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An Example of Absence of Turbulence for any Reynolds Number

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Abstract. We consider a viscous incompressible fluid moving in a two-dimensional flat torus. We show a particular external force $\underline{f}_0$ for which there is a globally attractive stationary state for any Reynolds number R. Moreover, for any fixed R, this stability property holds also for a neighbourhood of $\underline{f}_0$.

We consider a viscous incompressible fluid moving in a two-dimensional flat torus. The Navier-Stokes equations governing the motion are

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} = -\underline{\nabla} p + \underline{f} + \nu\Delta\underline{u}, \qquad \underline{u}(0) = \underline{u}_0, \tag{1}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \tag{2}$$

$$\int_{T^2} \underline{u}\, d\underline{x} = 0, \qquad \int_{T^2} \underline{f}\, d\underline{x} = 0, \tag{3}$$

$$T^2 = [0, 2\pi] \times [0, 2\pi], \qquad \underline{x} \equiv (x, y) = x\underline{e}_1 + y\underline{e}_2 \in T^2,$$

where $\underline{u}(\underline{x}, t)$ is the velocity, $p(\underline{x}, t) \in \mathbb{R}$ the pressure, $\nu > 0$ the viscosity, $\underline{f}(\underline{x})$ the external force. All functions involved are periodic in x, y of period 2π.

In our problem we fix a time scale and we assume as a reasonable Reynolds number

$$R = \sup_{\underline{x} \in T^2} |\underline{f}(\underline{x})|/\nu\,.$$

In general the behavior of the solutions depends on R: if R is small there exists a stationary state stable and attractive. When R increases this state loses its stability and, for large R, the motion becomes chaotic. This fact is related with the turbulence. (On this subject there is a lot of literature: see for instance [1].)

In this paper we want to show particular forces $\underline{f}_0(\underline{x})$ for which the stationary state remains attractive for every Reynolds number R. These forces are not

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completely exceptional in the sense that they have a neighbourhood (depending on R) for which this stability property holds.

We assume $\underline{f}$ smooth

$$\underline{f}=\underline{f}_0+\underline{f}_1\,, \tag{4}$$

where

$$\begin{aligned}\underline{f}_0=&\underline{c}_1[\nu(A_1\cos y+A_2\sin y)+(A_3\cos x+A_4\sin x)(-A_1\sin y+A_2\cos y)]\\&+\underline{c}_2[\nu(A_3\cos x+A_4\sin x)+(A_1\cos y+A_2\sin y)(-A_3\sin x+A_4\cos x)]\,,\\&A_1,A_2,A_3,A_4\in\mathbb{R}\end{aligned} \tag{5}$$

We define

$$R_0=|A_1|+|A_2|+|A_3|+|A_4|\,, \tag{6}$$

$$r_1=\int_{T^2}|\underline{f}_1|^2/\nu^2\,d\underline{x}\,, \tag{7}$$

$$r_2=\int_{T^2}F_1^2/\gamma^2\,d\underline{x}\,, \tag{8}$$

where

$$F_1=\partial_x f_{1,x}-\partial_y f_{1,x}\,. \tag{9}$$

The result of this paper is stated in the following theorem:

Theorem. *For any R_0, there exist $\varepsilon_1(R_0)>0$, $\varepsilon_2(R_0)>0$ such that for any $r_1<\varepsilon_1$, $r_2<\varepsilon_2$ there is a stable stationary state which attracts exponentially each solution.*

More precisely we put

$$\underline{u}=\underline{\bar{u}}+\underline{v}\,, \tag{10}$$

where $\underline{\bar{u}}$ is the stationary state.

Then

$$E(t)=\tfrac{1}{2}\int_{T^2}\underline{v}\cdot\underline{v}\,d\underline{x}\xrightarrow[t\to\infty]{}0\quad\textit{exponentially}\,. \tag{11}$$

Proof. For sake of simplicity we first give the proof for $\underline{f}_1=0$. Then we consider the general case.

