all y, satisfying condition $d(x, y)=1$, we shall say that φ is in the phase $p \neq 0$ at the point x. If at the point $x \in \mathbb{Z}^v$ the configuration φ is in none of the phases $0,1, \ldots, q$, then x will be called an incorrect point of the configuration φ. The union of all incorrect points of the configuration φ is called the preboundary of φ and denoted by $B^*(\varphi)$. The set $\{x \mid d(B^*(\varphi), x) \leqq 1\}$ is called the boundary of the configuration φ and denoted by $B(\varphi)$. We consider only the configurations for which $|B(\varphi)|<\infty$. A set $X \subset \mathbb{Z}^v$ is called connected if given any $x', x'' \in X$ there is a sequence $x_1, \ldots, x_n$ of points $x_i \in X$, $i=1, \ldots, n$, so that $x_1=x'$, $x_n=x''$, and $d(x_i, x_{i+1})=1$, $i=1,2, \ldots, n-1$. Let $B(\varphi)=\cup B_i(\varphi)$ be a decomposition of $B(\varphi)$ into its maximal connected components. Each of the sets $\partial B_i(\varphi)$ is in turn the union of its connected components. One of them is external and denoted by $\partial B_i^{(\mathrm{ext})}(\varphi)$, and the others are the boundaries of some bounded domains $O_{i, s}$ (they will be called internal domains) and denoted by $\partial B_{i, s}^{(\mathrm{int})}(s=1, \ldots, r(i))$, where $r(i)$ is the number of these domains. At each point of $\partial B_i(\varphi)$ the configuration φ is in some phase. At different points of the same connected component of $\partial B_i^{(\mathrm{ext})}$ or of $\partial B_{i, s}^{(\mathrm{int})}$ this phase is the same and coincides with the phase in which φ is at the points that are at distance 1 from this component and belong to the complement of $B_i(\varphi)$.

Definition 1.2. A contour $\gamma^{(p)}$ is a pair $(b^{(p)}, \psi(b^{(p)}))$, where $b^{(p)}$ is a connected component of $B(\varphi)$ for some configuration φ, which is in phase p $(p=0,1, \ldots, q)$ at the points of $(\partial b^{(p)})^{(\mathrm{ext})}$, and $\psi(b^{(p)})$ is the restriction of this configuration to $b^{(p)}$. The set $b^{(p)}$ is called the support of the contour $\gamma^{(p)}$ and denoted by $\operatorname{supp}\gamma^{(p)}$. The union of all internal domains O_s is called the interior of $\gamma^{(p)}$ and denoted by $\operatorname{int}\gamma^{(p)}$. Put

$$V(\gamma^{(p)})=\operatorname{supp}\gamma^{(p)}\cup\operatorname{int}\gamma^{(p)}, \qquad \operatorname{Ext}(\gamma^{(p)})=\mathbb{Z}^{\nu}\setminus V(\gamma^{(p)}).$$

The outer contours are defined as usual [5]. Given fixed $V\subset\mathbb{Z}^{\nu}$, $|V|<\infty$, denote by $\mathfrak{A}_0^{(p)}(V,\varphi_0)$, $p=0,1,\ldots,q$, the set of all configurations that are in phase p at each point of V and coincide with the configuration φ_0 on $\partial_1 V$. Here $\varphi_0(x)=p$ for all $x\in\partial_1 V$, if $p\neq 0$, and $\varphi_0(x)\neq\varphi_0(y)$ for all $x,y\in\partial_1 V$, such that $d(x,y)=1$ at $p=0$. We shall consider only such boundary conditions without mentioning it further. Introduce the following partition function

$$\Xi_0^{(p)}(V;\varphi_0,\beta)=\sum_{\varphi\in\mathfrak{A}_0^{(p)}(V,\varphi_0)}\exp\{-\beta H_V(\varphi)\}, \tag{1.1}$$

where

$$H_V(\varphi)=\sum_{\langle x,y\rangle\subset V}\delta_{\varphi(x),\varphi(y)}+\sum_{\langle x,y\rangle:x\in V,\,y\notin V}\delta_{\varphi(x),\varphi(y)}.$$

