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Homogeneous Kähler Manifolds: Paving the Way Towards New Supersymmetric Sigma Models

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Abstract. Homogeneous Kähler manifolds give rise to a broad class of supersymmetric sigma models containing, as a rather special subclass, the more familiar supersymmetric sigma models based on Hermitian symmetric spaces. In this article, all homogeneous Kähler manifolds with semisimple symmetry group G are constructed, and are classified in terms of Dynkin diagrams. Explicit expressions for the complex structure and the Kähler structure are given in terms of the Lie algebra $\mathfrak{g}$ of G. It is shown that for compact G, one can always find an Einstein-Kähler structure, which is unique up to a constant multiple and for which the Kähler potential takes a simple form.

1. Introduction and Summary of Results

Non-linear sigma models are natural candidates for effective low-energy theories, and they play an important rôle in our present understanding of symmetry breaking. In fact, whenever a field-theoretical model exhibits a (global) symmetry under a Lie group G which is spontaneously or dynamically broken down to a closed subgroup K, then independently of the details of the underlying dynamics, the associated Goldstone bosons are, in the low-energy sector, described by the non-linear sigma model on the homogeneous space G/K[1]. A similar scenario applies when all models are replaced by their supersymmetric extensions – at least as long as supersymmetry remains unbroken.

Now it is well known that the definition of a supersymmetric non-linear sigma model (with ordinary $N=1$ supersymmetry in four dimensions or with extended $N=2$ supersymmetry in two dimensions) requires the corresponding "field space" to be a Kähler manifold [1]. In fact, in four dimensions and in terms of superfields, the Lagrangian of the model (to be integrated over superspace) is simply the so-

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1 The term "homogeneous space" is synonymous for "coset space," and similarly, the term "Hermitian symmetric space" is synonymous for "symmetric Kähler manifold"

called Kähler potential, from which the Kähler metric can be derived. Thus combining supersymmetry with the symmetry breaking picture, one arrives naturally at the notion of homogeneous Kähler manifolds which, perhaps surprisingly, have been used only sporadically in the physics literature [2, 3] – quite in contrast to the more special class of Hermitian symmetric spaces [4, Vol. 2; 5]. On the other hand, there is, at least in four space-time dimensions, no reason to require the coset space in question to be symmetric. (This is not so in two space-time dimensions, where the symmetric space property is crucial for the integrability of the model [6].)

As an illustration of the extent to which homogeneous Kähler manifolds are more general than symmetric Kähler manifolds, consider the following simple example. Take $G=\mathrm{SU}(N)$ and $K=\mathrm{S}(\mathrm{U}(N_1)\times\ldots\times\mathrm{U}(N_p))$, where p and $N_1,\ldots,N_p$ are integers >0 such that $N_1+\ldots+N_p=N$, and consider the coset space $M=G/K$, which is the so-called generalized flag manifold

$$\mathrm{Fl}(N_1,\ldots,N_p)=\mathrm{SU}(N)/\mathrm{S}(\mathrm{U}(N_1)\times\ldots\times\mathrm{U}(N_p)). \tag{1.1}$$

Then M is a Kähler manifold. [The fact that M is a complex manifold can be seen, e.g., by rewriting M in the form $M=\tilde{G}/\tilde{K}$, where $\tilde{G}=\mathrm{SL}(N,\mathbb{C})$ is the complexification of $G=\mathrm{SU}(N)$ and $\tilde{K}=\Delta(N_1,\ldots,N_p)$ is the complex subgroup

$$\Delta(N_1,\ldots,N_p)=\left\{ g\in\mathrm{SL}(N,\mathbb{C}) \,\middle|\, g=\begin{pmatrix} * & * & \ldots & * \\ 0 & * & \ldots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & * \end{pmatrix} \right\} \tag{1.2}$$

consisting of unimodular matrices with zeros below the block diagonal.] Moreover, M appears as the orbit, under the adjoint representation of $\mathrm{SU}(N)$ on its Lie algebra $\mathrm{su}(N)$, through any generator Z_0 in $\mathrm{su}(N)$ of the form

$$Z_0=\begin{pmatrix} i\lambda_1 1_{N_1} & & & \\ & i\lambda_2 1_{N_2} & & \\ & & \ddots & \\ & & & i\lambda_p 1_{N_p} \end{pmatrix}, \tag{1.3}$$

with $\{\lambda_1,\ldots,\lambda_p\}$ some set of mutually different real numbers satisfying

$$N_1\lambda_1+\ldots+N_p\lambda_p=0. \tag{1.4}$$

On the other hand, M is known to be symmetric, rather than just homogeneous, if and only if $p=2$: this, of course, gives the complex Grassmannians.

The close connection between a) the Kähler structure and b) the adjoint orbit structure that shows up in the preceding example is far from accidental. Quite to the contrary, it provides the key to a complete and explicit classification of all homogeneous Kähler manifolds which admit a semisimple symmetry group G. For the compact case, the result is that these manifolds are precisely the orbits, under the adjoint representation of G on its Lie algebra $\mathfrak{g}$, through generators Z_0 in $\mathfrak{g}$, i.e., the sets

$$M_{Z_0}^G=\{gZ_0g^{-1}|g\in G\}. \tag{1.5}$$

For the non-compact case, the situation is more complicated; we shall have more to say on this later on.

It should be pointed out that these results seem to be well known to mathematicians [7, 8], although some of the proofs may be new. However, our main intention in this article is to make the whole subject of homogeneous Kähler manifolds accessible to physicists, and this requires a much more detailed and explicit presentation than what can be found in the mathematical literature.

The paper is organized as follows:

Section 2 starts with a short introduction to the notions of Kähler manifolds and of homogeneous spaces; this is meant to make the presentation selfcontained and to fix some notations. [Briefly, a Kähler manifold is a Riemann manifold with a complex structure which is compatible with the Riemannian metric in a sense to be specified. In particular, every Kähler manifold comes with a symplectic structure, i.e., a closed two-form ω which is nowhere degenerate.] The main result, stated explicitly in the form of a theorem towards the end of the section, is that homogeneous Kähler manifolds with semisimple symmetry group are coset spaces

$$M_{Z_0}^G = G/G_{Z_0}, \tag{1.6}$$

where G is a semisimple Lie group and G_{Z_0} is the stability group (centralizer) of a suitable generator Z_0 in the Lie algebra $\mathfrak{g}$ of G:

$$G_{Z_0} = \{g \in G | g Z_0 g^{-1} = Z_0\}. \tag{1.7}$$

The proof of this statement is based on an explicit determination of all G-invariant closed two-forms ϕ (in particular, of all G-invariant symplectic structures ω) on homogeneous spaces $M = G/K$, where G is connected semisimple and K is compact. In conclusion, a way towards the construction of Kähler potentials in terms of suitable frame fields is outlined.

Section 3 is devoted to the study of the coset spaces $M = G/K$ as given by (1.6) and (1.7), making use of the structure theory of semisimple Lie algebras. The outcome of this investigation is an explicit construction of

a) all possible G-invariant complex structures,

b) all possible G-invariant pseudo-Kählerian metrics

on any such coset space, both for compact and non-compact G, in terms of an appropriately chosen root system. Moreover, it is analyzed under what conditions the pseudo-Kählerian metric can in fact be chosen to be Kählerian, i.e., positive definite. As it turns out, this is possible, e.g., if $M = G/K$ is compact, and also if it is symmetric. In general, there exist whole families of metrics, depending on as many parameters as there are independent generators in the centre of the stability algebra $\mathfrak{k}$. In particular, the Killing form on the symmetry algebra $\mathfrak{g}$ gives rise to one such Kähler metric if $M = G/K$ is symmetric and – as must be emphasized – *only* if $M = G/K$ is symmetric. Once again, the main results are summarized in a theorem at the end of the section.

In Sect. 4, we compute the Ricci tensor for homogeneous Kähler manifolds. Surprisingly enough, it turns out to be independent of the metric. Moreover, the explicit expression obtained shows that if, and only if, $M = G/K$ is compact or symmetric, there exists, up to a constant positive multiple, a unique Einstein-Kähler metric, i.e., a Kähler metric for which the Ricci tensor is proportional to the metric tensor itself. (In the symmetric case, this metric is, up to a constant multiple, the one given by the Killing form on the symmetry algebra $\mathfrak{g}$.) Here, our main motivation for studying these Einstein-Kähler metrics is that they lead to a simpler expression for the Kähler potential. There is, however, also a physical reason for the special rôle played by Einstein manifolds (as opposed to general Riemann manifolds): namely that for the corresponding two-dimensional supersymmetric non-linear sigma models, all on-shell divergences can be absorbed into a renormalization of the overall scale of the metric. (For the 1-loop and 2-loop counterterms, this has been proved in [9], but it is presumably true to all orders; cf. [10].)

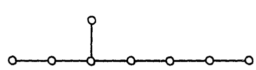
Section 5 contains a complete classification of all homogeneous Kähler manifolds with semisimple symmetry group, both compact and non-compact, in terms of Dynkin diagrams. For the compact case, the result can be resumed in the following cookbook recipe:

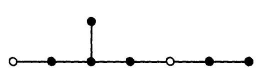
1. Draw the Dynkin diagram for the compact semisimple algebra $\mathfrak{g}$.
2. Paint any subset of its vertices black.
3. The unbroken subalgebra $\mathfrak{k}$ is then obtained as the direct sum

$$\mathfrak{k} = \mathrm{u}(1) \oplus \ldots \oplus \mathrm{u}(1) \oplus \mathfrak{k}', \tag{1.8}$$

where each white root gives rise to one u(1)-summand, and the set of black roots, together with the connecting lines between them, yields the Dynkin diagram of $\mathfrak{k}'$.

As an example, we consider the exceptional algebra $\mathfrak{g} = \mathfrak{e}_8$:

1. 

2. 

3. $\mathfrak{k} = \mathrm{u}(1) \oplus \mathrm{u}(1) \oplus \mathfrak{k}', \quad \mathfrak{k}' = \mathrm{su}(3) \oplus \mathrm{so}(8)$.

For the non-compact case, the classification problem is reduced, in a straightforward manner, to that for the compact case, together with that for the Hermitian symmetric spaces of the non-compact type. The results are collected in several tables at the end of the paper.

Certain classes of homogeneous Kähler manifolds arising from the classical groups, including explicit Kähler potentials, have also been treated in [2].

In conclusion, we want to mention another class of Kähler manifolds, namely the so-called hyperkähler manifolds, which are widely discussed in the context of supersymmetric non-linear sigma models because they allow for a doubling in the number of supersymmetries (extended $N=2$ supersymmetry in four dimensions or extended $N=4$ supersymmetry in two dimensions). An additional, perhaps more physical reason for the special rôle played by hyperkähler manifolds, and more generally, by Ricci-flat Riemann manifolds, is that the corresponding two-dimensional supersymmetric non-linear sigma models are ultraviolet finite to all orders of perturbation theory [9, 10]. Note, however, that (as a consequence of the result of Sect. 4) these manifolds cannot possibly be homogeneous under a semisimple symmetry group, and we shall therefore not discuss hyperkähler manifolds in this paper – except for formulating the following general conjecture:

The tangent bundle TM (or equivalently the cotangent bundle T^*M) of a homogeneous Kähler manifold $M=G/K$ with a semisimple symmetry group G can be made into a hyperkähler manifold in a natural way.

This conjecture has recently been proved locally, i.e., in a neighbourhood of the zero section, in TM (or equivalently T^*M), and globally on TM (or equivalently T^*M) for Hermitian symmetric spaces $M=G/K$, with the help of twistor space techniques [11].

2. Homogeneous Kähler Manifolds

We begin our discussion of the general mathematical situation by collecting a few definitions from the theory of complex manifolds and the theory of homogeneous spaces.

First, a Kähler manifold can be viewed as a real manifold M on which the following additional structures are given:

a) a Riemannian metric g,

b) an almost complex structure I which is isometric with respect to g:

$$g(Iu, Iv) = g(u, v)\,. \tag{2.1}$$

Recall [4, Vol. 2, p. 121; 5, p. 352] that an *almost complex structure* on M is simply a tensor field I of type $\binom{1}{1}$ on M satisfying $I^2=-1$ which, at every point m in M, represents multiplication by i in the tangent space T_mM to M at that point. From g and I, one constructs the so-called *fundamental two-form* ω on M by setting

$$\omega(u, v) = g(Iu, v)\,, \tag{2.2}$$

and one can combine g and ω into a Hermitian metric $\langle \cdot, \cdot \rangle$ which has g and ω as its real and imaginary part, respectively:

$$\langle u, v \rangle = g(u, v) + i\omega(u, v)\,. \tag{2.3}$$

The definition of a *Kähler manifold* is completed by requiring that I be *integrable* (a complex structure rather than just an almost complex structure) and that ω be *closed* (a symplectic form).

In any case, it is standard practice to extend g, I, and ω, without any change in notation, from the real tangent bundle TM to the complexified tangent bundle T^cM [2]. This means that one continues to write, at every point m in M, g_m and ω_m for the unique extensions of the real bilinear forms g_m and ω_m on T_mM, respectively, to complex bilinear forms on T_m^cM, and I_m for the unique extension of the real linear transformation I_m on T_mM to a complex linear transformation on T_m^cM.

2 The superscript $\cdot^c$ will always denote complexification

[Equations (2.1), (2.2) and the condition $I_m^2 = -1$ are, of course, preserved under these extensions.] The main point in performing this complexification is that due to $I_m^2 = -1$, the complex I_m has eigenvalues $\pm i$, while the real I_m has no eigenvalues. (Of course, one has to keep in mind that the complex I_m arises from a real I_m by complexification, or equivalently, that it commutes with the *conjugation* in $T_m^c M$ with respect to $T_m M$, which is usually denoted by a bar [4, Vol. 2, pp. 116/117].) Correspondingly, the complexified tangent space

$$T_m^c M = \{w = u + iv | u, v \in T_m M\}, \tag{2.4}$$

and its dual, which is identical with the complexified cotangent space

$$T_m^{*c} M = \{w^* = u^* + iv^* | u^*, v^* \in T_m^* M\}, \tag{2.5}$$

both admit natural direct decompositions

$$T_m^c M = T_m^{(1,0)} M \oplus T_m^{(0,1)} M, \qquad T_m^{*c} M = T_m^{*(1,0)} M \oplus T_m^{*(0,1)} M \tag{2.6}$$

into mutually conjugate complex subspaces, defined as follows:

$$\begin{aligned} T_m^{(1,0)} M &= \{w \in T_m^c M | I_m w = +iw\}, \\ T_m^{(0,1)} M &= \{w \in T_m^c M | I_m w = -iw\}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} T_m^{*(1,0)} M &= \{w^* \in T_m^{*c} M | w^*(w) = 0 \text{ for } w \in T_m^{(0,1)} M\}, \\ T_m^{*(0,1)} M &= \{w^* \in T_m^{*c} M | w^*(w) = 0 \text{ for } w \in T_m^{(1,0)} M\} \end{aligned} \tag{2.8}$$

(see [4, Vol. 2, p. 117]). These decompositions are useful in connection with complex (local) co-ordinates z^μ on M (which exist due to our assumption that I is integrable, i.e., has no torsion [4, Vol. 2, p. 124]). Namely, $T_m^{(1,0)} M$ and $T_m^{(0,1)} M$ are spanned by the complex tangent vectors $(\partial/\partial z^\mu)_m$ and $(\partial/\partial \bar{z}^\mu)_m$, respectively, while $T_m^{*(1,0)} M$ and $T_m^{*(0,1)} M$ are spanned by the complex cotangent vectors $(dz^\mu)_m$ and $(d\bar{z}^\mu)_m$, respectively. In terms of such co-ordinates, we have

$$g(\partial/\partial z^\mu, \partial/\partial z^\nu) = 0 = g(\partial/\partial \bar{z}^\mu, \partial/\partial \bar{z}^\nu), \tag{2.9}$$

and writing

$$g_{\mu\bar{\nu}} = g(\partial/\partial z^\mu, \partial/\partial \bar{z}^\nu), \tag{2.10}$$

so that

$$ds^2 = g_{\mu\bar{\nu}}\, dz^\mu\, d\bar{z}^\nu, \tag{2.11}$$

we get

$$\omega = i g_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu. \tag{2.12}$$