When the external force reduces to $\underline{f}_0$ the stationary state is

$$\begin{aligned}&\underline{\bar{u}}\equiv\underline{u}_0=\underline{c}_1(A_1\cos y+A_2\sin y)+\underline{c}_2(A_3\cos x+A_4\sin x)\,,\\&p=\text{const}\,.\end{aligned} \tag{12}$$

We introduce the vorticity

$$\omega=\partial_x u_y-\partial_y u_x\,. \tag{13}$$

Equation (1) becomes

$$\frac{\partial\omega}{\partial t}+(\underline{u}\cdot\underline{\nabla})\omega=F+\nu\Delta\omega\,, \tag{14}$$

where

$$F = \partial_x f_y - \partial_y f_x . \tag{15}$$

For the stationary state $\underline{\bar{u}}$,

$$\bar{\omega} \equiv \omega_0 = -A_3 \sin x + A_4 \cos x + A_1 \sin y - A_2 \cos y \tag{16}$$

We define

$$N = \tfrac{1}{2} \int_{T^2} \delta^2 \, d\underline{x} , \tag{17}$$

where

$$\delta = \omega - \bar{\omega} . \tag{18}$$

We study the variation in time of E and N. By a direct computation we have

$$\frac{d}{dt} E = -\int_{T^2} v_x v_y (-A_1 \sin y + A_2 \cos y - A_3 \sin x + A_4 \cos x) d\underline{x} - \nu \int_{T^2} (\underline{\nabla} \underline{v})^2 d\underline{x} , \tag{19}$$

$$\frac{dN}{dt} = -\int_{T^2} v_x v_y (-A_1 \sin y + A_2 \cos y - A_3 \sin x + A_4 \cos x) d\underline{x} - \nu \int_{T^2} (\underline{\nabla} \delta)^2 d\underline{x} . \tag{20}$$

Hence

$$\frac{d}{dt}(N - E) = -\nu \int_{T^2} [(\underline{\nabla} \delta)^2 - (\underline{\nabla} \underline{v})^2] d\underline{x} . \tag{21}$$

We study the right-hand side of (21) and we show that

$$\int_{T^2} [(\underline{\nabla} \delta)^2 - (\underline{\nabla} \underline{v})^2] d\underline{x} \geqq 4(N - E) . \tag{22}$$

To prove this inequality, we develop v_x, v_y in Fourier series

$$\begin{aligned} \underline{v} = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \underline{a}_{mn} \cos mx \cos ny + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \underline{b}_{mn} \cos mx \sin ny \\ & + \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \underline{c}_{mn} \sin mx \cos ny + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \underline{d}_{mn} \sin mx \sin ny . \end{aligned} \tag{23}$$

Condition (3) and Eq. (2) give

$$\begin{aligned} \underline{a}_{00} = 0 ; \quad m a_{x,mn} = n d_{y,mn} ; \quad m b_{x,mn} = -n c_{y,mn} ; \\ m c_{x,mn} = -n f_{y,mn} ; \quad m d_{x,mn} = n a_{y,mn} . \end{aligned} \tag{24}$$

Hence

$$\begin{aligned} E = & \frac{\pi^2}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (|\underline{a}_{mn}|^2 + |\underline{b}_{mn}|^2 + |\underline{c}_{mn}|^2 + |\underline{d}_{mn}|^2) \\ & + \pi^2 \sum_{m=1}^{\infty} (a_{y,m0}^2 + c_{y,m0}^2) + \pi^2 \sum_{n=1}^{\infty} (a_{x,0n}^2 + b_{x,0n}^2) . \end{aligned} \tag{25}$$

In a similar way we compute the other term in (22),

$$N=\frac{\pi^2}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[(ma_{y,mn}+nd_{x,mn})^2 +(-mb_{y,mn}+nc_{x,mn})^2+(mc_{y,mn}-nb_{x,mn})^2 +(md_{y,mn}+na_{x,mn})^2]+\pi^2\sum_{m=1}^{\infty}m^2(a_{y,m0}^2+c_{y,m0}^2) +\pi^2\sum_{n=1}^{\infty}n^2(a_{x,0n}^2+b_{x,0n}^2)^2, \tag{26}$$

$$\int_{T^2}(\underline{\nabla}\underline{v})^2d\underline{x}=2N, \tag{27}$$

$$\int_{T^2}(\underline{\nabla}\delta)^2d\underline{x}=\pi^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2)\{(ma_{y,mn}+nd_{x,mn})^2 +(-mb_{y,mn}+nc_{x,mn})^2+(mc_{y,mn}-nb_{x,mn})^2 +(md_{y,mn}+na_{x,mn})^2\}+2\pi^2\left\{\sum_{m=1}^{\infty}m^4(a_{y,m0}^2+c_{y,m0}^2) +\sum_{n=1}^{\infty}n^4(a_{x,0n}^2+b_{x,0n}^2)\right\}. \tag{28}$$