Fix p, $0\leqq p\leqq q$, and consider the arbitrary collection $\gamma_i^{(p)}=(b_i^{(p)},\ \psi(b_i^{(p)}))$, $i=1,2,\ldots,n$, of pairwise outer contours. Let $V(\gamma_i^{(p)})\subset V$, $i=1,2,\ldots,n$, for some V. Denote by $\mathfrak{A}^{(p)}(\{\gamma_i^{(p)}\},V,\varphi_0)$ the set of all configurations $\varphi(V\cup\partial_1 V)$ such that both $\varphi(\partial_1 V)=\varphi_0(\partial_1 V)$ and the set of contours $\gamma_i^{(p)}(i=1,2,\ldots,n)$ coincides with the set of all outer contours of $B(\varphi)$. Introduce the partition function

$$\Xi^{(p)}(V|\{\gamma_i^{(p)}\}_{i=1}^n,\beta,\varphi_0)=(\Xi_0^{(p)}(V,\beta,\varphi_0))^{-1}\sum_{\varphi\in\mathfrak{A}^{(p)}(\gamma_i^{(p)},V,\varphi_0)}\exp\{-\beta H_V(\varphi)\}, \tag{1.2}$$

and put

$$\Xi^{(p)}(\{\gamma_i^{(p)}\}_{i=1}^n,\beta)=\lim_{V\to\infty}\Xi^{(p)}(V|\{\gamma_i^{(p)}\}_{i=1}^n,\beta,\varphi_0), \tag{1.3}$$

where $V\to\infty$ in the Van Hove sense. $\Xi^{(p)}(\gamma^{(p)},\beta)$ will be called a crystallic partition function. Given any $W\subset\mathbb{Z}^{\nu}$, $|W|<\infty$ define the dilute partition function

$$\Xi^{(p)}(W,\beta)=\sum_{\{\gamma_1^{(p)},\ldots,\gamma_n^{(p)}\}\subset W}\Xi^{(p)}(\{\gamma_i^{(p)}\}_{i=1}^n,\beta), \tag{1.4}$$

where the sum is taken over all such collections of outer contours $\{\gamma_i^{(p)}\}_{i=1}^n$, that $V(\gamma_i^{(p)})\subset W$ $\forall i$ and $d(\partial W,\cup V(\gamma_i))>1$. Let $W\subset V$, $|V|<\infty$. We consider also the dilute partition function

$$\Xi^{(p)}(V|W,\beta,\varphi_0)=\sum_{\{\gamma_1^{(p)},\ldots,\gamma_n^{(p)}\}}\Xi^{(p)}(V|\{\gamma_i^{(p)}\}_{i=1}^n,\beta,\varphi_0), \tag{1.5}$$

where the sum is taken, as above, over all such collections of outer contours $\gamma_i^{(p)}$, $i=1,\ldots,n$, that $V(\gamma_i^{(p)})\subset W$ and $d(\partial W,\cup V(\gamma_i))>1$. It is not difficult to see that

$$\Xi^{(p)}(V|\gamma^{(p)},\beta,\varphi_0)=\sum_{\varphi\in\mathfrak{A}^{(p)}(\gamma^{(p)},V,\varphi_0)}\exp\left\{-\frac{\beta}{2}\sum_{x\in V}\alpha(x,\varphi)\right\}, \tag{1.6}$$

when $p\neq 0$, where $\alpha(x,\varphi)=0$ for all $x\notin V(\gamma^{(p)})$ and partition function (1.6) does not depend on V. In particular from this one can get both the existence of the limit (1.3) for $p \neq 0$ and the formulas

$$\Xi^{(p)}(\gamma^{(p)}, \beta) = \Xi^{(p)}(V|\gamma^{(p)}, \beta, \varphi_0), \tag{1.7}$$

$$\Xi^{(p)}(\{\gamma_i^{(p)}\}_{i=1}^n, \beta) = \prod_{i=1}^{n} \Xi^{(p)}(\gamma_i^{(p)}, \beta). \tag{1.8}$$

For $p=0$ the existence of the limit (1.3) is obtained in [2]. Finally, let us formulate the results of E. Dinaburg and Ya. Sinai [2], which will be helpful for us later on (see also [3]).

