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The U(1) Higgs Model

I. The Continuum Limit

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Abstract. By using rigorous renormalization group methods we construct the continuum limit of the finite-volume lattice U(1) Higgs model in two and three dimensions. The method relies on a proof of the convergence of the effective action.

1. Introduction

Recently, renormalization group methods have been used to study lattice regularizations of Euclidean quantum field theories. In particular, Balaban has proved ultra-violet stability for the finite volume lattice Higgs model in three dimensions [Ba 1–4], obtaining bounds independent of the lattice spacing. In this paper we construct the continuum limit of this model in two and three dimensions. In a succeeding paper, the infinite volume limit will also be constructed, and some of the Osterwalder-Schrader axioms verified. This model was constructed previously in two dimensions by Brydges et al. [BFS 1–3].

The U(1) Higgs model is an interacting theory of a vector field $A_\mu(x)$ coupled in a gauge covariant way to a N-component scalar field $\phi(x)$. The classical (Euclidean) action of the model in d dimensions is

$$S(A,\phi)=\int d^dx\left\{1/4\sum_{\mu,\nu=1}^{d}|F_{\mu\nu}(x)|^2+1/2\sum_{\mu=1}^{d}|D_\mu\phi(x)|^2+1/2m^2|\phi(x)|^2+\lambda|\phi(x)|^4\right\}. \tag{1.1}$$

The field strength tensor is $F_{\mu\nu}(x)=\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)$, and the covariant derivative of the scalar field is

$$(D_\mu\phi)_i(x)=\partial_\mu\phi_i(x)-eA_\mu(x)\,(q\phi)_i(x)\,. \tag{1.2}$$

The coupling constants of the theory are e, λ and q is an antisymmetric $N\times N$ matrix. The action (1.1) is invariant under local gauge transformations: define

$$A_\mu^\chi=A_\mu(x)-\partial_\mu\chi(x)\,,\qquad \phi^\chi(x)=\exp[-eq\chi(x)]\phi(x)\,, \tag{1.3}$$

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where $\chi(x)$ is an arbitrary real function on R^d. Then

$$S(A,\phi)=S(A^\chi,\phi^\chi)\,. \tag{1.4}$$

We study the quantum field theory corresponding to the action (1.1) by using the lattice approximation introduced by Wilson [W 1]. The model is considered on a finite regular lattice T_ε, with spacing ε, and (1.1) is replaced by a (gauge-invariant) lattice action $S^\varepsilon(T_\varepsilon, A, \phi)$ used to define a functional integral

$$Z^\varepsilon(T_\varepsilon)=\int (dA)\,(d\phi)\exp[-S^\varepsilon(T_\varepsilon, A,\phi)]\,.$$

These cutoff functional integrals can be analyzed using a phase cell expansion, first introduced by Glimm and Jaffe [GJ 1]. On the lattice, the expansion is implemented using a block spin transformation. The lattice is divided into disjoint blocks, each containing L^d sites, for some small integer L. Every field configuration $\{A,\phi\}$ is separated into an average part $\{A_1,\phi_1\}$, which is constant over each block, and a fluctuation part $\{A',\phi'\}$, which varies within blocks. The integral over $\{A',\phi'\}$ is estimated using a standard weak coupling expansion, producing a new effective action $S^{(1),L\varepsilon}(T_{L\varepsilon}, A_1, \phi_1)$ for the average fields. Contributions from configurations where fields are large can be estimated, and a perturbative expansion derived for $S^{(1),L\varepsilon}$ in the small field region ("large" and "small" fields will be defined later). By repeating the block spin transformation k times, we produce an effective action $S^{(k),L^k\varepsilon}$ on the $L^k\varepsilon$-lattice. This program has been carried out by Balaban for the Higgs model in dimensions $d=2,3$, giving the ultraviolet stability bound [Ba 1–4].

The renormalization transformation can be understood from a different viewpoint if we re-scale every effective action $S^{(k)}$ to the unit lattice. When $k=0$, this is the original action; the re-scaling replaces the parameters m^2, λ, e of the theory by $m^2\varepsilon^2$, $\lambda\varepsilon^{4-d}$, $e\varepsilon^{2-d/2}$. So for $d<4$, the action is a small perturbation of the massless gaussian action given by

$$S_G(A,\phi)=\sum_x\left\{1/4\sum_{\mu,\nu=1}^{d}|\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)|^2+1/2\sum_{\mu=1}^{d}|\partial_\mu\phi(x)|^2\right\} \tag{1.5}$$

(∂_μ is the difference derivative on the unit lattice). We can imagine a space of unit lattice actions as in Fig. 1; the renormalization transformation takes every point along a trajectory in this space. The massless gaussian (1.5) at G is driven to a fixed point P in the subspace V.

In our case, the initial action $S^{(0)}$ is at B, outside the gaussian subspace, and is driven along a trajectory away from V (this is ultraviolet asymptotic freedom). Consider a fixed lattice spacing ε_0; when $L^k\varepsilon=\varepsilon_0$, this trajectory is at B', and the effective action (rescaled to the ε_0-lattice) is $S^{(k),\varepsilon_0}$. It is clear that if the initial action $S^{(0)}$ is defined on a smaller lattice, it is given by a point C closer to G. So taking the limit $\varepsilon\to 0$ produces a sequence of starting points converging to G. Corresponding to these, there is a sequence of effective actions $\{S^{(k),\varepsilon_0}\}$, $k=0,\ldots,\infty$, on the ε_0-lattice.

To construct the continuum theory, we need to know that $Z^\varepsilon(T_\varepsilon)$ converges to a finite limit as the initial points $B, C, \ldots$ converge to G. By using the renormalization transformation, this is equivalent to proving the convergence of $\{S^{(k),\varepsilon_0}\}$ as $k\to\infty$,

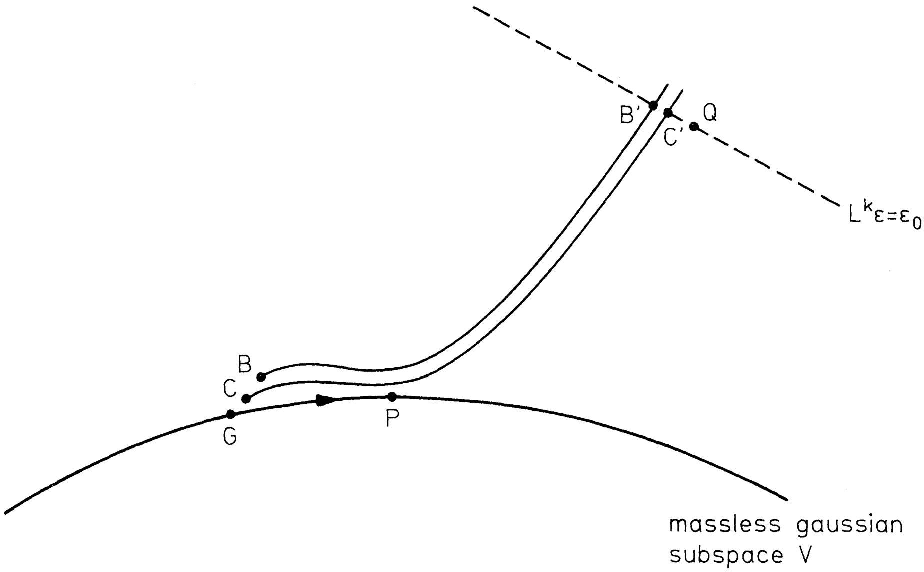


Fig. 1. Renormalization transformation trajectories in the space of unit lattice actions

or that the points B', C'... converge to some point Q. In this paper we prove that the perturbation expansions for $\{S^{(k),\varepsilon_0}(A,\phi)\}$ converge provided the fields A, ϕ are small. By taking ε_0 arbitrarily small this allows us to derive convergence of $Z^\varepsilon(T_\varepsilon)$ as $\varepsilon\to 0$.

We use the non-compact form of the action for the vector field, and also introduce an explicit vector field mass μ_0. However, the methods used in this paper apply to any other model analyzed using the renormalization transformation, for example to the compact formulation of the Higgs model under consideration by Balaban et al. [BBIJ 1], [BIJ 1].

The paper is organized as follows. Section 2 provides a summary of the renormalization transformation notation, and a statement of the main result. Section 3 contains the proof of convergence of the effective action, and hence of the partition function, and in Sect. 4, we derive the technical estimates used in the proof.

2. Notation and Results

2.1. Definition of the Model. Let Ω be a subset of εZ^d; the set of oriented bonds connecting nearest neighbor sites in Ω is denoted Ω^*. If $b\in\Omega^*$ has endpoints x and y, we write $b=\langle x,y\rangle=\langle b_-,b_+\rangle$. Then a vector field configuration on Ω is a map $A:\Omega^*\to R$; it can also be interpreted as a map $A:\Omega\to R^d$. We identify these meanings by $A(\langle x,x+\varepsilon e_\mu\rangle)=A_\mu(x)$, where e_μ is the unit vector in R^d parallel to the x_μ-axis. We assume that $A_{-b}=-A_b$. A scalar field configuration on Ω is a map $\phi:\Omega\to R^N$ for some $N\geqq 2$. The scalar and vector fields interact in Ω via a unitary operator on R^N: $U(A)=\exp(\varepsilon eqA)$, where e is the electric charge (coupling constant) and q is an $N\times N$ anti-symmetric matrix with $\|q\|=1$. The covariant derivative of the scalar field is

$$(D^{\varepsilon}_{A}\phi)(b)=\varepsilon^{-1}(U(A_b)\phi(b_+)-\phi(b_-)) \tag{2.1}$$

(we denote by ∂^{ε} the usual derivative on εZ^d). We introduce an inner product on Ω for n-component functions, any n:

$$\langle f,g\rangle=\sum_{i=1}^{n}\sum_{x\in\Omega}\varepsilon^d f_i(x)g_i(x). \tag{2.2}$$

With this notation, the covariant Laplacian on Ω is

$$1/2\langle\phi,(-\Delta^{\varepsilon,\Omega}_{A})\phi\rangle=1/2\sum_{b\in\Omega^*}\varepsilon^d|(D^{\varepsilon}_{A}\phi)(b)|^2. \tag{2.3}$$

The regularized action which we consider is

$$\begin{aligned}S^{\varepsilon}(\Omega,A,\phi)=&1/2\langle A,(-\Delta^{\varepsilon,\Omega}+\mu_0^2)A\rangle+1/2\langle\phi,(-\Delta^{\varepsilon,\Omega}_{A}+m^2)\phi\rangle\\&+\sum_{x\in\Omega}\varepsilon^d(1/2\delta m^2(x)|\phi(x)|^2+\lambda|\phi(x)|^4)+E_0+E_1,\end{aligned} \tag{2.4}$$

and the corresponding partition function is

$$Z^{\varepsilon}(\Omega)=\int(dA)(d\phi)\exp[-S^{\varepsilon}(\Omega,A,\phi)]. \tag{2.5}$$

Explicit expressions for the counterterms δm^2 and E_1 will be given later. E_0 is the normalization for the Gaussian measure:

$$\exp[E_0]=\int(dA)(d\phi)\exp[-1/2\langle A,(-\Delta^{\varepsilon,\Omega}+\mu_0^2)A\rangle-1/2\langle\phi,(-\Delta^{\varepsilon,\Omega}+m^2)\phi\rangle]. \tag{2.6}$$

Since we are interested in gauge invariant Schwinger functions, we shall introduce sources coupled to gauge invariant operators. Denote by Ω^{**} the set of plaquettes on Ω; a plaquette p is an elementary square of the lattice, written $p=\langle x,x+e\varepsilon_\mu,x+\varepsilon e_\mu+\varepsilon e_\nu,x+\varepsilon e_\nu\rangle$ with $\mu<\nu$. The field strength tensor on Ω is a map $F:\Omega^{**}\to R$ given by $F(p)=\sum_{b\in\partial p}\varepsilon^{-1}A_b$. For a function $g:\Omega^{**}\to R$, we define

$$F(g)=\sum_{p\in\Omega^{**}}\varepsilon^d F(p)g(p). \tag{2.7}$$

The gauge invariant operator for the scalar field is $|\phi|^2$; of course this must be normal-ordered in the generating functional. So for $h:\Omega\to R$, we have

$$:|\phi|^2:(h)=\sum_{x\in\Omega}\varepsilon^d h(x)\sum_{i=1}^{N}[\phi_i(x)^2-(-\Delta^{\varepsilon,\Omega}+m^2)^{-1}(x,x)], \tag{2.8}$$

and the generating functional with source g, h is

$$Z^{\varepsilon}(\Omega,g,h)=\int(dA)(d\phi)\exp[-S^{\varepsilon}(\Omega,A,\phi)+F(g)+:|\phi|^2:(h)]. \tag{2.9}$$

We shall assume that g, h are C^∞ functions on R^d with compact support. With abuse of notation, we also denote their restrictions to Ω by g, h. Throughout this paper, we shall use C to denote a positive constant which is either $O(1)$ or depends only on $\|g\|$, $\|h\|$.

2.2. Block Spin Transformations. Throughout this paper we will use the block spin transformations defined in [Ba 1], and the reader is assumed to be familiar with the idea of this method. We present below a collection of relevant notation and formulae connected with the method.

