The Spectrum of a Schrödinger Operator in
$L_p(\mathbb{R}^n)$ is $p$-Independent

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Abstract. Let $H_p = -\frac{1}{2} \Delta + V$ denote a Schrödinger operator, acting in $L_p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. We show that $\sigma(H_p) = \sigma(H_2)$ for all $p \in [1, \infty]$, for rather general potentials $V$.

Introduction. In [12, 13], B. Simon conjectured that $\sigma(H_p)$ is $p$-independent, where $H_p = -\frac{1}{2} \Delta + V$ is a general Schrödinger operator in $L_p(\mathbb{R}^n)$. Partial results on this problem are contained in Simon [12], Sigal [10], Hempel, Voigt [5].

In the notations of Sect. 1, our main result reads as follows.

Theorem. Let $V = V_+ - V_-$, $V_+ \geq 0$, where $V_+$ is admissible, and $V_- \in \mathcal{K}_c$ with $c_\nu(V_-) < 1$. Then $\sigma(H_p) = \sigma(H_2)$ for $1 \leq p \leq \infty$.

In addition, if $\lambda$ is an isolated eigenvalue of finite algebraic multiplicity $k$ of $H_p$, for some $p \in [1, \infty]$, then the same is true for all $p \in [1, \infty]$.

The proof of this result is contained in Propositions 2.1, 3.1, and 2.2.

In Sect. 2 we prove the inclusion $\sigma(H_2) \subseteq \sigma(H_p)$, following ideas of Simon and Davies.

In Sect. 3 we show that the integral kernel of $(H_2 - z)^{-n}$, for $n \in \mathbb{N}$, $n > v/2$, defines an analytic $\mathcal{B}(L_p(\mathbb{R}^n))$-valued function on $\rho(H_2)$, which coincides with $(H_p - z)^{-n}$ for $z$ real and sufficiently negative. This implies $\sigma(H_p) \subset \sigma(H_2)$, by unique continuation.

A different situation, where an integral kernel determines operators with $p$-dependent spectrum, can be found in Jörgens [6; IV, Aufg. 12.11 (b)]; note that the kernel in Jörgens' example is the resolvent kernel of the differential operator

$$-rac{d}{dx}x^2\frac{d}{dx}$$
on $$(0, \infty), \text{ at } z = -2.$$
(analogously, \( C_c^\infty \equiv C_c^\infty (\mathbb{R}^n) \), etc.). For \( t \in \mathbb{C} \), \( \text{Re } t > 0 \), we define \( k_t \in L_1 \) by
\[
k_t(x) \equiv (2\pi t)^{-n/2} \exp(-|x|^2/2t).
\]
For \( 1 \leq p \leq \infty \) we define \( U_{0,p}(t) \in \mathcal{B}(L_p) \) \((t \in \mathbb{C}, \text{Re } t > 0)\) by
\[
U_{0,p}(t)f \equiv k_t * f \quad (f \in L_p),
\]
and further \( U_{0,p}(0) = I \). For \( 1 \leq p < \infty \), \( U_{0,p}(-) \) is a holomorphic semigroup of angle \( \pi/2 \); let \(-H_{0,p}\) denote its generator. Further denote \( H_{0,\infty} = H_{0,1} \).

Next we introduce the class of potentials \( V \) to be considered in this paper. Following Voigt [14], we define classes of potentials by
\[
\mathcal{K}_\nu := \{ V \in L_1, \text{loc}; \text{ess sup } \int_{|x-y| \leq 1} |g_\nu(x-y)||V(y)|dy < \infty \},
\]
where \( g_\nu \) is the usual fundamental solution of \( \Delta \). Note that this class is slightly larger than the class \( \mathcal{K}_\nu \) in Aizenman, Simon [1], Simon [13]. For \( V \in \mathcal{K}_\nu \) we define
\[
c_\nu(V) \equiv \lim_{\alpha \to 0} \text{(ess sup)} \int_{|x-y| \leq \alpha} |g_\nu(x-y)||V(y)|dy.
\]
Obviously \( \mathcal{K}_\nu \subset L_1, \text{loc, unif} \) for all \( \nu \in \mathbb{N} \), \( \mathcal{K}_1 \equiv L_1, \text{loc, unif} \), and \( c_\nu(V) = 0 \) for all \( V \in \mathcal{K}_1 \).