1.1. Interacting Contour Models

Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a finite collection of the contours, such that $\mathrm{supp}\gamma_i \cap \mathrm{supp}\gamma_j = \emptyset$ for all $i \neq j$ and $\Gamma' \subset \Gamma$. Γ' is called the maximal permissible subcollection of Γ, if given any two contours $\gamma'_1, \gamma'_2 \in \Gamma'$, neither of them is inside of one other, there does not exist such a contour $\gamma \in \Gamma$, such that $\mathrm{supp}\gamma'_1 \subset \mathrm{int}\gamma$, $\mathrm{supp}\gamma'_2 \subset \mathrm{Ext}\gamma$, and it is impossible to extend Γ', keeping the above mentioned properties. It is supposed that contour Hamiltonian is written as

$$H(\Gamma) = \sum_{\gamma \in \Gamma} F(\gamma) + G(\Gamma), \tag{1.9}$$

where $G(\Gamma)$ is the interaction energy of contours of Γ and has the special form

$$G(\Gamma) = \sum_{\Gamma' \subset \Gamma} G(\Gamma'|\Gamma), \qquad |\Gamma'| > 1, \tag{1.10}$$

and the sum is taken over all the maximal permissible subcollections Γ' of the collection Γ. Then $G(\Gamma)$ and $F(\gamma)$ are supposed to be invariant with respect to any shift of the lattice. It is supposed, moreover, that the estimates

$$\sum_{\gamma : \mathrm{supp}\gamma = C} \exp(-F(\gamma)) \leqq \exp(-k|C|) \tag{1.11}$$

hold with some constant $k > 0$. Other properties of the interacting contour models wouldn't be immediatly used in this paper, that is why we omit them (see [2]). We shall write $\Gamma \subset V$, if $\mathrm{supp}\gamma \subset V$ for any $\gamma \in \Gamma$. Let $V \subset \mathbb{Z}^\nu$, $|V| < \infty$. Dilute (contour) partition function in V is written as follows:

$$Z(V|F, G) = \sum_{\Gamma \subset V} \exp(-H(\Gamma)),$$

and

$$Z(\gamma|F, G) = \sum_{\Gamma \subset \mathrm{int}\gamma} \exp(-H(\gamma \cup \Gamma))$$

is called the crystallic partition function of the contour γ.

1.2. The Interaction G for the Potts Model

Let $V \subset \mathbb{Z}^\nu$, $|V| < \infty$ and let $\gamma_i^{(0)} = (b_i^{(0)}, \psi(b_i^{(0)}))$, $i = 1, 2, \ldots, n$, be such pairwise outer contours, that $V(\gamma_i^{(0)}) \subset V$ for any $i = 1, 2, \ldots, n$. Then the functions

$$G^{(0)}(\{\gamma_i^{(0)}\}_{i=1}^n) = \ln \Xi^{(0)}(\{\gamma_i^{(0)}\}_{i=1}^n, \beta) - \sum_{i=1}^{n} \ln \Xi^{(0)}(\gamma_i^{(0)}, \beta), \tag{1.12}$$

$$G^{(0)}(\{\gamma_i^{(0)}\}_{i=1}^n; V; \varphi_0) = \ln \Xi^{(0)}(V|\{\gamma_i^{(0)}\}_{i=1}^n, \beta, \varphi_0) - \sum_{i=1}^{n} \ln \Xi^{(0)}(V|\gamma_i^{(0)}, \beta, \varphi_0) \tag{1.13}$$

don't depend on β. Let us mention the connection between (1.10) and (1.12). If the collection Γ of contours satisfies the condition of point 1.1, then

$$G^{(0)}(\{\gamma_i^{(0)}\}_{i=1}^n) = G(\Gamma'|\Gamma),$$

where $\Gamma' = (\gamma_1^{(0)}, \ldots, \gamma_n^{(0)})$ is the set of all outer contours of the collection Γ.

1.3. Description of Gibbs States, Some Estimates

There exists $q(\nu)>0$ such that for all $q>q(\nu)$, one finds $\beta_c(q)>0$ $\left(\beta_c(q) = \frac{\ln q}{\nu} + O(q^{-1})\right)$, the contour functional $F^{(0)}$, the interaction $G^{(0)}$ and q contour functionals $F^{(p)}$, $p=1,2,\ldots,q$, so that

$$\Xi^{(0)}(\gamma^{(0)}, \beta_c(q)) = Z(\gamma^{(0)}|F^{(0)}, G^{(0)}), \tag{1.14}$$

$$\Xi^{(p)}(\gamma^{(p)}, \beta_c(q)) = Z(\gamma^{(p)}|F^{(p)}), \quad p=1,\ldots,q. \tag{1.15}$$

Moreover

$$\sum_{\gamma^{(p)}:\,\mathrm{supp}\,\gamma^{(p)}=C} \exp(-F^{(p)}|(\gamma^{(p)})) \leqq \exp(-k(q)|C|), \quad p=0,\ldots,q, \tag{1.16}$$

where $k(q)\to 0$ when $q\to\infty$. Note finally that based on the contour definition one can prove the existence of the constant $c(q)$, such that the estimate

$$F^{(p)}(\gamma^{(p)}) \leqq c(q)\,|\mathrm{supp}\,\gamma^{(p)}| \tag{1.17}$$

holds when $\beta = \beta_c(q)$.