The block size is L, a small integer, and for any $\Omega \subset \varepsilon Z^d$ we write $\Omega^{(k)} = \Omega \cap L^k \varepsilon Z^d$. To each point $y \in L^k \varepsilon Z^d$ there corresponds a block of L^{kd} sites on εZ^d, which we define as $B^k(y) = \{x \in \varepsilon Z^d : y_i \leqq x_i < y_i + L^k \varepsilon,\ i = 1, \ldots, d\}$. The averaging operators used to map configurations on Ω into configurations on $\Omega^{(k)}$ are

$$Q_k A_\mu(y) = L^{-kd} \sum_{x \in B^k(y)} A_\mu(x), \tag{2.10}$$

$$(Q_k(A)\phi)(y) = L^{-kd} \sum_{x \in B^k(y)} U(A(\Gamma^{(k)}_{y,x}))\phi(x), \tag{2.11}$$

where the contour $\Gamma^{(k)}_{y,x}$ connects y to x in the following way; writing x_j as the site for which $x \in B^j(x_j)$, we have $\Gamma^{(k)}_{y,x} = \Gamma_{y,x_{k-1}} \cup \Gamma_{x_{k-1},x_{k-2}} \cup \ldots \Gamma_{x_1,x}$ and the contour $\Gamma_{z,w}$ is a union of bonds on εZ^d given by

$$\begin{aligned}\Gamma_{z,w} = &\langle z, (z_1, \ldots, z_{d-1}, w_d)\rangle \cup \langle (z_1, \ldots, z_{d-1}, w_d), (z_1, \ldots, z_{d-2}, w_{d-1}, w_d)\rangle \\ &\cup \ldots \cup \langle (z_1, \ldots, w_d), w\rangle .\end{aligned} \tag{2.12}$$

Furthermore, $A(\Gamma) = \sum_{b \in \Gamma} \varepsilon A_b$ and $U(A(\Gamma)) = \exp[eqA(\Gamma)]$. The block spin transformation proceeds by holding fixed these average fields and integrating over fluctuations around them. The covariance of the scalar fluctuation field over blocks of size L^{kd} on Ω is

$$G^\varepsilon_k(\Omega, A) = (-\Delta^{\varepsilon,\Omega}_A + m^2 + a_k(L^k\varepsilon)^{-2} Q_k(A)^* Q_k(A))^{-1}, \tag{2.13}$$

where $a_k = a(1 - L^{-2})(1 - L^{-2k})^{-1}$ and a is $O(1)$. The effective Laplacian of the average scalar field on $\Omega^{(k)}$ is

$$\Delta^{(k), L^k\varepsilon}(\Omega, A) = a_k(L^k\varepsilon)^{-2} I - a_k^2 (L^k\varepsilon)^{-4} Q_k(A) G^\varepsilon_k(\Omega, A) Q_k(A)^* . \tag{2.14}$$

We will also need the normalization factor for (2.13):

$$Z^\varepsilon_k(\Omega, A) = \left(\frac{a_k(L^k\varepsilon)^{d-2}}{2\pi}\right)^{N/2(L^k\varepsilon)^{-d}|\Omega^{(k)}|} \int (d\phi) \exp[-1/2\langle \phi, G^\varepsilon_k(\Omega, A)^{-1}\phi\rangle], \tag{2.15}$$

where $(L^k\varepsilon)^{-d}|\Omega^{(k)}|$ is the number of lattice sites in $\Omega^{(k)}$. The covariance of the scalar fluctuation field over blocks of size L^d on $\Omega^{(k)}$ is

$$C^{(k), L^k\varepsilon}(\Omega, A) = (\Delta^{(k), L^k\varepsilon}(\Omega, A) + a(L^{k+1}\varepsilon)^{-2} Q(A)^* Q(A))^{-1}, \tag{2.16}$$

where $C^{(0),\varepsilon}(\Omega, A) = G^\varepsilon_1(\Omega, A)$. The operator (2.13) can be decomposed into a sum of contributions from scales between Ω and $\Omega^{(k)}$:

$$\begin{aligned}G^\varepsilon_k(\Omega, A) = C^{(0),\varepsilon}(\Omega, A) + &\sum_{j=1}^{k-1} a_j^2 (L^j\varepsilon)^{-4} G^\varepsilon_j(\Omega, A) Q_j(A)^* \\ &\cdot C^{(j), L^j\varepsilon}(\Omega, A) Q_j(A) G^\varepsilon_j(\Omega, A) \equiv \sum_{j=0}^{k-1} G^\varepsilon_{(j)}(\Omega, A),\end{aligned} \tag{2.17}$$

with the obvious definition of $G^\varepsilon_{(j)}(\Omega, A)$. All the corresponding expressions for the vector field are obtained from (2.13)–(2.17) by setting $m^2 = \mu_0^2$, $N = d$ and $A = 0$.

After k steps of the block spin transformation, the original vector field A on Ω is replaced by a sum of $k+1$ independent fields, coming from fluctuation fields A'_j

defined on $\Omega^{(j)}$, and the average field A_k on $\Omega^{(k)}$:

$$A=\sum_{j=0}^{k-1} A'^{(j),\varepsilon}+A^{(k),\varepsilon}=A'+A^{(k),\varepsilon}, \tag{2.18}$$

where $A'^{(0),\varepsilon}=A'_0$, $A'^{(j),\varepsilon}=a_j(L^j\varepsilon)^{-2}G^\varepsilon_j(\Omega)Q^*_jA'_j$ for $1\leqq j\leqq k-1$, and $A^{(k),\varepsilon}=a_k(L^k\varepsilon)^{-2}G^\varepsilon_k(\Omega)Q^*_kA_k$. When A is given by (2.18), we adopt the following convention:

$$A(\Gamma^{(k)}_{y,x})=\sum_{j=0}^{k-1} A'^{(j),\varepsilon}(\Gamma^{(j+1)}_{x_{j+1},x})+A^{(k),\varepsilon}(\Gamma^{(k)}_{y,k}), \tag{2.19}$$

In Sect. 3.3 we will describe in detail the perturbative expansion of the effective action obtained for the fields ϕ_k and A_k on $\Omega^{(k)}$ after applying k steps of the block spin transformation to the original action on Ω. We call this effective action $S^{(k),L^k\varepsilon}(\Omega^{(k)}, A_k, \phi_k, g, h)$, or if $\Omega^{(k)}$ is rescaled to have lattice spacing equal to one, it is written $S^{(k),1}$.

Finally we note how operators transform under rescalings of the lattice. If we rescale εZ^d to ηZ^d, the following relations hold between the operators on those lattices:

$$\begin{aligned}&\{G^\varepsilon_k, \Delta^{(k),L^k\varepsilon}, C^{(k),L^k\varepsilon}\}(x,y)\\&=\{(\varepsilon/\eta)^{2-d}G^\eta_k, (\varepsilon/\eta)^{-(2+d)}\Delta^{(k),L^k\eta}, (\varepsilon/\eta)^{2-d}C^{(k),L^k\eta}\}\\&\quad\cdot(\eta x/\varepsilon, \eta y/\varepsilon).\end{aligned} \tag{2.20}$$

When $\eta=L^{-k}$ we will omit the lattice superscript.

2.3. Statement of Results. We shall consider the model on a torus T defined by

$$T=\{x\in R^d: -L_\mu\leqq x_\mu<L_\mu, \mu=1,\dots,d\}, \tag{2.21}$$

i.e. we use periodic boundary conditions on T. We require that $2L_\mu M^{-1}\in N$, each $\mu=1,\dots,d$, where M is a large integer (this is the same integer that is specified in [Ba 4]). We then define lattice spacings $\{\varepsilon_K=L^{-K}\}$, $K=0,\dots,\infty$; so for each $K\geqq 0$, T can be fitted exactly by neighboring disjoint blocks of M^d sites on the lattice $\varepsilon_K Z^d$ (henceforth called "large blocks"). We write $T_\varepsilon=T\cap\varepsilon Z^d$.

In [Ba 1–4] uniform upper and lower bounds were derived for $Z^{\varepsilon_K}(T_{\varepsilon_K})$ in $d=2,3$. We prove the existence of the continuum limit as $K\to\infty$.

Theorem 2.1. *For the torus T defined by* (2.21) *in dimensions $d=2,3$,*

(i) $$\exists \lim_{K\to\infty} Z^{\varepsilon_K}(T_{\varepsilon_K}, g, h)=Z(T,g,h), \tag{2.22}$$

(ii) $$|\ln Z(T,g,h)|\leqq C|T|, \tag{2.23}$$

where C depends on g, h and $|T|=\prod_{\mu=1}^{d} 2L_\mu$.

3. Covergence in a Finite Volume

3.1. Summary of Previous Results. In this section we will recall some results from the papers [Ba 1–4]. We use the following quantity to define "large" and "small" fields:

$$p(\varepsilon)=b_0(1+\log\varepsilon^{-1})^p, \tag{3.1}$$

where p is a small integer and b_0 is a constant $O(1)$. After rescaling to the unit lattice $T_1^{(k)}$, the effective action is written $S^{(k),1}(T_1^{(k)}, A_k, \phi_k, g, h)$ with A_k, ϕ_k unit lattice fields. Let us denote by $\chi_k(A_k, \phi_k)$ the function which is one when all the following inequalities hold and zero otherwise:

$$\begin{gathered}
|A_{k,\mu}(y)|\leqq p(L^{k-1}\varepsilon)(\mu_0 L^{k-1}\varepsilon)^{-1},\\
|\partial A_{k,\mu}(b)|\leqq p(L^{k-1}\varepsilon),\\
|\phi_k(y)|\leqq p(L^{k-1}\varepsilon)\,[\lambda(L^{k-1}\varepsilon)^{4-d}]^{-1/4},\\
|D_{A_k}\phi_k(b)|\leqq p(L^{k-1}\varepsilon),
\end{gathered} \tag{3.2}$$

for all y, b and μ, and where

$$D_{A_k}\phi_k(b)=\exp[e(L^k\varepsilon)^{2-d/2}qA_k(b)]\phi_k(b_+)-\phi_k(b_-).$$

We can now formulate the basic results of the papers [Ba 1–2].

Theorem 3.1. *Let T be defined by* (2.28) *in dimension $d=2,3$. Then for some fixed integer J, for all $K\geqq J$, and all $k\leqq K-J$, and for some $\sigma>0$,*

$$Z^{\varepsilon_K}(T_{\varepsilon_K}, g, h)\geqq \int (dA_k)(d\phi_k)\chi_k(A_k,\phi_k)\exp[-S^{(k),1}(T_1^{(k)}, A_k, \phi_k, g, h)+C(L^k\varepsilon_K)^\sigma|T|], \tag{3.3}$$

$$\begin{aligned}Z^{\varepsilon_K}(T_{\varepsilon_K}, g, h)\leqq \int (dA_k)(d\phi_k)\chi_k(A_k,\phi_k)\exp[-S^{(k),1}(T_1^{(k)}, A_k, \phi_k, g, h)+C(L^k\varepsilon_K)^\sigma|T|]\\ +\exp[-p(L^k\varepsilon_K)^2+C|T|].\end{aligned} \tag{3.4}$$

Once again, C depends on g, h and $|T|=\prod_{\mu=1}^{d} 2L_\mu$. The bounds on the fields (3.2) are different from those used in [Ba 1–2]. This change is justified in the Appendix, and also the inclusion of sources g, h is presented. The reader is referred to [Ba 1–4] for the proof of Theorem 3.1. We will also use extensively the decay and regularity results obtained in [Ba 4] for the operators (2.13), (2.14), and (2.16). When $\Omega=T_\varepsilon$, the torus, all operators are defined with periodic boundary conditions, and for covenience we will omit their volume dependence. When Ω is a proper subset of T_ε and $\Omega^{(k)}$ is a union of large blocks on $T_{L^k\varepsilon}^{(k)}$, we will define $G_k^\varepsilon(\Omega, A)$, $\Delta^{(k),L^k\varepsilon}(\Omega, A)$ and $C^{(k),L^k\varepsilon}(\Omega, A)$ using Neumann boundary conditions on $\partial\Omega$.

Definition 3.2. Let A be a vector field configuration on Ω_η, where $\Omega^{(k)}\subset T_1^{(k)}$ is a union of large blocks and $\eta=L^{-k}$. Then A is *regular* on Ω_η if

$$|\partial_\mu^\eta A_\nu(x)|\leqq C(e(L^k\varepsilon)^{2-d/2})^{\beta-1} \tag{3.5}$$

for all $x\in\Omega_\eta$, μ, $\nu=1,\ldots,d$ and some $\beta>0$ (e is the electric charge).

The following theorem summarises the decay properties of the operators (2.13), (2.14), and (2.16).

Theorem 3.3. *Let A be regular on Ω_η, with $\eta=L^{-k}$, and let $\Omega^{(k)}$ be a rectangular parallelepiped which is a union of large blocks on $T_1^{(k)}$. Then for all $k\leqq K$, some*

$\delta_0>0$, $0<\alpha<1$, *and for* $f:\Omega_\eta\to R^N$,

$$|\Delta^{(k)}(\Omega,A)(x,y)|\,,\quad |C^{(k)}(\Omega,A)(x,y)|\leqq C\exp[-\delta_0|x-y|]\,, \tag{3.6}$$

$$|G_k(\Omega,A)f(x)|\,,\quad |D^\eta_{A,\mu}G_k(\Omega,A)f(x)|\leqq C\exp[-\delta_0\,\mathrm{dist}(x,\mathrm{supp}\,f)]\,\|f\|, \tag{3.7}$$

$$\frac{1}{|x-y|^\alpha}|U(A(\Gamma_{y,x}))D^\eta_{A,\mu}G_k(\Omega,A)f(x)-D^\eta_{A,\mu}G_k(\Omega,A)f(y)|$$
$$\leqq C\exp[-\delta_0\,\mathrm{dist}(\{x,y\},\mathrm{supp}\,f)]\,\|f\|\,. \tag{3.8}$$

Finally, define $\delta C^{(k)}(\Omega,A)=C^{(k)}(\Omega,A)-C^{(k)}(A)$ and $\delta G_k(\Omega,A)=G_k(\Omega,A)-G_k(A)$; then for $x,y\in\Omega_\eta$, $\delta C^{(k)}(\Omega,A)$ and $\delta G_k(\Omega,A)$ satisfy the bounds (3.6) and (3.7)–(3.8) respectively, with the additional factors

$$\exp[-\delta_0\,\mathrm{dist}(\{x,y\},\partial\Omega)]$$

and

$$\exp[-\delta_0\,\mathrm{dist}(\{x,y\},\partial\Omega)-\delta_0\,\mathrm{dist}(\mathrm{supp}\,f,\partial\Omega)]$$

respectively on the right-hand sides.

Theorem 3.3 is proved in [Ba 4]. It implies the same bounds for the vector field operators (of course with $A=0$).

3.2. Convergence of the Partition Function. We will use the convergence of the effective action in the small field region to deduce convergence of the partition function. Consider the model defined on the lattices T_{ε_K} and $T_{\varepsilon_{K+n}}$ for some integer n. Then by applying the renormalization transformation k and $k+n$ times respectively we generate two models on the same unit lattice $T_1^{(k)}$, with effective actions $S^{(k),1}$ and $S^{(k+n),1}$. The following theorem is the core of this paper.

Theorem 3.4.

$$\chi_k(A_k,\phi_k)|S^{(k),1}(T_1^{(k)},A_k,\phi_k,g,h)-S^{(k+n),1}(T_1^{(k)},A_k,\phi_k,g,h)|$$
$$\leqq C(L^{-\gamma k}(L^k\varepsilon_K)^{-\beta}+(L^k\varepsilon_K)^\sigma)\,|T|\,, \tag{3.9}$$

where $0<\gamma<1$, $0<\sigma$, β *and* C *depends on* g, h.