Hence, using (2), we have

$$\int_{T^2}[(\underline{\nabla}\delta)^2-(\underline{\nabla}\underline{v})^2]d\underline{x}=\frac{\pi^2}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2-1) \times\left[\left(m+\frac{n}{m}\right)^2(a_{y,mn}^2+b_{y,mn}^2+c_{y,mn}^2+d_{y,mn}^2) +\left(n+\frac{m}{n}\right)^2(a_{x,mn}^2+b_{x,mn}^2+c_{x,mn}^2+d_{x,mn}^2)\right] +2\pi^2\left[\sum_{m=1}^{\infty}m^2(m^2-1)(a_{y,m0}^2+c_{y,m0}^2)+\sum_{n=1}^{\infty}n^2(n^2-1)(a_{x,0n}^2+b_{x,0n}^2)\right], \tag{29}$$

and

$$N-E=\frac{\pi^2}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\left[\tfrac{1}{2}\left(m+\frac{n}{m}\right)^2-1\right](a_{y,mn}^2+b_{y,mn}^2+c_{y,mn}^2+d_{y,mn}^2) +\left[\tfrac{1}{2}\left(n+\frac{m}{n}\right)^2-1\right](a_{x,mn}^2+b_{x,mn}^2+c_{x,mn}^2+d_{x,mn}^2)\right\} +\pi^2\left[\sum_{m=1}^{\infty}(m^2-1)(a_{y,m0}^2+c_{y,m0}^2)+\sum_{n=1}^{\infty}(n^2-1)(a_{x,0n}^2+b_{x,0n}^2)\right]. \tag{30}$$

A comparison between (29) and (30) gives inequality (22).
We put (22) in (21), we observe that $N-E\geqq 0$, and we obtain

$$\frac{d}{dt}(N-E)\leqq -4\nu(N-E). \tag{31}$$

A second inequality can be obtained by (19) controlling its right-hand side. We have

$$\int_{T^2} (\nabla v)^2 dx \geqq 2E. \tag{32}$$

We write

$$\begin{aligned} v_x &= a_{x,01}\cos y + b_{x,01}\sin y + \varphi(x,y), \\ v_y &= a_{y,10}\cos x + c_{y,10}\sin x + \psi(x,y), \end{aligned} \tag{33}$$

where

$$\int_{T^2} \varphi^2 dx \leqq 2(N-E), \tag{34}$$

$$\int_{T^2} \psi^2 dx \leqq 2(N-E). \tag{35}$$

Hence

$$\begin{aligned} \frac{d}{dt}E &\leqq R_0\left[\left(\int_{T^2}\psi^2 dx\right)^{1/2}\left(\int_{T^2} v_x^2\, dx\right)^{1/2} + \left(\int_{T^2}\varphi^2 dx\right)^{1/2}\left(\int_{T^2} v_y^2 dx\right)^{1/2}\right] \\ &\quad - 2\nu E \leqq 4R_0(N-E)^{1/2}E^{1/2} - 2\nu E. \end{aligned} \tag{36}$$

Differential inequality (31) and (36) are linear in $(N-E)^{1/2}$ and $E^{1/2}$, can be easily solved, and give the statement of the theorem.

General Case. First we discuss the stationary state. We prove that

$$\sup_{x\in T^2} |\bar{u}| = H_1 < \infty, \tag{37}$$

$$\int_{T^2} |\bar{u}|^2 dx = H_2 < \infty, \tag{38}$$

$$\sup_{x\in T^2} |\bar{\omega}| = \sup_{x\in T^2} |\partial_x \bar{u}_y - \partial_y \bar{u}_x| = H_3 < \infty. \tag{39}$$

In fact

$$\int_{T^2} \bar{\omega}[-(\bar{u}\cdot\nabla)\bar{\omega} + F + \nu\Delta\bar{\omega}]dx = 0, \tag{40}$$

hence

$$\nu\int_{T^2} (\nabla\omega)^2 dx = \int_{T^2} \bar{\omega}F\, dx \leqq c_1 H_3\left[\int_{T^2} F^2\, dx\right]^{1/2}. \tag{41}$$

By the Cauchy-Schwartz inequality

$$\int_{T^2} (\nabla\bar{\omega})^2 dx \geqq c_2\left(\int_{T^2} |\nabla\bar{\omega}| dx\right)^2 \geqq c_3 H_3^2. \tag{42}$$

So

$$H_3 \leqq c_4\left(\int_{T^2} F^2/\nu^2\, dx\right)^{1/2}. \tag{43}$$

From now on we indicate with c_i a numerical constant.

Equation (37) is a consequence of (43) and (27). Equation (38) can be proved in a similar way using (1).

Now we put

$$\underline{\bar{u}}=\underline{u}_0+\underline{u}_1\,; \qquad \underline{u}_1=\underline{\hat{u}}_0+\underline{\hat{u}}\,, \tag{44}$$

where $\underline{u}_0$ is defined in (12).