A potential \( V \geq 0 \) will be called admissible if \( Q(H_{0,2}) \cap Q(V) \) is dense in \( L_2 \); cf. Voigt [14]. In particular, \( V \geq 0 \) is admissible if \( V \in L_1, \text{loc}(G) \), where \( G = \mathbb{R}^n \setminus \mathbb{R}^n \) is such that \( \mathbb{R}^n \setminus G \) is a (closed) set of Lebesgue measure zero.

Throughout this paper we shall assume
\[
V = V_+ - V_-, \quad V_\pm \geq 0, \quad V_- \in \mathcal{K}_\nu \quad \text{with } c_\nu(V_-) < 1, \quad V_+ \text{ admissible.}
\]
In the following proposition we denote the truncation of \( V \) by
\[
V^{(n)} := (\text{sgn } V)(|V| \wedge n) \quad (n \in \mathbb{N}).
\]

1.1. Proposition. Let \( V \) satisfy (1.1), and let \( 1 \leq p < \infty \). Then, for \( t \geq 0 \), the limit
\[
U_p(t) := s - \lim_{n \to \infty} \exp(-t(H_{0,p} + V^{(n)}))
\]
exists, and \((U_p(t); t \geq 0)\) is a \( C_0 \)-semigroup on \( L_p \). The Feynman-Kac formula
\[
U_p(t)f(x) = E_x \left\{ \exp \left( - \int_0^t V(b(s))ds \right) f(b(t)) \right\}
\]
holds for all \( f \in L_p \).

Here, \( E_x \) and \( b(\cdot) \) are as in Simon [13]; cf. Reed, Simon [9], Simon [11]. The proof of this proposition can be found in Voigt [14; Proposition 5.8(a), Proposition 2.8, Remark 5.2(b), Proposition 3.2, Proposition 6.1(c)].

We denote the generator of \((U_p(t); t \geq 0)\) by \(-H_p\), for \( 1 \leq p < \infty \), and we shall henceforth write \( U_p(t) = \exp(-tH_p) \). Also, \( H_\infty = H^* \). More detailed information about the operators \( H_p \), in particular for \( p = 1, p = 2 \) can be found in Voigt [14].
Note that \( H_2 \) is the form sum of \(-\frac{1}{2}\Delta\) and \( V\); cf. Voigt [14; Remark 6.2(c)]. (It follows from Devinatz [3; Lemma 4] that \( V_- \) is \( H_{0,2} \)-form small.)

2. \( \sigma(H_2) \subset \sigma(H_p) \)

In this section we show that interpolation, duality, and \( p-q \)-smoothing lead to the following result.

2.1. Proposition. Let \( V \) satisfy (1.1). Then \( \rho(H_p) \subset \rho(H_2) \) for all \( p \in [1, \infty] \), and

\[
(H_p - z)^{-1} \mid_{L_p \cap L_2} = (H_2 - z)^{-1} \mid_{L_p \cap L_2} \quad (z \in \rho(H_p)).
\]

This result was stated in Simon [12, 13]. The argument given there was based on interpolation between the resolvents \((H_p - z)^{-1}\) and \((H_p - z)^{-1}\), for \( z \in \rho(H_p) = \rho(H_p) \). It is not immediate, however, that these resolvents coincide on \( L_p \cap L_q \), as can be seen from Jörgens’ example mentioned in the introduction. This gap in Simon’s argument was closed by E. B. Davies (private communication). Compare also Hempel, Voigt [5; Proposition 3.1].