We can now prove Theorem 2.1. We apply the renormalization transformation k times to $Z^{\varepsilon_K}(T_{\varepsilon_K},g,h)$ and $k+n$ times to $Z^{\varepsilon_{K+n}}(T_{\varepsilon_{K+n}},g,h)$. Theorem 3.1 gives upper and lower bounds on these partition functions in terms of $S^{(k),1}$ and $S^{(k+n),1}$. Since $L^k\varepsilon_K=L^{k+n}\varepsilon_{K+n}$, the fields A_k, ϕ_k have the same bounds in each integral. Hence

$$|Z^{\varepsilon_K}(T_{\varepsilon_K},g,h)-Z^{\varepsilon_{K+n}}(T_{\varepsilon_{K+n}},g,h)|\leqq \int(dA_k)\,(d\phi_k)\chi_k(A_k,\phi_k)$$
$$\cdot|\exp[-S^{(k),1}+C(L^k\varepsilon_K)^\sigma|T|]-\exp[-S^{(k+n),1}+C(L^k\varepsilon_K)^\sigma|T|]|$$
$$+\exp[-p(L^k\varepsilon_K)^2+C|T|]\,. \tag{3.10}$$

Using Theorem 3.4 and the bound $|e^x-e^y|\leqq|x-y|\,(e^x+e^y)$,

$$(3.10)\leqq C(L^{-\gamma k}(L^k\varepsilon_K)^{-\beta}+(L^k\varepsilon_K)^\sigma)\,|T|(Z^{\varepsilon_K}+Z^{\varepsilon_{K+n}})$$
$$\cdot\exp(C(L^k\varepsilon_K)^\sigma|T|)+\exp[-p(L^k\varepsilon_K)^2+C|T|]$$
$$\leqq C(L^{-\gamma k}(L^k\varepsilon_K)^{-\beta}+(L^k\varepsilon_K)^\sigma)\exp C|T|\,, \tag{3.11}$$

where we have used the ultra-violet stability bound and $\exp[-p(\varepsilon)^2] \leqq \varepsilon^\sigma$ for ε small, since $p \geqq 1$. Of course we are free to choose k as we like. To show that (3.11) vanishes as $K \to \infty$, we can take

$$k = [2\beta K(2\beta+\gamma)^{-1}], \tag{3.12}$$

in which case for any n

$$(3.11) \leqq C(L^{-\beta\gamma K/2\beta+\gamma} + L^{-\sigma\gamma K/2\beta+\gamma})e^{C|T|}. \tag{3.13}$$

Hence $\{Z^{\varepsilon_K}(T_{\varepsilon_K}, g, h)\}$ is a Cauchy sequence and converges to a unique limit as $K \to \infty$. The uniform bound in Theorem 2.1 is just the statement of ultra-violet stability.

3.3. The Effective Action. The effective action at the k^{th} step can be written in the following way:

$$\begin{aligned} S^{(k),1}(T^{(k)}, A_k, \phi_k, g, h) = &\ 1/2\langle A_k, \Delta^{(k)} A_k\rangle - \ln Z_k + 1/2\langle \phi_k, \Delta^{(k)}(A^{(k)})\phi_k\rangle - \ln Z_k(A^{(k)}) \\ &+ E_0 + P^{(k),1}(T^{(k)}, A_k, \phi_k, g, h), \end{aligned} \tag{3.14}$$

where $A^{(k)}$ and $Z_k(A)$ are the rescaled versions of $A^{(k),\varepsilon}$ and $Z_k^\varepsilon(A)$, and

$$\begin{aligned} E_0 = \ln\{ \textstyle\int (dA)(d\phi) \exp[&-1/2\langle A, (-\Delta^{\eta, T_\eta} + \mu_0^2(L^k\varepsilon)^2)A\rangle \\ &-1/2\langle \phi, (-\Delta^{\eta, T_\eta} + m^2(L^k\varepsilon)^2)\phi\rangle]. \end{aligned} \tag{3.15}$$

Henceforth we will always denote $\eta = L^{-k}$, $\eta' = L^{-n}\eta = L^{-(k+n)}$ and $|T^{(k)}| = |T| = \prod_{\mu=1}^{d} 2L_\mu$. The interaction term $P^{(k),1}$ in (3.14) is given by derivatives of a function E_k', given below. The scalar field ϕ is separated into an average and a fluctuation part as follows:

$$\phi = \phi' + a_k G_k(A^{(k)}) Q_k(A^{(k)})^* \phi_k = \phi' + \phi^{(k)}. \tag{3.16}$$

Then we define E_k' to be

$$E_k'(e', \lambda', \tau, \theta, A_k, \phi_k, g, h) = -\ln\left\{ \int \prod_{j=0}^{k-1} d\mu_{C^{(j)}, L^j\eta}(A_j') d\mu_{G_k(A^{(k)})}(\phi') e^{V^{(0),\eta}} \right\}, \tag{3.17}$$

where $V^{(0),\eta}$ depends on A', ϕ' and A_k, ϕ_k, as well as on the parameters e', λ', τ, θ. The interaction $P^{(k),1}$ is a sum of derivatives of (3.17) taken with respect to e', λ', τ, θ; it is sufficient to take derivatives up to order $\bar{n} = 13$. These derivatives are represented by connected graphs on T_η; the vertices are those present in $V^{(0),\eta}$, and the propagators are the covariances of A_j' and ϕ' in (3.17). $V^{(0),\eta}$ contains the following vertices (for convenience we write Ω in place of T_η):

$$-\lambda'\lambda(L^k\varepsilon)^{4-d} \sum_{x\in\Omega} \eta^d |\phi'(x) + \phi^{(k)}(x)|^4, \tag{3.18}$$

$$-1/2(L^k\varepsilon)^2 \sum_{x\in\Omega} \eta^d \delta m^2(x; e'e, \lambda'\lambda)\, |\phi'(x) + \phi^{(k)}(x)|^2, \tag{3.19}$$

$$\tau(L^k\varepsilon)^{d/2} \sum_{p\in\Omega^{**}} \eta^d g(p) \left(\eta^{-1} \sum_{b\in\partial p} (A_b' + A_b^{(k)})\right), \tag{3.20}$$

$$\theta(L^k\varepsilon)^2 \sum_{x\in\Omega} \eta^d h(x) \sum_{i=1}^{N} [(\phi_i'(x) + \phi_i^{(k)}(x))^2 - (-\Delta^{\eta,\Omega} + m^2(L^k\varepsilon)^2)^{-1}(x,x)] \tag{3.21}$$

$$-(e(L^k\varepsilon)^{2-d/2})^m \frac{\eta^{m-1}}{m!} \sum_{b\in\Omega^*} \eta^d [D^\eta_{A^{(k)}}(\phi' + \phi^{(k)})(b)$$

$$U(A_b^{(k)})q^m(\phi'+\phi^{(k)})(b_+)](e'A_b')^m \quad \text{with} \quad 1 \leqq m \leqq \bar{n}, \tag{3.22}$$

$$(e(L^k\varepsilon)^{2-d/2})^m \frac{\eta^{m-2}}{m!} \sum_{b\in\Omega^*} \eta^d[(\phi'+\phi^{(k)})(b_+)q^m(\phi'+\phi^{(k)})(b_+)](e'A_b')^m$$

with m even,

$$2 < m \leqq \bar{n}. \tag{3.23}$$

The remaining vertices occur when the renormalization transformation of the scalar field is expanded about $A'=0$:

$$\phi_k(y), \tag{3.24}$$

$$-Q_k(A^{(k)})(\phi'+\phi^{(k)})(y), \tag{3.25}$$

$$-(e(L^k\varepsilon)^{2-d/2})^m(m!)^{-1} \sum_{x\in B^k(y)} \eta^d U(A^{(k)}(\Gamma_{y,x}^{(k)}))(e'A'(\Gamma_{y,x}^{(k)}))^m$$

$$\cdot q^m(\phi'+\phi^{(k)})(x) \quad \text{with} \quad 1 \leqq m \leqq \bar{n}. \tag{3.26}$$

These vertices are made by multiplying two of the expressions (3.24)–(3.26), one of which must be (3.26) for some m, at the same $y\in\Omega^{(k)}$. If the expressions are different, the product is multiplied by $-a_k$; if the same, by $-(1/2)a_k$. The resulting vertices are summed over $y\in\Omega^{(k)}$.

Finally, the vacuum energy counterterm $E_1(e'e, \lambda'\lambda)$ is also a part of $V^{(0),\eta}$. We shall see that E_1 is also represented by connected graphs on T_η built from the vertices (3.18), (3.19), and (3.22), (3.23).

Except for the vertex (3.26), the vector fields A_j' enter $V^{(0),\eta}$ in the combination A'. Therefore, in a graph in $P^{(k),1}$, we can sum over the propagators for these fields using (2.17), and get the full propagator G_k for the "field" A'. When vertex (3.26) occurs in a graph, we can use (2.17) and (2.19) to write the propagator as

$$G_{(j)}^\eta(\Gamma_{x_{j+1},x}^{(j+1)}, b) = \sum_{b'\in\Gamma_{x_{j+1},x}^{(j+1)}} \eta G_{(j)}^\eta(b', b). \tag{3.27}$$

When the external field $\phi^{(k)}$ occurs at a vertex v in a graph in $P^{(k),1}$, we can use (3.16) to write it as

$$\phi^{(k)}(v) = \sum_{y\in T^{(k)}} a_k(G_k(A^{(k)})Q_k(A^{(k)})^*)(v, y)\phi_k(y). \tag{3.28}$$

If there are s such fields in a graph G, the contribution to $P^{(k),1}$ from G can be written

$$E^{(k)}(G) = \sum_{a_1,\ldots,a_s=1}^{N} \sum_{y_1,\ldots,y_s\in T^{(k)}} \phi_k^{a_1}(y_1)\ldots\phi_k^{a_s}(y_s)E_{a_1\ldots a_s}^{(k)}(G, A^{(k)}; \{y_i\}). \tag{3.29}$$

We will omit indices on $E^{(k)}$ from now on. To facilitate our analysis, we will use graphical notation. As usual, we use a wavy line for a vector field propagator, and a straight line for a scalar field propagator. A derivative acting on a propagator is denoted by an arrow at the appropriate vertex. We also introduce the "full" propagators for the fields,

$$C_v = (-\Delta^{\eta,\Omega} + \mu_0^2(L^k\varepsilon)^2)^{-1}, \qquad C_s = (-\Delta^{\eta,\Omega} + m^2(L^k\varepsilon)^2)^{-1},$$

and we define $C_v = G_k + F_{k,v}$ and $C_s = G_k(0) + F_{k,s}$. The propagator C_v (or $F_{k,v}$) will be denoted by a wavy line with one dash (or two dashes) through it, and similarly for C_s and $F_{k,s}$.

In order to isolate divergences in graphs, we need the notion of degree of a graph. A line in a graph is *internal* if it carries one of the propagators G_k, $G_k(A^{(k)})$, C_v, C_s or their derivatives. A line is external if it carries anything else, for example $a_k G_k(A^{(k)}) Q_k(A^{(k)})^*$. An *internal vertex* is the endpoint of at least one internal line. Then the degree of an internal vertex v in a graph G is defined by

$$\begin{aligned} D_G(v) = {} & (1-d/2)\ (\text{number of ends of internal lines at } v \text{ without derivatives}) \\ & + (-d/2)\ (\text{number of ends of internal lines at } v \text{ with derivatives}) \\ & + (\text{number of factors } \eta \text{ at } v) \\ & + (\text{value of } m \text{ when } v \text{ contains (3.26)}). \end{aligned} \tag{3.30}$$

There is always one factor η^d at v, coming from the integration over T_η; the vertices (3.18)–(3.26) may produce other factors. The degree of a connected graph G is then

$$D(G) = \sum_{v \in G} D_G(v) - d\,, \tag{3.31}$$

and the degree of an arbitrary graph is the sum of the degrees of its connected components. The divergent graphs are those with non-positive degree.

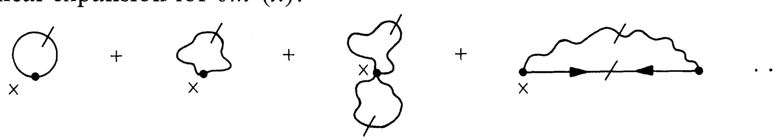
We write $(-\delta m^2 + \Gamma^\varepsilon)$ for the full one particle irreducible graph with two (undifferentiated) external scalar field lines. The divergent graphs in the graphical expansion of $(-\delta m^2 + \Gamma^\varepsilon)$ have order less than or equal to four in the couplings e and λ, and so we define

$$\delta m^2(x) = \sum_{2 \leqq \alpha + 2\beta \leqq 4} e^\alpha \lambda^\beta \delta m^2(x; \alpha, \beta)\,. \tag{3.32}$$

This definition is inserted into Γ^ε, and terms of the same order in e, λ are collected together; this allows us to define $\delta m^2(x; \alpha, \beta)$ by the equation

$$-\delta m^2(x; \alpha, \beta) + \sum_{y \in \Omega} \varepsilon^d \Gamma^\varepsilon(x, y; \alpha, \beta) = 0\,. \tag{3.33}$$

Equation (3.33) is solved recursively beginning with $\beta = 1$, and it implies a graphical expansion for $\delta m^2(x)$:



(3.34)

The vacuum energy counterterm E_1 is defined by

$$E_1(e', \lambda') = \sum_{1 \leqq \alpha + \beta \leqq \bar{n}} \frac{e'^\alpha \lambda'^\beta}{\alpha!\,\beta!} \frac{\partial^{\alpha+\beta}}{\partial e^\alpha \partial \lambda^\beta} \cdot (\ln \textstyle\int (dA)(d\phi) \exp[-S^\varepsilon(\Omega, A, \phi) + E_1])_{e=\lambda=0}\,, \tag{3.35}$$

where $S^\varepsilon(\Omega, A, \phi)$ is given by (2.4). Clearly E_1 is represented by a sum of vacuum energy diagrams on T; the vertices are (3.18), (3.19), (3.22), (3.23) with $A^{(k)} = 0$, $\phi^{(k)} = 0$, and propagators C_v^η, C_s^η.

3.4. Convergence of a Diagram. We now establish Theorem 3.4 for a diagram H with s external scalar field legs, as in (3.29), and t internal vertices. Introduce the functions $\{\chi_z\}$, $z \in T^{(k)}$, where $\chi_z(x) = \prod_{\mu=1}^{d} \chi(x_\mu - z_\mu)$. The positive real function χ is

C^∞, supported in $[-2/3, 2/3]$, equal to unity on $[-1/3, 1/3]$, and chosen so that

$$\sum_{z \in T^{(k)}} \chi_z(x) = 1\,, \quad \text{each} \quad x \in T_\eta\,. \tag{3.36}$$

By inserting (3.36) at each internal vertex of H, we can rewrite (3.29) as

$$E^{(k)}(H) = \sum_{y_1, \ldots, y_s \in T^{(k)}} \sum_{z_1, \ldots, z_t \in T^{(k)}} \phi_k(y_1) \ldots \phi_k(y_s) E^{(k)}(H, A^{(k)}; \{y_i\}, \{z_q\})\,. \tag{3.37}$$

The next theorem expresses the results of this section. We define $\text{dist}(\{v_i\})$ to be the length of the shortest tree graph connecting $\{v_i\}$.