We consider the Fourier development of $\underline{u}_1$. $\underline{\hat{u}}_0$ is given by the first terms of the form

$$\begin{gathered}\underline{\hat{u}}_0=\underline{\hat{a}}_{10}\cos x+\underline{\hat{a}}_{01}\cos y+\underline{b}_{01}\sin y+\underline{c}_{10}\sin x\,,\\ \operatorname{div}\underline{\hat{u}}_0=0\,,\end{gathered} \tag{45}$$

and $\underline{\hat{u}}$ contains all remaining terms.

We note that

$$\sup_{\underline{x}\in T^2,\,i,\,j}|\partial_i\hat{u}_{0j}|=G<\infty\,, \tag{46}$$

as we can see by (38) and the explicit form of $\underline{\hat{u}}_0$.

Moreover

$$\sup_{\underline{x}\in T^2}|\underline{\hat{u}}|=D_1\,, \tag{47}$$

$$\sup_{\underline{x}\in T^2}|\hat{\eta}|=\sup_{\underline{x}\in T^2}|\partial_x\hat{u}_y-\partial_y\hat{u}_x|=D_2\,, \tag{48}$$

and D_1, D_2 go to zero when r_1, r_2 vanish.

We prove (48).

$$\int_{T^2}\underline{u}_1\cdot[-(\underline{\bar{u}}\cdot\underline{\nabla})\underline{\bar{u}}+\underline{f}+\nu\Delta\underline{\bar{u}}]d\underline{x}=0\,. \tag{49}$$

Hence

$$\nu\int_{T^2}(\underline{\nabla}\underline{u}_1)^2d\underline{x}=\int_{T^2}\underline{u}_1\cdot\underline{f}_1\,d\underline{x}-\int_{T^2}[\underline{u}_1\cdot(\underline{u}_1\cdot\underline{\nabla})\underline{u}_0]d\underline{x}\,. \tag{50}$$

For the vorticity we obtain

$$\int_{T^2}\eta[-(\underline{\bar{u}}\cdot\underline{\nabla})\bar{\omega}+F+\nu\Delta\bar{\omega}]d\underline{x}=0\,, \tag{51}$$

where

$$\eta=\partial_x u_{1y}-\partial_y u_{1x}\,. \tag{52}$$

Hence

$$\nu\int_{T^2}(\underline{\nabla}\eta)^2d\underline{x}=-\int_{T^2}\eta(\underline{u}_1\cdot\underline{\nabla})\omega_0\,d\underline{x}+\int_{T^2}\eta F_1\,d\underline{x}\,. \tag{53}$$

We subtract (50) from (53):

$$\begin{aligned}\nu\int_{T^2}[(\underline{\nabla}\eta)^2-(\underline{\nabla}\underline{u}_1)^2]d\underline{x}&=\int_{T^2}[\eta F_1-\underline{u}_1\cdot\underline{f}_1]d\underline{x}\\ &\leqq c_5(2R_0+G+H_3)\left[\left(\int_{T^2}F_1^2\,d\underline{x}\right)^{1/2}+\left(\int_{T^2}\underline{f}_1^2\right)^{1/2}\right],\end{aligned} \tag{54}$$

hence

$$\int_{T^2}(\underline{\nabla}\hat{\eta})^2d\underline{x}\leqq c_6(2R_0+G+H_3)\,(r_1^{1/2}+r_2^{1/2})\,, \tag{55}$$

and then

$$D_2 \leqq c_7[(2R_0+G+H_3)(r_1^{1/2}+r_2^{1/2})]^{1/2}. \tag{56}$$

So (48) is proved. Equation (47) is a consequence of (56) and (27).

We consider now the non equilibrium problem. By a direct computation

$$\frac{dE}{dt} = -\int_{T^2} \underline{v}\cdot(\underline{v}\cdot\underline{\nabla})\underline{u}\,d\underline{x} - \nu\int_{T^2}(\underline{\nabla}\underline{v})^2 d\underline{x}, \tag{57}$$

$$\frac{dN}{dt} = -\int_{T^2} \delta(\underline{v}\cdot\underline{\nabla})\omega\,d\underline{x} - \nu\int_{T^2}(\nabla\delta)^2 d\underline{x}. \tag{58}$$