Proof of Proposition 2.1. (i) (due to E. B. Davies) Let \( 1 \leq p < q \leq \infty \), \( t > 0 \). Then \( e^{-tH_p} \in \mathcal{B}(L_p, L_q) \); cf. Voigt [14; Proposition 6.3]. This implies

\[
e^{-tH_p}H_p \subset H_qe^{-tH_p}. \tag{2.1}
\]

Assume additionally \( \lambda \in \rho(H_p) \cap \rho(H_q) \). Then (2.1) implies

\[
(H_q - \lambda)^{-1} e^{-tH_p} = e^{-tH_p}(H_p - \lambda)^{-1}.
\]

For \( t \to 0 \) we obtain

\[
(H_p - \lambda)^{-1} \mid_{L_p \cap L_q} = (H_q - \lambda)^{-1} \mid_{L_p \cap L_q}. \tag{2.2}
\]

(This holds also for \( q = \infty \) because \( e^{-tH_q}f \) is \( \sigma(L_{\infty}, L_1) \)-continuous for \( f \in L_p \cap L_\infty \).)

(ii) Let \( 1 \leq p \leq 2 \), \( 1/p + 1/p' = 1 \), and let \( \lambda \in \rho(H_p) = \rho(H_p) \). Then \((H_p - \lambda)^{-1} \mid_{L_p \cap L_q} = (H_p - \lambda)^{-1} \mid_{L_p \cap L_p'} \), by (2.2). The Riesz–Thorin convexity theorem implies that \((H_p - \lambda)^{-1} \) is continuous as an operator \( R_\lambda \) on \( L_2 \).

For \( f \in L_2 \cap L_p' \), (2.1) implies

\[
(H_2 - \lambda)e^{-tH_q}(H_p - \lambda)^{-1} f = e^{-tH_p}f.
\]

For \( t \to 0 \) we obtain \((H_2 - \lambda) (H_p - \lambda)^{-1} f = f \). This implies \((H_2 - \lambda)R_\lambda = I\), and hence \( \lambda \in \rho(H_2) \).\]

2.2. Proposition. Let \( V \) satisfy (1.1), and let \( 1 \leq p \leq \infty \). Assume that \( \lambda \) is an isolated point of \( \sigma(H_p) \). Then \( \lambda \) is an eigenvalue of \( H_p \) with finite algebraic multiplicity if and only if the same is true for \( H_2 \). In this case, \( \lambda \) is real and a pole of first order of the resolvents of \( H_p \) and \( H_2 \), and the multiplicities of \( \lambda \) as an eigenvalue of \( H_p \) and \( H_2 \) coincide.

Proof. Without restriction \( p < \infty \). (Duality for \( p = \infty \).) Note first that the selfadjoint operator \( H_2 \) can only have real eigenvalues which are poles of first order of the resolvent of \( H_2 \). Now the assertions follow from Proposition 2.1 and Auterhoff [2; Theorem 1.5]; see also Hempel, Voigt [5; Theorem 1.3].\]
3. \( \sigma(H_p) \subset \sigma(H_2) \)

In this section we shall derive properties of the integral kernel of \((H_2 - z)^{-n}\), for \(n \in \mathbb{N}, n > v/2\), in order to show the following result.

3.1. Proposition. Let \( V \) satisfy (1.1). Then \( \rho(H_2) \subset \rho(H_p) \), for all \( p \in [1, \infty] \).

The proof relies on the following two auxiliary results which will be proved below.

3.2. Lemma. Let \( X \) be a Banach space, \( T \) a closed operator in \( X \), \( \rho(T) \neq \emptyset \). Then \( \rho(T) \) is the domain of holomorphy of \((T - z)^{-n}\), for \( n = 1, 2, \ldots \).

3.3. Proposition. Let \( V \) satisfy (1.1), and let \( n \in \mathbb{N}, n > v/2 \).

(a) Then \((H_2 - z)^{-n}\) is an integral operator, for \( z \in \rho(H_2) \).