Theorem 3.5. *With the notation of* (3.37), *and for some* $n_1, \delta, \gamma, \sigma > 0$ *depending only on* $\bar{n}$, d, *and* $\varrho \geqq s/4$,

(i) $|E^{(\theta)}(H, A^{(\theta)}; \{y_i\}, \{z_q\})|$

$$\leqq C(L^k\varepsilon)^{\varrho}(p(L^k\varepsilon))^{n_1} \exp[-\delta\, \text{dist}(\{y_i\}, \{z_q\})]\,, \tag{3.38}$$

where θ *is* k *or* $k+n$;

(ii) *when* $\text{dist}(\{y_i\}, \{z_q\}) \leqq p(L^k\varepsilon)$,

$$\begin{aligned} &|E^{(k+n)}(H, A^{(k+n)}; \{y_i\}, \{z_q\}) - E^{(k)}(H, A^{(k)}; \{y_i\}, \{z_q\})| \\ &\quad \leqq C(L^k\varepsilon)^{\varrho - 1/2}(p(L^k\varepsilon))^{n_1}[L^{-\gamma k} + (L^k\varepsilon)^{d+\sigma+1/2}] \\ &\quad\quad \cdot \exp[-\delta\, \text{dist}(\{y_i\}, \{z_q\})]\,. \end{aligned} \tag{3.39}$$

Using Theorem 3.5, we can now establish Theorem 3.4 for the graph H. From (3.37) we have

$$\begin{aligned} E^{(k+n)}(H) - E^{(k)}(H) = \sum_{\{y_i\}, \{z_q\} \in T^{(k)}} &\phi_k(y_1) \ldots \phi_k(y_s) \{E^{(k+n)}(H, A^{(k+n)}; \{y_i\}, \{z_q\}) \\ &- E^{(k)}(H, A^{(k)}; \{y_i\}, \{z_q\})\}\,. \end{aligned} \tag{3.40}$$

Define $R = \text{dist}(\{y_i\}, \{z_q\})$; when $R > p(L^k\varepsilon)$, we have

$$\exp[-\delta R] \leqq C(L^k\varepsilon)^{d+\sigma} \exp[-1/2\delta R] \tag{3.41}$$

for any σ, since $p > 1$. Therefore using Theorem 3.5 and the bounds (3.2), we have

$$\begin{aligned} &|E^{(k+n)}(H) - E^{(k)}(H)| \\ &\leqq C \sum_{\{y_i\}, \{z_q\} \in T^{(k)}} (L^k\varepsilon)^{\varrho - s/4}(p(L^k\varepsilon))^{n_1} \{\chi(R > p(L^k\varepsilon))\, (L^k\varepsilon)^{d+\sigma} \exp[-1/2\delta R] \\ &\quad + \chi(R \leqq p(L^k\varepsilon))\, (L^k\varepsilon)^{-1/2}[L^{-\gamma k} + (L^k\varepsilon)^{d+\sigma+1/2}] \exp[-\delta R]\} \\ &\leqq C(L^k\varepsilon)^{-1/2}(p(L^k\varepsilon))^{n_1}[L^{-\gamma k} + (L^k\varepsilon)^{d+\sigma+1/2}] \cdot \sum_{\{y_i\}, \{z_q\} \in T^{(k)}} \exp[-1/2\delta R] \\ &\leqq C(L^k\varepsilon)^{-1/2}(p(L^k\varepsilon))^{n_1}[L^{-\gamma k} + (L^k\varepsilon)^{d+\sigma+1/2}]\, (L^k\varepsilon)^{-d} |T| \\ &\leqq C[L^{-\gamma k}(L^k\varepsilon)^{-\beta} + (L^k\varepsilon)^{\sigma'}]\, |T|\,. \end{aligned} \tag{3.42}$$

The proof of Theorem 3.5 is similar to the proofs of renormalizability in [Ba 3]. We shall first prove (3.39). There is a smallest cube $\square$ on $T(k)$ which is a union of large blocks and which contains every vertex y_i and z_q. We denote by $\square'$, $\square''$ the smallest cubes on $T^{(k)}$ containing the large blocks closer than $2p(L^k\varepsilon)$, $p(L^k\varepsilon)$ to $\square$. Let B be the value of A_k at some point x_0 in $\square'$, and define $\tilde{A}^{(k)}_\mu(x)=A^{(k)}_\mu(x)-B_\mu$ for x in $\square'$. Using Eq. (A.4) from the Appendix we have

$$B_\mu=(1+\mu_0^2(L^k\varepsilon)^2a_k^{-1})a_kG_kQ_k^*B_\mu\,. \tag{3.43}$$

Therefore writing $\tilde{A}_{k,\mu}(w)=A_{k,\mu}(w)-(1)+\mu_0^2(L^k\varepsilon)^2a_k^{-1})B_\mu$, we have

$$\tilde{A}^{(k)}_\mu(x)=\sum_{w\in T^{(k)}}(a_kG_kQ_k^*)(x,w)\tilde{A}_{k,\mu}(w)\,. \tag{3.44}$$

From the bounds (3.2) and using Theorem 3.3, we deduce that

$$\begin{aligned}&|\tilde{A}_{k,\mu}(w)|\leqq Cp(L^k\varepsilon)\,[|w-x_0|+L^k\varepsilon]\,,\\ &|\tilde{A}^{(k)}_\mu(x)|\,,\quad |\partial^\eta_\nu\tilde{A}^{(k)}_\mu(x)|\leqq Cp(L^k\varepsilon)|x-x_0|\,.\end{aligned} \tag{3.45}$$

In the expression $E^{(k)}(H,A^{(k)};\{y_i\},\{z_q\})$, we replace each propagator $G_k(A^{(k)})$ by the sum $G_k(\square',A^{(k)})-\delta G_k(\square',A^{(k)})$. Multiplying out the result gives a sum of graphs of the same form as H, but with appropriately changed propagators. A line carrying $\delta G_k(\square',A^{(k)})$ (or its derivative) is treated as external. We now make a gauge transformation of the external field $A^{(k)}$ in each propagator and vertex function in the graph, except in the propagators $\delta G_k(\square',A^{(k)})$. $A^{(k)}_\mu(x)$ is replaced by $\tilde{A}^{(k)}_\mu(x)$ for each $x\in\square'$; by gauge covariance this is equivalent to a unitary transformation at each vertex connected to an external field or an external line. Specifically, for each graph produced by the expansion, there is a subset of vertices $\{v_j\}$ such that the only dependence of the graph on B is the following factor at each vertex v_j:

$$U(B(\Gamma^{(k)}_{x_0,v_j}))=\exp[e(L^k\varepsilon)^{2-d/2}qB(\Gamma^{(k)}_{x_0,v_j})]\,. \tag{3.46}$$

We next want to write each graph as a (non-local) polynomial in the field $\tilde{A}^{(k)}$. We do this by expanding around $\theta\tilde{A}^{(k)}=0$; θ is a C^∞ function with $\theta(x)=1$ for $\mathrm{dist}(\square'',x)\leqq 1/3p(L^k\varepsilon)$, and $\theta(x)=0$ for $\mathrm{dist}(\square'',x)>2/3p(L^k\varepsilon)$. For future reference, we give the expansion for a general fluctuation field A' about a background field B. The covariant derivative becomes

$$D^\eta_{A'+B}\phi(b)=D^\eta_B\phi(b)+F_{1,k}(A')U(B)\phi(b_+)\,, \tag{3.47}$$

where

$$F_{1,k}(A')=\sum_{j=1}^{\bar{n}}\frac{\eta^{j-1}}{j!}(e(L^k\varepsilon)^{2-d/2}qA')^j+R_{\bar{n}+1}(e(L^k\varepsilon)^{2-d/2}qA')\,. \tag{3.48}$$

The remainder term can be written as $R_n(x)=(x^n/n!)V_n(x)$, and $V_n(z)$ is an analytic function of z given by $V_n(z)=n\int_0^1(1-t)^{n-1}e^{tz}dt$. Therefore taking $A'=\theta\tilde{A}^{(k)}$ and using (3.45),

$$|R_{\bar{n}+1}(e(L^k\varepsilon)^{2-d/2}qA')|\leqq\frac{C}{(\bar{n}+1)!}(e(L^k\varepsilon)^{2-d/2}p(L^k\varepsilon)^2)^{\bar{n}+1}\,. \tag{3.49}$$

Similarly the averaging function becomes

$$(Q_k(A'+B)\phi)(y)=(Q_k(B)\phi)(y)+(F_{2,k}(A',B)\phi)(y)\,, \tag{3.50}$$

where

$$(F_{2,k}(A',B)\phi)(y)=\sum_{x\in B^k(y)}\eta^{d+1}F_{1,k}(A'(\Gamma^{(k)}_{y,x}))U(B(\Gamma^{(k)}_{y,x}))\phi(x). \quad (3.51)$$

Together these imply an expansion for $G_k(\square', A'+B)$:

$$G_k(\square', A'+B)=\sum_{m=0}^{\bar{n}} G_k(\square', B)[V_k(A',B)G_k(\square',B)]^m+G_k(\square',B)$$
$$\cdot[V_k(A',B)G_k(\square',B)]^{\bar{n}}V_k(A',B)G_k(\square',A'+B), \quad (3.52)$$

where the vertex function is

$$-V_k(A',B)=D_B^{\eta *}F_{1,k}(A')U(B)+U^*(B)F_{1,k}(A')^*D_B^\eta+|F_{1,k}(A')|^2$$
$$+a_kF_{2,k}(A',B)^*Q_k(B)+a_kQ_k(B)^*F_{2,k}(A',B)+a_kF_{2,k}(A',B)^*F_{2,k}(A',B). \quad (3.53)$$

These equations are used to expand the propagators, vertex functions and covariant derivatives, taking $A'=\theta\tilde{A}^{(k)}$ and $B=0$. The result is a sum of new graphs, all with vertices inside $\square'$, and with vector fields $\tilde{A}^{(k)}$ at the new vertices. Note that these graphs have the same structure as those already present in $P^{(k),1}$. Finally, we write every scalar field propagator as a sum

$$G_k(\square',(1-\theta)\tilde{A}^{(k)})=G_k(0)+\delta G_k(\square'',0)+\delta G_k(\square',\square'',(1-\theta)\tilde{A}^{(k)}), \quad (3.54)$$

where $\delta G_k(\square',\square'',A)=G_k(\square',A)-G_k(\square'',A)$. Using (3.44) we can hold the fields $\tilde{A}_k$ fixed at some points $\{w_l\}$ in $T^{(k)}$. So if we denote by $\{\tilde{H}\}$ the graphs obtained from H by all the previous expansions, we can write

$$E^{(k)}(H,A^{(k)};\{y_i\},\{z_q\})=\sum_{\{\tilde{H}\}}\Bigg\{\sum_{w_1,\dots,w_r\in T^{(k)}}\tilde{A}_k(w_1)\dots\tilde{A}_k(w_r)$$
$$\cdot E^{(k)}(\tilde{H};\{y_i\},\{z_q\},\{w_l\})\Bigg\}. \quad (3.55)$$

There is a similar expansion of $E^{(k+n)}(H,A^{(k+n)};\{y_i\},\{z_q\})$ with the same graphs $\tilde{H}$, so the proof reduces to the following proposition.

Proposition 3.6. *With the notation of* (3.55),

$$|E^{(k+n)}(\tilde{H};\{y_i\},\{z_q\},\{w_l\})-E^{(k)}(\tilde{H};\{y_i\},\{z_q\},\{w_l\})|\leqq C(L^k\varepsilon)^{\varrho-1/2}$$
$$\cdot(p(L^k\varepsilon))^{n_2}[L^{-\gamma k}+(L^k\varepsilon)^{d+\sigma+1/2}]\exp[-\delta\,\mathrm{dist}(\{y_i\},\{z_q\},\{w_l\})]. \quad (3.56)$$

To see that (3.39) follows from (3.56) consider

$$|E^{(k+n)}(H,A^{(k+n)};\{y_i\},\{z_q\})-E^{(k)}(H,A^{(k)};\{y_i\},\{z_q\})|$$
$$\leqq\sum_{\{\tilde{H}\}}\Bigg\{\sum_{\{w_l\}\in T^{(k)}}|\tilde{A}_k(w_1)|\dots|\tilde{A}_k(w_r)|\cdot|E^{(k+n)}(\tilde{H};\{y_i\},\{z_q\},\{w_l\})$$
$$-E^{(k)}(\tilde{H};\{y_i\},\{z_q\},\{w_l\})|\Bigg\}. \quad (3.57)$$

Using the bounds (3.45) and recalling that $x_0\in\square'$, so $|x_0-y_i|$, $|x_0-z_q|\leqq Cp(L^k\varepsilon)$ each i, q, we can use the exponential tree decay in (3.56) to sum over $\{w_l\}$ and get (3.39) with $n_1=n_2+2r$ (the number of graphs $\tilde{H}$ depends only on $\bar{n}$ and d).