Hence

$$\begin{aligned}\frac{d}{dt}(N-E) &= -\int_{T^2}[\delta(\underline{v}\cdot\underline{\nabla})\hat{\eta} - \underline{v}\cdot(\underline{v}\cdot\underline{\nabla})\hat{\underline{u}}]d\underline{x} - \nu\int_{T^2}[(\underline{\nabla}\delta)^2-(\underline{\nabla}\underline{v})^2]d\underline{x}\\ &= \int_{T^2}[\hat{\eta}(\underline{v}\cdot\underline{\nabla})\delta - \hat{u}_x(v_x\partial_x v_x + v_y\partial_y v_x)\\ &\quad - \hat{u}_y(v_x\partial_x v_y + v_y\partial_y v_y)]d\underline{x} - \nu\int_{T^2}[(\underline{\nabla}\delta)^2-(\underline{\nabla}\underline{v})^2]d\underline{x}.\end{aligned} \tag{59}$$

Using (47), (48), (27), and (22), we have

$$\begin{aligned}\frac{d}{dt}(N-E) &\leqq c_8 D_2 E^{1/2}\left[\int_{T^2}(\underline{\nabla}\delta)^2 d\underline{x}\right]^{1/2} + c_9 D_1 E^{1/2}N^{1/2} - 4(\nu - c_8 D_2)(N-E)\\ &\quad - c_8 D_2\int_{T^2}(\underline{\nabla}\delta)^2 d\underline{x} + 2c_8 D_2 N\\ &\leqq c_{10}(D_1+D_2)N - 4(\nu - c_8 D_2)(N-E).\end{aligned} \tag{60}$$

We divide $\underline{v}$ as in (44),

$$\begin{aligned}\left|\int_{T^2}\delta(\underline{v}\cdot\underline{\nabla})(\omega_0+\hat{\eta}_0)d\underline{x}\right| &= \left|\int_{T^2}\underline{v}\cdot(\underline{v}\cdot\underline{\nabla})(\underline{u}_0+\hat{\underline{u}}_0)d\underline{x}\right|\\ &= \left|\int_{T_2} v_x v_y[\partial_y(u_{0x}+\hat{u}_{0x}) + \partial_x(u_{0y}+\hat{u}_{0y})]d\underline{x}\right|\\ &\leqq c_{11}(R_0+G)E^{1/2}(N-E)^{1/2}.\end{aligned} \tag{61}$$

Hence

$$\begin{aligned}\frac{dN}{dt} &\leqq c_{11}(R_0+G)E^{1/2}(N-E)^{1/2} + c_8 D_2 E^{1/2}\left[\int_{T^2}(\underline{\nabla}\delta)^2 d\underline{x}\right]^{1/2}\\ &\quad - \nu\int_{T^2}(\underline{\nabla}\delta)^2 d\underline{x} \leqq c_{11}(R_0+G)N^{1/2}(N-E)^{1/2} - 2(\nu - c_8 D_2)N.\end{aligned} \tag{62}$$

When D_1, D_2 are small enough differential inequalities, (60) and (62) imply $N\to 0$ and $(N-E)\to 0$ exponentially. For a proof we note that the more difficult case is realized when the equality is reached. We combine the two equations so obtained,

$$\begin{aligned}\frac{d}{dt}[N+\alpha(N-E)] &= \alpha c_{10}(D_1+D_2)N + c_{11}(R_0+G)N^{1/2}(N-E)^{1/2}\\ &\quad - 2(\nu - c_8 D_2)[N+2\alpha(N-E)].\end{aligned} \tag{63}$$

We can choose $\alpha>0$ such that $\exists\gamma>0$,

$$\frac{d}{dt}[N+\alpha(N-E)]\leqq\alpha c_{10}(D_1+D_2)[N+\alpha(N-E)]-\gamma[N+\alpha(N-E)]$$

for

$$\nu>c_8 D_2 . \tag{64}$$

For D_1, D_2 small enough the theorem is proved. □

In conclusion, we have proved that this model has no turbulence for a particular force $\underline{f}_0$. Moreover, for any fixed R, the stability property remains valid for a neighborhood of $\underline{f}_0$. Of course this does not exclude that for fixed $\underline{f}\neq\underline{f}_0$ and large R chaotic motion may appear. For instance, for truncated Navier-Stokes equation numerical studies proved that our model with a convenient force, although simple and without boundary, can produce a rich phenomenology [2].

Remark. The same result of Theorem 1 can be obtained in an asymmetric flat torus $[0,L]\times[0,2\pi]$ when $L\leqq 2\pi$ and $\underline{f}=\underline{c}_1\nu(A_1\cos y+A_2\sin y)$ $A_1, A_2\in\mathbb{R}$. The proof is similar to the previous one. Note that with our technique the condition $L\leqq 2\pi$ is essential for the nonnegative definiteness of $N-E$.

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Note added in proof. For $L<2\pi$ it is possible to show a set of attractive stationary states of size and radius of attraction independent of R. The proof will be given elsewhere.