(b) Let \( G^{(n)}(x, y; z) \) denote the integral kernel of \((H_2 - z)^{-n}\). Then, for any \( K \subset \subset \rho(H_2)^1 \) there exist constants \( C, \eta > 0 \) such that

\[
|G^{(n)}(x, y; z)| \leq Ce^{-\eta|x-y|} \quad (z \in K, x, y \in \mathbb{R}^n).
\]

Proof of Proposition 3.1. By duality, it is sufficient to consider the case \( 1 \leq p \leq 2 \). Fix \( n \in \mathbb{N}, n > v/2 \), and let \( G^{(n)}(x, y; z) \) be as in Proposition 3.3.

First we show that \( G^{(n)}(\cdot, \cdot; z) \) defines an analytic \( \mathcal{B}(L_p) \)-valued function \( G^{(n)}(z) \) on \( \rho(H_2) \). To prove this, we remark that for any \( \phi, \psi \in C_0^\infty \), the mapping

\[
\rho(H_2) \ni z \mapsto \mathcal{F} \int \int G^{(n)}(x, y; z)\phi(y)\psi(x)dx\,dy
\]

is holomorphic. Furthermore, for any \( K \subset \subset \rho(H_2) \), there exists a constant \( C' \) such that

\[
\|G^{(n)}(z)\|_{\mathcal{B}(L_p)} \leq C' \quad (z \in K),
\]

by the estimates in Proposition 3.3(b) and Young's inequality (cf. Reed, Simon [9; p. 32]).

Next, the fact that \( e^{-\mathcal{H}_p} \) coincides with \( e^{-\mathcal{H}_2} \) on \( L_p \cap L_2 \) implies that \( G^{(n)}(z) \) coincides with \((H_p - z)^{-n}\) for \( z \) real and sufficiently negative.

It follows by unique continuation that the domain of holomorphy of \((H_p - z)^{-n}\) contains \( \rho(H_2) \). Hence, \( \rho(H_p) \supset \rho(H_2) \), by Lemma 3.2 above. \( \blacksquare \)

Let us now prove the auxiliary results.

Proof of Lemma 3.2. Clearly, \((T - z)^{-n}\) is holomorphic on \( \rho(T) \). Let \( \text{spr}(A) \) denote the spectral radius of an operator \( A \in \mathcal{B}(X) \). From the well-known facts (cf. Kato [7; p. 27, p. 37])

\[
\text{spr}((T - \zeta)^{-1}) = \inf_{n \in \mathbb{N}} \| (T - \zeta)^{-n} \|^{1/n},
\]

\[
\text{spr}((T - \zeta)^{-1}) \geq \text{dist}(\zeta, \sigma(T))^{-1} \quad (\zeta \in \rho(T)),
\]

it is clear that \( \| (T - \zeta)^{-n} \| \geq \text{dist}(\zeta, \sigma(T))^{-n} \) (\( \zeta \in \rho(T) \)). \( \blacksquare \)

For several reasons, we include a proof of Proposition 3.3 (instead of simply referring to Simon [13; Theorem B.7.1 (c')]): The estimate given in [13; loc. cit.] is

\[ K \subset \subset \rho(H_2) \text{ means: } K \text{ compact and } K < \rho(H_2) \]
not uniform for $z \in K \subset \rho(H_2)$ (although one might be willing to believe that it must be true). Also, the proof of the (essential) Lemma B.7.11 in [13] is very sketchy, and it is our aim to give a complete proof of reasonable length. Finally, our proof will show that it is advantageous to consider $(H_p - z)^{-n}, n > \nu/2, n \in \mathbb{N}$, instead of arguing with $(H_p - z)^{-1}$ directly (which would be possible, but involve more estimates, like [13; Theorem B.7.2 (1), (2), (4)]).

Since we shall have to consider $e^{-tH_p}$ as an operator from $L_p$ to $L_q$, $q \geq p$, we shall frequently drop the subscript $p$ and simply write $H = -\frac{1}{2}\Delta + V$, in the sequel. The proof will involve several steps, following rather closely the outline given in [13; proof of Lemma B.7.11]. For the remainder of this section, the assumptions of Proposition 3.3 are always assumed to hold.