We now prove Proposition 3.6. Every internal line in $\tilde{H}$ carries a propagator G_k, $G_k(0)$ or one of their derivatives, which may be decomposed by writing $G_k^\eta = \sum_{j=0}^{k-1} G_{(j)}^\eta$, see (2.17). The resulting product of sums is multiplied out and rewritten as a sum over orderings $\tilde{l}=\{l(1), \ldots, l(m)\}$ of the m internal lines. This is followed by a sum over integers $\{j_{l(p)}\}$, $p=1, \ldots, m$, compatible with this ordering, meaning

$$j_{l(1)} \leqq j_{l(2)} \leqq \cdots \leqq j_{l(m)} . \tag{3.58}$$

A term with two or more integers equal is arbitrarily assigned to one of the orderings with which it is compatible. Denoting by $\tilde{H}(j)$ the graph with specified integers $j=\{j_{l(p)}\}$ on each internal line, and omitting the external vertices, we have

$$E^{(k)}(\tilde{H}) = \sum_{\tilde{l}} \sum_{\substack{j \text{ compatible} \\ \text{with } \tilde{l}}} E^{(k)}(\tilde{H}(j)) . \tag{3.59}$$

There is a similar representation for $E^{(k+n)}(\tilde{H})$; for convenience we write the decomposition of $G_{k+n}^{\eta'}$ in the form $G_{k+n}^{\eta'} = \sum_{j=-n}^{k-1} G_{(j)}^{\eta'}$, where

$$G_{(-n)}^{\eta'} = C^{(0),\eta'} , \quad G_{(j)}^{\eta'} = a_{j+n}^2 (L^j \eta)^{-4} G_{j+n}^{\eta'} Q_{j+n}^* C^{(j+n), L^j \eta} Q_{j+n} G_{j+n}^{\eta'} . \tag{3.60}$$

We divide the internal lines into two sets S and S^c; in S all integers are zero or positive, while in S^c all integers are negative. Hence we can write

$$E^{(k+n)}(\tilde{H}) = \sum_{S^c} E_{S^c}^{(k+n)}(\tilde{H}) , \tag{3.61}$$

where $E_{S^c}^{(k+n)}(\tilde{H})$ contains a sum over orderings and integers compatible with S. Before proceeding, we state the required bounds on propagators. For a propagator $G(x, y)$, we define a "Holder derivative" by

$$(\partial_\alpha(x, y) G)(z) = |x-y|^{-\alpha} \{G(x, z) - G(y, z)\} . \tag{3.62}$$

Proposition 3.7. *For* $x, y, z \in T_\eta$, $0 < \alpha < 1$, *and all* $0 \leqq j \leqq k-1$,

$$\begin{aligned} |G_{(j)}^\eta(x, y)| , \quad |\partial_\mu^\eta G_{(j)}^\eta(x, y)| &\leqq C\{(L^j \eta)^{2-d}, (L^j \eta)^{1-d}\} \\ &\quad \cdot \exp[-\delta_0 (L^j \eta)^{-1} |x-y|] , \end{aligned} \tag{3.63}$$

$$|G_{(j)}^\eta(\Gamma_{x_{j+1}, x}^{(j+1)}, b)| \leqq C(L^j \eta)^{3-d} \exp[-\delta_0 (L^j \eta)^{-1} \operatorname{dist}(B^j(x), b)] , \tag{3.64}$$

$$\begin{aligned} &|\partial_\alpha(x, y) G_{(j)}^\eta(z)| , \quad |\partial_\alpha(x, y) \partial_\mu^\eta G_{(j)}^\eta(z)| \\ &\leqq C\{(L^j \eta)^{2-d-\alpha}, (L^j \eta)^{1-d-\alpha}\} \exp[-\delta_0 (L^j \eta)^{-1} \operatorname{dist}(\{x, y\}, z] . \end{aligned} \tag{3.65}$$

Furthermore, if we replace $G_{(j)}^\eta$ by $G_{(j)}^{\eta'}$ throughout, and replace $\Gamma_{x_{j+1}, x}^{(j+1)}$ by $\Gamma_{x_{j+n+1}, x}^{(j+n+1)}$, then the bounds are valid for $-n \leqq j \leqq k-1$.

Proposition 3.7 follows immediately from Theorem 3.3 and the scaling properties of the operators. We will first establish (3.56) for a graph with no remainder terms from the expansion (3.48), no final terms from (3.52) and no propagators δG_k. These restrictions will be dropped later. For a fixed ordering $\tilde{l}$, we define a sequence of subgraphs $H_1, \ldots, H_m$ as follows:

$$\begin{aligned} H_1 &= \{\text{the line } l(1) \text{ and its two vertices}\} , \\ H_{i+1} &= H_i \cup \{\text{the line } l(i+1) \text{ and its two vertices}\} . \end{aligned} \tag{3.66}$$

Some of the subgraphs H_i may be divergent. In Sect. 3.5, we will show how graphs may be added together to form renormalised graphs, in which every subgraph has positive degree. We will assume below that this has been done already, and that we are analysing a renormalised graph.

Consider first a term on the right-hand side of (3.61) with S^c non-empty. We get an upper bound for this expression by replacing every propagator by the bounds given in Proposition 3.7 and Theorem 3.3, and bounding vertex functions appropriately (the sources, g, h are bounded by C). By extracting a small part of each propagator, we get the exponential decay on the right-hand side of (3.56). Let x, y be the endpoints of the graph $H_1 = l(1)$; there is a factor $\exp[-\delta_0 L^{k-j_{l(1)}}|x-y|]$ present. All the other propagators attached to y carry similar exponential factors, but with $j_{l(1)}$ replaced by some $j_{l(p)} \geqq j_{l(1)}$. These propagators are "transferred" to x by using the inequality

$$\exp[-\delta_0 L^{k-j_{l(p)}}|y-z| - \delta_1 L^{k-j_{l(1)}}|x-y|] \leqq \exp[-\delta_1 L^{k-j_{l(p)}}|x-z|], \quad (3.67)$$

where δ_1 is a fraction of δ_0 chosen so that after all these transfers there is some decay left on $l(1)$. We then sum over y, giving

$$\sum_y \eta^d \chi(y) \exp[-\delta_1 L^{k-j_{l(1)}}|x-y|] \leqq C[L^{k-j_{l(1)}}]^{-d}. \quad (3.68)$$

Combining this factor with the power of $L^{j_{l(1)}-k}$ already present from the propagator on $l(1)$ (and any extra factors η at x and y), we get altogether the exponent $D(H_1)$. If $l(2) \in S^c$, we can sum over $j_{l(1)} \leqq j_{l(2)}$, since $D(H_1) > 0$:

$$\sum_{j=-n}^{j_{l(2)}} (L^{j-k})^{D(H_1)} \leqq C(L^{j_{l(2)}-k})^{D(H_1)}. \quad (3.69)$$

Graphically, we have shrunk $l(1)$ to a point in $\tilde{H}$; the remaining vertices are summed with the constraint $-n \leqq j_{l(2)} \leqq j_{l(3)} \leqq \ldots \leqq j_{l(m)} \leqq k-1$. We continue doing this until S^c is exhausted, so that $l(i+1) \in S$. This gives

$$\sum_{j=-n}^{0} (L^{j-k})^{D(H_i)} \leqq C(L^{-k})^{D(H_i)} \leqq CL^{-\gamma k}(L^{j_{l(i+1)}-k})^{D(H_i)-\gamma}. \quad (3.70)$$

For γ small enough, $D(H_i) - \gamma > 0$ and we continue the process, eventually shrinking $\tilde{H}$ to one point x. The final sum over $j_{l(m)}$ is then bounded by C, and the sum over x by $|\square'| \leqq C(p(L^k\varepsilon))^d$. The vertices in $\tilde{H}$ give the factor $(L^k\varepsilon)^\varrho$, and (3.70) gives $L^{-\gamma k}$. Finally, there is a sum over orderings of the lines, again depending only on $\bar{n}$.

So we reduce to the case with S^c empty, i.e. we must bound $|E^{(k)}(\tilde{H}) - E_\phi^{(k+n)}(\tilde{H})|$. This is done by replacing one by one every factor in $E_\phi^{(k+n)}(\tilde{H})$ by the corresponding factor in $E^{(k)}(\tilde{H})$, and bounding the error at each step. First, we replace the propagators on the external lines, using the following proposition proved in Sect. 4. When $x' \in T_{\eta'}$, we denote by x that point in T_η for which $x' \in B^n(x)$.

Proposition 3.8. *For $x', y' \in T_{\eta'}$, $0 < \alpha < 1$, and γ sufficiently small,*

$$\begin{gathered} |a_{k+n} G_{k+n}^{\eta'} Q_{k+n}^*(x', z) - a_k G_k^\eta Q_k^*(x, z)|, \\ |a_{k+n} \partial_\mu^{\eta'} G_{k+n}^{\eta'} Q_{k+n}^*(x', z) - a_k \partial_\mu^\eta G_k^\eta Q_k^*(x, z)|, \\ |(\partial_\alpha(x', y') a_{k+n} G_{k+n}^{\eta'} Q_{k+n}^*)(z) - (\partial_\alpha(x, y) a_k G_k^\eta Q_k^*)(z)|, \\ |(\partial_\alpha(x', y') a_{k+n} \partial_\mu^{\eta'} G_{k+n}^{\eta'} Q_{k+n}^*)(z) - (\partial_\alpha(x, y) a_k \partial_\mu^\eta G_k^\eta Q_k^*)(z)| \\ \leqq CL^{-\gamma k} \exp[-\delta_0 \{|x-z|, \mathrm{dist}(\{x, y\}, z)\}]. \end{gathered} \quad (3.71)$$

If we replace such a propagator in $E_\phi^{(k+n)}(\tilde{H})$, the error is the same graph with a difference or propagators on one line. Using the method presented, this error is bounded, and Proposition 3.8 gives the desired factor $L^{-\gamma k}$. All such propagators are replaced in this way. The sources $g_{\mu\nu}(x')$, $h(x')$ and the partition of unity function $\chi(x')$ can be replaced by $g_{\mu\nu}(x)$, $h(x)$ and $\chi(x)$ respectively, since their derivatives are uniformly bounded. The operator (3.46) can be replaced using the following bound:

$$|U(B(\Gamma^{(k+n)}_{x_0,x'})) - U(B(\Gamma^{(k)}_{x_0,x}))| \leqq Ce(L^k\varepsilon)^{2-d/2}L^{-k}\frac{p(L^k\varepsilon)}{\mu_0 L^k\varepsilon}. \tag{3.72}$$

Now we replace propagators on the internal lines, using the following proposition (also proved in Sect. 4):

Proposition 3.9. *For* $0 \leqq j \leqq k-1$, $0<\alpha<1$, *and* γ *sufficiently small,*

$$|G^{\eta'}_{(j)}(x',y') - G^{\eta}_{(j)}(x,y)|\,, \quad |\partial^{\eta'}_\mu G^{\eta'}_{(j)}(x',y') - \partial^\eta_\mu G^\eta_{(j)}(x,y)|$$
$$\leqq CL^{-\gamma k}\{(L^j\eta)^{2-d-\gamma}, (L^j\eta)^{1-d-\gamma}\}\exp[-\delta_0(L^j\eta)^{-1}|x-y|]\,, \tag{3.73}$$

$$|G^{\eta'}_{(j)}(\Gamma^{(j+n+1)}_{x'_{j+n+1},x'}, b') - G^\eta_{(j)}(\Gamma^{(j+1)}_{x_{j+1},x}, b)|$$
$$\leqq CL^{-\gamma k}(L^j\eta)^{3-d-\gamma}\exp[-\delta_0(L^j\eta)^{-1}\operatorname{dist}(B^j(x), b)]\,, \tag{3.74}$$

$$|(\partial_\alpha(x',y')G^{\eta'}_{(j)})(z') - (\partial_\alpha(x,y)G^\eta_{(j)})(z)|\,,$$
$$|(\partial_\alpha(x',y')\partial^{\eta'}_\mu G^{\eta'}_{(j)})(z') - (\partial_\alpha(x,y)\partial^\eta_\mu G^\eta_{(j)})(z)|$$
$$\leqq CL^{-\gamma k}\{(L^j\eta)^{2-d-\alpha-\gamma}, (L^j\eta)^{1-d-\alpha-\gamma}\}\exp[-\delta_0\operatorname{dist}(\{x,y\},z)]\,. \tag{3.75}$$

First we fix the ordering $\tilde{l}$ of the internal lines. If we replace a propagator $G^{\eta'}_{(j)}(x',y')$ by $G^\eta_{(j)}(x,y)$, the error is the same graph with a difference of propagators on one line. Redoing the analysis, we see that the degrees of some subgraphs have been reduced by γ; for γ small enough, the exponents $D(H_i)-\gamma$ are still positive, and the bound proceeds as before. This replacement is made for every internal line, and every ordering $\tilde{l}$. Having done this, we can replace the sums of internal vertices x' over $T_{\eta'}$ by sums over $x \in T_\eta$, and this gives exactly $E^{(k)}(\tilde{H})$, proving Proposition 3.6

The remainder terms we neglected give large positive powers of $L^k\varepsilon$, so it is sufficient to bound $E^{(k)}(\tilde{H})$ and $E^{(k+n)}(\tilde{H})$ separately. We note that Proposition 3.7 also holds for $G^\eta_{(j)}(\square')$ and $G^\eta_{(j)}(\square', \tilde{A}^{(k)})$, since Theorem 3.3 gives bounds on these operators also. The remainders from the expansions (3.48) and (3.52) give factors $((L^k\varepsilon)^{2-\alpha-d/2})^{\bar{n}+1}$, for any $\alpha>0$. When a graph contains a vertex with extra powers of η or η', we can extract $\eta^\gamma = L^{-\gamma k}$. The propagator $\delta G_k(\square', A^{(k)})(x,y)$ is bounded by $C\exp[-\delta_0|x-y| - \delta_0 p(L^k\varepsilon)]$, since $x, y \in \square$. The propagators $\delta G_k(\square', \square'', (1-\theta)\tilde{A}^{(k)})$ and $\delta G_k(\square'', 0)$ give a factor $\exp[-\delta_0\operatorname{dist}(\{x,y\}, \partial\square'')]$. If $x, y \in \square$, this gives $\exp[-\delta_0 p(L^k\varepsilon)]$. If $x, y \notin \square$, then δG_k must come from the expansion (3.52), so there is a string of propagators, one of which is δG_k, which begins and ends in $\square$. By extracting a small exponential factor from each of these propagators, we get $\exp[-\delta\operatorname{dist}(\square, \partial\square')]$.

So we have established Proposition 3.6. The factor $(L^k\varepsilon)^\varrho$ comes from the vertices (3.18)–(3.26). Examining these vertices, we see that in each of them we may give a factor $(L^k\varepsilon)^{1-d/4}$ to every field $\phi^{(k)}(x)$. Therefore $\varrho \geqq s/4$. The bound (3.72) gives $(L^k\varepsilon)^{-1/2}$.

In order to derive (3.38) we can bound each propagator $G(x, y)$ which is larger than $3\sqrt{d}$ (say) by $\exp[-\delta_0|x-y|]$, and treat it as an external line. The graph then factorizes into a product of localised subgraphs, in each of which the methods outlined yield the desired bound.

3.5. Cancellation of Divergences. Divergent subdiagrams vanish either because of some symmetry requirement (gauge covariance or the approximate Euclidean symmetry of the lattice), or because they are cancelled by similar divergent diagrams in the counterterms δm^2 and E_1. We consider together all graphs of the same order and type in $P^{(k),1}$; for each ordering $\tilde{l}$, these graphs can be combined to form renormalised graphs, which are convergent. This property, which is a consequence of the renormalizability of the model, is proved in [Ba 3].