Moreover, the fact that ω is closed ($d\omega = 0$) is equivalent to the requirement that (on the domain of definition of the co-ordinates z^μ) the coefficients $g_{\mu\bar{\nu}}$ can be derived from a real-valued function F, called the *Kähler potential* [4, Vol. 2, pp. 155–158]:

$$g_{\mu\bar{\nu}} = \frac{\partial^2}{\partial z^\mu \partial \bar{z}^\nu} F. \tag{2.13}$$

This potential is, of course, not unique, but may be subjected to "gauge transformations" of the form $F(z,\bar{z}) \to F(z,\bar{z}) + f(z)$ and/or $F(z,\bar{z}) \to F(z,\bar{z}) + f(\bar{z})$, with arbitrary functions f. (For the corresponding supersymmetric non-linear sigma model, the resulting change in the Lagrangian vanishes after integration over the odd variables.) One possible choice for the Kähler potential is given by explicit integration:

$$F(z,\bar{z}) = \int_{z^{(0)}}^{z} d\zeta^\mu \int_{\bar{z}^{(0)}}^{\bar{z}} d\bar{\zeta}^\nu \, g_{\mu\bar{\nu}}(\zeta,\bar{\zeta}) \,. \tag{2.14}$$

For later use, we also note that – barring all considerations of positivity or non-degeneracy – we can apply the same procedure as before with the metric g replaced by the Ricci tensor Ric and the fundamental two-form ω replaced by the Ricci two-form ϱ: this is possible since in analogy with (2.1), (2.2) and the equation $d\omega = 0$, we have

$$\mathrm{Ric}(Iu, Iv) = \mathrm{Ric}(u, v) \tag{2.15}$$

[4, Vol. 2, p. 149],

$$\varrho(u,v) = \mathrm{Ric}(Iu, v)\,, \tag{2.16}$$

and the equation $d\varrho = 0$ [4, Vol. 2, p. 153]. In particular, we have

$$\mathrm{Ric}(\partial/\partial z^\mu, \partial/\partial z^\nu) = 0 = \mathrm{Ric}(\partial/\partial \bar{z}^\mu, \partial/\partial \bar{z}^\nu)\,, \tag{2.17}$$

and writing

$$R_{\mu\bar{\nu}} = \mathrm{Ric}(\partial/\partial z^\mu, \partial/\partial \bar{z}^\nu)\,, \tag{2.18}$$

we get

$$\varrho = iR_{\mu\bar{\nu}}\, dz^\mu \wedge d\bar{z}^\nu \,. \tag{2.19}$$

A remarkable property of the Ricci tensor is that (on the domain of definition of the co-ordinates z^μ) its coefficients $R_{\mu\bar{\nu}}$ can be derived from an explicitly known real-valued function, namely the logarithm of the invariant volume element [4, Vol. 2, pp. 155–158]:

$$R_{\mu\bar{\nu}} = -\frac{\partial^2}{\partial z^\mu \partial \bar{z}^\nu} \ln\det(g_{\kappa\bar{\lambda}})\,. \tag{2.20}$$

This gives an especially convenient choice for the Kähler potential of an Einstein-Kähler manifold, i.e., a Kähler manifold whose Ricci tensor is simply a constant multiple of the metric. (Compare [4, Vol. 1, pp. 292–294].)

In order to bring to bear group theory, we shall assume henceforth that the Kähler manifold M in question is *homogeneous*, i.e., that there exists a Lie group G which acts *transitively* on M by holomorphic isometries. Then the stability group of a given reference point $\circ$ in M (which is fixed once and for all) is a closed subgroup K of G, and the manifold M can be identified with the homogeneous space G/K such that $\circ$ appears as the left coset $1K = K$ of 1 in G. For the sake of simplicity, we shall also demand that the symmetry group G is connected and semisimple and acts effectively on M, i.e., only $1 \in G$ acts trivially on M[3]. As a consequence of this, the stability group K is compact and connected, as will be shown in the sequel.

We note first that the homogeneous space $M = G/K$ must be reductive, i.e., that the Lie algebra $\mathfrak{g}$ of G can be decomposed into the direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{2.21}$$

of the Lie algebra $\mathfrak{k}$ of K and a complementary subspace $\mathfrak{m}$ which is $\mathrm{Ad}(K)$-invariant[4]; this $\mathrm{Ad}(K)$-invariance implies (and for connected K is equivalent to) the commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}. \tag{2.22}$$

Namely, G being semisimple, the Killing form of $\mathfrak{g}$[4] is non-degenerate on $\mathfrak{g}$. We can set

$$\mathfrak{m} = \mathfrak{k}^{\perp} \quad \text{with respect to Kill}, \tag{2.23}$$

and the Killing form of $\mathfrak{g}$ will then be non-degenerate on $\mathfrak{m}$ as well (thus $\mathfrak{m}$ being actually complementary to $\mathfrak{k}$ in $\mathfrak{g}$), because it turns out to be negative definite on $\mathfrak{k}$. [Indeed, this can be proved in a more general context (without assuming $\mathfrak{g}$ to be semisimple) as follows. First, factoring out the redundancy group N of the action, which is assumed finite[3], we can regard G/N as a (not necessarily closed) Lie subgroup of the group $J(M)$ of all isometries on M; then K/N becomes a (not necessarily closed) Lie subgroup of the group $J_0(M)$ of all isometries on M leaving the point $\circ$ fixed. This last group being compact [4, Vol. 1, p. 239], one can use a standard averaging procedure to construct a positive definite scalar product on the Lie algebra of $J(M)$ which is invariant under the adjoint action of $J_0(M)$ [4, Vol. 2, p. 199]. The restriction of this scalar product to $\mathfrak{g}$ is also positive definite and is $\mathrm{Ad}(K)$-invariant. In an orthonormal basis of $\mathfrak{g}$, the linear transformation $\mathrm{ad}(X)$, for any $X \in \mathfrak{k}$, is thus expressed by a skew-symmetric matrix. Now the Killing form of $\mathfrak{g}$ is easily seen to be negative definite on $\mathfrak{k}$, since ad vanishes precisely on the centre of $\mathfrak{g}$ and this has trivial intersection with $\mathfrak{k}$[3], cf. [5, p. 133].] Going back to (2.21), we shall write

$$X = X_{\mathfrak{k}} + X_{\mathfrak{m}}, \tag{2.24}$$

for the decomposition of elements X in $\mathfrak{g}$ corresponding to (2.21). Moreover, we shall find it useful to identify the tangent space $T_0 M$ to M at the distinguished point $\circ \in M$ with $\mathfrak{m}$. Explicitly, this identification is given by $X = X_M(\circ)$ for $X \in \mathfrak{m}$, or more generally $X_{\mathfrak{m}} = X_M(\circ)$ for X in $\mathfrak{g}$, where X_M denotes the fundamental vector field on M generated by an element X in $\mathfrak{g}$:

$$X_M(m) = \frac{d}{dt}(\exp tX \cdot m)\Big|_{t=0} \quad \text{for} \quad m \in M. \tag{2.25}$$

3 More generally, it suffices to demand that the redundancy group of the action, i.e., the closed normal subgroup N of G consisting of those elements in G that act trivially on M, is finite. For later use, note that $N \subset K$ always and that if $Z(G)$ denotes the centre of G, which, for G semisimple, is necessarily discrete [5, p. 132], then $K \cap Z(G) \subset N$, so $K \cap Z(G)$ is also finite

4 We write Ad respectively ad for the adjoint representations of G respectively $\mathfrak{g}$, as well as for the various representations of K respectively $\mathfrak{k}$ on $\mathfrak{k}$ or $\mathfrak{m}$ obtained from these by appropriate restrictions, and Kill will denote the Killing form of $\mathfrak{g}$: $\mathrm{Kill}(X, Y) = \mathrm{trace}\, \mathrm{ad}(X)\, \mathrm{ad}(Y)$ for $X, Y \in \mathfrak{g}$

In passing, we mention that the homogeneous space $M=G/K$ is called (locally) *symmetric* if, in addition to (2.22), the following commutation relation also holds:

$$[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}. \tag{2.26}$$

Next, we exploit the G-invariance of the metric g, the almost complex structure I and the fundamental two-form ω on M to identify these, via left translation under elements of G, with an $\mathrm{Ad}(K)$-invariant positive definite symmetric bilinear form g_0 on $\mathfrak{m}$, an $\mathrm{Ad}(K)$-invariant linear transformation I_0 on $\mathfrak{m}$ satisfying $I_0^2=-1$ and an $\mathrm{Ad}(K)$-invariant non-degenerate antisymmetric bilinear form ω_0 on $\mathfrak{m}$, respectively. Similar identifications are performed for the Ricci tensor Ric and the Ricci form ϱ. It can then be shown [4, Vol. 2, p. 219] that I is integrable, i.e., has no torsion, if and only if for $X, Y\in\mathfrak{m}$,

$$[I_0X,I_0Y]_m-[X,Y]_m-I_0[X,I_0Y]_m-I_0[I_0X,Y]_m=0. \tag{2.27}$$

Moreover, given any G-invariant two-form ϕ on M, e.g., $\phi=\omega$ or $\phi=\varrho$, a simple calculation [evaluation of $d\phi$ on three fundamental vector fields on M generated by elements in $\mathfrak{m}$ according to (2.25)] proves that ϕ is closed if and only if for $X, Y, Z\in\mathfrak{m}$,

$$\phi_0([X,Y]_\mathfrak{m},Z)+\phi_0([Y,Z]_\mathfrak{m},X)+\phi_0([Z,X]_\mathfrak{m},Y)=0. \tag{2.28}$$

Now since the Killing form of $\mathfrak{g}$ is non-degenerate on $\mathfrak{m}$, we can express ϕ_0 in terms of a unique linear transformation Φ_0 on $\mathfrak{m}$, which is necessarily antisymmetric with respect to the Killing form: namely, for $X, Y\in\mathfrak{m}$,

$$\phi_0(X,Y)=\mathrm{Kill}(\Phi_0X,Y). \tag{2.29}$$

In these terms, (2.27) states that for $X, Y, Z\in\mathfrak{m}$,

$$\mathrm{Kill}(\Phi_0[X,Y]_\mathfrak{m},Z)+\mathrm{Kill}(\Phi_0[Y,Z]_\mathfrak{m},X)+\mathrm{Kill}(\Phi_0[Z,X]_\mathfrak{m},Y)=0, \tag{2.30}$$

or equivalently,

$$\begin{aligned}\mathrm{Kill}(\Phi_0[X,Y]_\mathfrak{m},Z) &= \mathrm{Kill}([Y,Z]_\mathfrak{m},\Phi_0X)+\mathrm{Kill}([Z,X]_\mathfrak{m},\Phi_0Y)\\ &\underset{(2.23)}{=} \mathrm{Kill}([Y,Z],\Phi_0X)+\mathrm{Kill}([Z,X],\Phi_0Y)\\ &= \mathrm{Kill}([\Phi_0X,Y],Z)+\mathrm{Kill}([X,\Phi_0Y],Z),\end{aligned} \tag{2.31}$$

where in the last equation, we have used the $\mathrm{ad}(\mathfrak{g})$-invariance of the Killing form. But the $\mathrm{ad}(\mathfrak{k})$-invariance of ϕ_0 means that for $X, Y\in\mathfrak{m}$, $Z\in\mathfrak{k}$,

$$\begin{aligned}&\mathrm{Kill}([\Phi_0X,Y],Z)+\mathrm{Kill}([X,\Phi_0Y],Z)\\ &=-\mathrm{Kill}(\Phi_0X,[Z,Y])+\mathrm{Kill}([Z,X],\Phi_0Y)\\ &=-\phi_0(X,[Z,Y])-\phi_0([Z,X],Y)\\ &=0,\end{aligned} \tag{2.32}$$

where in the first equation, we have used the $\mathrm{ad}(\mathfrak{g})$-invariance of the Killing form. Combining (2.31) and (2.32), we see that for $X, Y\in\mathfrak{m}$,

$$\Phi_0[X,Y]_\mathfrak{m}=[\Phi_0X,Y]+[X,\Phi_0Y]. \tag{2.33}$$

Therefore, extending Φ_0 from a linear transformation on $\mathfrak{m}$ to a linear transformation on $\mathfrak{g}$ (we use the same notation for both transformations) by requiring

$$\Phi_0 = 0 \quad \text{on} \quad \mathfrak{k}, \tag{2.34}$$

and exploiting the ad($\mathfrak{k}$)-invariance of Φ_0 (which is equivalent to that of ϕ_0) once again, we see that Φ_0 must be a derivation, i.e., we have for $X, Y \in \mathfrak{g}$

$$\Phi_0[X, Y] = [\Phi_0 X, Y] + [X, \Phi_0 Y]. \tag{2.35}$$

Since G is semisimple, Φ_0 is necessarily of the form $\Phi_0 = \mathrm{ad}(Z^\Phi)$ with $Z^\Phi \in \mathfrak{g}$ [5, p. 132], and so we have for $X, Y \in \mathfrak{m}$

$$\phi_0(X, Y) = \mathrm{Kill}([Z^\Phi, X], Y) = \mathrm{Kill}(Z^\Phi, [X, Y]). \tag{2.36}$$

Moreover, the Ad(K)-invariance of ϕ_0 is expressed through the condition that Z^Φ is K-invariant, i.e., that for $k \in K$,

$$\mathrm{Ad}(k) Z^\Phi = Z^\Phi. \tag{2.37}$$

Indeed, (2.36) implies that for $k \in K$ and $X, Y \in \mathfrak{m}$,

$$\begin{aligned}
\mathrm{Kill}([\mathrm{Ad}(k) Z^\Phi, X], Y) &= \mathrm{Kill}(\mathrm{Ad}(k)[Z^\Phi, \mathrm{Ad}(k)^{-1} X], \mathrm{Ad}(k)\,\mathrm{Ad}(k)^{-1} Y) \\
&= \phi_0(\mathrm{Ad}(k)^{-1} X, \mathrm{Ad}(k)^{-1} Y) \\
&= \phi_0(X, Y) \\
&= \mathrm{Kill}([Z^\Phi, X], Y),
\end{aligned}$$

and from non-degeneracy of the Killing form on $\mathfrak{m}$, plus the fact that $\mathrm{ad}(Z^\Phi) = \Phi_0$ and $\mathrm{ad}(\mathrm{Ad}(k) Z^\Phi) = \mathrm{Ad}(k)\Phi_0\,\mathrm{Ad}(k)^{-1}$ ($k \in K$) map $\mathfrak{m}$ into $\mathfrak{m}$, we conclude that for $k \in K$ and $X \in \mathfrak{m}$,

$$\mathrm{ad}(\mathrm{Ad}(k) Z^\Phi) X = \mathrm{ad}(Z^\Phi) X.$$

This formula also holds for $k \in K$ and $X \in \mathfrak{k}$ [both sides are then equal to zero; cf. (2.34)], and since the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ is faithful ($\mathfrak{g}$, being semisimple, has trivial centre [5, p. 132]), we arrive at (2.37).