2. Construction of Pure Phases in the Case $\beta \neq \beta_c(q)$

All constructions in the case $\beta \neq \beta_c(q)$ are based on some inequalities which are similar to that of R. Minlos and Ya. Sinai [6] for the Ising model.

Definition 2.1. The point $x\in\mathbb{Z}^\nu$ is called a stable point of the configuration φ under either of the following conditions:

i) $\beta > \beta_c(q)$, φ is in phase $p\neq 0$ at the point x,

ii) $\beta = \beta_c(q)$, φ is in phase p, $p=0,1,\ldots,q$, at the point x,

iii) $\beta < \beta_c(q)$, φ is in phase 0 at the point x. In all other cases the point x is called the unstable point of the configuration φ.

Lemma 2.1. *For any $\nu \geqq 2$ there exists $q_1(\nu)>0$ such that for each $q>q_1(\nu)$ and $\beta \geqq \beta_c(q)$ one can construct q contour functionals $\{F^{(p)}(\gamma^{(p)},\beta)\}$, $p=1,\ldots,q$, so that*

$$\Xi^{(p)}(\gamma^{(p)}, \beta) = Z\{\gamma^{(p)}|F^{(p)}(\cdot,\beta)\}, \quad p=1,\ldots,q. \tag{2.1}$$

Moreover, for both arbitrary fixed p, $1\leqq p\leqq q$, and contour $\gamma^{(p)}$, the function $F^{(p)}(\gamma^{(p)},\beta)$ is monotone increasing with respect to β when $\beta\geqq\beta_c(q)$.

Proof. Let $p\neq 0$ and let $\mathfrak{A}^{(p)}(\gamma^{(p)})$ be the set of configurations that have only one outer contour $\gamma^{(p)}$. Applying the relation (1.6) we get

$$-\frac{\partial}{\partial\beta}\Xi^{(p)}(\gamma^{(p)},\beta) = \sum_{\varphi\in\mathfrak{A}^{(p)}(\gamma^{(p)})} \frac{1}{2}\left(\sum_x \alpha(x,\varphi)\right)\exp\left(-\frac{\beta}{2}\sum_x \alpha(x,\varphi)\right). \tag{2.2}$$

As far as $\alpha(x,\varphi)=0$ for $x\notin V(\gamma^{(p)})$, and $0\leqq\alpha(x,\varphi)\leqq 2v$ for all other x, we have

$$\left|\frac{\partial}{\partial\beta}\ln\Xi^{(p)}(\gamma^{(p)},\beta)\right|\leqq v|V(\gamma_i)|\,. \tag{2.3}$$

Then

$$\begin{aligned}\frac{\partial}{\partial\beta}\Xi^{(p)}(V,\beta)&=\sum_{\{\gamma_1,\ldots,\gamma_n\}}\left(\sum_{k=1}^{n}\frac{\partial}{\partial\beta}\ln\Xi^{(p)}(\gamma_k^{(p)},\beta)\right)\prod_{i=1}^{n}\Xi^{(p)}(\gamma_i^{(p)},\beta)\\&=\sum_{\gamma\subset V}\left[\frac{\partial}{\partial\beta}\ln\Xi^{(p)}(\gamma^{(p)},\beta)\right]\Xi^{(p)}(\gamma^{(p)},\beta)\,\Xi^{(p)}(V\backslash V(\gamma^{(p)},\beta))\,. \end{aligned}\tag{2.4}$$

Choose now $k_0(v)$ so that

$$\sum_{0\in C} v|V(C)|\exp(-k_0|C|)<\tfrac{1}{4}\,, \tag{2.5}$$

where the sum is taken over all connected sets C, containing the point 0, $V(C)=C\cup\operatorname{int}C$, $C\neq\emptyset$. Based on relation (1.16), choose $q_1(v)$ so that for all $q>q_1(v)$ the inequalities

$$\sum_{\gamma:\,\operatorname{supp}\gamma=C}\exp(-F^{(p)}(\gamma^{(p)},\beta))\leqq\exp(-k_0(v)|C|)\,,\qquad p=1,\ldots,q\,, \tag{2.6}$$

hold at the point $\beta=\beta_c(q)$. Suppose $\beta\geqq\beta_c(q)$. Let $\gamma^{(p)}$ be a contour and O_m $(m=1,2,\ldots,r)$ be connected components of the set $\operatorname{int}\gamma^{(p)}$. Put

$$F^{(p)}(\gamma^{(p)},\beta)=\sum_{m=1}^{r}\ln\Xi^{(p)}(O_m,\beta)-\ln\Xi^{(p)}(\gamma^{(p)},\beta)\,. \tag{2.7}$$