We shall present below a more precise version of this statement. Recall that graphs in δm^2 and E_1 are defined with the propagators C_v^η and C_s^η; these may be decomposed as $C^\eta = \sum_{j=0}^{K} G_{(j)}^\eta$, where $G_{(j)}^\eta$ is given by (2.17) for $0 \leqq j \leqq K-1$, and $G_{(K)}^\eta = C^\eta - G_K^\eta$. In Sect. 4 we will show that $G_{(j)}^\eta$ and $G_{(j)}^{\eta'}$ satisfy Propositions (3.7) and (3.9) for $k \leqq j \leqq K$. Therefore in any graph containing propagators C^η, we may introduce this decomposition and again order the lines. For any ordering $\tilde{l}$, we again get a sum of terms; each term has the property that for some $m_1 \leqq m$, we sum over integers with the constraint

$$0 \leqq j_{l(1)} \leqq \ldots \leqq j_{l(m_1)} \leqq k-1 < j_{l(m_1+1)} \leqq \ldots \leqq j_{l(m)} \leqq K. \tag{3.76}$$

Then the renormalised graphs $\{H_{\text{ren}}\}$ produced after cancelling divergences for this ordering $\tilde{l}$ have the following properties;

(i)
$$D(H_i) > 0 \quad \text{for} \quad 1 \leqq i \leqq m_1, \tag{3.77}$$

(ii) for $m_1 < m$, and some $\{m_i\}$ with $m_1 < m_2 < \ldots < m_n = m$, $D(H_{m_i+l}) > D(H_{m_i})$ each $l = 1, \ldots, m_{i+1} - m_i - 1$, each $i = 1, \ldots, n-1$,

$$D(H_{m_{i+1}}) \leqq D(H_{m_i}) \quad \text{each} \quad i = 1, \ldots, n-1. \tag{3.78}$$

The integers $\{m_i\}$ correspond to divergent subgraphs encountered between integers k and K. We can now extend the bound (3.38) to a graph H_{ren}. Using the methods of Sect. 3.4, we shrink lines until H_{m_1} is one vertex, leaving the sum

$$\sum_{j=0}^{k-1} (L^{j-k})^{D(H_{m_1})} \leqq C. \tag{3.79}$$

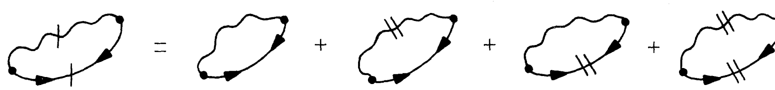
Continuing the procedure, we shrink lines until H_{m_2} is one vertex, giving the sum

$$\sum_{j=k}^{j_{l(m_2+1)}} (L^{j-k})^{D(H_{m_2}) - D(H_{m_1})} \leqq \begin{cases} C & \text{if} \quad D(H_{m_2}) < D(H_{m_1}) \\ C \ln(L^k \varepsilon)^{-1} & \text{if} \quad D(H_{m_2}) = D(H_{m_1}). \end{cases} \tag{3.80}$$

We get a similar bound for each m_i, $i = 2, \ldots, n$. Therefore the graph H_{ren} is bounded as before, with the possible addition of some power of $\ln(L^k \varepsilon)^{-1}$. Notice that when $j \geqq k$, we cannot "transfer" external propagators. However, this never arises for graphs in E_1, and graphs in δm^2 are attached to external propagators at one vertex only.

Before proceeding, we must specify how the partition of unity (3.36) is to be introduced in the counterterms. First we decompose C^η on each line to get a set of graphs satisfying (3.76). When all the integers in a graph in δm^2 are less than m_1, we introduce (3.36) at each vertex of the graph. If any integer on a line is bigger than $k-1$, we do not introduce (3.36) at any vertex in the graph. For E_1, we again use (3.36) everywhere if $m_1=m$; if $m_1<m$, we introduce (3.36) only at one vertex connected to $l(m)$. Of course the exponential tree decay in (3.38) and (3.39) involves only the external vertices and those internal vertices with partitions of unity.

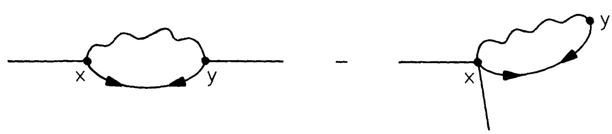
We will prove (3.77) and (3.78) for one linearly divergent self-energy diagram for the scalar field. This will show the idea of the general proof; the reader is referred to [Ba 3] for an exhaustive list of how divergences cancel. The graph in δm^2 corresponding to the diagram is the following:



(3.81)

When δm^2 occurs in a graph, the last three diagrams on the right-hand side of (3.81) contribute to terms with $m_1<m$. After shrinking to a vertex all lines with intergers less than k, the remaining subgraphs (composed of lines with two dashes) have non-positive degree. So (3.78) holds in these cases. It is not hard to extend this reasoning to show that (3.78) holds for every other graph in δm^2 and E_1.

The first graph on the right-hand side of (3.81) combines with a divergent diagram in $P^{(k),1}$ to give the following difference:

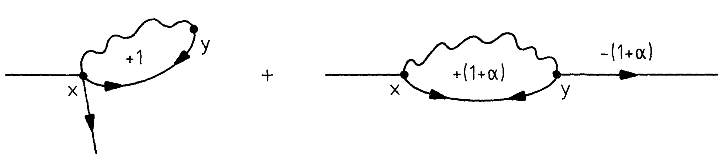


(3.82)

We can expand the external scalar field $\phi(y)$ as follows:

$$\phi(y)=\phi(x)+\sum_{\mu=1}^{d}(y_\mu-x_\mu)(\partial_\mu^\eta\phi)(x)+\sum_{b\in\Gamma_{x,y}}\eta|b_x-b|^\alpha(\partial_\alpha(b,b_x)\partial^\eta\phi), \quad (3.83)$$

where $0<\alpha<1$, and b_x is the bond at x parallel to b. Substituting this into (3.82) gives



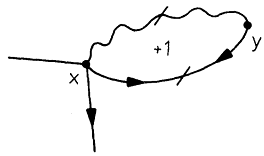
(3.84)

We have indicated the presence of additional convergence factors for the graphs, and additional derivatives on propagators. When the field $\phi(y)$ is $\chi(y)G^\eta_{(j)}(y,z)$ for some j and z, we can use (3.65) to bound the last term in (3.84). The localization function gives terms involving (3.63), so the bound is

$$C|x-y|^{1+\alpha}\{(L^j\eta)^{1-d-\alpha}+(L^j\eta)^{1-d}+(L^j\eta)^{2-d}\} \cdot\exp[-\delta_0(L^j\eta)^{-1}\,\mathrm{dist}(\Gamma_{x,y},z)]\,. \quad (3.85)$$

The factor $|x-y|^{1+\alpha}$ means that the last graph in (3.84) has degree $+\alpha$, and so it is convergent. Of course this extra degree of convergence has been obtained at the price of a derivative of order $(1+\alpha)$ on an external line. However, this means that we have postponed consideration of this divergence until it occurs in a bigger subgraph. Because the model is superrenormalizable, sufficiently large subgraphs always have positive degree; therefore the divergence will cause no problems at a later stage.

If we consider the following graph



(3.86)

we see that it contains the factor

$$\sum_{\mu=1}^{d} \partial^{\eta}_{\mu,x}\partial^{\eta}_{\mu,y} C^{\eta}_{s}(x,y)\,(y_{\mu'}-x_{\mu'})$$
$$= \delta^{\eta}(x-y)\,(y_{\mu'}-x_{\mu'}) - m^2(L^k\varepsilon)^2 C^{\eta}_{s}(x,y)\,(y_{\mu'}-x_{\mu'})\,, \tag{3.87}$$

where δ^η is the lattice δ-function. So whenever a graph contains the first term in (3.84), we may introduce a similar graph with (3.86) at the vertex x, and the error is the last term in (3.87). Clearly, the graph (3.86) cancels the first graph in (3.84), and since (3.86) has degree zero, it satisfies (3.78). Furthermore the error term in (3.87) has degree two, but the factor $(L^k\varepsilon)^2$ means that (3.78) still holds.

The inequalities (3.77) and (3.78) can be established for all renormalized graphs in $P^{(k),1}$. We would now like to see how this allows us to deduce Theorem 3.5 for such a graph. The bound (3.38) relies solely on the cancellation of divergences. In this connection, it should be noted that only through the expansion (3.52) can vertices approach $\partial\square'$. But this expansion never produces divergent subgraphs, so the vertices in a divergent subgraph never see a "sharp" boundary. In order to establish (3.39) we must extract convergence factors as before. Since some subgraphs have degree α, we can extract only $L^{-\gamma k}$ with $\gamma<\alpha$. The only new factors to be replaced in $E^{(k+n)}_{\phi}$ are the convergence factors $|x-y|^{\alpha}$, etc. These must be replaced *after* all the other factors. To do so, we use the following bounds:

$$\begin{aligned} \big||x'-y'|^{\alpha}-|x-y|^{\alpha}\big| &\leqq CL^{-\gamma k}\{|x'-y'|^{\alpha-\gamma}+|x-y|^{\alpha-\gamma}\}\,,\\ |(x'_{\mu}-y'_{\mu})-(x_{\mu}-y_{\mu})| &\leqq CL^{-\gamma k}\{|x'-y'|^{1-\gamma}+|x-y|^{1-\gamma}\}\,,\\ \Big|\sum_{b'\in b}\eta'|b'_{-}-x'|^{\alpha}-\eta|b_{-}-x|^{\alpha}\Big| &\leqq CL^{-\gamma k}\sum_{b'\in b}\{\eta'|b'_{-}-x'|^{\alpha-\gamma}+\eta|b_{-}-x|^{\alpha-\gamma}\}\,. \end{aligned} \tag{3.88}$$

Then by redoing the analysis of Sect. 3.4, we can establish Theorem 3.4 for any renormalized graph in $P^{(k),1}$.

3.6. Convergence of $S^{(k)}$. To complete the proof of Theorem 3.4, we now consider the remaining terms in (3.14). The normalization factor E_0 can be written

$$\begin{aligned} \exp(E_0) &= Z^{\eta}_{k}Z^{\eta}_{k}(0)\int(dA_k)\exp[-1/2\langle A_k,\Delta^{(k)}A_k\rangle]\int(d\phi_k)\exp[-1/2\langle\phi_k,\Delta^{(k)}(0)\phi_k\rangle]\\ &= Z^{\eta}_{k}Z^{\eta}_{k}(0)N_kN_k(0)\,. \end{aligned} \tag{3.89}$$

Hence (3.14) can be written

$$
\begin{aligned}
&S^{(k),1}(T^{(k)}, A_k, \phi_k, g, h) \\
&\quad = 1/2\langle A_k, \Delta^{(k)} A_k\rangle + \ln N_k + 1/2\langle \phi_k, \Delta^{(k)}(A^{(k)})\phi_k\rangle + \ln N_k(0) \\
&\quad\quad - \ln[Z_k(A^{(k)})Z_k(0)^{-1}] + P^{(k),1}(T^{(k)}, A_k, \phi_k, g, h)\,. \qquad (3.90)
\end{aligned}
$$

Examining the representation (2.14) for the quadratic term $\langle \phi_k, \Delta^{(k)}(A^{(k)})\phi_k\rangle$, we see that it has the form of a graph in $P^{(k),1}$ with two external scalar fields, and so the results of Sect. 3.4 apply.

The operator $\Delta^{(k)}$ is diagonal in the Fourier representation on $T^{(k)}$, and in Sect. 4 we prove the following proposition concerning its Fourier transform $\Delta^{(k)}(p)$.

Proposition 3.10. $|\Delta^{(k)}(p)| \leqq C$ *uniformly in* k,

$$
|\Delta^{(k)}(p) - \Delta^{(k+n)}(p)| \leqq CL^{-2k}|\Delta^{(k)}(p)|\,. \qquad (3.91)
$$

From Proposition 3.10 we get immediately

$$
|\langle A_k, (\Delta^{(k)} - \Delta^{(k+n)})A_k\rangle| \leqq CL^{-2k}(p(L^k\varepsilon))^2(\mu_0 L^k\varepsilon)^{-2}(L^k\varepsilon)^{-d}|T|\,, \qquad (3.92)
$$

where we used the bounds (3.2). Furthermore,

$$
\begin{aligned}
|\ln N_k - \ln N_{k+n}| &= 1/2d\left|\sum_p \ln[\Delta^{(k+n)}(p)\Delta^{(k)}(p)^{-1}]\right| \\
&\leqq C\sum_p \ln[1 + (\Delta^{(k+n)}(p) - \Delta^{(k)}(p))\Delta^{(k)}(p)^{-1}] \\
&\leqq C\sum_p L^{-2k} \leqq CL^{-2k}(L^k\varepsilon)^{-d}|T|\,. \qquad (3.93)
\end{aligned}
$$

The convergence of $\ln N_k(0)$ follows similarly. Finally we consider the term $\ln[Z_k(A^{(k)})Z_k(0)^{-1}]$. We introduce the fields $(j/R)A^{(k)}$, $j = 1, \ldots, R$, where $R = 1 + [(\mu_0 L^k\varepsilon)^{-1}]$. Then we have

$$
\begin{aligned}
\ln[Z_k(A^{(k)})Z_k(0)^{-1}] &= 1/2N \ln\det[G_k(A^{(k)})G_k(0)^{-1}] \\
&= 1/2N \sum_{j=1}^{R} \ln\det[G_k(j/RA^{(k)})G_k(j-1/RA^{(k)})^{-1}]\,. \qquad (3.94)
\end{aligned}
$$

We will expand $G_k(j/RA^{(k)})$, using (3.52) with $B = (j-1/R)A^{(k)}$ and $A' = (1/R)A^{(k)}$. The field A' is bounded by $Cp(L^k\varepsilon)$, so we have

$$
\begin{aligned}
&\ln\det[G_k(j/RA^{(k)})G_k(j-1/RA^{(k)})^{-1}] \\
&\quad = \ln\det[I + G_k^{1/2}(j/RA^{(k)})V_k(A', B)G_k^{1/2}(j/RA^{(k)})]\,. \qquad (3.95)
\end{aligned}
$$

For a symmetric operator D with $\|D\| < 1$, we have

$$
\begin{aligned}
\ln(I + D) &\leqq \sum_{p=1}^{n} (-1)^{p-1}p^{-1}\,\mathrm{tr}\,D^p\,, \quad n\ \text{odd} \\
&\geqq \sum_{p=1}^{n} (-1)^{p-1}p^{-1}\,\mathrm{tr}\,D^p\,, \quad n\ \text{even}\,. \qquad (3.96)
\end{aligned}
$$

We use (3.96) to expand (3.95) up to order $2\bar{n}$. A general term in the expansion is

$$
(-1)^{p-1}p^{-1}\,\mathrm{tr}[V_k(A', B)G_k(j/RA^{(k)})]^p\,. \qquad (3.97)
$$

Graphically, this is a scalar loop with external vector fields. We know from Sects. 3.4 and 3.5 that all such graphs converge (in fact the Ward-Takahashi identities guarantee that there are no divergences produced; see [Ba 3] for details). Therefore we have

$$
\begin{aligned}
&|\ln[Z_k(A^{(k)})Z_k(0)^{-1}]-\ln[Z_{k+n}(A^{(k+n)})Z_{k+n}(0)^{-1}]| \\
&\leq C\sum_{j=1}^{R}\left\{\sum_{p=1}^{2\bar{n}} p^{-1}|\mathrm{tr}[V_kG_k(j/RA^{(k)})]^p-\mathrm{tr}[V_{k+n}G_{k+n}(j/RA^{(k+n)})]^p|\right. \\
&\quad\left.+(2\bar{n}+1)^{-1}\,\mathrm{tr}[V_kG_k(j/RA^{(k)})]^{2\bar{n}+1}\right\} \\
&\leq C\sum_{j=1}^{R}\left\{\sum_{p=1}^{2\bar{n}} p^{-1}(L^{-\gamma k}(L^k\varepsilon)^{-\beta}+(L^k\varepsilon)^{\sigma+1})|T|+(2\bar{n}+1)^{-1}(L^k\varepsilon)^{\sigma+1}|T|\right\} \\
&\leq C(L^{-\gamma k}(L^k\varepsilon)^{-\beta-1}+(L^k\varepsilon)^{\sigma})|T|\,.
\end{aligned}
\tag{3.98}
$$

This completes the proof of Theorem 3.4.