To summarize, we have shown that a closed G-invariant two-form ϕ on M can be expressed, according to (2.36), in terms of a K-invariant element Z^Φ in $\mathfrak{g}$. Infinitesimally, the invariance condition (2.37) becomes

$$\mathfrak{k} \subset \ker \mathrm{ad}(Z^\Phi), \tag{2.38}$$

and the equality will hold if and only if ϕ is non-degenerate. In that case Z^Φ must belong to the centre of the stability algebra $\mathfrak{k}$, and the stability group K can be shown to be the centralizer of a torus $\tilde{T}$ in G; in particular, K is compact connected, and

$$K = \{g \in G \mid \mathrm{Ad}(g) Z^\Phi = Z^\Phi\}. \tag{2.39}$$

For the proof of these statements, we follow the argument of Koszul [12, p. 56]. First of all, let $\tilde{K}$ denote the isotropy group of Z^Φ under Ad, i.e.,

$$\tilde{K} = \{g \in G \mid \mathrm{Ad}(g) Z^\Phi = Z^\Phi\}, \tag{2.40}$$

and let $\tilde{T}$ be the closure of the one-parameter subgroup $\{\exp tZ^{\phi}/t \in \mathbb{R}\}$ of G generated by Z^{ϕ}. Then applying the exponential map in (2.40), together with a continuity argument, we see that $\tilde{K}$ is the centralizer of $\tilde{T}$ in G. On the other hand, (2.37) states that $K \subset \tilde{K}$. But K and $\tilde{K}$ have the same Lie algebra, namely ker ad(Z^{ϕ}). Therefore, K is an open, thus also closed, subgroup of $\tilde{K}$. Similarly, Ad(K) must be an open, thus also closed, subgroup of Ad$(\tilde{K})$, while Ad$(\tilde{K})$ is obviously a closed subgroup of Ad(G), which itself, by the semisimplicity of $\mathfrak{g}$, is a closed subgroup of the group GL$(\mathfrak{g})$ of all non-singular linear transformations on $\mathfrak{g}$ [5, pp. 126 and 135]. Now we have seen [cf. the proof following (2.23)] that Ad(K) leaves invariant a positive definite scalar product on $\mathfrak{g}$. Therefore, Ad(K) is a closed subgroup of the corresponding orthogonal group $O(\mathfrak{g})$, hence compact. But Ad$(K) \cong K/K \cap Z(G)$ under Ad [5, p. 129], and $K \cap Z(G)$ is finite[3], so K must be compact as well. This, in turn, implies that $\tilde{T}$ is a torus, so that according to a classical theorem [5, p. 287], the centralizer $\tilde{K}$ of $\tilde{T}$ in G must be connected: this proves that $\tilde{K} = K$ is compact connected and that (2.39) holds. [Strictly speaking, the aforementioned classical theorem can be applied directly only if G itself is compact. However, the non-compact case can be reduced to the compact case by fixing a maximal compact subgroup L of G containing K [5, p. 256], and then proving that $\tilde{K}$ must be contained in L as well: this proof is based on using the polar decomposition $g = (\exp X)\, l$ of group elements $g \in G$ [5, pp. 252/253], with $X \in \mathfrak{p}$ and $l \in L$, to show that $g \in \tilde{K}$ forces $X = 0$. We leave it to the reader to work out the details.]

In particular, since $M = G/K$ is supposed to be a Kähler manifold, the fundamental two-form ω is non-degenerate. We have therefore proved the following

Theorem. *Let M be a connected homogeneous Kähler manifold, and assume that M admits a symmetry group G which is a connected semisimple Lie group and which acts effectively on M, i.e., only $1 \in G$ acts trivially on M*[3]*. Then the stability group K of a given point $\circ$ in M is a compact connected subgroup of G and is the centralizer of some torus $\tilde{T}$ in G. Moreover, $M = G/K$ can be identified with an orbit under the adjoint representation of G on the corresponding Lie algebra $\mathfrak{g}$.*

For later use, we note that in this situation, the stability algebra $\mathfrak{k}$ will contain a maximal Abelian subalgebra of $\mathfrak{g}$, so the element Z^{ϕ} in $\mathfrak{g}$ corresponding to an arbitrary closed G-invariant two-form ϕ on M must belong to the centre of the stability algebra $\mathfrak{k}$, even if ϕ is degenerate.

The form ϕ given by (2.36) plays a prominent rôle in symplectic geometry: it is the *Kirillov-Kostant-Souriau form* associated with the G-orbit through Z^{ϕ} under the adjoint representation (assuming that the Killing form of $\mathfrak{g}$ has been used to identify coadjoint orbits with adjoint orbits). See [13] for details.

In the second part of this section, and throughout the next section, the various tensor fields of interest on M are always evaluated at the special point $\circ$; they are then extended to all of M by making use of their G-invariance. In order to make this last step more explicit, and also to make contact with expressions in terms of complex (local) co-ordinates z^{μ} on M that were used in the first part of this section, we introduce complex (local) frame fields on M as follows: Let $\sigma: U \to G$ be a (local) section of the principal K-bundle $\pi: G \to M$, defined on a suitable open neighbourhood U of $\circ$ in M, and such that $\sigma(\circ) = 1$ [4, Vol. 1, p. 55]. Then given $X \in \mathfrak{m}^{c\,2}$, we define a complex vector field X^σ on U by setting

$$X^\sigma(m) = \sigma(m) \cdot X = \frac{d}{dt}(\sigma(m) \exp tX \cdot \circ)\Big|_{t=0} \quad \text{for} \quad m \in U. \tag{2.41}$$

[Note that X^σ is simply the inverse image, under σ, of the restriction to $\sigma(U) \subset G$ of the left invariant complex vector field on G generated by X.] The G-invariance of the complex structure I on M implies that

$$\begin{aligned} X \in \mathfrak{m}^{(1,0)} &\Rightarrow X^\sigma(m) \in T_m^{(1,0)}M \quad \text{for} \quad m \in U, \\ X \in \mathfrak{m}^{(0,1)} &\Rightarrow X^\sigma(m) \in T_m^{(0,1)}M \quad \text{for} \quad m \in U. \end{aligned} \tag{2.42}$$

Therefore, choosing any basis of vectors E_α in $\mathfrak{m}^{(1,0)}$, we obtain complex vector fields E_α^σ and $\bar{E}_\alpha^\sigma$ on U which form a complex frame field on U, and it is easy to see that the components of any G-invariant tensor field on M with respect to this complex frame field are constant functions on U which, in addition, do not depend on the choice of σ, either. In particular,

$$\begin{aligned} g(E_\alpha^\sigma, E_\beta^\sigma) &= 0 = g(\bar{E}_\alpha^\sigma, \bar{E}_\beta^\sigma), \\ \mathrm{Ric}(E_\alpha^\sigma, E_\beta^\sigma) &= 0 = \mathrm{Ric}(\bar{E}_\alpha^\sigma, \bar{E}_\beta^\sigma) \end{aligned} \tag{2.43}$$

[cf. (2.9), (2.17)], and

$$\begin{aligned} g(E_\alpha^\sigma, \bar{E}_\beta^\sigma) &= g_0(E_\alpha, \bar{E}_\beta) = g_{\alpha\bar{\beta}}, \\ \mathrm{Ric}(E_\alpha^\sigma, \bar{E}_\beta^\sigma) &= \mathrm{Ric}_0(E_\alpha, \bar{E}_\beta) = R_{\alpha\bar{\beta}} \end{aligned} \tag{2.44}$$

[cf. (2.10)–(2.12), (2.17)–(2.19)]. Finally, the transition from this complex (local) frame field to complex (local) co-ordinates is performed, as usual, by writing

$$\begin{aligned} \partial/\partial z^\mu = e_\mu^\alpha E_\alpha^\sigma \quad &\text{and (hence)} \quad \partial/\partial \bar{z}^\mu = \bar{e}_\mu^\alpha \bar{E}_\alpha^\sigma, \\ E_\alpha^\sigma = e_\alpha^\mu \partial/\partial z^\mu \quad &\text{and (hence)} \quad \bar{E}_\alpha^\sigma = \bar{e}_\alpha^\mu \partial/\partial \bar{z}^\mu, \end{aligned} \tag{2.45}$$

where $\bar{e}_\mu^\alpha$ and $\bar{e}_\alpha^\mu$ are the complex conjugates of e_μ^α and e_α^μ, respectively, and

$$e_\mu^\alpha e_\beta^\mu = \delta_\beta^\alpha, \qquad e_\alpha^\mu e_\nu^\alpha = \delta_\nu^\mu. \tag{2.46}$$

(For simplicity, we have omitted reference to the choice of σ in the coefficient functions e, $\bar{e}$.) Then, for example,

$$\begin{aligned} g_{\mu\bar{\nu}} &= g_{\alpha\bar{\beta}} e_\mu^\alpha \bar{e}_\nu^\beta, \\ R_{\mu\bar{\nu}} &= R_{\alpha\bar{\beta}} e_\mu^\alpha \bar{e}_\nu^\beta. \end{aligned} \tag{2.47}$$

3. Root Systems and Kähler Structures on Semisimple Adjoint Orbits

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Given any element Z_0 in $\mathfrak{g}$, the G-orbit through Z_0 under the adjoint representation can be identified with the homogeneous space G/K, where the stability group

$$K = \{g \in G | \mathrm{Ad}(g) Z_0 = Z_0\} \tag{3.1}$$

is the centralizer of Z_0 in G, and the corresponding stability algebra

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid [X, Z_0] = 0\} \tag{3.2}$$

is the centralizer of Z_0 in $\mathfrak{g}$. In accordance with the discussion in Sect. 2, we shall demand the stability group K to be compact (rather than just closed in G), and as before, the Killing form of $\mathfrak{g}$ will be non-degenerate on $\mathfrak{g}$ and negative definite on $\mathfrak{k}$, hence

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{3.3}$$

with

$$\mathfrak{m} = \mathfrak{k}^{\perp} \quad \text{with respect to Kill}. \tag{3.4}$$

Note that $\mathrm{ad}(Z_0)$ being antisymmetric with respect to Kill, we can also write

$$\mathfrak{k} = \ker \mathrm{ad}(Z_0), \qquad \mathfrak{m} = \mathrm{im}\, \mathrm{ad}(Z_0). \tag{3.5}$$

Anyway, we have the commutation relations

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \tag{3.6}$$

but not necessarily the commutation relation $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$; this means that G/K is a homogeneous space but not necessarily a symmetric space. Next, the connected one-component $\hat{T}$ of the centre of K is a torus in K, and the centre $\hat{\mathfrak{t}}$ of $\mathfrak{k}$ is the Lie algebra of $\hat{T}$. For the following, we let T be a maximal torus in K containing $\hat{T}$, and we write $\mathfrak{t}$ for the Lie algebra of T, so that $\hat{T} \subset T \subset K$ and

$$Z_0 \in \hat{\mathfrak{t}} \subset \mathfrak{t} \subset \mathfrak{k}. \tag{3.7}$$

Finally, K is necessarily connected, because it is the centralizer of the torus $\hat{T}$ – or more generally, any torus $\tilde{T}$ satisfying $\{\exp t Z_0 / t \in \mathbb{R}\} \subset \tilde{T} \subset \hat{T}$ – in G; cf. the discussion in Sect. 2. Moreover, K must contain the (necessarily discrete) centre of G.

Before going on, we should mention the fact that we suffer no loss of generality by assuming, wherever this may seem convenient, that apart from being semisimple, G is simply connected and/or simple. Indeed, if G is not simply connected, and if $\tilde{G}$ is the universal covering group of G, then $G/K \cong \tilde{G}/\tilde{K}$, where $\tilde{K}$ is the centralizer of Z_0 in $\tilde{G}$. (This uses the fact that the kernel of the covering homomorphism from $\tilde{G}$ to G is contained in the centre of $\tilde{G}$, and hence in $\tilde{K}$.) Similarly, if G is simply connected but not simple, and if $G^{(1)}, \ldots, G^{(r)}$ are the closed normal subgroups of G generated by the simple ideals $\mathfrak{g}^{(1)}, \ldots, \mathfrak{g}^{(r)}$ in $\mathfrak{g}$, then $G/K \cong G^{(1)}/K^{(1)} \times \ldots \times G^{(r)}/K^{(r)}$, where $K^{(i)}$ is the centralizer of $Z_0^{(i)}$ in $G^{(i)}$ $(1 \leqq i \leqq r)$, and

$$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \ldots \oplus \mathfrak{g}^{(r)}, \qquad Z_0 = Z_0^{(1)} + \ldots + Z_0^{(r)}.$$

We may therefore consider each factor separately.

Independently of whether G is simply connected or simple, the non-compact case requires further conventions. Namely, if G is non-compact, we let L be a maximal compact subgroup of G containing K, and we write $\mathfrak{l}$ for the Lie algebra of L. Note that L is connected [5, p. 256] and that once again, the Killing form of $\mathfrak{g}$ is non-degenerate on $\mathfrak{g}$ and negative definite on $\mathfrak{l}$, hence

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p} \tag{3.8}$$

with

$$\mathfrak{p} = \mathfrak{l}^{\perp} \quad \text{with respect to Kill}. \tag{3.9}$$

In fact, (3.8) is a Cartan decomposition of $\mathfrak{g}$ [5, pp. 182–185 and 256], i.e., we have the commutation relations

$$[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}, \quad [\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}, \tag{3.10}$$

while the Killing form of $\mathfrak{g}$ is negative definite on $\mathfrak{l}$ and positive definite on $\mathfrak{p}$; this means that G/L is a Riemannian symmetric space of the non-compact type [5, pp. 252/253]. Again, the connected one-component C of the centre of L is a torus in L, and the centre $\mathfrak{c}$ of $\mathfrak{l}$ is the Lie algebra of L. Thus

$$\mathfrak{c} \subset \hat{\mathfrak{t}} \subset \mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{l}, \quad \mathfrak{p} \subset \mathfrak{m}. \tag{3.11}$$

We note here that for G simple, we have $\mathfrak{c} \neq \{0\}$ if and only if G/L is a Hermitian symmetric space of the non-compact type [5, p. 381].

Returning to the general situation, we extract further information by invoking the structure theory of semisimple Lie algebras.

First, $\mathfrak{h} = \mathfrak{t}^c$ is a Cartan subalgebra of the complex semisimple Lie algebra $\mathfrak{g}^c$ [2]; we let Δ denote the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{h}$, and write

$$\mathfrak{g}^c = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{(\alpha)} \tag{3.12}$$

for the corresponding root space decomposition [5, pp. 165/166], with

$$X_\alpha \in \mathfrak{g}^{(\alpha)} \Leftrightarrow [H, X_\alpha] = \alpha(H) X_\alpha \quad \text{for} \quad H \in \mathfrak{h}. \tag{3.13}$$

For later use, we introduce the generators $H_\alpha \in \mathfrak{h}$ ($\alpha \in \Delta$), uniquely determined by the condition

$$\mathrm{Kill}(H_\alpha, H) = \alpha(H) \quad \text{for} \quad H \in \mathfrak{h}, \tag{3.14}$$

as well as generators $E_\alpha \in \mathfrak{g}^{(\alpha)}$ ($\alpha \in \Delta$), satisfying the commutation relations

$$[E_\alpha, E_{-\alpha}] = H_\alpha \quad \text{for} \quad \alpha \in \Delta, \tag{3.15}$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad \text{for} \quad \alpha, \beta \in \Delta, \quad \alpha + \beta \neq 0, \tag{3.16}$$

with coefficients $N_{\alpha,\beta}$ such that $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Delta$ and $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$ if $\alpha + \beta \in \Delta$ [5, p. 176]; this implies the normalization conditions

$$\mathrm{Kill}(E_\alpha, E_{-\alpha}) = 1 \quad \text{for} \quad \alpha \in \Delta, \tag{3.17}$$

$$\mathrm{Kill}(E_\alpha, E_\beta) = 0 \quad \text{for} \quad \alpha, \beta \in \Delta, \quad \alpha + \beta \neq 0 \tag{3.18}$$

[5, pp. 166/167]. Note that T being compact, the Killing form of $\mathfrak{g}$ is negative definite on $\mathfrak{t}$ [5, p. 133]; this implies that the generators iH_α ($\alpha \in \Delta$) belong to $\mathfrak{t}$ (in fact, they span $\mathfrak{t}$), and that all roots $\alpha \in \Delta$ take imaginary values on $\mathfrak{t}$ [5, pp. 170 and 171]. Moreover, it follows that for all $\alpha \in \Delta$, the conjugation in $\mathfrak{g}^c$ with respect to the given real form $\mathfrak{g}$ [5, pp. 180], which we shall denote by a bar, maps $\mathfrak{g}^{(\alpha)}$ into $\mathfrak{g}^{(-\alpha)}$, and we may even assume, without loss of generality, that

$$\text{either} \quad \bar{E}_\alpha = -E_{-\alpha} \quad \text{or} \quad \bar{E}_\alpha = +E_{-\alpha}. \tag{3.19}$$