Suppose that for some $\beta_0\geqq\beta_c$ the inequalities

$$F^{(p)}(\gamma^{(p)},\beta_0)\geqq F^{(p)}(\gamma^{(p)},\beta_c)\,,\qquad p=1,\ldots,q\,, \tag{2.8}$$

hold. From the condition (2.8) and from (2.2)–(2.4) one can get, in a standard fashion (see [6]), that for $\beta=\beta_0$,

$$\frac{\partial}{\partial\beta}\ln\Xi^{(p)}(V,\beta)|_{\beta=\beta_0}=a(F^{(p)}(\cdot\,,\beta_0))|V|+b(\partial V,F^{(p)})\,, \tag{2.9}$$

where $|a(F^{(p)}(\cdot\,,\beta_0))|<\frac{1}{4}$, $|b(\partial V,F^{(p)})|<\frac{1}{4}|\partial V|$. Let $\gamma^{(p)}=(b^{(p)},\psi(b^{(p)}))$ be a contour. Given any configuration $\varphi\in\mathfrak{A}^{(p)}(\gamma^{(p)})$, denote by $\tilde{O}^{(u)}(\varphi)$ the largest connected subset of unstable points such that $b^{(p)}\cap\tilde{O}^{(u)}(\varphi)\neq\emptyset$. The complement of $\tilde{O}^{(u)}(\varphi)$ with respect to $V'(\gamma^{(p)})=V(\gamma^{(p)})\backslash\partial^{(\mathrm{ext})}b^{(p)}$ is split into the connected components $\tilde{O}_n^{(p_n)}(\varphi)$, $n=1,\ldots,\tilde{r}(\varphi)$, where $p_n\neq 0$ is the value of the phase on $\partial\tilde{O}_n^{(p_n)}(\varphi)$. Then

$$\Xi^{(p)}(\gamma^{(p)},\beta)=\sum_{\varphi\in\mathfrak{A}(\gamma^{(p)})}\exp\left(-\frac{\beta}{2}\sum_{x\in\tilde{O}^{(u)}(\varphi)}\alpha(x,\varphi)\right)\prod_{n=1}^{\tilde{r}(\varphi)}\Xi^{(p_n)}(\tilde{O}_n^{(p_n)},\beta)\,. \tag{2.10}$$

Inserting this expression in (2.7) and computing the derivative $\frac{\partial}{\partial\beta}F^{(p)}(\gamma^{(p)},\beta)$ at the point β_0, we have

$$\frac{\partial}{\partial\beta}F^{(p)}(\gamma^{(p)},\beta)|_{\beta=\beta_0}\geqq 0\,,$$

provided

$$\sum_{x\in O^{(u)}(\varphi)} \alpha(x,\varphi) - \sum_{n=1}^{\tilde{r}(\varphi)} \frac{\partial}{\partial\beta} \ln \Xi^{(p_n)}(\tilde{O}_n^{(p_n)},\beta) + \sum_m \frac{\partial}{\partial\beta} \ln \Xi^{(p)}(O_m,\beta) \geqq 0\,.$$

By virtue of the symmetry of the Potts model

$$\Xi^{(p')}(V,\beta) = \Xi^{(p'')}(V,\beta)$$

holds for $V\subset\mathbb{Z}^\nu$, $|V|<\infty$ and any p', p'', $1\leqq p'\leqq q$, $1\leqq p''\leqq q$. From this and also from representation (2.9) and the inequality $\alpha(x,\varphi)\geqq 1$ that holds for all $x\in\tilde{O}(\varphi)$, it is not difficult to make sure that the latter inequality holds for all configurations $\varphi\in\mathfrak{A}^{(p)}(\gamma^{(p)})$. Since this discussion is valid for $\beta=\beta_c$, it remains valid for all $\beta>\beta_c$, Q.E.D.