4. Technical Estimates

By using multiple reflection representations, the propagators G_k^η and $G_k^\eta(\Omega)$ can be written in terms of the operator defined by (2.13) with free boundary conditions (and $A=0$, of course), as long as Ω is a rectangular parallelepiped which is a union of blocks of L^{kd} sites. Such representations are given explicitly in [Ba 4], so it is sufficient to prove Propositions 3.8 and 3.9 for the operator with free boundary conditions, which we write as G_k for simplicity. The basis for our proof is an explicit Fourier representation for $a_kG_kQ_k^*$. We introduce a Fourier transform on ηZ^d by

$$\tilde{f}(p)=\sum_x \eta^d e^{-ipx}f(x)\,,\qquad f(x)=(2\pi)^{-d}\int_{|p|\leq\pi/\eta} e^{ixp}\tilde{f}(p)\,. \tag{4.1}$$

By applying this to the equation defining G_k, it follows in a straightforward way (see [Ba 4]) that

$$(a_kG_kQ_k^*)\,(x,y)=(2\pi)^{-d}\int_{|p'|\leq\pi} dp'\Delta^{(k)}(p')\sum_l e^{i(p'+l)(x-y)}\frac{u_k^\eta(p'+l)}{\Delta^\eta(p'+l)}\,, \tag{4.2}$$

where $x\in\eta Z^d$, $y\in Z^d$, $p'\in[-\pi,\pi)$, $l\in 2\pi Z^d$ and $-\pi(L^k-1)\leq l_\mu\leq\pi(L^k-1)$ for L odd, while $-\pi L^k\leq l_\mu\leq\pi L^k$ for L even. Also

$$u_k^\eta(p)=\prod_{\mu=1}^{d}[(e^{-ip_\mu}-1)\eta(e^{-i\eta p_\mu}-1)^{-1}]\,, \tag{4.3}$$

$$\Delta^\eta(p)=4\eta^{-2}\sum_{\mu=1}^{d}\sin^2(1/2\eta p_\mu)+m^2(L^k\varepsilon)^2\,, \tag{4.4}$$

$$\Delta^{(k)}(p')=\left(a_k^{-1}+\sum_l |u_k^\eta(p'+l)|^2\Delta^\eta(p'+l)^{-1}\right)^{-1}. \tag{4.5}$$

We will first prove a bound on the difference between $\Delta^{(k)}(p')$ and the following operator:

$$\bar{\Delta}^{(k)}(p')=\left(a_k^{-1}+\sum_l |u_\kappa^\eta(p'+l)|^2D^{-1}(p'+l)\right)^{-1}, \tag{4.6}$$

where the operator D satisfies

(i) $$|D^{-1}(p)-(|p|^2+m^2(L^k\varepsilon)^2)^{-1}|\leqq CL^{-2k}, \tag{4.7}$$

(ii) $$0\leqq C|p|^2\leqq D(p). \tag{4.8}$$

Lemma 4.1.

$$|\Delta^{(k)}(p')-\bar{\Delta}^{(k)}(p')|\leqq CL^{-2k}\Delta^{(k)}(p'). \tag{4.9}$$

Proof. We first check that $\Delta^\eta(p)$ satisfies (4.7) and (4.8). By using $x^2\geqq\sin^2 x\geqq x^2-x^4/3$, we have

$$|\Delta^\eta(p)-(|p|^2+m^2(L^k\varepsilon)^2)|\leqq C\eta^2\sum_{\mu=1}^{d}|p_\mu|^4\leqq CL^{-2k}|p|^4, \tag{4.10}$$

from which (4.7) follows. The bound (4.8) is obvious. Since Δ^η and D are positive, we have $\Delta^{(k)}(p')$, $\bar{\Delta}^{(k)}(p')\leqq a_k$. Hence

$$\begin{aligned}|\Delta^{(k)}(p')-\bar{\Delta}^{(k)}(p')|&\leqq C\Delta^{(k)}(p')|\Delta^{(k)}(p')^{-1}-\bar{\Delta}^{(k)}(p')^{-1}|\\&\leqq C\Delta^{(k)}(p')\sum_l|u_k^\eta(p'+l)|^2L^{-2k},\end{aligned} \tag{4.11}$$

where we used (4.7). Since $\sum_l|u_k^\eta(p'+l)|^2=1$, the bound (4.9) follows.

To prove convergence of $\Delta^{(k)}(p')$, we notice that the composition law for renormalization transformations allows us to write $\Delta^{(k+n)}(p')$ in the form (4.6), with

$$D^{-1}(p)=a_n^{-1}L^{-2k}+\sum_{l'}|u_n^{\eta'}(p+l')|^2\Delta^{\eta'}(p+l')^{-1}, \tag{4.12}$$

where $l'\in 2\pi(L^k/2)Z^d$, and $|l'_\mu|\leqq\pi(L^{k+n}-L^k)$ for L odd, while $|l'_\mu|\leqq\pi L^{k+n}$ for L even. Also

$$u_n^{\eta'}(p)=\prod_{\mu=1}^{d}[\eta^{-1}(e^{-i\eta p_\mu}-1)\eta'(e^{-i\eta' p_\mu}-1)]. \tag{4.13}$$

Lemma 4.2. *The operator* (4.12) *satisfies* (4.7) *and* (4.8).

Proof.

$$\begin{aligned}D^{-1}(p)-(|p|^2+m^2(L^k\varepsilon)^2)^{-1}&=a_n^{-1}L^{-2k}+|u_n^{\eta'}(p)|^2\Delta^{\eta'}(p)^{-1}\\&\quad-(|p|^2+m^2(L^k\varepsilon)^2)^{-1}+\sum_{l'\neq 0}|u_n^{\eta'}(p+l')|^2\Delta^\eta(p+l')^{-1},\end{aligned} \tag{4.14}$$

when $l'\neq 0$, $\Delta^{\eta'}(p+l')\geqq C|p+l'|^2\geqq CL^{2k}$. Therefore

$$\begin{aligned}|1-|u_n^{\eta'}(p)|^2|&\leqq CL^{-2nd}\sum_{x,w\in B^n(y)}\sin^2 1/2p(x-w)\\&\leqq CL^{-2nd}\sum_{\mu,\nu=1}^{d}p_\mu p_\nu\left\{\sum_{x,w\in B^n(y)}(x-w)_\mu(x-w)_\nu\right\}\\&\leqq CL^{-2nd}|p|^2L^{nd}L^{n(d-1)}L^{-2(k+n)}L^{3n}\\&\leqq CL^{-2k}|p|^2.\end{aligned} \tag{4.15}$$

Therefore using (4.7) for $\Delta^\eta(p)$ and (4.15),

$$\begin{aligned}|D^{-1}(p)+(|p|^2+m^2(L^k\varepsilon)^2)^{-1}|&\leqq CL^{-2k}\left\{1+\sum_{l'\neq 0}|u_n^{\eta'}(p+l')|^2\right\}\\&\leqq CL^{-2k}.\end{aligned} \tag{4.16}$$

To prove (4.8) we see that

$$D^{-1}(p) \leqq a_n^{-1} L^{-2k} + C|p|^{-2} \sum_{l'} |u_n^{\eta'}(p+l')|^2$$
$$\leqq |p|^{-2}[C + a_n^{-1}|p|^2 L^{-2k}] \leqq C|p|^{-2}. \tag{4.17}$$

Combining Lemmas 4.1 and 4.2 gives

Lemma 4.3.

$$|\Delta^{(k)}(p') - \Delta^{(k+n)}(p')| \leqq CL^{-2k}\Delta^{(k)}(p'). \tag{4.18}$$

Using the representation (4.2), we shall now establish the last bound in (3.71). The rest of Proposition 3.8 is simpler. We have the identity

$$\begin{aligned}&(\partial_\alpha(x',y')\partial_\mu^{\eta'} a_{k+n} G_{k+n}^{\eta'} Q_{k+n}^*)(z)\\ &\quad = (2\pi)^{-d} \int dp' \sum_l \sum_m e^{i(p'+l+m)(x'-z)} |x'-y'|^{-\alpha}\{1-\exp[i(p'+l+m)(y'-x')]\}\\ &\quad\cdot (\eta)^{-1}\{\exp[i\eta'(p'+l+m)_\mu]-1\}\{\Delta^{(k+n)}(p') u_{k+n}^{\eta'}(p'+l+m)\Delta^{\eta'}(p'+l+m)^{-1}\},\end{aligned} \tag{4.19}$$

where $l \in 2\pi Z^d$ is the same as in (4.2). Also $m \in 2\pi L^k Z^d$ and $|m_\mu| \leqq \pi L^k(L^n - 1)$ for L odd, while $m \in 2\pi(L^k+1)Z^d$ and $|m_\mu| \leqq \pi(L^k+1)(L^n-1)$ for L even. The corresponding expression for G_k^η is obtained from (4.19) by replacing x', y', η' by x, y, η and $(k+n)$ by (k), and taking $m=0$. We have the following bounds:

$$|u_{k+n}^{\eta'}(p'+l+m)| < C \prod_{\mu=1}^{d} |p'_\mu| \, |(p'+l+m)_\mu|^{-1}, \tag{4.20}$$

$$|\Delta^{(k+n)}(p')\Delta^{\eta'}(p'+l+m)^{-1}| \leqq C|p'|^2 \, |p'+l+m|^{-2}. \tag{4.21}$$

Also

$$(\eta')^{-1} |\exp[i\eta'(p'+l+m)_\mu] - 1| \leqq C|(p'+l+m)_\mu|$$

and

$$|x'-y'|^{-\alpha}|1-\exp[i(p'+l+m)(y'-x')]| \leqq C|p'+l+m|^\alpha,$$

so the sum over l, m is bounded by

$$\sum_{l,m} |p'+l+m|^{\alpha-1} \prod_\mu |(p'+l+m)_\mu|^{-1} \leqq C \quad \text{for} \quad \alpha < 1. \tag{4.22}$$

We first bound the terms in (4.19) with $m \neq 0$ as follows:

$$\begin{aligned}&|(4.19); m \neq 0|\\ &\quad \leqq C(2\pi)^{-d} \int dp' \sum_{m \neq 0} \sum_l |p'+l+m|^{\alpha-1} |p'|^2 \prod_\mu |p'_\mu| |(p'+l+m)_\mu|^{-1}\\ &\quad \leqq CL^{-\gamma k}(2\pi)^{-d} \int dp' \sum_{m \neq 0} \sum_l |p'+l+m|^{\alpha-1+\gamma} |p'|^2 \prod_\mu |p'_\mu| \, |(p'+l+m)_\mu|^{-1}\\ &\quad \leqq CL^{-\gamma k} \quad \text{for} \quad \alpha+\gamma < 1.\end{aligned} \tag{4.23}$$

To analyze the $m=0$ term in (4.19), we successively replace each factor by the corresponding one in the expression for $(\partial_\alpha(x,y)\partial_\mu^\eta a_k G_k^\eta Q_k^*)(z)$ and bound the error. We must always be careful to keep enough negative powers of momentum so that the sum over l is bounded. First we have

$$\begin{aligned}&|\exp[i(p'+l)x]-\exp[i(p'+l)x']|\\&\quad\leqq C|p'+l|^{\gamma}|x-x'|^{\gamma}\leqq CL^{-\gamma k}|p'+l|^{\gamma}.\end{aligned}\tag{4.24}$$

So keeping $\gamma+\alpha<1$, the error produced by the above replacement is bounded by $CL^{-\gamma k}$. Next, we have

$$\begin{aligned}&|(\eta')^{-1}[\exp[i\eta'(p'+l)_\mu]-1]-\eta^{-1}[\exp[i\eta(p'+l)_\mu]-1]|\\&\leqq(\eta')^{-1}\sin^2 1/2\eta'(p'+l)_\mu+\eta^{-1}\sin^2 1/2\eta(p'+l)_\mu\\&\quad+|(\eta')^{-1}\sin\eta'(p'+l)_\mu-\eta^{-1}\sin\eta(p'+l)_\mu|\\&\leqq C|p'+l|^2L^{-k}\leqq C|p'+l|^{1+\gamma}L^{-\gamma k}.\end{aligned}\tag{4.25}$$

For the Hölder derivatives, suppose $|x-y|\leqq L^{-k}$. Then

$$|x-y|^{-\alpha}|1-\exp[i(p'+l)(y-x)]|\leqq C|p'+l|^{\alpha+\gamma}|x-y|^{\gamma}\leqq CL^{-\gamma k}|p'+l|^{\alpha+\gamma}.\tag{4.26}$$

Furthermore, when $|x-y|>L^{-k}$, we have

$$||x-y|-|x'-y'||\leqq L^{-k}.\tag{4.27}$$

Therefore it follows easily that

$$\begin{aligned}&||x-y|^{-\alpha}\{1-\exp[i(p'+l)(y-x)]\}-|x'-y'|^{-\alpha}\{1-\exp[i(p'+l)(y'-x')]\}|\\&\qquad\leqq CL^{-\gamma k}|p'+l|^{\alpha+\gamma},\end{aligned}\tag{4.28}$$

where we have assumed $\alpha+\gamma<1$ and $\gamma\leqq\alpha$. Next we need a lemma.

Lemma 4.4.

$$|u_k^\eta(p'+l)-u_{k+n}^{\eta'}(p'+l)|\leqq CL^{-\gamma k}|p'+l|^{\gamma}|u_k^\eta(p'+l)|.\tag{4.29}$$

Proof.

$$\begin{aligned}&u_{k+n}^{\eta'}(p'+l)-u_k^\eta(p'+l)\\&=u_k^\eta(p'+l)\prod_{\mu=1}^{d}[\eta'(e^{-i\eta'(p'+l)_\mu}-1)^{-1}]\\&\quad\cdot\left[\prod_{\mu=1}^{d}\eta^{-1}(e^{-i\eta(p'+l)_\mu}-1)-\prod_{\mu=1}^{d}(\eta')^{-1}(e^{-i\eta'(p'+l)_\mu}-1)\right].\end{aligned}\tag{4.30}$$

By repeated use of the identity $xy-zw=1/2(x-y)(z+w)+1/2(x+y)(z-w)$, we can write the difference inside the last bracket of (4.30) as a sum of 2^{d-1} terms. Each term is a product of d factors, at least one of which is the left-hand side of (4.25). So inserting the appropriate bounds, we get

$$\begin{aligned}|(4.30)|&\leqq C|u_k^\eta(p'+l)|\prod_{\mu=1}^{d}|(p'+l)_\mu|^{-1}L^{-\gamma k}\sum_{\nu=1}^{d}|(p'+l)_\nu|^{1+\gamma}\prod_{\mu'\neq\nu}|(p'+l)_{\mu'}|\\&\leqq CL^{-\gamma k}|p'+l|^{\gamma}|u_k^\eta(p'+l)|,\end{aligned}$$

as required.