[Indeed, the ansatz $\bar{E}_\alpha = c_\alpha E_{-\alpha}$ with $c_\alpha \in \mathbb{C}$ implies

$$E_\alpha = \bar{\bar{E}}_\alpha = \bar{c}_\alpha \bar{E}_{-\alpha} = \bar{c}_\alpha c_{-\alpha} E_\alpha \Rightarrow \bar{c}_\alpha c_{-\alpha} = 1,$$

$$H_\alpha = -\bar{H}_\alpha = -[\bar{E}_\alpha, \bar{E}_{-\alpha}] = -c_\alpha c_{-\alpha}[E_{-\alpha}, E_\alpha] = c_\alpha c_{-\alpha} H_\alpha \Rightarrow c_\alpha c_{-\alpha} = 1,$$

$$c_{\alpha+\beta} N_{\alpha,\beta} E_{-\alpha-\beta} = \bar{N}_{\alpha,\beta} \bar{E}_{\alpha+\beta} = [\bar{E}_\alpha, \bar{E}_\beta] = -c_\alpha c_\beta N_{\alpha,\beta} E_{-\alpha-\beta} \Rightarrow c_{\alpha+\beta} = -c_\alpha c_\beta,$$

where we have used $N_{-\alpha,-\beta} = -N_{\alpha,\beta}$ and $N^2_{\alpha,\beta} > 0$, so $\bar{N}_{\alpha,\beta} = N_{\alpha,\beta}$. In other words, the coefficients c_α must be real and must satisfy $c_{-\alpha} = c_\alpha^{-1}$ and $|c_{\alpha+\beta}| = |c_\alpha c_\beta|$. On the other hand, the ansatz $E'_\alpha = a_\alpha E_\alpha$ with $a_\alpha \in \mathbb{C}$ and $a_\alpha a_{-\alpha} = 1$ gives $[E'_\alpha, E'_{-\alpha}] = H_\alpha$, $[E'_\alpha, E'_\beta] = N'_{\alpha,\beta} E'_{\alpha+\beta}$ and $\bar{E}'_\alpha = c'_\alpha E'_{-\alpha}$ with $N'_{\alpha,\beta} = a_{-\alpha-\beta} a_\alpha a_\beta N_{\alpha,\beta}$ and $c'_\alpha = |a_\alpha|^2 c_\alpha$, so we can achieve $N'_{\alpha,\beta} = N_{\alpha,\beta}$ and $c'_\alpha = \pm 1$ by setting $a_\alpha = |c_\alpha|^{-1/2}$.]
Note that

$$\begin{aligned} \bar{E}_\alpha = -E_{-\alpha} &\Rightarrow i(E_\alpha + E_{-\alpha}), E_\alpha - E_{-\alpha} \text{ span } \mathfrak{g} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}) \\ &\Rightarrow \text{Kill negative definite on } \mathfrak{g} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}), \end{aligned} \tag{3.20}$$

$$\begin{aligned} \bar{E}_\alpha = +E_{-\alpha} &\Rightarrow E_\alpha + E_{-\alpha}, i(E_\alpha - E_{-\alpha}) \text{ span } \mathfrak{g} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}) \\ &\Rightarrow \text{Kill positive definite on } \mathfrak{g} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}), \end{aligned} \tag{3.21}$$

which provides the motivation for calling roots $\alpha \in \Delta$ *compact* respectively *non-compact* if they satisfy (3.20) respectively (3.21). If G itself is compact, all roots $\alpha \in \Delta$ are compact, and conversely, compactness of all roots $\alpha \in \Delta$ implies compactness of G [5, p. 133]. If, however, G is non-compact, and we use the conventions from the beginning of this section that are relevant to this case, then[2]

$$\mathfrak{l}^c = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha \in \Delta \\ \alpha \text{ compact}}} \mathfrak{g}^{(\alpha)}, \quad \mathfrak{p}^c = \bigoplus_{\substack{\alpha \in \Delta \\ \alpha \text{ noncompact}}} \mathfrak{g}^{(\alpha)}. \tag{3.22}$$

Second, the stability algebra $\mathfrak{k}$ and the complementary subspace $\mathfrak{m}$ [cf. (3.3) and (3.4)] can also be described explicitly in terms of the root system Δ, simply because the commutation relation

$$[Z_0, E_\alpha] = \alpha(Z_0) E_\alpha \quad \text{for} \quad \alpha \in \Delta \tag{3.23}$$

states that the complexification $\mathfrak{k}^c$ of the centralizer $\mathfrak{k}$ of Z_0 contains precisely those root vectors E_γ for which $\gamma(Z_0) = 0$. This leads us to a splitting of Δ into two pieces,

$$\Delta = \Delta' \cup \hat{\Delta}: \quad \begin{aligned} \Delta' &= \{\gamma \in \Delta | \gamma(Z_0) = 0\}, \\ \hat{\Delta} &= \{\alpha \in \Delta | \alpha(Z_0) \neq 0\}, \end{aligned} \tag{3.24}$$

and also to a direct decomposition of $\mathfrak{t}$ into two subspaces

$$\mathfrak{t} = \hat{\mathfrak{t}} \oplus \mathfrak{t}': \quad \begin{aligned} \hat{\mathfrak{t}} &= \{X \in \mathfrak{t} | \gamma(X) = 0 \text{ for } \gamma \in \Delta'\}, \\ \mathfrak{t}' &= \text{linear span of } \{iH_\gamma | \gamma \in \Delta'\}, \end{aligned} \tag{3.25}$$

which are orthogonal under the Killing form of $\mathfrak{g}$. Then

$$\mathfrak{k} = \hat{\mathfrak{t}} \oplus \mathfrak{k}', \tag{3.26}$$

where (in accordance with our previous notation) $\hat{\mathfrak{t}}$ is the centre of $\mathfrak{k}$ and $\mathfrak{k}'$ is a semisimple ideal in $\mathfrak{k}$. Moreover, $\mathfrak{h}' = \mathfrak{t}'^c$ is a Cartan subalgebra of the complex semisimple Lie algebra $\mathfrak{k}'^{c\,2}$, Δ' is the root system of $\mathfrak{k}'^c$ with respect to $\mathfrak{h}'$, and

$$\mathfrak{k}'^c = \mathfrak{h}' \oplus \bigoplus_{\gamma \in \Delta'} \mathfrak{g}^{(\gamma)} \tag{3.27}$$

is the corresponding root space decomposition [5, p. 191, Ex.B.1]. Finally,

$$\mathfrak{m}^c = \bigoplus_{\alpha \in \hat{\Delta}} \mathfrak{g}^{(\alpha)} . \tag{3.28}$$

More explicitly, all roots $\gamma \in \Delta'$ are compact, while roots $\alpha \in \hat{\Delta}$ may be compact or non-compact, and

$$\begin{array}{llll} i(E_\gamma + E_{-\gamma}), E_\gamma - E_{-\gamma} & \text{span} & \mathfrak{k} \cap (\mathfrak{g}^{(\gamma)} \oplus \mathfrak{g}^{(-\gamma)}) & \text{for } \gamma \in \Delta', \\ i(E_\alpha + E_{-\alpha}), E_\alpha - E_{-\alpha} & \text{span} & \mathfrak{m} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}) & \text{for } \alpha \in \hat{\Delta} \quad \text{compact}, \\ E_\alpha + E_{-\alpha}, i(E_\alpha - E_{-\alpha}) & \text{span} & \mathfrak{m} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}) & \text{for } \alpha \in \hat{\Delta} \quad \text{noncompact}. \end{array} \tag{3.29}$$

Third, the notions of positive and negative roots, of simple roots, of Weyl chambers, etc., can all be extended and modified in such a way as to apply specifically to the present situation.

We begin by recalling [14, p. 280] that an *ordering* in Δ can be defined by singling out a subset Δ^+ of Δ, whose elements are called *positive*, subject to the following two conditions:

$$\Delta^+ \cup \Delta^- = \Delta , \quad \Delta^+ \cap \Delta^- = \emptyset , \quad \text{where} \quad \Delta^- = -\Delta^+ , \tag{3.30}$$

$$\alpha, \beta \in \Delta^+ , \quad \alpha + \beta \in \Delta \Rightarrow \alpha + \beta \in \Delta^+ . \tag{3.31}$$

Now consider the splitting $\Delta = \Delta' \cup \hat{\Delta}$ of Δ given by (3.24), and observe that $\hat{\Delta}$ is invariant under Δ', i.e.,

$$\alpha \in \hat{\Delta} , \quad \gamma \in \Delta' , \quad \alpha + \gamma \in \Delta \Rightarrow \alpha + \gamma \in \hat{\Delta} . \tag{3.32}$$

(In view of (3.25)–(3.28), this is simply one way of rewriting (the complexification of) the commutation relation $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$.) Then although $\hat{\Delta}$ is not a root system by itself, an *invariant ordering* in $\hat{\Delta}$ can be defined by singling out a subset $\hat{\Delta}^+$ of $\hat{\Delta}$, whose elements are called *positive*, subject to the following three conditions:

$$\hat{\Delta}^+ \cup \hat{\Delta}^- = \hat{\Delta} , \quad \hat{\Delta}^+ \cap \hat{\Delta}^- = \emptyset , \quad \text{where} \quad \hat{\Delta}^- = -\hat{\Delta}^+ , \tag{3.33}$$

$$\alpha, \beta \in \hat{\Delta}^+ , \quad \alpha + \beta \in \hat{\Delta} \Rightarrow \alpha + \beta \in \hat{\Delta}^+ , \tag{3.34}$$

$$\alpha \in \hat{\Delta}^+ , \quad \gamma \in \Delta' , \quad \alpha + \gamma \in \Delta \Rightarrow \alpha + \gamma \in \hat{\Delta}^+ . \tag{3.35}$$

[Note that due to (3.33) and (3.35), one cannot have $\alpha, \beta \in \hat{\Delta}^+$ but $\alpha + \beta \in \Delta'$ in (3.34).] On the other hand, an ordering in Δ will be called *compatible* (with the splitting $\Delta = \Delta' \cup \hat{\Delta}$) if (3.35) is satisfied when we set

$$\Delta'^{\pm} = \Delta^{\pm} \cap \Delta' , \quad \hat{\Delta}^{\pm} = \Delta^{\pm} \cap \hat{\Delta} , \tag{3.36}$$

and it is clear that the choice of a compatible ordering in Δ amounts to the choice of an ordering in Δ' together with an invariant ordering in $\hat{\Delta}$, with

$$\Delta'^{\pm} \cup \hat{\Delta}^{\pm} = \Delta^{\pm} , \quad \Delta'^{\pm} \cap \hat{\Delta}^{\pm} = \emptyset . \tag{3.37}$$

Next, given any such ordering, one defines $\alpha \in \Delta^+$ to be *simple* if it cannot be written in the form $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in \Delta^+$ and $\alpha_2 \in \Delta^+$. Then the difference of two simple roots in Δ^+ does not belong to Δ, which implies that the set B of simple roots in Δ^+ is linearly independent [5, pp. 177/178 and 456–458], while an iterative argument [5, p. 178] shows that B is a basis of Δ, i.e., every root $\beta \in \Delta$ can be uniquely represented in the form

$$\beta = \sum_{\alpha \in B} n_\alpha \alpha \tag{3.38}$$

with integer coefficients n_α which are all $\geqq 0$ (at least one being >0) if $\beta \in \Delta^+$ and all $\leqq 0$ (at least one being <0) if $\beta \in \Delta^-$; in particular, $\{iH_\alpha / \alpha \in B\}$ is a basis of t[5]. Moreover, setting

$$B' = B \cap \Delta', \qquad \hat{B} = B \cap \hat{\Delta} \tag{3.39}$$

one obtains a basis B' of Δ' together with a "basis" $\hat{B}$ of $\hat{\Delta}$, or more correctly, of Δ modulo Δ'; in particular, $\{iH_\gamma / \gamma \in B'\}$ is a basis of $\mathfrak{t}'$ and $\{i\alpha|_{\hat{\mathfrak{t}}} / \alpha \in \hat{B}\}$ is a basis of the dual space $\hat{\mathfrak{t}}^*$ of $\hat{\mathfrak{t}}$[5]. (It should perhaps be noted that $\hat{B}$ cannot be defined solely in terms of $\hat{\Delta}$: in fact, explicit examples show that $\hat{B}$ will in general depend on the choice of ordering in Δ'.) Finally, it is known [5, p. 458] that orderings in Δ are in one-to-one correspondence with *Weyl chambers* in $\mathfrak{t}$, which are defined as the connected components of the open dense subset $\mathfrak{t}^\circ$ of $\mathfrak{t}$ obtained by removing all hyperplanes $\mathfrak{t}_\alpha = \{X \in \mathfrak{t} / \alpha(X) = 0\}$ $(\alpha \in \Delta)$. Namely, $\alpha \in \Delta$ is positive with respect to some given ordering if and only if it takes strictly positive values on the corresponding Weyl chamber C, which means that for an arbitrary vector $X \in C$, we have[5]

$$\operatorname{sign} \alpha = \operatorname{sign}(i\alpha(X)) \quad \text{for} \quad \alpha \in \Delta, \tag{3.40}$$

where, of course, $\operatorname{sign} \alpha = \pm 1$ means $\alpha \in \Delta^\pm$. Similarly, one can show that invariant orderings in $\hat{\Delta}$ are in one-to-one correspondence with *Weyl chambers* in $\hat{\mathfrak{t}}$, which are defined as the connected components of the open dense subset $\hat{\mathfrak{t}}^\circ$ of $\hat{\mathfrak{t}}$ obtained by removing all hyperplanes $\hat{\mathfrak{t}}_\alpha = \{X \in \hat{\mathfrak{t}} / \alpha(X) = 0\}$ $(\alpha \in \hat{\Delta})$. Namely, $\alpha \in \hat{\Delta}$ is positive with respect to some given invariant ordering if and only if it takes strictly positive values on the corresponding Weyl chamber $\hat{C}$, which means that for an arbitrary vector $X \in \hat{C}$, we have[5]

$$\operatorname{sign} \alpha = \operatorname{sign}(i\alpha(X)) \quad \text{for} \quad \alpha \in \hat{\Delta}, \tag{3.41}$$

where, of course, $\operatorname{sign} \alpha = \pm 1$ means $\alpha \in \hat{\Delta}^\pm$. Finally, it is clear that combining an invariant ordering in $\hat{\Delta}$ with an ordering in Δ' to yield a compatible ordering in Δ, one will have

$$B' \cup \hat{B} = B, \qquad B' \cap \hat{B} = \emptyset \tag{3.42}$$

for the set of simple roots and

$$C'^{cl} = C^{cl} \cap \mathfrak{t}', \qquad \hat{C}^{cl} = C^{cl} \cap \hat{\mathfrak{t}} \tag{3.43}$$

for the closures of the Weyl chambers C in $\mathfrak{t}$, C' in $\mathfrak{t}'$, and $\hat{C}$ in $\hat{\mathfrak{t}}$.

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5 Strictly speaking, it is $i\Delta$, rather than Δ itself, which forms a root system in the abstract sense [5, pp. 455/456], so that in some formulae, we must insert factors of i in a consistent manner

With all these preliminaries out of the way, we can now discuss the construction of G-invariant Kähler structures on the homogeneous space $M=G/K$, which is the G-orbit through Z_0 under the adjoint representation.

We begin with the invariant complex structure I. According to the discussion in Sect. 2, and using that K is connected, such a structure can be represented by an $\mathrm{ad}(\mathfrak{k}^c)$-invariant complex linear transformation I_0 on $\mathfrak{m}^c$ which commutes with the conjugation $\bar{\ }$ in $\mathfrak{g}^c$ with respect to $\mathfrak{g}$ and which satisfies $I_0^2=-1$ as well as (the complexified version of) the integrability condition (2.22). In particular, I_0 must commute with all $\mathrm{ad}(H)$ $(H\in\mathfrak{h})$, and therefore,

$$I_0E_\alpha=i\varepsilon_\alpha E_\alpha \quad \text{for} \quad \alpha\in\hat{\Delta} \tag{3.44}$$

with coefficients ε_α $(\alpha\in\hat{\Delta})$ which must satisfy

$$\varepsilon_\alpha=\pm1 \quad \text{for} \quad \alpha\in\hat{\Delta} \tag{3.45}$$

in order that $I_0^2=-1$ and

$$\varepsilon_{-\alpha}=-\varepsilon_\alpha \quad \text{for} \quad \alpha\in\hat{\Delta} \tag{3.46}$$

in order that I_0 commute with the conjugation $\bar{\ }$. [Here we have used (3.19).] Moreover, (3.26) and (3.27) tell us that I_0 must also commute with all $\mathrm{ad}(E_\gamma)$ $(\gamma\in\Delta')$, which means that

$$\varepsilon_{\alpha+\gamma}=\varepsilon_\alpha \quad \text{for} \quad \alpha\in\hat{\Delta}, \quad \gamma\in\Delta' \quad \text{such that} \quad \alpha+\gamma\in\hat{\Delta}. \tag{3.47}$$

Finally, the integrability condition gives the constraint

$$\varepsilon_{\alpha+\beta}(\varepsilon_\alpha+\varepsilon_\beta)=\varepsilon_\alpha\varepsilon_\beta+1 \quad \text{for} \quad \alpha,\beta\in\hat{\Delta} \quad \text{such that} \quad \alpha+\beta\in\hat{\Delta}. \tag{3.48}$$

This is automatically satisfied if $\varepsilon_\alpha+\varepsilon_\beta=0$, and hence (3.48) reduces to

$$\alpha,\beta\in\hat{\Delta}, \quad \alpha+\beta\in\hat{\Delta}, \quad \varepsilon_\alpha=\varepsilon_\beta \Rightarrow \varepsilon_\alpha=\varepsilon_{\alpha+\beta}=\varepsilon_\beta. \tag{3.49}$$

But now comparison between (3.33), (3.34), (3.35) and (3.46), (3.47), (3.49) reveals that ε_α must be the sign of α with respect to a certain invariant ordering in $\hat{\Delta}$, which, in turn, corresponds to a certain Weyl chamber $\hat{C}^I$ in $\hat{\mathfrak{t}}$. In other words,

$$\varepsilon_\alpha=\mathrm{sign}(i\alpha(Z^I)) \quad \text{for} \quad \alpha\in\hat{\Delta}, \tag{3.50}$$

where Z^I is any element in the centre $\hat{\mathfrak{t}}$ of the stability algebra $\mathfrak{k}$ which belongs to the Weyl chamber $\hat{C}^I$, and

$$\begin{aligned} E_\alpha&\in\mathfrak{m}^{(1,0)} \quad \text{for} \quad \alpha\in\hat{\Delta}^+,\\ E_\alpha&\in\mathfrak{m}^{(0,1)} \quad \text{for} \quad \alpha\in\hat{\Delta}^-, \end{aligned} \tag{3.51}$$

with respect to this ordering. In particular, the element Z_0 itself gives rise to a distinguished complex structure by setting $Z^I=Z_0$ in (3.50).