Lemma 2.2. *Given any $\nu\geqq 2$ $q_2(\nu)>0$ may be found such that for all $q>q_2(\nu)$ and $\beta<\beta_c$ one can construct the contour functional $\{F^{(0)}(\gamma^{(0)},\beta)\}$ and the interaction $G^{(0)}$ so that*

$$\Xi^{(0)}(\gamma^{(0)},\beta) = Z(\gamma^{(0)}|F^{(0)},G^{(0)})\,. \tag{2.11}$$

Here the function $G^{(0)}$ does not depend on β, and $F^{(0)}(\gamma^{(0)},\beta)$ is monotone decreasing with respect to β provided $\beta\leqq\beta_c$.

Proof. The proof of this lemma differs only a little from the previous one. Let us mention the distinctions between them. First of all choose $G^{(0)}(\{\gamma_i^{(0)}\}_{i=1}^n)$ according to (1.12) and note that $G^{(0)}$ does not depend on β. Comparing this with (1.13) we obtain

$$\frac{\partial}{\partial\beta}\ln\Xi^{(0)}(V|\{\gamma_i^{(0)}\}_{i=1}^n,\beta,\varphi_0) = \sum_{i=1}^n \frac{\partial}{\partial\beta}\ln\Xi^{(0)}(V|\gamma_i^{(0)},\beta,\varphi_0)\,.$$

Denote by

$$P_W(\mathfrak{A}(\{\gamma_i^{(0)}\}_{i=1}^n,V,\varphi_0)|\beta) = \frac{\Xi^{(0)}(V|\{\gamma_i^{(0)}\}_{i=1}^n,\beta,\varphi_0)}{\Xi^{(0)}(V|W,\beta,\varphi_0)}$$

the probability distribution $P_W(\cdot\,|\beta)$ on the set of all $\mathfrak{A}(\{\gamma_i^{(0)}\}_{i=1}^n,V,\varphi_0)$ such that $\operatorname{supp}\gamma_i^{(0)}\subset W$. Taking into consideration this notation and the previous equality we obtain [just as in demonstration of (2.4)]

$$\begin{aligned}&\frac{\partial}{\partial\beta}\ln\Xi^{(0)}(V|W,\beta,\varphi_0)\\ &\quad=\sum_{\operatorname{supp}\gamma\subset W}\left[\frac{\partial}{\partial\beta}\ln\Xi^{(0)}(V|\gamma^{(0)},\beta,\varphi_0)\right]P_W(\gamma^{(0)}\in\mathfrak{A}(\{\gamma_i^{(0)}\}_{i=1}^n,V,\varphi_0)|\beta)\,.\end{aligned}$$

Note that the probability $P_W(\gamma^{(0)}\in\mathfrak{A}|\beta)$ of $\gamma^{(0)}$ to be an outer contour, arising here, satisfies the Peierls' [2, 3] inequality

$$\begin{aligned}&P_W(\gamma^{(0)}\in\mathfrak{A}(\{\gamma_i^{(0)}\}_{i=1}^n,V,\varphi_0)|\beta_c)\\ &\quad\leqq\exp\left\{-F^{(0)}(\gamma^{(0)},\beta_c(q))+O\left(\frac{1}{q}\right)|\operatorname{supp}\gamma^{(0)}|\right\}\end{aligned} \tag{2.12}$$

when $\beta=\beta_c(q)$. Then, obviously,

$$\Xi_0^{(0)}(V;\varphi_0)\cdot\frac{\partial}{\partial\beta}\ln\Xi^{(0)}(V|\gamma^{(0)},\beta,\varphi_0)$$
$$=\sum_{\varphi\in\mathfrak{A}^{(0)}(\gamma^{(0)},V,\varphi_0)}\sum_{x\in V}\left[\nu-\frac{\alpha(x,\varphi)}{2}\right]\exp\{-\beta H_V(\varphi)\},$$

and since $\alpha(x,\varphi)=2\nu$ for all $x\notin V(\gamma^{(0)})$, we obtain

$$\left|\frac{\partial}{\partial\beta}\ln\Xi^{(0)}(V|\gamma^{(0)},\beta,\varphi_0)\right|\leqq\nu|V(\gamma^{(0)})|. \tag{2.13}$$

Put as in (2.7)

$$F^{(0)}(\gamma^{(0)},\beta)=\sum_{m=1}^{r}\ln\Xi^{(0)}(O_m,\beta)-\ln\Xi^{(0)}(\gamma^{(0)},\beta).$$

Since the estimates (2.12) and (2.13) are uniform with respect to all V and φ_0, we are able to repeat the reasoning of the previous lemma. This proves that $F^{(0)}(\gamma^{(0)},\beta)$ is monotone decreasing when $\beta\leqq\beta_c(q)$.