3.4. Lemma. Let $1 \leq p \leq q \leq \infty, \varepsilon_0 > 0$. Then there exist constants $C = C(p, q, \varepsilon_0)$, $A = A(p, q, \varepsilon_0)$, such that for $\varepsilon \in \mathbb{R}^\nu, |\varepsilon| \leq \varepsilon_0, t > 0$, we have

$$\| e^{\varepsilon^t \Delta} e^{-\varepsilon^t} \|_{p, q} \leq Ct^{-\gamma} e^{At},$$

where $\gamma = (\nu/2)(p^{-1} - q^{-1})$.

Proof (compare Simon [13; Lemma B.6.1]). Let $\varepsilon \in \mathbb{R}^\nu, |\varepsilon| \leq \varepsilon_0$. Clearly,

$$K_\varepsilon(x, y; t) := (2\pi t)^{-\nu/2} e^{\varepsilon^t (x - y)} \exp\left(-\frac{|x - y|^2}{2t}\right),$$

is the kernel of $e^{\varepsilon^t \Delta} e^{-\varepsilon^t}$. By Young’s inequality (cf. Reed, Simon [9; p. 32]), it is enough to estimate $\| K_\varepsilon(0, \cdot; t) \|_s$, for $s = (1 + q^{-1} - p^{-1})^{-1}$. Now,

$$\| K_\varepsilon(0, \cdot; t) \|_s \leq ct^{-\gamma(1-s)^{-1}} \left[ \int_{\mathbb{R}^\nu} e^{\nu \eta_1^t |\eta|^2} d\eta \right]^{1/s},$$

and the term in square brackets can be estimated by

$$\int_{|\eta| \leq 4\varepsilon_0 \sqrt{t}} e^{\eta_1^t |\eta|^2} d\eta + \int_{|\eta| > 4\varepsilon_0 \sqrt{t}} e^{-(s/4)|\eta|^2} d\eta \leq c't^{-\gamma} e^{At} + c''.$$

3.5. Proposition (compare [13; Eq. (B11)]). For all $1 \leq p \leq q \leq \infty$ there exist constants $C = C(p, q)$, $A = A(p, q)$ such that for all $t > 0$ we have

$$\| e^{-tH} \|_{p, q} \leq Ct^{-\gamma} e^{At},$$

where $\gamma = (\nu/2)(p^{-1} - q^{-1})$.

Proof. This follows from Devinatz [3; Lemma 2] combined with duality and interpolation as described in Voigt [14; proof of Proposition 6.3]. Under the slightly stronger assumption $c_\nu(V) = 0$ a simpler proof can be found in Simon [13; loc. cit.]

3.6. Lemma (compare [13; Lemma B.6.2(b)]). Let $1 < c < c_\nu(V)^{-1}, 1/c + 1/c' = 1$. Then, for any $\varepsilon \in \mathbb{R}^\nu$,

$$\| e^{\varepsilon^t} e^{-tH} e^{-\varepsilon^t} \|_{p, q} \leq \| e^{-(1/2)\Delta + cV} \|_{1/c} \| e^{c' \varepsilon^t} e^{(t/2)\Delta} e^{-\varepsilon^t} \|_{1/c'}.$$

Proof. Let $\varepsilon \in \mathbb{R}^\nu$ and write $w(x) = e^{\varepsilon^t}$. Also, let $h \in C_0^\infty, g := w^{-1} h$. Factorizing
\[ |g| = |h|^{1/c} |w^{-\varepsilon}h|^{1/c}, \]
it follows by Hölder’s inequality in function space that
\[ |(e^{-t\varepsilon H} g)(x)| \leq \left[ (e^{-t\varepsilon (1/2) A + cV}) |h||x|^{1/c} \left[ (e^{t\varepsilon (1/2) A} w^{-\varepsilon} h) \right](x) \right]^{1/c}. \]