Before going on, let us briefly consider the special situation where $M=G/K$ is not only homogeneous but is (locally) symmetric. First of all, we note that I_0 necessarily leaves the Killing form invariant, i.e., we have for $X, Y\in\mathfrak{m}$

$$\mathrm{Kill}(I_0X, I_0Y)=\mathrm{Kill}(X, Y). \tag{3.52}$$

[This follows by combining (3.17), (3.18) with (3.46).] Now if $M=G/K$ is (locally) symmetric, we can combine this formula with the commutation relation $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}$ to show that for $X, Y\in\mathfrak{m}$,

$$[I_0X, I_0Y]=[X, Y], \tag{3.53}$$

or equivalently

$$[I_0X, Y]+[X, I_0Y]=0. \tag{3.54}$$

Indeed, for $X, Y\in\mathfrak{m}$, both sides of (3.53) belong to $\mathfrak{k}$, and if $Z\in\mathfrak{k}$, the ad($\mathfrak{k}$)-invariance of I_0 on $\mathfrak{m}$ gives

$$\begin{aligned}\mathrm{Kill}([I_0X, I_0Y], Z)&=\mathrm{Kill}(I_0Y, [Z, I_0X])=\mathrm{Kill}(I_0Y, I_0[Z, X])\\&=\mathrm{Kill}(Y, [Z, X])=\mathrm{Kill}([X, Y], Z).\end{aligned}$$

Therefore, extending I_0 from a linear transformation on $\mathfrak{m}$ to a linear transformation on $\mathfrak{g}$ (we use the same notation for both transformations) by requiring

$$I_0=0 \quad \text{on } \mathfrak{k}, \tag{3.55}$$

and exploiting the ad($\mathfrak{k}$)-invariance of I_0 once again, we infer from (2.22), (2.26), and (3.54) that I_0 must be a derivation, i.e., we have for $X, Y\in\mathfrak{g}$,

$$I_0[X, Y]=[I_0X, Y]+[X, I_0Y]. \tag{3.56}$$

Since G is semisimple, I_0 is necessarily of the form $I_0=\mathrm{ad}(Z_0^I)$ with $Z_0^I\in\mathfrak{g}$ [5, p. 132], and so we have for $X\in\mathfrak{m}$,

$$I_0X=[Z_0^I, X]. \tag{3.57}$$

Moreover, (2.22), (2.26), (3.55), and the fact that $I_0^2=-1$ on $\mathfrak{m}$, imply that Z_0^I must belong to the centre $\hat{\mathfrak{k}}$ of the stability algebra $\mathfrak{k}$ and that

$$\alpha(Z_0^I)^2=-1 \quad \text{for} \quad \alpha\in\hat{\Delta}. \tag{3.58}$$

Conversely, combining (3.55) and the fact that $I_0^2=-1$ on $\mathfrak{m}$ into the formulae

$$I_0^3=-I_0 \quad \text{and} \quad \mathfrak{k}=\ker I_0, \quad \mathfrak{m}=\mathrm{im}\, I_0 \tag{3.59}$$

[cf. (3.5)], we can easily convince ourselves that if I_0 is a derivation, then $M=G/K$ must be (locally) symmetric. Indeed, (3.59) and (3.56) imply that for $X, Y\in\mathfrak{g}$,

$$\begin{aligned}I_0[X, Y]&=-I_0^3[X, Y]=-I_0^2([I_0X, Y]+[X, I_0Y])\\&=-I_0([I_0^2X, Y]+2[I_0X, I_0Y]+[X, I_0^2Y])\\&=-[I_0^3X, Y]-3[I_0^2X, I_0Y]-3[I_0X, I_0^2Y]-[X, I_0^3Y]\\&=[I_0X, Y]+[X, I_0Y]-3I_0[I_0X, I_0Y]\\&=I_0[X, Y]-3I_0[I_0X, I_0Y],\end{aligned}$$

from which the commutation relation $[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{k}$ follows immediately.

Returning to the general case, let us now consider the invariant metric g and the invariant fundamental two-form ω. According to the discussion in Sect. 2, and using that K is connected, these can be represented by ad($\mathfrak{k}^c$)-invariant complex bilinear forms g_0 and ω_0 on $\mathfrak{m}^c$, respectively, which are given by

$$g_0(E_\alpha, E_{-\alpha}) = -i\varepsilon_\alpha \alpha(Z^\omega), \quad \omega_0(E_\alpha, E_{-\alpha}) = \alpha(Z^\omega) \quad \text{for} \quad \alpha \in \hat{\Delta}, \tag{3.60}$$

$$g_0(E_\alpha, E_\beta) = 0, \quad \omega_0(E_\alpha, E_\beta) = 0 \quad \text{for} \quad \alpha, \beta \in \hat{\Delta}, \quad \alpha + \beta \neq 0, \tag{3.61}$$

where Z^ω is an appropriate element in the centre $\hat{\mathfrak{t}}$ of the stability algebra $\mathfrak{k}$ [cf. (2.36)]. Obviously, non-degeneracy of g_0 and ω_0 requires $\alpha(Z^\omega) \neq 0$ for all $\alpha \in \hat{\Delta}$, so Z^ω must belong to a certain Weyl chamber $\hat{C}^\omega$ in $\hat{\mathfrak{t}}$. Similar statements hold for the Ricci tensor Ric and the Ricci two-form ϱ, i.e.,

$$\mathrm{Ric}_0(E_\alpha, E_{-\alpha}) = -i\varepsilon_\alpha \alpha(Z^\varrho), \quad \varrho_0(E_\alpha, E_{-\alpha}) = \alpha(Z^\varrho) \quad \text{for} \quad \alpha \in \hat{\Delta}, \tag{3.62}$$

$$\mathrm{Ric}_0(E_\alpha, E_\beta) = 0, \quad \varrho_0(E_\alpha, E_\beta) = 0 \quad \text{for} \quad \alpha, \beta \in \hat{\Delta}, \quad \alpha + \beta \neq 0, \tag{3.63}$$

where Z^ϱ is an appropriate element in the centre $\hat{\mathfrak{t}}$ of the stability algebra $\mathfrak{k}$ [cf. (2.36)]. This element will be computed explicitly in Sect. 4.

The main problem that remains to be solved is to find out under what conditions the invariant metric g can be made positive definite. Now g_0 will be positive definite on $\mathfrak{m}$ if and only if

$$g_0(E_\alpha, \bar{E}_\alpha) > 0 \quad \text{for} \quad \alpha \in \hat{\Delta}, \tag{3.64}$$

or [cf. (3.19)–(3.21) and (3.60)]

$$\begin{aligned} i\varepsilon_\alpha \alpha(Z^\omega) &> 0 \quad \text{for} \quad \alpha \in \hat{\Delta}, \quad \alpha \text{ compact}, \\ i\varepsilon_\alpha \alpha(Z^\omega) &< 0 \quad \text{for} \quad \alpha \in \hat{\Delta}, \quad \alpha \text{ noncompact}. \end{aligned} \tag{3.65}$$

Let us define

$$\begin{aligned} \tilde{\varepsilon}_\alpha &= +\varepsilon_\alpha \quad \text{for} \quad \alpha \in \hat{\Delta}, \quad \alpha \text{ compact}, \\ \tilde{\varepsilon}_\alpha &= -\varepsilon_\alpha \quad \text{for} \quad \alpha \in \hat{\Delta}, \quad \alpha \text{ noncompact}. \end{aligned} \tag{3.66}$$

Then g_0 will be positive definite on $\mathfrak{m}$ if and only if

$$\tilde{\varepsilon}_\alpha = \mathrm{sign}(i\alpha(Z^\omega)) \quad \text{for} \quad \alpha \in \hat{\Delta}, \tag{3.67}$$

and so the coefficients $\tilde{\varepsilon}_\alpha$ $(\alpha \in \hat{\Delta})$ given by (3.66) must define an invariant ordering in $\hat{\Delta}$, namely the invariant ordering that corresponds to the Weyl chamber $\hat{C}^\omega$ in $\hat{\mathfrak{t}}$ which contains Z^ω.

In the compact case, existence of a positive definite metric is therefore a trivial consequence of the fact that all roots are then compact [so $\tilde{\varepsilon}_\alpha = \varepsilon_\alpha$, $\hat{C}^\omega = \hat{C}^I$, $Z^\omega = Z^I$; cf. (3.50), (3.67)]. In the non-compact case, however, existence of a positive definite metric, rather than being automatic, imposes a severe restriction. Namely, in terms of the conventions from the beginning of this section that are relevant to this case, the G-invariant complex structure I on G/K must induce a G-invariant complex structure J on G/L, and the space G/L, when equipped with the metric induced by the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ (which is positive definite), becomes a Hermitian symmetric space – rather than just a Riemannian symmetric space – of the non-compact type [5, p. 373].

To prove this last statement, note first that just as before, such a structure can be represented by an $\mathrm{ad}(\mathfrak{l}^c)$-invariant complex linear transformation J_0 on $\mathfrak{p}^c$

which commutes with the conjugation $\bar{\ }$ in $\mathfrak{g}^c$ with respect to $\mathfrak{g}$ and which satisfies $J_0^2 = -1$, while the appropriate analogue of (the complexified version of) the integrability condition (2.22) is an automatic consequence of the commutation relation $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{l}$. But the condition that I induces J means that J_0 arises from I_0 by restriction, i.e.,

$$J_0 E_\alpha = i\varepsilon_\alpha E_\alpha \quad \text{for} \quad \alpha \in \hat{\Delta}\,, \quad \alpha \text{ noncompact} \tag{3.68}$$

[cf. (3.44)], and this restriction will have all the desired properties provided that its $\mathrm{ad}(\mathfrak{k}^c)$-invariance extends to an $\mathrm{ad}(\mathfrak{l}^c)$-invariance, i.e., provided that the coefficients ε_α $(\alpha \in \hat{\Delta})$ satisfy, in addition to (3.47), the condition

$$\varepsilon_{\alpha+\beta} = \varepsilon_\alpha \quad \text{for} \quad \alpha \in \hat{\Delta} \quad \text{noncompact}\,, \quad \beta \in \hat{\Delta} \quad \text{compact}$$
$$\text{such that} \quad \alpha + \beta \in \hat{\Delta}\,. \tag{3.69}$$

Therefore, our statement will be proved if we can show that this is equivalent to the requirement that the coefficients $\tilde{\varepsilon}_\alpha$ $(\alpha \in \hat{\Delta})$ given by (3.66) define an invariant ordering in $\hat{\Delta}$, which means that they must satisfy the condition

$$\alpha, \beta \in \hat{\Delta}\,, \qquad \alpha + \beta \in \hat{\Delta}\,, \qquad \tilde{\varepsilon}_\alpha = \tilde{\varepsilon}_\beta \;\Rightarrow\; \tilde{\varepsilon}_\alpha = \tilde{\varepsilon}_{\alpha+\beta} = \tilde{\varepsilon}_\beta\,. \tag{3.70}$$

[The Eqs. (3.46) and (3.47) for $\tilde{\varepsilon}$, rather than ε, are automatic consequences of the corresponding equations for ε and of the definition (3.66).]

Thus assume that (3.70) is satisfied, and choose $\alpha, \beta \in \hat{\Delta}$ such that $\alpha + \beta \in \hat{\Delta}$ and such that α is non-compact while β is compact; we have to show $\varepsilon_{\alpha+\beta} = \varepsilon_\alpha$. Using that $\alpha + \beta$ must be non-compact [cf. (3.10), (3.22)], we distinguish two cases:

a) $\varepsilon_\alpha = \varepsilon_\beta$. Then $\varepsilon_{\alpha+\beta} = \varepsilon_\alpha$ by (3.49).

b) $\varepsilon_\alpha + \varepsilon_\beta = 0$. Then $\tilde{\varepsilon}_\alpha = \tilde{\varepsilon}_\beta$, so $\varepsilon_{\alpha+\beta} = -\tilde{\varepsilon}_{\alpha+\beta} = -\tilde{\varepsilon}_\alpha = \varepsilon_\alpha$ by (3.70).

Conversely, assume that (3.69) is satisfied, and choose $\alpha, \beta \in \hat{\Delta}$ such that $\alpha + \beta \in \hat{\Delta}$ and $\tilde{\varepsilon}_\alpha = \tilde{\varepsilon}_\beta$; we have to show $\tilde{\varepsilon}_{\alpha+\beta} = \tilde{\varepsilon}_\alpha$, say. We distinguish three cases:

a) α and β are compact. Then $\alpha + \beta$ is compact [cf. (3.10), (3.22)], and $\varepsilon_\alpha = \varepsilon_\beta$, so $\tilde{\varepsilon}_{\alpha+\beta} = \varepsilon_{\alpha+\beta} = \varepsilon_\alpha = \tilde{\varepsilon}_\alpha$ by (3.49).

b) α is non-compact and β is compact, say. Then $\alpha + \beta$ is non-compact [cf. (3.10), (3.22)], so $\tilde{\varepsilon}_{\alpha+\beta} = -\varepsilon_{\alpha+\beta} = -\varepsilon_\alpha = \tilde{\varepsilon}_\alpha$ by (3.69).

c) α and β are non-compact. This is impossible since we have assumed $\tilde{\varepsilon}_\alpha = \tilde{\varepsilon}_\beta$, hence $\varepsilon_\alpha = \varepsilon_\beta$. Indeed, the equation

$$[J_0 E_\alpha, J_0 E_\beta] = [E_\alpha, E_\beta] \quad \text{for} \quad \alpha, \beta \in \hat{\Delta} \quad \text{noncompact} \tag{3.71}$$

[cf. (3.53)] shows that

$$\varepsilon_\alpha = \varepsilon_\beta \;\Rightarrow\; \alpha + \beta \notin \Delta \quad \text{for} \quad \alpha, \beta \in \hat{\Delta} \quad \text{noncompact}\,. \tag{3.72}$$

Let us summarize the results of this section in a

Theorem. *Let $M = G/K$ be a semisimple adjoint orbit as explained at the beginning of this section, i.e., G is a connected semisimple Lie group, K is a compact connected subgroup of G and is the centralizer of an element $Z_0 \in \mathfrak{g}$, or equivalently, of a torus $\tilde{T} \subset G$, in G (cf. the theorem at the end of Sect. 2). Then*

(i) *M is a complex manifold, and the set of all G-invariant complex structures I on M is in one-to-one correspondence with the set of all Weyl chambers $\hat{C}^I$ in the centre $\hat{\mathfrak{t}}$ of $\mathfrak{k}$ (cf. (3.44), (3.47)).*

(ii) a) *If M is compact, M is a Kähler manifold.* b) *If M is non-compact and L is a maximal compact subgroup of G containing K, then M is a Kähler manifold if and only if G/L is a Hermitian symmetric space.*

In both cases, fixing a complex structure I on M, the set of all G-invariant Kählerian metrics g on M is in one-to-one correspondence with a certain Weyl chamber $\hat{C}^{\omega}$ in the centre $\hat{\mathfrak{k}}$ of $\mathfrak{k}$ (cf. (3.60)*). In the compact case, $\hat{C}^{\omega}$ equals $\hat{C}^{I}$, while in the non-compact case, the relation is in general much more complicated.*

4. Calculation of the Ricci Tensor

In this section, we shall compute the Ricci tensor for the semisimple adjoint orbits $M=G/K$ of Sect. 3. It will turn out that the Ricci tensor Ric does not depend at all on the choice of the metric g, and that it is always non-degenerate. In the compact case, it is positive definite and can itself be used as a metric. This metric, or any positive multiple thereof, will turn $M=G/K$ into an Einstein-Kähler manifold, and as has already been indicated in Sect. 2, the corresponding Kähler potential is then simply a negative multiple of the logarithm of the invariant volume element on $M=G/K$ [cf. (2.13), (2.20)]. In the non-compact case, analogous statements (with opposite signs) can be made if and only if $M=G/K$ is Hermitian symmetric.