Therefore we can replace $u_{k+n}^{\eta'}(p'+l)$ by $u_k^\eta(p'+l)$ and bound the error. Lemma 4.3 allows us to replace $\Delta^{(k+n)}(p')$ by $\Delta^{(k)}(p')$. Finally, from (4.7)

$$|\Delta^{\eta'}(p'+l)^{-1}-\Delta^{\eta}(p'+l)^{-1}|\leqq CL^{-2k}\leqq CL^{-\gamma k}|p'+l|^{-2+\gamma}.\tag{4.31}$$

Therefore every term in (4.19) for $m=0$ can be replaced, and so combining our bounds with Theorem 3.3 we deduce (3.71).

To prove Proposition 3.9 we use the representation (2.17) and Proposition 3.8. We also need convergence properties of the operator $C^{(k)}$ defined in (2.16). Introduce the operator

$$C_\Omega^{(k)}(s)=(s\Delta^{(k)}+(1-s)\Delta^{(k+n)}+aL^{-2}Q^*Q)^{-1}, \tag{4.32}$$

where $0\leqq s\leqq 1$, $\Omega\subset Z^d$ is a rectangular parallelepiped composed of blocks of L^d sites, and where $\Delta^{(k)}$, $\Delta^{(k+n)}$ are defined with periodic boundary conditions on Ω. Using the general results of Chap. 5 of [Ba 4], $C_\Omega^{(k)}(s)$ has uniform exponential decay if the following conditions hold:

$$\text{(i)}\qquad C_\Omega^{(k)}(s)^{-1}\geqq\gamma_0 I, \tag{4.33}$$

$$\text{(ii)}\qquad |C_\Omega^{(k)}(s)^{-1}(x,y)|\leqq C\exp[-\delta_0|x-y|]. \tag{4.34}$$

The condition (4.34) is immediate, since Q^*Q is a short-range operator and $\Delta^{(k)}$ has uniform exponential decay (see Theorem 3.3). To prove (4.33), we use a Fourier representation on Ω. Since Ω is a torus, the allowed momenta satisfy $p'_\mu\in 2\pi|\Omega_\mu|^{-1}Z$, $|p'_\mu|\leqq\pi$, where Ω_μ is the dimension of Ω in the μ^{th} coordinate direction. So $\Delta^{(k)}$ has the representation

$$\langle\phi,\Delta^{(k)}\phi\rangle=|\Omega|^{-1}\sum_{p'}|\tilde\phi(p')|^2\Delta^{(k)}(p'), \tag{4.35}$$

where $|\Omega|=\prod_\mu|\Omega_\mu|$. It is easy to see that $\Delta^{(k)}(p')\geqq C|p'|^2$. Furthermore,

$$\begin{aligned}\langle\phi,Q^*Q\phi\rangle&=|\Omega|^{-1}\sum_{p'}|\tilde\phi(p')|^2\prod_{\mu=1}^{d}\frac{4L^{-2}\sin^2(1/2Lp'_\mu)}{4\sin^2(1/2p'_\mu)}\\&\geqq C|\Omega|^{-1}\sum_{|p'|\leqq\pi/L}|\tilde\phi(p')|^2.\end{aligned} \tag{4.36}$$

Therefore we have

$$\begin{aligned}&\langle\phi,(s\Delta^{(k)}+(1-s)\Delta^{(k+n)}+aL^{-2}Q^*Q)\phi\rangle\\&\geqq C|\Omega|^{-1}\sum_{p'}|\tilde\phi(p')|^2|p'|^2+C|\Omega|^{-1}\sum_{|p'|\leqq\pi/L}|\tilde\phi(p')|^2\geqq C\langle\phi,\phi\rangle,\end{aligned} \tag{4.37}$$

as required. We now prove the required convergence properties of $C^{(k)}$.

Lemma 4.5.

$$|C^{(k)}(x,y)-C^{(k+n)}(x,y)|\leqq CL^{-k}e^{-\delta_0|x-y|}. \tag{4.38}$$

Proof. We prove (4.38) in a finite volume Ω with periodic boundary conditions; the result for free boundary conditions then holds by continuity. First,

$$\begin{aligned}C^{(k)}(x,y)-C^{(k+n)}(x,y)&=\int_0^1 ds\,d/ds\,C_\Omega^{(k)}(s)\,(x,y)=1/2\int_0^1 ds\\&\cdot\left\{\langle\phi(x)\phi(y);\sum_{z,w\in\Omega}\phi(z)\,[\Delta^{(k+n)}(z,w)-\Delta^{(k)}(z,w)]\phi(w)\rangle_s\right\},\end{aligned} \tag{4.39}$$

where the expectation $\langle\cdot\rangle_s$ is taken with respect to the covariance $C_\Omega^{(k)}(s)$. Explicit computation gives

$$\sum_{z,w\in\Omega}\{C_\Omega^{(k)}(s)(x,z)[\Delta^{(k+n)}(z,w)-\Delta^{(k)}(z,w)]C_\Omega^{(k)}(s)(w,y)\}. \tag{4.40}$$

Combining Lemma 4.3 and the uniform exponential decay of $\Delta^{(k)}$ and $C_\Omega^{(k)}(s)$ gives

$$\begin{aligned}|(4.40)|&\leqq CL^{-k}\sum_{z,w\in\Omega}\exp[-\delta_0|x-z|-\delta_0|z-w|-\delta_0|w-y|]\\&\leqq CL^{-k}\exp[-\delta_0|x-y|].\end{aligned} \tag{4.41}$$

We are finally ready to prove Proposition 3.9. We will prove the first bound in (3.73); the other bounds follow in the same way. For $j>0$, we have the expansion

$$\begin{aligned}G_{(j)}^{\eta'}(x',y')=a_{j+n}^2(L^j\eta)^{-4}&\sum_{z,w\in L^j\eta Z^d}(L^j\eta)^{2d}G_{j+n}^{\eta'}Q_{j+n}^*(x',z)\\&\cdot C^{(j+n),L^j\eta}(z,w)Q_{j+n}G_{j+n}^{\eta'}(w,y').\end{aligned} \tag{4.42}$$

We now replace $a_{j+n}Q_{j+n}G_{j+n}^{\eta'}(w,y')$ by $a_jQ_jG_j^{\eta}(w,y)$ in (4.42); using Proposition 3.8 and the scaling of operators (2.20), the error from this replacement is bounded by

$$\begin{aligned}&C(L^j\eta)^{-4}\sum_{z,w}(L^j\eta)^{2d}(L^j\eta)^{2-d}\exp[-\delta_0(L^j\eta)^{-1}|x'-z|](L^j\eta)^{2-d}\\&\quad\cdot\exp[-\delta_0(L^j\eta)^{-1}|z-w|]\cdot L^{-\gamma j}(L^j\eta)^{2-d}\exp[-\delta_0(L^j\eta)^{-1}|w-y'|]\\&\leqq C(L^j\eta)^{2-3d}L^{-\gamma j}\sum_{z,w}(L^j\eta)^{2d}\exp[-\delta_0(L^j\eta)^{-1}\{|x'-z|+|z-w|+|w-y'|\}]\\&\leqq C(L^j\eta)^{2-d}L^{-\gamma j}\exp[-\delta_0(L^j\eta)^{-1}|x'-y'|]\\&\leqq CL^{-\gamma k}(L^j\eta)^{2-d-\gamma}\exp[-\delta_0(L^j\eta)^{-1}|x-y|].\end{aligned} \tag{4.43}$$

Clearly we can replace $a_{j+n}G_{j+n}^{\eta'}Q_{j+n}^*(x'z)$ by $a_jG_j^\eta Q_j^*(x,z)$, and $C^{(j+n),L^j\eta}(z,w)$ by $C^{(j),L^j\eta}(z,w)$, and bound the error in the same way. Hence we deduce the required bound. When $j=0$, we bound each term separately using Proposition 3.7, and write the factor η^{2-d} as $L^{-\gamma k}L^{-k(2-d-\gamma)}$.

Finally, we must establish Propositions 3.7 and 3.9 for $G_{(K)}^\eta=C^\eta-G_K^\eta$. It can be written

$$C^\eta-G_K^\eta=C^\eta(a_K(L^K\eta)^{-2}Q_K^*Q_K)G_K^\eta=(L^K\eta)^{-2}C^\eta Q_K^*a_KQ_KG_K^\eta. \tag{4.44}$$

Furthermore we obtain the Fourier respresentation for $C^\varepsilon Q_K^*$ from (4.2) by setting $k=K$, $\eta=L^{-K}$ and $a=0$:

$$C^\varepsilon Q_K^*(x,y)=(2\pi)^{-d}\int_{|p'|\leqq\pi}dp'\sum_l e^{i(p'+l)(x-y)}u_K^\varepsilon(p'+l)\Delta^\varepsilon(p'+l)^{-1}, \tag{4.45}$$

where $u_K^\varepsilon(p)=\prod_{\mu=1}^d\{(\exp[-ip_\mu]-1)\varepsilon(\exp[-i\varepsilon p_\mu]-1)^{-1}\}$ and $l\in2\pi Z^d$ with $|l_\mu|\leqq\pi/\varepsilon$. We can extend $\Delta^\varepsilon(p)^{-1}$ to an analytic function for $|\mathrm{Im}\,p_\mu|\leqq m/2$, each $\mu=1,\dots,d$. The function $u_K^\varepsilon(p)$ also has such an extension, and therefore we may extract a decay factor $\exp[-\delta_0|x-y|]$ from (4.45). The sum over l is bounded as before. In the same way, we get exponential decay for $(\partial_\alpha(x,y)\partial_\mu^\varepsilon C^\varepsilon Q_K^*)(z)$. We can easily extend Proposition 3.8 to include (4.45), since the integrand is even simpler than in (4.2). Finally, Proposition 3.9 holds for $G^{\eta}_{(K)}$ from the convergence of $C^{\eta}Q^*_K$ and $a_K Q_K G^{\eta}_K$, and the scalings of the operators.

Appendix

We wish to modify the proof of the lower bound presented in [Ba 1] by using the following bounds on the fields at each step:

$$\begin{aligned} &|A^{(k)}_{\mu}(x)|\,, \quad |\Delta^{\eta} A^{(k)}_{\mu}(x)| \leqq C_1 p(L^k\varepsilon)\,(\mu_0(L^k\varepsilon))^{-1}\,,\\ &|\phi^{(k)}(x)|\,, \quad |\Delta^{\eta}_{A^{(k)}}\phi^{(k)}(x)| \leqq C_1 p(L^k\varepsilon)\,[\lambda(L^k\varepsilon)^{4-d}]^{-1/4}\,, \end{aligned} \tag{A.1}$$

where C_1 is a fixed constant $O(1)$. Examining the paper [Ba 1], we see that the expansion (3.59) must be taken to order $\bar{n}=13$, and that (3.52) is replaced by

$$|\psi(y)| \leqq Cp(L^k\varepsilon)\,[\lambda(L^k\varepsilon)^{4-d}]^{-1/4}\,,$$

but the relations (3.54) are still sufficient. With these modifications the lower bound holds as before.

Next we want to justify replacing the bound on $D_{\bar{A}^{(k)}}\phi_{(k)}(b)$ in [Ba 2] by (3.2). Recall that

$$\bar{A}^{(k)}_{\mu}(x) = \sum_{j=0}^{L^k-1} \eta A^{(k)}(\langle x+j\eta e_{\mu}, x+(j+1)\eta e_{\mu}\rangle)\,. \tag{A.2}$$

Furthermore, denoting by 1 the constant function, we have

$$\begin{aligned} a_k G_k Q^*_k \cdot 1 &= a_k G_k \cdot 1\\ &= G_k(-\Delta^{\eta} + \mu_0^2(L^k\varepsilon)^2 + a_k Q^*_k Q_k)\cdot 1 - \mu_0^2(L^k\varepsilon)^2 G_k \cdot 1\,, \end{aligned} \tag{A.3}$$

and therefore

$$a_k G_k Q^*_k \cdot 1 = (1+\mu_0^2(L^k\varepsilon)^2 a_k^{-1})Q^*_k \cdot 1\,. \tag{A.4}$$

So for $x\in B^k(y)$, it follows from (A.4) and the bounds (3.2) that

$$\begin{aligned} A^{(k)}_{\mu}(x) &= a_k G_k Q^*_k \cdot A_{k,\mu}(y) + a_k G_k Q^*_k (A_{k,\mu}(x) - A_{k,\mu}(y))\\ &= A_{k,\mu}(y) + C(L^k\varepsilon)^{\alpha} + Cp(L^k\varepsilon)\,, \end{aligned} \tag{A.5}$$

where $\alpha>0$. Therefore from (A.2) we see that $\bar{A}^{(k)}_{\mu}(x) = A_{k,\mu}(y) + Cp(L^k\varepsilon)$ for $x\in B^k(y)$, and so the expansion (3.47) and the bounds (3.2) imply

$$D_{\bar{A}^{(k)}}\phi_k(b) = D_{A_k}\phi_k(b) + C(L^k\varepsilon)^{\alpha}\,, \qquad \alpha>0\,. \tag{A.6}$$

Finally we wish to show that the bounds (3.2) imply (A.1), allowing us to replace (A.1) by (3.2) in the lower bound of Theorem 3.1. For instance, we get

$$(-\Delta^{\eta}_{A^{(k)}})\phi^{(k)} = a_k Q^*_k(A^{(k)})\phi_k - m^2(L^k\varepsilon)^2\phi^{(k)} - a_k Q^*_k(A^{(k)})Q_k(A^{(k)})\phi^{(k)}\,. \tag{A.7}$$

So from (3.2) and Theorem 3.3 we get

$$|\Delta^{\eta}_{A^{(k)}}\phi^{(k)}| \leqq Cp(L^k\varepsilon)\,[\lambda(L^k\varepsilon)^{4-d}]^{-1/4} + C(L^k\varepsilon)^{\alpha}\,. \tag{A.8}$$

Therefore for C_1 large enough, we get the desired result. The other bounds in (A.1) follow similarly.

The inclusion of sources g, h is straightforward. From (3.20), (3.21) we see that they are small perturbations in the effective action, and so may be bounded when making a perturbative expansion. Furthermore, in the small field region they are proportional to $(L^k\varepsilon)^\alpha$, $\alpha > 0$, and so they do not affect the leading order positivity of the action, which is provided by (3.18).

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