Now, multiplying by \(|w(x)|\), taking \(q\)th powers and integrating, we obtain
\[
\int |w e^{-t\varepsilon H} w^{-1} h|^q dx \leq \int \left| \left( e^{-t\varepsilon (1/2) A + cV} |h||x|^{1/c} \left[ (e^{t\varepsilon (1/2) A} w^{-\varepsilon} h) \right] \right|^q dx \leq \left\{ \left[ (e^{-t\varepsilon (1/2) A + cV} |h||x|^{1/c} \left[ (w e^{t\varepsilon (1/2) A} w^{-\varepsilon} h) |h| \right]^q dx \right] \right\}^{1/c},
\]
which implies
\[ \| w e^{-t\varepsilon H} w^{-1} h \|_q \leq \| e^{-t\varepsilon (1/2) A + cV} \|^{1/c} \| h \|^{1/c} \| w e^{t\varepsilon (1/2) A} w^{-\varepsilon} h \|^{1/c}. \]

3.7. Proposition (compare [13; Theorem B.6.3]). Let \(1 \leq p \leq q \leq \infty, \alpha > \gamma = (v/2)(p^{-1} - q^{-1})\), and \(\varepsilon_0 > 0\). Then, for \(z\) real and sufficiently negative, there exists a constant \(C\) such that
\[ \left\| e^{-t\varepsilon} (H - z)^{-\alpha} e^{-\varepsilon x} \right\|_{p,q} \leq C \quad (\varepsilon \in \mathbb{R}, |\varepsilon| \leq \varepsilon_0). \]

Proof. For \(\phi \in C_c^\infty\), we have (with \(w := e^{-\varepsilon x}\))
\[ (H - z)^{-\alpha} (w^{-1} \phi) = c_2 \int_0^\infty e^{tz} t^{\alpha - 1} e^{-tH(w^{-1} \phi)} dt, \]
and hence
\[ \| w(H - z)^{-\alpha} w^{-1} \phi \|_q \leq c_2 \int_0^\infty \| w e^{-tH} w^{-1} \|_{p,q} e^{tz} t^{\alpha - 1} dt \cdot \| \phi \|_p \]
\[ \leq c_2 \int_0^\infty \| e^{-t\varepsilon (1/2) A + cV} \|^{1/c} \| w e^{t\varepsilon (1/2) A} w^{-\varepsilon} h \|^{1/c} \| e^{tz} t^{\alpha - 1} dt \cdot \| \phi \|_p \]
(by Lemma 3.6)
\[ \leq c_3 \int_0^\infty \| C_1 t^{-\gamma} e^{At} \|^{1/c} \| C_2 t^{-\gamma} e^{At} \|^{1/c} e^{tz} t^{\alpha - 1} dt \cdot \| \phi \|_p \]
(by Proposition 3.5 and Lemma 3.4)
\[ \leq C_3 \int_0^\infty t^{-\gamma + \alpha - 1} e^{At + tz} dt \cdot \| \phi \|_p \leq C_4 \cdot \| \phi \|_p, \]
provided \(A + z < 0\).

3.8. Proposition. For any \(K \subset \subset \rho(H_2)\), there exist \(\varepsilon_0 = \varepsilon_0(K) > 0\) and a constant \(C = C(K, \varepsilon_0)\) such that \(K \subset \subset \rho(e^{t\varepsilon X} H_2 e^{-t\varepsilon X})\) for \(|\varepsilon| \leq \varepsilon_0\), and
\[ \| e^{t\varepsilon X} (H_2 - z)^{-1} e^{-t\varepsilon X} \| = \| (e^{t\varepsilon X} H_2 e^{-t\varepsilon X} - z)^{-1} \| \leq C \quad (|\varepsilon| \leq \varepsilon_0, z \in K). \]

Proof. As \(W_{1/2}^1\) contains the form domain of \(H_2\), the operators \(A_j\) are \(|H_2|^{1/2}\)-bounded and hence \(H_2\)-bounded with relative bound zero \((j = 1, \ldots, v)\). This implies
\[ e^{t\varepsilon X} H_2 e^{-t\varepsilon X} = H_2 + \varepsilon \cdot \nabla - \frac{1}{2} \varepsilon^2, \]
for all \(\varepsilon \in \mathbb{R}\). Now the identity
\[ (H_2 + \varepsilon \cdot \nabla - \frac{1}{2} \varepsilon^2 - z) = (I + (\varepsilon \cdot \nabla - \frac{1}{2} \varepsilon^2)(H_2 - z)^{-1}) (H_2 - z) \]
implies the desired conclusion.