We proceed to the proof. For arbitrary vector fields ξ, η, ζ on M, the Riemannian curvature tensor R is defined by

$$R(\xi,\eta)\zeta=\nabla_{\xi}\nabla_{\eta}\zeta-\nabla_{\eta}\nabla_{\xi}\zeta-\nabla_{[\xi,\eta]}\zeta\,, \tag{4.1}$$

where ∇ denotes the Riemannian connection corresponding to the Kählerian metric g [5, pp. 43 and 48]. (In passing, we note that for the calculation of the Ricci tensor, the positive definiteness of g is irrelevant: it is sufficient to assume g non-degenerate.) Moreover, for Kähler manifolds, the Ricci tensor Ric and the Ricci form ϱ [cf. (2.16)] take a particularly simple form [4, Vol. 2, p. 149]:

$$\varrho(\xi,\eta)=\mathrm{Ric}(I\xi,\eta)=\tfrac{1}{2}\mathrm{trace}(IR(\xi,\eta))\,. \tag{4.2}$$

In our case, R, Ric, and ϱ are G-invariant, so it suffices to evaluate them at the distinguished point $\circ$. This is done by introducing, for every $X\in\mathfrak{m}$, a linear transformation $\Lambda_{\mathfrak{m}}(X)$ on $\mathfrak{m}$ by setting, for $Y\in\mathfrak{m}$,

$$\Lambda_{\mathfrak{m}}(X)\cdot Y=(\nabla_{X_M}Y_M)(0)-[X_M,Y_M](0)\,. \tag{4.3}$$

(Here, and below, X_M and Y_M denote the fundamental vector fields on M generated by X and Y, respectively; cf. (2.25) and [4, Vol. 2, pp. 188 and 191].) In terms of $\Lambda_{\mathfrak{m}}$, one finds [4, Vol. 2, p. 192] that the curvature tensor at the point $\circ$, evaluated on $X, Y, Z\in\mathfrak{m}$, is

$$R_0(X,Y)Z=\Lambda_{\mathfrak{m}}(X)\Lambda_{\mathfrak{m}}(Y)Z-\Lambda_{\mathfrak{m}}(Y)\Lambda_{\mathfrak{m}}(X)Z-\Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}})Z-[[X,Y]_{\mathfrak{k}},Z]\,. \tag{4.4}$$

The important observation is now that the trace in (4.2) can actually be computed by passing to the complexification, due to the following two basic facts:

a) The complex structure I is simultaneously G-invariant and covariantly constant with respect to ∇ [4, Vol. 2, p. 148], which means that for $X, Y \in \mathfrak{m}$ (or even $\mathfrak{g}$),

$$[X_M, IY_M] = I[X_M, Y_M] \quad \text{and} \quad \nabla_{X_M}(IY_M) = I\nabla_{X_M} Y_M\,.$$

But this implies that, for every $X \in \mathfrak{m}$, $\Lambda_{\mathfrak{m}}(X)$ commutes with I_0, so that for $X, Y \in \mathfrak{m}$,

$$\operatorname{trace}(I_0 \Lambda_{\mathfrak{m}}(X)\Lambda_{\mathfrak{m}}(Y)) = \operatorname{trace}(\Lambda_{\mathfrak{m}}(X) I_0 \Lambda_{\mathfrak{m}}(Y)) = \operatorname{trace}(I_0 \Lambda_{\mathfrak{m}}(Y)\Lambda_{\mathfrak{m}}(X))\,,$$

i.e., the contributions from the first two terms in (4.4) cancel under the trace in (4.2).

b) According to (3.63), it suffices to evaluate the trace in (4.2) with $X = E_\alpha$ and $Y = E_{-\alpha}$ $(\alpha \in \hat{\Delta})$. But then $[E_\alpha, E_{-\alpha}] = H_\alpha \in \mathfrak{k}^c$ [cf. (3.15)], so $[E_\alpha, E_{-\alpha}]_{\mathfrak{m}^c} = 0$ and $[E_\alpha, E_{-\alpha}]_{\mathfrak{k}^c} = H_\alpha$.

We are therefore left with

$$\varrho_0(E_\alpha, E_{-\alpha}) = -\tfrac{1}{2}\operatorname{trace}_{\mathfrak{m}^c}(I_0 \operatorname{ad}(H_\alpha)) \quad \text{for} \quad \alpha \in \hat{\Delta}\,. \tag{4.5}$$

But on $\mathfrak{m}^c$, both I_0 and $\operatorname{ad}(H_\alpha)$ have the root vectors E_β $(\beta \in \hat{\Delta})$ as eigenvectors, with $i\varepsilon_\beta$ and $\beta(H_\alpha)$ as eigenvalues [cf. (3.44) and (3.13)], so

$$\begin{aligned}\varrho_0(E_\alpha, E_{-\alpha}) &= -\frac{i}{2}\sum_{\beta \in \hat{\Delta}} \varepsilon_\beta \beta(H_\alpha)\\ &= -\frac{i}{2}\Big(\sum_{\beta \in \hat{\Delta}^+} \beta(H_\alpha) - \sum_{\beta \in \hat{\Delta}^-} \beta(H_\alpha)\Big)\\ &= -i\sum_{\beta \in \hat{\Delta}^+} \beta(H_\alpha) = -i\alpha\Big(\sum_{\beta \in \hat{\Delta}^+} H_\beta\Big) \quad \text{for} \quad \alpha \in \hat{\Delta}\,,\end{aligned}$$

where the invariant ordering in $\hat{\Delta}$ is, of course, that given by the complex structure I [cf. (3.50), (3.51)]. Thus we have once again proved (3.62), i.e.,

$$\operatorname{Ric}_0(E_\alpha, E_{-\alpha}) = -i\varepsilon_\alpha \alpha(Z^\varrho)\,, \qquad \varrho_0(E_\alpha, E_{-\alpha}) = \alpha(Z^\varrho) \quad \text{for} \quad \alpha \in \hat{\Delta}\,, \tag{4.6}$$

or in other words,

$$\operatorname{Ric}_0(E_\alpha, E_{-\alpha}) = -2\varepsilon_\alpha \operatorname{Kill}(H_{\hat{\delta}}, H_\alpha), \quad \varrho_0(E_\alpha, E_{-\alpha}) = -2i\operatorname{Kill}(H_{\hat{\delta}}, H_\alpha) \quad \text{for} \quad \alpha \in \hat{\Delta}\,, \tag{4.7}$$

where

$$\hat{\delta} = \tfrac{1}{2}\sum_{\beta \in \hat{\Delta}^+} \beta\,, \tag{4.8}$$

while $H_{\hat{\delta}}$ is uniquely determined by the condition

$$\operatorname{Kill}(H_{\hat{\delta}}, H) = \hat{\delta}(H) \quad \text{for} \quad H \in \mathfrak{h}\,, \tag{4.9}$$

or equivalently

$$H_{\hat{\delta}} = \tfrac{1}{2}\sum_{\beta \in \hat{\Delta}^+} H_\beta \tag{4.10}$$

[cf. (3.14)], and $Z^\varrho = -2iH_{\hat{\delta}}$.

With these explicit expressions for Ric and ϱ – which, remarkably enough, do not depend on the choice of the metric g – at our disposal, we can now prove that the Ricci tensor is non-degenerate, and can investigate its signature. As will become clear below, all the necessary information can be extracted from the following formula:

$$\mathrm{Kill}(H_{\hat{\delta}}, H_\alpha) > 0 \quad \text{for} \quad \alpha \in \hat{\Delta}^+ , \qquad \mathrm{Kill}(H_{\hat{\delta}}, H_\alpha) < 0 \quad \text{for} \quad \alpha \in \hat{\Delta}^- . \tag{4.11}$$

For the proof of (4.11), we recall from Sect. 3 that the given invariant ordering in $\hat{\Delta}$ can be combined with an arbitrary ordering in Δ' to yield a compatible ordering in Δ [cf. (3.33), (3.37)]; then

$$\delta = \delta' + \hat{\delta} , \tag{4.12}$$

where

$$\delta = \tfrac{1}{2} \sum_{\beta \in \Delta^+} \beta , \qquad \delta' = \tfrac{1}{2} \sum_{\gamma \in \Delta'^+} \gamma . \tag{4.13}$$

Now combining (3.38) with the fact that for the simple roots $\alpha \in B$,

$$\mathrm{Kill}(H_\delta, H_\alpha) = (\delta, \alpha) = \tfrac{1}{2}(\alpha, \alpha) ,$$

[5, p. 461], we see that the vector δ, which plays a fundamental rôle in representation theory, has positive scalar products with all positive roots; in particular,

$$\mathrm{Kill}(H_{\hat{\delta}}, H_\alpha) + \mathrm{Kill}(H_{\delta'}, H_\alpha) > 0 \quad \text{for} \quad \alpha \in \hat{\Delta}^+ . \tag{4.14}$$

But since the ordering in Δ' was arbitrary, and independent of the invariant ordering in $\hat{\Delta}$, the same result holds if the ordering in Δ' is reversed ($\Delta'^+ \to \Delta'^-$, $\Delta'^- \to \Delta'^+$), i.e., if δ' is replaced by $-\delta'$:

$$\mathrm{Kill}(H_{\hat{\delta}}, H_\alpha) - \mathrm{Kill}(H_{\delta'}, H_\alpha) > 0 \quad \text{for} \quad \alpha \in \hat{\Delta}^+ . \tag{4.15}$$

Adding (4.14) and (4.15) gives the first formula in (4.11), which trivially implies the second one.

Returning to the Ricci tensor, we see directly from (4.7) and (4.11) that it must be non-degenerate. More explicitly, we can combine (3.29) with (4.7) and (4.11) to show that

$$\begin{aligned} &\mathrm{Ric}_0 \text{ positive definite on } \quad \mathfrak{m} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}) \quad \text{for} \quad \alpha \in \hat{\Delta} \quad \text{compact} , \\ &\mathrm{Ric}_0 \text{ negative definite on } \quad \mathfrak{m} \cap (\mathfrak{g}^{(\alpha)} \oplus \mathfrak{g}^{(-\alpha)}) \quad \text{for} \quad \alpha \in \hat{\Delta} \quad \text{noncompact} . \end{aligned} \tag{4.16}$$

This implies that the Ricci tensor will be positive definite if $M = G/K$ is compact, negative definite if $M = G/K$ is a Hermitian symmetric space of the non-compact type, and indefinite in all other cases.

5. Classification of Homogeneous Kähler Manifolds

In this section, we shall give a classification of the semisimple adjoint orbits $M = G/K$ of Sect. 3 in terms of Dynkin diagrams, thus providing a scheme for the possible ways of breaking an internal symmetry from G down to K in such a way that the coset space $M = G/K$ is a Kähler manifold.

In the non-compact case, this symmetry breaking proceeds in two steps, namely by choosing some maximal compact subgroup L of G containing K and breaking 1. from G down to L and 2. from L down to K; the homogeneous space G/K is then a fibre bundle with base space G/L and typical fibre L/K. Now on the one hand, we have seen in Sect. 3 that the space $M=G/K$ being a Kähler manifold forces the space G/L to be a Hermitian symmetric space of the non-compact type, or equivalently, a bounded symmetric domain [5, pp. 382/383]. But then G/L must be the direct product of irreducible Hermitian symmetric spaces of the non-compact type [5, pp. 374 and 376], and for a complete list of the latter, we refer the reader to Table 1, reproduced from [5, p. 518]. Next, let C be the centre of L and $\mathfrak{c}$ be the centre of $\mathfrak{l}$ [cf. (3.11)]; note that C is necessarily connected [5, pp. 381/382], so $C\subset T\subset K$. Then, on the other hand, the space

$$L/K=L_s/K_s \quad \text{with} \quad L_s=L/C, \qquad K_s=K/C \tag{5.1}$$

is one of the semisimple adjoint orbits of Sect. 3, but of the compact type. Therefore, the classification of the non-compact Kählerian orbits is reduced to that of the compact ones.

In view of this situation, and of arguments presented at the beginning of Sect. 3, we may assume without loss of generality that the symmetry group G is simply connected, compact, and simple. Then imitating a procedure due to I. Satake, which has been invented in connection with the classification problem for symmetric spaces [5, pp. 530–535], we can characterize the orbit $M=G/K$ by what we shall call a *painted Dynkin diagram*, which is obtained as follows:

1. Draw an ordinary connected Dynkin diagram, which is a connected graph consisting of vertices representing the simple roots for the symmetry algebra $\mathfrak{g}$. Thus every pair of vertices is connected by 0, 1, 2 or 3 lines according to whether the angle between the corresponding simple roots is 90°, 120°, 135° or 150°, respectively, and in the last two cases, i.e., for double or triple lines, an arrow is attached, pointing from the longer to the shorter root.

2. Paint this Dynkin diagram by letting black vertices represent the simple roots for the semisimple part $\mathfrak{k}'$ of the stability algebra $\mathfrak{k}$, while white vertices represent the remaining simple roots. [In particular, the subdiagram formed by the black roots is automatically an ordinary (not necessarily connected) Dynkin diagram in itself.]

More concretely, in the notation of Sect. 3, the construction proceeds by fixing a maximal torus T in G containing the torus $\hat{T}$ (which, by definition, is the connected one-component of the centre of K), together with an ordering in the resulting root system Δ which is compatible with the splitting $\Delta=\Delta'\cup\hat{\Delta}$. This uniquely determines a basis B in Δ such that $B=B'\cup\hat{B}$, and thus gives rise to a division of the r simple roots in B ($r=\operatorname{rank}\mathfrak{g}=\dim\mathfrak{t}$) into r' simple black roots lying in B' ($r'=\operatorname{rank}\mathfrak{k}'=\dim\mathfrak{t}'$) and $\hat{r}$ simple white roots lying in $\hat{B}$ ($\hat{r}=\dim\hat{\mathfrak{t}}$). It should be noted that the resulting painted Dynkin diagram does not depend on the choice of T, and hence of Δ. In fact, G being a compact connected Lie group, any two maximal tori in G, and hence the corresponding root systems, are conjugate under an element of G [5, p. 248], and if both maximal tori contain the same torus $\hat{T}$, this element can be chosen to normalize $\hat{T}$, and hence K. (The last statement can be deduced from a group-theoretical version [5, pp. 297–300] of a proposition in [5,

p. 285].) On the other hand, the necessity of making a choice for the compatible ordering in Δ does lead to a certain ambiguity in the painting process. However, this is really an advantage, rather than a drawback, because it reduces the number of painted Dynkin diagrams; we shall return to this aspect below.

The converse procedure of (re)constructing the orbit $M=G/K$ from the painted Dynkin diagram that corresponds to it, uniquely up to isomorphisms of G respecting all additional structures, does not present any problems, either. In fact, given any painted Dynkin diagram, we begin by reducing it to an ordinary one. Then we can (re)construct the root system Δ, and from that, the complex Lie algebra $\mathfrak{g}^c$, with prescribed Cartan subalgebra $\mathfrak{h}$ and root space decomposition (3.12), uniquely up to isomorphisms [5, p. 481]. Moreover, the real Lie algebra $\mathfrak{g}$, being a compact real form of $\mathfrak{g}^c$, is also unique up to isomorphisms (in fact, up to inner automorphisms of $\mathfrak{g}^c$) [5, p. 184], so that G, being simply connected, is unique up to isomorphisms as well, while $\mathfrak{t}$ becomes a prescribed maximal Abelian subalgebra in $\mathfrak{g}$ and T becomes a prescribed maximal torus in G; we shall therefore, in the following, consider G, $\mathfrak{g}$ and T, $\mathfrak{t}$ as being fixed. Next, we can make use of the splitting $B=B'\cup\hat{B}$, which is precisely the additional information coded into the painted Dynkin diagram (as opposed to the ordinary one), to derive the splitting $\Delta=\Delta'\cup\hat{\Delta}$ [from (3.22) and (3.38)] and the orthogonal direct decomposition $\mathfrak{t}=\mathfrak{t}'\oplus\hat{\mathfrak{t}}$ [from $\mathfrak{t}=\mathfrak{g}\cap\mathfrak{h}$ and (3.25)]. This defines $\mathfrak{k}^c$ as a subalgebra of $\mathfrak{g}^c$ and $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$ in a unique manner, and finally K will be the unique connected Lie subgroup of G that corresponds to the Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$.