3. Proof of Theorem

In the case $\beta=\beta_c$, the proof of the theorem is similar to that for the Ising model [7].

Let $\beta\neq\beta_c$. We shall study the properties of the Gibbs state in V with the boundary conditions φ_0 on $\partial_1 V$, assuming that $\varphi_0(x)\neq\varphi_0(y)$ for any $x,y\in\partial_1 V$, $d(x,y)=1$ if $\beta>\beta_c$, and demanding of the boundary conditions φ_0 that $\varphi_0(\partial_1 V)=p$, $1\leqq p\leqq q$ in the case $\beta<\beta_c$. The passage to the case of arbitrary boundary conditions is simple enough (see, for example, [8, 9]), so the assumptions about the boundary conditions discussed above are to be fulfiled later without mentioning it.

Let $V\subset\mathbb{Z}^\nu$, $|V|<\infty$, be a connected set. Consider the configuration φ, the restriction of which to $\partial_1 V$ has the properties mentioned at the beginning of this section. The connected component of the set of unstable points of the configuration φ in $V\cup\partial_1 V$, containing $\partial_1 V$, will be denoted by $V^{(u)}(\varphi)$. Put

$$\mathfrak{A}^N=\{\varphi(V\cup\partial_1 V)|\varphi(\partial_1 V)=\varphi_0(\partial_1 V),|V^{(u)}(\varphi)|=N\}$$

for $N\in\mathbb{Z}^+$ and consider the partition function

$$\Xi^{(p),N}(V|\beta,\varphi_0)=\sum_{\varphi\in\mathfrak{A}^N}\exp\{-\beta H_V(\varphi)\},\qquad p=0,1,\ldots,q. \tag{3.1}$$

Here it is supposed that $p=0$ when $\beta>\beta_c(q)$, and $p=1,\ldots,q$ when $\beta<\beta_c(q)$. The proof of the theorem follows from the estimate (see [8, 9])

$$\frac{\Xi^{(p),N}(V|\beta,\varphi_0)}{\Xi^{(p)}(V|\beta,\varphi_0)}\leqq\exp\{-c(\beta)N+c_1(\beta)|\partial V|\}, \tag{3.2}$$

which is of main importance in this paper. To establish the inequality (3.2) we consider the set of contours γ, which has the properties

i) $\mathrm{supp}\,\gamma \subset V$,

ii) the restrictions of the configuration φ on $\partial^{(\mathrm{int})}\gamma$ and on $\partial^{(\mathrm{ext})}\gamma$ are in the opposite phases (i.e. in the case $\beta > \beta_c$ the points of the set $\partial^{(\mathrm{ext})}\gamma$ are in phase 0, and the points of the set $\partial^{(\mathrm{int})}\gamma$ are in either of the phases $1, 2, \ldots, q$, and vice versa in the case $\beta < \beta_c$). Let us choose in this class of contours the contour γ with the largest $|\mathrm{int}\,\gamma|$, and denote it by γ_V. It is clear that

$$\Xi^{(p)}(V|\beta, \varphi_0) \geqq \Xi^{(p)}(V|\beta, \gamma_V^{(p)}, \varphi_0). \tag{3.3}$$

Let

$$V \backslash V^{(u)}(\varphi) = V_1^{(s)}(\varphi) \cup \ldots \cup V_k^{(s)}(\varphi) \tag{3.4}$$

be the decomposition of the set $V \backslash V^{(u)}(\varphi)$ into connected components. Note that for any $m = 1, \ldots, k$ all points of the set $\partial V_m^{(s)}(\varphi)$ are in phase 0 if $\beta < \beta_c$, and are in phase $p_m \neq 0$ if $\beta > \beta_c$. Taking this into account we rewrite the partition function $\Xi^{(p),N}(V, \beta, \varphi_0)$ as

$$\begin{aligned}&\Xi^{(p),N}(V|\beta, \varphi_0)\\ &= \sum_{\varphi \in \mathfrak{A}^N} \exp\{-\beta H(\varphi_{V^{(u)}})\} \times \prod_{m=1}^{k} \Xi_0^{(p_m)}(V_m^{(s)} \backslash \partial V_m^{(s)}, \beta, \varphi(\partial V_m^{(s)}))\\ &\cdot \prod_{m=1}^{k} \Xi^{(p_m)}(V_m^{(s)} \backslash \partial V_m^{(s)}, \beta, \varphi(\partial V_m^{(s)})),\end{aligned} \tag{3.5}$$