We can now finally proceed to the proof of Proposition 3.3.

**Proof of Proposition 3.3.** Fix $n \in \mathbb{N}$, $n > n/2$, and choose $w$ real and so negative that, by Proposition 3.7,

$$\| e^{εx}(H - w)^{-n/2} e^{-εx} \|_{1, 2} + \| e^{εx}(H - w)^{-n/2} e^{-εx} \|_{2, \infty} \leq C$$  

(3.1)

for all $|ε| \leq 1$, with some constant $C$.

Now let $K \subset \rho(H^2)$ and $z \in K$. Taking $n^{th}$ powers of the resolvent equation

$$(H^2 - z)^{-1} = (H^2 - w)^{-1} + (z - w)(H^2 - w)^{-1}(H^2 - z)^{-1},$$

we obtain

$$(H^2 - z)^{-n} = (H^2 - w)^{-n} \sum_{j=0}^{n} \binom{n}{j} (z - w)^j (H^2 - z)^{-j}.$$  

(3.2)

To prove Proposition 3.3, it is clearly enough to show that, for any $0 \leq j \leq n$, the operator

$$(H^2 - w)^{-n}(H^2 - z)^{-j} = (H^2 - w)^{-n/2}(H^2 - z)^{-j/2}(H^2 - w)^{-n/2}$$  

(3.3)

is an integral operator with kernel $G_{nj}(x, y; z)$ satisfying

$$|G_{nj}(x, y; z)| \leq C_n e^{-ζ_{nj}<x-y>^2} (z \in K, x, y \in \mathbb{R}^∗),$$  

(3.4)

with some positive constants $C_n$, $ζ_{nj}$.

So let $0 \leq j \leq n$. By Proposition 3.8, there exists $ε_0 > 0$ such that

$$\| e^{εx}(H^2 - z)^{-j} e^{-εx} \| \leq C' (|ε| \leq ε_0, z \in K).$$  

(3.5)

By (3.3) we have

$$e^{εx}(H^2 - w)^{-n}(H^2 - z)^{-j} e^{-εx} = (e^{εx}(H^2 - w)^{-n/2} e^{-εx})(e^{εx}(H^2 - z)^{-j/2}) e^{-εx},$$

and hence it follows from (3.1), (3.5), that

$$\| e^{εx}(H^2 - w)^{-n}(H^2 - z)^{-j} e^{-εx} \|_{1, \infty} \leq C'' (|ε| \leq ε_0, z \in K).$$

Now it follows from a classical theorem of Dunford and Pettis ([4; Theorem 2.2.5, p. 348]; see also Simon [13; Cor. A.1.2]), that the operator $e^{εx}(H^2 - w)^{-n}(H^2 - z)^{-j} e^{-εx}$ is an integral operator, and its kernel $G_{nj,ε}(x, y; z)$ satisfies the estimate

$$\| G_{nj,ε}(\cdot, \cdot, z) \|_{1, \infty} \leq C'' (|ε| \leq ε_0, z \in K).$$  

(3.6)

In particular, the above statements apply to $ε = 0$, and we see that $(H^2 - w)^{-n} \times (H^2 - z)^{-j}$ is an integral operator with $L_∞$-kernel $G_{nj}(\cdot, \cdot, z)$; clearly,

$$e^{ε(x - y)} G_{nj}(x, y, z) = G_{nj,ε}(x, y, z).$$

Therefore (3.6) implies

$$e^{ζ_0 (x - y)}|G_{nj}(x, y, z)| \leq C'' (z \in K).$$  

$\blacksquare$
Acknowledgement. One of the authors (R. H.) would like to thank P. Deift (Courant Institute, New York) for several useful discussions.

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Communicated by B. Simon

Received July 22, 1985; in revised form September 26, 1985