As indicated before, an important rôle is played by the observation that differently painted Dynkin diagrams may give rise to the same orbit and should therefore be considered equivalent. Namely, the homogeneous spaces G/K_1 and G/K_2 are identical as orbits in $\mathfrak{g}$, differing from one another simply by the choice of their reference points $Z_{0,1}$ and $Z_{0,2}$, if and only if the stability groups K_1 and K_2, or equivalently, the tori $\hat{T}_1$ and $\hat{T}_2$, are conjugate under an element of G. But since we are considering the maximal torus T in G as being fixed, so $\hat{T}_1\subset T\subset K_1$ and $\hat{T}_2\subset T\subset K_2$, they must then be conjugate under an element of G which normalizes T. (Once again, this statement can be deduced from a group-theoretical version [5, pp. 297–300] of a proposition in [5, p. 285].) In other words, we are left with the freedom of performing Weyl group transformations – the Weyl group $W(G)$ of G being the quotient of the normalizer of T modulo the centralizer of T [5, pp. 284 and 297–300]. More specifically, we may use a Weyl group transformation w to transform the subspace $\hat{\mathfrak{t}}$ of $\mathfrak{t}$ to a new subspace $w^T\hat{\mathfrak{t}}$ (where T denotes transpose of linear maps), or equivalently, the splitting $\Delta=\Delta'\cup\hat{\Delta}$ of Δ to a new splitting $\Delta=w\Delta'\cup w\hat{\Delta}$; the ordering, however, is kept fixed. [Equivalently, one could transform the ordering and leave the subspace, or equivalently, the splitting, fixed. This procedure, which gives rise to the ambiguity mentioned in the penultimate paragraph, would however change the basis B itself, rather than just the way in which it is painted, and that is why we prefer the other point of view.] Now for the diagram, application of a Weyl group transformation w makes sense (if and) only if the new splitting $\Delta=w\Delta'\cup w\hat{\Delta}$ shares the property of the original splitting $\Delta=\Delta'\cup\hat{\Delta}$ of being compatible with the given ordering. Formulated as a condition on w, this amounts to demanding that if w preserves (switches) the sign of a root $\alpha\in\Delta$, then it must also preserve (switch) the sign of any root in Δ that can be written in the form $\alpha+\gamma$ with $\gamma \in \Delta'$; in this case, we shall say that w is *admissible*. Thus it is the admissible Weyl group transformations which, out of a given painted Dynkin diagram, produce a repainted Dynkin diagram that should be considered equivalent to the original one. [Note that products of reflections along the black roots $\alpha \in B'$, which constitute the Weyl group of the semisimple part of the stability group K, are, of course, admissible, but they are uninteresting since they do not lead to any repainting. Reflections along the white roots $\alpha \in \hat{B}$, on the other hand, are in general not admissible.] Observe finally that if two differently painted Dynkin diagrams are supposed to arise from one another by repainting, as above, then the two subdiagrams formed by the black roots must be isomorphic (since they generate isomorphic – in fact, conjugate – centralizer subalgebras). However, this necessary condition is far from being sufficient (cf. the discussion below).

Before proceeding further, we want to illustrate the concepts introduced so far on the typical and important example of the generalized flag manifolds considered in Sect. 1, for which $G=\mathrm{SU}(N)$ and $K=\mathrm{S}(\mathrm{U}(N_1)\times\ldots\times\mathrm{U}(N_p))$ [cf. (1.1)–(1.4)]. The Lie algebra $\mathfrak{g}$ consists of all traceless antihermitian complex $(N\times N)$-matrices, and the maximal Abelian subalgebra $\mathfrak{t}$ of all traceless purely imaginary diagonal $(N\times N)$-matrices. The corresponding root system is $\Delta=\{\alpha_{i,j}|1\leqq i,j\leqq N, i\neq j\}$ with

$$\alpha_{i,j}(\mathrm{diag}(\lambda_1,\ldots,\lambda_N))=\lambda_i-\lambda_j \quad \text{for} \quad \lambda_1,\ldots,\lambda_N\in\mathbb{C} \quad \text{such that} \quad \lambda_1+\ldots+\lambda_N=0, \tag{5.2}$$

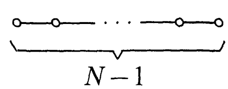
and we choose the ordering where $\alpha_{i,j}$ is positive (negative) if $i<j$ $(i>j)$. That this is indeed an ordering can be deduced, for example, from the relations

$$-\alpha_{i,j}=\alpha_{j,i} \tag{5.3}$$

and

$$\alpha_{i,j}+\alpha_{k,l}\in\Delta \;\Leftrightarrow\; \begin{Bmatrix} \text{either} & i\neq l,\ j=k, & \text{then} & \alpha_{i,j}+\alpha_{k,l}=\alpha_{i,l} \\ \text{or} & i=l,\ j\neq k, & \text{then} & \alpha_{i,j}+\alpha_{k,l}=\alpha_{k,j} \end{Bmatrix}. \tag{5.4}$$

This gives the basis $B=\{\alpha_{i,i+1}|1\leqq i\leqq N-1\}$ as the corresponding set of simple roots, depicted by the ordinary Dynkin diagram

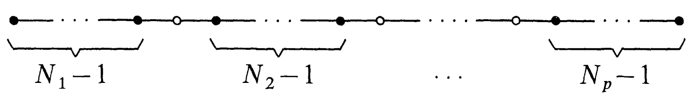


(5.5)

(cf. [5, pp. 186/187 and 462]). Now the generator Z_0 in (1.3), and the $(p-1)$-dimensional Abelian subalgebra $\mathfrak{t}$, whose common centralizer is the stability algebra $\mathfrak{k}=\mathfrak{s}(\mathfrak{u}(N_1)\times\ldots\times\mathfrak{u}(N_p))$ of block diagonal matrices in $\mathfrak{g}=\mathfrak{su}(N)$, give rise to the splitting $\Delta=\Delta'\cup\hat{\Delta}$ with

$$\begin{aligned} \Delta'&=\{\alpha_{i,j}\in\Delta|i,j \text{ are in the same block}\}, \\ \hat{\Delta}&=\{\alpha_{i,j}\in\Delta|i,j \text{ are in different blocks}\}. \end{aligned} \tag{5.6}$$

It is then immediate that the corresponding painted Dynkin diagram is



(5.7)

This becomes especially transparent if we use the thumb rule that in

$$Z_0 = \mathrm{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{N_1}, \underbrace{\lambda_2, \ldots, \lambda_2}_{N_2}, \ldots, \underbrace{\lambda_p, \ldots, \lambda_p}_{N_p}), \tag{5.8}$$

the simple roots are placed on top of the commas and are painted black (white) if the adjacent eigenvalues are equal (different). (Some of the N_i may, of course, be 1, in which case the corresponding blocks of black roots are absent.) Next, the Weyl group of $G = SU(N)$ is the symmetric group of permutations w of the set $\{1, \ldots, N\}$, which acts on Δ according to

$$w\alpha_{i,j} = \alpha_{w(i), w(j)}. \tag{5.9}$$

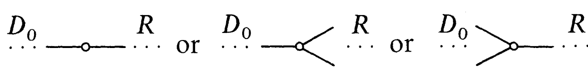
Then using (5.3), (5.4), and (5.6), it is easy to see that w is admissible if and only if it is a block permutation, i.e., if and only if for any two different blocks and for indices $1 \leqq i, j, k, l \leqq N$ with i, j in the first block and k, l in the second block, we have

$$w(i) < w(k) \Rightarrow w(j) < w(l). \tag{5.10}$$

Thus in (5.7), the repainting procedure amounts to a permutation of the integers N_i, which in (5.8) corresponds to a block permutation of the different eigenvalues λ_i: this modifies Z_0 but does indeed leave the orbit as a whole invariant.

Returning to the general case, we are now going to develop a technique for extending admissible Weyl group transformations from smaller painted Dynkin diagrams to larger ones. For convenience, we formulate this as a

Lemma. *Let D be a painted Dynkin diagram and D_0 be a painted Dynkin subdiagram such that D_0 is joined to the rest R of D by a single white root:*



Assume that w is a product of reflections along the simple roots belonging to D_0. Then, if w defines an admissible Weyl group transformation on D_0, it also defines an admissible Weyl group transformation on D which, in addition, leaves the connecting root white and the rest R of D unchanged.

For the proof, let us denote the connecting white root in D by ε. Then any two roots $\beta, \gamma \in \Delta$ have the representation

$$\beta = \beta_0 + n_\varepsilon \varepsilon + \beta_R, \quad \beta_0 = \sum_{\alpha \in D_0} n_\alpha \alpha, \quad \beta_R = \sum_{\alpha \in R} n_\alpha \alpha,$$

$$\gamma = \gamma_0 + m_\varepsilon \varepsilon + \gamma_R, \quad \gamma_0 = \sum_{\alpha \in D_0} m_\alpha \alpha, \quad \gamma_R = \sum_{\alpha \in R} m_\alpha \alpha,$$

where all coefficients are integers of the same sign. Now for the reflection s_σ along any simple root σ belonging to D_0, we have $s_\sigma \alpha = \alpha$ for $\alpha \in R$ (since σ is orthogonal to R) and $s_\sigma \varepsilon = \varepsilon - n_{\sigma,\varepsilon} \sigma$ with $n_{\sigma,\varepsilon} = 2\mathrm{Kill}(H_\sigma, H_\varepsilon)/\mathrm{Kill}(H_\sigma, H_\sigma) = 0, \pm 1, \pm 2$ or ± 3. Hence the inverse image of β, γ under any product w of such reflections takes the form

$$w^{-1}\beta = w^{-1}\beta_0 + n_\varepsilon \varepsilon + \beta_R, \quad w^{-1}\beta_0 = \sum_{\alpha \in D_0} n_\alpha^w \alpha,$$

$$w^{-1}\gamma = w^{-1}\gamma_0 + m_\varepsilon \varepsilon + \gamma_R, \quad w^{-1}\gamma_0 = \sum_{\alpha \in D_0} m_\alpha^w \alpha,$$

where again all coefficients are integers of the same sign. Note also that if the ε-coefficient vanishes (i.e., $n_\varepsilon=0$ or $m_\varepsilon=0$), then the partial sums over D_0 and R (i.e., β_0, $w^{-1}\beta_0$, and β_R or γ_0, $w^{-1}\gamma_0$, and γ_R) must also belong to Δ, since the subdiagrams are orthogonal. But the statement that w defines an admissible Weyl group transformation on D means that

$$\beta\in(w\hat{\Delta})^+, \quad \gamma\in w\Delta' \Rightarrow \beta+\gamma\in(w\hat{\Delta})^+ .$$

Now, for $\beta\in(w\hat{\Delta})^+$ and $\gamma\in w\Delta'$, we must have $n_\varepsilon\geqq 0$, $m_\varepsilon=0$, and $n_\alpha\geqq 0$ for all $\alpha\in R$. Hence for $n_\varepsilon>0$, it is automatic that $\beta+\gamma\in(w\hat{\Delta})^+$, while for $n_\varepsilon=0$, the proof follows from the requirement that w defines an admissible Weyl group transformation on D_0, i.e., that

$$\beta_0\in(w\hat{\Delta}_0)^+, \quad \gamma_0\in w\Delta_0' \Rightarrow \beta_0+\gamma_0\in(w\hat{\Delta}_0)^+ .$$

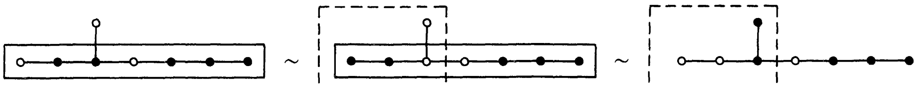
Indeed, this is clear if $n_\alpha=0$ for all $\alpha\in R$, and if $n_\alpha>0$ for some $\alpha\in R$, it follows that $w^{-1}\gamma\in\Delta'$ and $w^{-1}\beta\in\hat{\Delta}^+$, implying $w^{-1}\beta+w^{-1}\gamma\in\hat{\Delta}^+$, so $n_\alpha+m_\alpha\geqq 0$ for all $\alpha\in R$. But then either $n_\alpha+m_\alpha>0$ for some $\alpha\in R$, forcing $\beta+\gamma\in(w\hat{\Delta})^+$, or $n_\alpha+m_\alpha=0$ for all $\alpha\in R$, which gives $\beta_R+\gamma_R=0$ and $\beta+\gamma=\beta_0+\gamma_0$ with $\beta_0\in(w\hat{\Delta}_0)^+$, $\gamma_0\in w\Delta_0'$, hence $\beta_0+\gamma_0\in(w\hat{\Delta}_0)^+$, q.e.d..

We are left with the task of showing that w leaves the connecting root ε white. But since $\hat{D}_0\cup\{\varepsilon\}$ is a linearly independent system of linear forms on the centre $\hat{\mathfrak{t}}$ of $\mathfrak{k}$, one can find a vector Z_ε in $\hat{\mathfrak{t}}$ such that all roots in $\hat{D}_0$ (i.e., all white roots in D_0) take integer values on Z_ε, while $\varepsilon(Z_\varepsilon)$ is an irrational number. But then

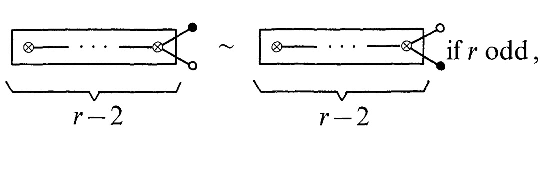
$$w^{-1}\varepsilon=\varepsilon+\varepsilon_w'+\hat{\varepsilon}_w ,$$

where ε_w' and $\hat{\varepsilon}_w$ are linear combinations of the roots in D_0' and $\hat{D}_0$ (the black and white roots in D_0), respectively, with integer coefficients. Here, $\varepsilon_w'(Z_\varepsilon)=0$, $\hat{\varepsilon}_w(Z_\varepsilon)$ is an integer, and $\varepsilon(Z_\varepsilon)$ is irrational, so $(w^{-1}\varepsilon)(Z_\varepsilon)\neq 0$ and hence $w^{-1}\varepsilon\in\hat{\Delta}$, which means that after repainting, $\varepsilon\in\hat{B}_w=B\cap w\hat{\Delta}$ is still white. This completes the proof.