where

$$H(\varphi_{V^{(u)}}) = -\sum_{\langle x, y\rangle} \delta_{\varphi(x), \varphi(y)}, \tag{3.6}$$

and the sum in the latter relation is taken over all the pairs of nearest neighbors, such that either $\langle x, y\rangle \subset V^{(u)}(\varphi) \cap V$ or $x \in V^{(u)}(\varphi) \cap V$, $y \notin V^{(u)}(\varphi) \cap V$. The remaining calculation will be carried out only for the case $\beta < \beta_c$. The case $\beta > \beta_c$ is similar. So let $p \neq 0$, $\varphi_0(\partial V) \equiv p$ and $\beta < \beta_c$. From (3.3) and (1.17) it follows

$$\frac{\Xi^{(p),N}(V|\beta_c, \varphi_0)}{\Xi^{(p)}(V|\beta_c, \varphi_0)} \leqq \frac{\Xi^{(p)}(V|\beta_c, \varphi_0)}{\Xi^{(p)}(V|\gamma_V, \beta_c, \varphi_0)} \leqq \exp(c_2(q)|\partial V|). \tag{3.7}$$

Having applied (3.5) for $\Xi^{(p)}(V|\gamma_V, \beta, \varphi_0)$, we obtain

$$\frac{\Xi^{(p),N}(V|\beta, \varphi_0)}{\Xi^{(p)}(V|\gamma_V, \beta, \varphi_0)} = \sum_{\varphi \in \mathfrak{A}^N} \exp\{\omega_V(\beta, \varphi)\}, \tag{3.8}$$

where

$$\begin{aligned}\omega_V(\beta, \varphi) &= \frac{\beta}{2} \sum_{x \in V^{(u)}(\varphi)} (2\nu - \alpha(x, \varphi)) - \frac{\beta}{2} \sum_{x \in V \backslash V(\gamma_V)} (2\nu - \alpha(x, \varphi))\\ &+ \sum_{m=1}^{k} \ln \Xi^{(0)}(V_m^{(s)} \backslash \partial V_m^{(s)}, \beta, \varphi(\partial V_m^{(s)}))\\ &- \ln \Xi^{(0)}(\mathrm{int}\,\gamma^{(0)} \backslash \partial\{\mathrm{int}\,\gamma^{(0)}\}, \beta, \varphi(\partial\{\mathrm{int}\,\gamma^{(0)}\})) + \omega_V^{(0)}(\varphi).\end{aligned}$$

Here

$$\omega_V^{(0)}(\varphi) = \sum_{m=1}^{k} \ln \Xi_0^{(0)}(V_m^{(s)} \backslash \partial V_m^{(s)}, \varphi(\partial V_m^{(s)})) - \ln \Xi_0^{(0)}(\operatorname{int}\gamma^{(0)} \backslash \partial\{\operatorname{int}\gamma^{(0)}\}, \varphi(\partial\{\operatorname{int}\gamma^{(0)}\}))$$

does not depend on β. From (3.8) and the considerations of previous section it follows (see Lemma 2.2), that for every configuration $\varphi \in \mathfrak{A}^N(V, \varphi_0)$,

$$\frac{\partial}{\partial\beta} \omega_V^{(0)}(\varphi, \beta) \geqq \frac{1}{4} N - \frac{v}{4} |V \backslash V(\gamma_V)| \geqq \frac{1}{4} N - c_3(v) |\partial V|,$$

if $q > q_2(v)$ and $\beta < \beta_c(q)$. Hence

$$\omega_V^{(0)}(\varphi, \beta) - \omega_V^{(0)}(\varphi, \beta_c) \leqq -\tfrac{1}{4}(\beta_c - \beta) N + c_3(v)(\beta_c - \beta)|\partial V|. \tag{3.9}$$

Choosing $c(\beta) = \frac{1}{4}(\beta_c - \beta)$, $c_1(\beta) = c_2(q) + c_3(v)(\beta_c - \beta)$, from (3.3), (3.7)–(3.9) we obtain the estimate (3.2). The theorem is proved.

Acknowledgements. It is a pleasure to thank Professors Ya.G. Sinai and E.I. Dinaburg for helpful discussions.

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Communicated by Ya. G. Sinai

Received February 15, 1985