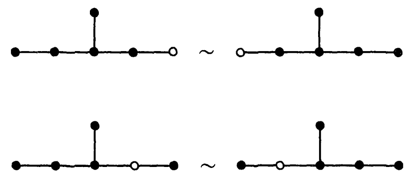
As it turns out, the lemma that we have just proved is the basic technical tool for establishing the equivalence of differently painted Dynkin diagrams. The main application is, of course, to suitably shift around connected blocks of black roots inside $\mathrm{su}(N)$-subdiagrams (for which the admissible Weyl group transformations are, as we have seen before, just the block permutations); we show an example for $\mathfrak{g}=\mathfrak{e}_8$ and $\mathfrak{k}=\mathrm{u}(1)\oplus\mathrm{u}(1)\oplus\mathrm{u}(1)\oplus\mathfrak{k}'$ with $\mathfrak{k}'=\mathrm{su}(3)\oplus\mathrm{su}(4)$ [as before, $\mathfrak{k}'$ denotes the semisimple part of the stability algebra $\mathfrak{k}$; cf. (3.26)]:



Two other types of equivalences, which are not covered by this strategy and which play a rôle in the classification of painted Dynkin diagrams for the exceptional groups E_6, E_7, E_8, are diagrammatically depicted as follows:

D_r: 

(5.11)

E_6:  (5.12)

(Here, ⊗ denotes a root that may be either black or white.) The proof is based on explicit Weyl group calculations, of which we shall only give a brief indication for the first case. There, $G = \mathrm{SO}(N)$ with $N = 2r$, so the Lie algebra $\mathfrak{g}$ consists of all antisymmetric real $(N \times N)$-matrices, and the maximal Abelian subalgebra $\mathfrak{t}$ will be chosen to consist of all matrices

$$D(\lambda_1, \ldots, \lambda_r) = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_r \\ & & & -\lambda_r & 0 \end{pmatrix} \tag{5.13}$$

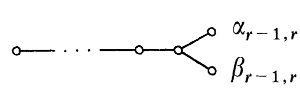
with $\lambda_1, \ldots, \lambda_r$ real. The corresponding root system is

$$\Delta = \{\pm \alpha_{i,j} | 1 \leqq i, j \leqq r\} \cup \{\pm \beta_{i,j} | 1 \leqq i < j \leqq r\}$$

with

$$\begin{aligned} \alpha_{i,j}(D(\lambda_1, \ldots, \lambda_r)) &= \lambda_i - \lambda_j\,, \\ \beta_{i,j}(D(\lambda_1, \ldots, \lambda_r)) &= \lambda_i + \lambda_j\,, \end{aligned} \tag{5.14}$$

and we choose the ordering where the $\alpha_{i,j}$ and $\beta_{i,j}$ $(1 \leqq i < j \leqq r)$ are positive. This gives the basis $B = \{\alpha_{i,i+1} | 1 \leqq i \leqq r-1\} \cup \{\beta_{r-1,r}\}$ as the corresponding set of simple roots, depicted by the ordinary Dynkin diagram



(5.15)

(cf. [5, pp. 187/188 and 464/465]). Now the generators $Z_0^{(1)}$ and $Z_0^{(2)}$ corresponding to the two painted Dynkin diagrams in (5.11) must have the form

$$Z_0^{(1)} = D(\lambda_1^{(1)}, \ldots, \lambda_{r-1}^{(1)}, \lambda_r^{(1)}) \quad \text{with} \quad \lambda_r^{(1)} = +\lambda_{r-1}^{(1)} \neq 0\,,$$

and

$$Z_0^{(2)} = D(\lambda_1^{(2)}, \ldots, \lambda_{r-1}^{(2)}, \lambda_r^{(2)}) \quad \text{with} \quad \lambda_r^{(2)} = -\lambda_{r-1}^{(2)} \neq 0\,,$$

respectively. On the other hand, the Weyl group of $G = \mathrm{SO}(N)$ is the semidirect product of the symmetric group of permutations σ of the set $\{1, \ldots, r\}$ with the multiplicative group of r-tuples $(\varepsilon_1 \ldots \varepsilon_r)$ of signs $\varepsilon_i = \pm$ satisfying $\varepsilon_1 \ldots \varepsilon_r = +$. More explicitly, for $w = (\sigma; \varepsilon_1 \ldots \varepsilon_r)$, w acts on $\mathfrak{t}$ according to the formula

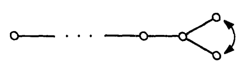
$$w^T(D(\lambda_1, \ldots, \lambda_r)) = D(\varepsilon_1 \lambda_{\sigma^{-1}(1)}, \ldots, \varepsilon_r \lambda_{\sigma^{-1}(r)})$$

(where T denotes transpose of linear maps). [In fact, the permutation group is generated by the reflections along the simple roots $\alpha_{1,2}, \ldots, \alpha_{r-1,r}$, while the product of the reflections along the two simple roots $\alpha_{r-1,r}$ and $\beta_{r-1,r}$ gives rise to the r-tuple $(+, \ldots, +, -, -)$.] But from all this, it is obvious that as long as r is odd, $w = (1; -, \ldots, -, +)$ belongs to the Weyl group of $\mathrm{SO}(N)$, with

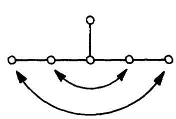
$$w\alpha_{i,i+1} = -\alpha_{i,i+1} \quad (1 \leqq i \leqq r-2), \qquad w\alpha_{r-1,r} = -\beta_{r-1,r}, \qquad w\beta_{r-1,r} = -\alpha_{r-1,r},$$

and yields the desired equivalence in (5.11).

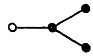
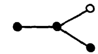
So far, we have made extensive use of Weyl group transformations, leaving aside automorphisms of the Dynkin diagrams themselves. The former, as we have seen, generate inner automorphisms of $\mathfrak{g}$ and leave the orbit $M = G/K$ invariant, while the latter generate outer automorphisms of $\mathfrak{g}$ and transform the orbit $M = G/K$ in a non-trivial way. This, however, is not really a problem, because two homogeneous spaces G/K_1 and G/K_2 with stability groups K_1 and K_2 related to one another by an outer automorphism of G could still be viewed as essentially identical; in particular, they are diffeomorphic as Kähler manifolds, i.e., both isometrically and holomorphically. And indeed, the reason why we use the complicated equivalences (5.11) and (5.12), rather than the standard automorphisms

D_r:  (5.16)

and

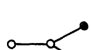
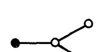
E_6:  (5.17)

of (ordinary) Dynkin diagrams, is a different one: it is that "outer equivalences" between differently painted Dynkin diagrams, which originate from the application of an automorphism to the common underlying ordinary Dynkin diagram, cannot always be extended from smaller diagrams to larger ones. As an illustration, consider once again the D-series. For $r = 4$ ($\mathfrak{g} = \mathrm{so}(8)$), the following painted Dynkin diagrams are equivalent under an outer automorphism:

 and 

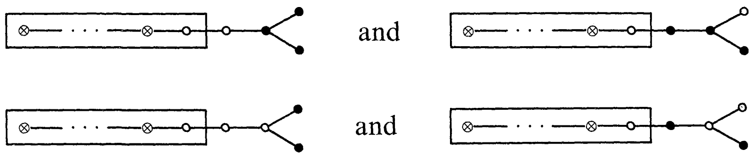
$(\mathfrak{k} = \mathrm{u}(1) \oplus \mathfrak{k}', \ \mathfrak{k}' = \mathrm{so}(6) = \mathrm{su}(4))$

(5.18)

 and 

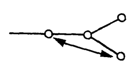
$(\mathfrak{k} = \mathrm{u}(1) \oplus \mathrm{u}(1) \oplus \mathfrak{k}', \ \mathfrak{k}' = \mathrm{so}(4) = \mathrm{su}(2) \oplus \mathrm{su}(2))$

For $r>4$ [$\mathfrak{g}=\mathrm{so}(N)$ with $N=2r>8$], however, the painted Dynkin diagrams obtained by attaching the same chain (ending in a white root) on the left-hand side, i.e.,



(5.19)

(where as before, ⊗ denotes a root that may be either black or white) are no longer equivalent, not even under an outer automorphism, simply because the automorphism of the root system for so(8) induced by the automorphism



of the ordinary Dynkin diagram for so(8) cannot be extended to an automorphism of the root system for so(N) with $N=2r>8$.

Applying the previously described techniques to classify the possible painted Dynkin diagrams, up to equivalence, we arrive at the results assembled in Tables 3 and 4, referring to the classical groups and to the exceptional groups, respectively. In some cases (A-Series, D-Series for $r_0 \geqq 4$, E_6, E_8), these diagrams are determined, uniquely up to equivalence, by the structure of the stability algebra $\mathfrak{k}$ alone. However, this circumstance can by no means be considered as a rule. In fact, it cannot possibly be a general statement for diagrams containing double or triple links (B-Series, C-Series, F_4, G_2), simply because stating that the semisimple part $\mathfrak{k}'$ of the stability algebra $\mathfrak{k}$ is an su(2)-subalgebra, say, does not convey any information on whether this subalgebra is generated by a long root or by a short root. Moreover, there are exceptions even when all roots have the same length.

Table 1. Hermitian symmetric spaces [5, pp. 518 and 531]

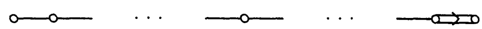
Series	Compact type	Non-compact type
A III	$\mathrm{SU}(N_1+N_2)/\mathrm{S}(\mathrm{U}(N_1)\times\mathrm{U}(N_2))$	$\mathrm{SU}(N_1,N_2)/\mathrm{S}(\mathrm{U}(N_1)\times\mathrm{U}(N_2))$
B I	$\mathrm{SO}(N)/\mathrm{SO}(N-2)\times\mathrm{SO}(2)$ N odd	$\mathrm{SO}(N-2,2)/\mathrm{SO}(N-2)\times\mathrm{SO}(2)$ N odd
C I	$\mathrm{Sp}(N)/\mathrm{U}(N)$	$\mathrm{Sp}(N,\mathbb{R})/\mathrm{U}(N)$
D I	$\mathrm{SO}(N)/\mathrm{SO}(N-2)\times\mathrm{SO}(2)$ N even	$\mathrm{SO}(N-2,2)/\mathrm{SO}(N-2)\times\mathrm{SO}(2)$ N even
D III	$\mathrm{SO}(2N)/\mathrm{U}(N)$	$\mathrm{SO}^*(2N)/\mathrm{U}(N)$
E III	$E_6/\mathrm{U}(1)\times\mathrm{SO}(10)$	$\tilde{E}_6/\mathrm{U}(1)\times\mathrm{SO}(10)$
E VII	$E_7/\mathrm{U}(1)\times E_6$	$\tilde{E}_7/\mathrm{U}(1)\times E_6$

Table 2. Dynkin diagrams, with coefficients of 2δ. (The numbers on each simple root indicate the coefficient with which it contributes to the vector 2δ, the sum of all positive roots, and $r = \operatorname{rank} \mathfrak{g}$)

A-Series: $\mathfrak{g} = \mathrm{su}(N)$ with $N = r+1$

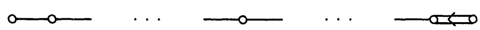
$1 \cdot r, 2(r-1), \quad \ldots, \quad k(r-k+1), \quad \ldots, \quad (r-1)2, r \cdot 1$

B-Series: $\mathfrak{g} = \mathrm{so}(N)$ with $N = 2r+1$ odd



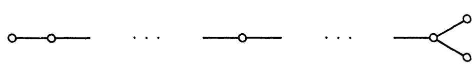
$1 \cdot (2r \cdot 1), 2(2r-2), \quad \ldots, \quad k(2r-k), \quad \ldots, \quad (r-1)(r+1), r^2$

C-Series: $\mathfrak{g} = \mathrm{sp}(N)$ with $N = r$



$1 \cdot 2r, 2(2r-1), \quad \ldots, \quad k(2r-k+1), \quad \ldots, \quad (r-1)(r+2), r(r+1)/2$

D-Series: $\mathfrak{g} = \mathrm{so}(N)$ with $N = 2r$ even



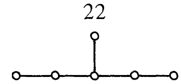
$(r-1)r/2$

$(r-1)r/2$

$1 \cdot (2r-2), 2(2r-3), \quad \ldots, \quad k(2r-k-1), \quad \ldots, \quad (r-2)(r+1)$

E_6:

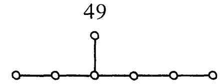
22



16 30 42 30 16

E_7:

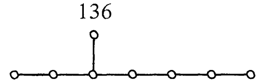
49



34 66 96 75 52 27

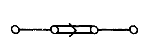
E_8:

136



92 182 270 220 168 114 58

F_4:



16 30 42 22

G_2:

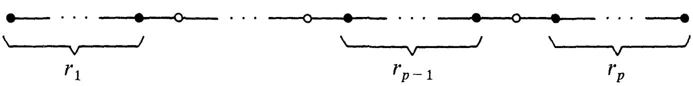
6 10

Table 3. Painted Dynkin diagrams for the classical groups

Conventions. $p, r, r_0, r_1, \ldots, r_p$ are integers $\geqq 0$, with $r = \operatorname{rank} \mathfrak{g}$ and $r_1 \leqq \ldots \leqq r_p$. As in the text, $\mathfrak{k}'$ denotes the semisimple part of the stability algebra $\mathfrak{k}$, and $\mathrm{u}(1) \oplus \ldots \oplus \mathrm{u}(1)$ is its centre, with dimension equal to the number of white roots

---

A-Series: $\mathfrak{g} = \mathrm{su}(N)$ with $N = r+1$



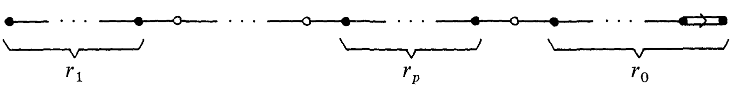
$\sum_{i=1}^{p} r_i + (p-1) = r$

($p-1$ white roots)

$\mathfrak{k}' = \mathrm{su}(N_1) \oplus \ldots \oplus \mathrm{su}(N_p)$ with $N_i = r_i + 1$.

Remark. G/K is Hermitian symmetric iff $p = 2$ (A III):

B-Series: $\mathfrak{g} = \mathrm{so}(N)$ with $N = 2r+1$ odd



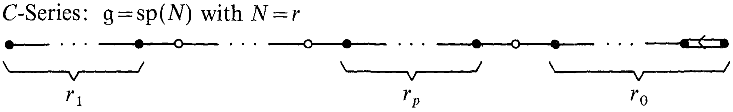
$r_0 + \sum_{i=1}^{p} r_i + p = r$

(p white roots)

$\mathfrak{k}' = \mathrm{su}(N_1) \oplus \ldots \oplus \mathrm{su}(N_p) \oplus \mathrm{so}(N_0)$ with $N_i = r_i + 1$, $N_0 = 2r_0 + 1$.

Remark. G/K is Hermitian symmetric iff $p = 1$, $N_1 = 1$, $N_0 = N-2$ (B I):

C-Series: $\mathfrak{g} = \mathrm{sp}(N)$ with $N = r$

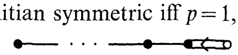


$r_0 + \sum_{i=1}^{p} r_i + p = r$

(p white roots)

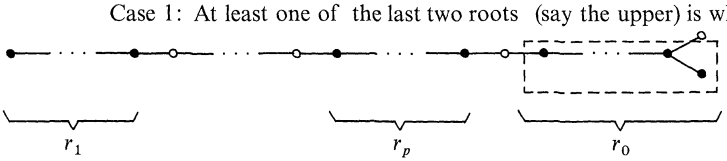
$\mathfrak{k}' = \mathrm{su}(N_1) \oplus \ldots \oplus \mathrm{su}(N_p) \oplus \mathrm{sp}(N_0)$ with $N_i = r_i + 1$, $N_0 = r_0$.

Remark. G/K is Hermitian symmetric iff $p = 1$, $N_1 = N$, $N_0 = 0$ (C I):



D-Series: $\mathfrak{g} = \mathrm{so}(N)$ with $N = 2r$ even

Case 1: At least one of the last two roots (say the upper) is white.

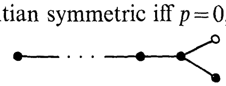


$r_0 + \sum_{i=1}^{p} r_i + (p+1) = r$

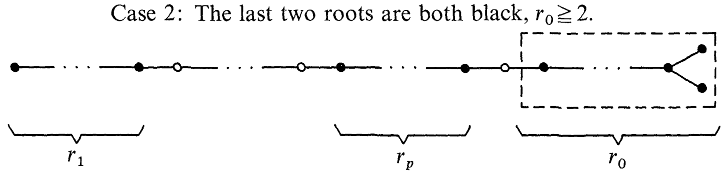
($p+1$ white roots)

$\mathfrak{k}' = \mathrm{su}(N_1) \oplus \ldots \oplus \mathrm{su}(N_p) \oplus \mathrm{su}(N_0)$ with $N_i = r_i + 1$, $N_0 = r_0 + 1$.

Remark. G/K is Hermitian symmetric iff $p = 0$, $N_0 = N/2$ (D III):



Case 2: The last two roots are both black, $r_0 \geqq 2$.

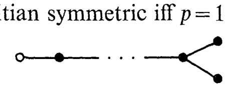


$r_0 + \sum_{i=1}^{p} r_i + p = r$

(p white roots)

$\mathfrak{k}' = \mathrm{su}(N_1) \oplus \ldots \oplus \mathrm{su}(N_p) \oplus \mathrm{so}(N_0)$ with $N_i = r_i + 1$, $N_0 = 2r_0$.

Remark. G/K is Hermitian symmetric iff $p = 1$, $N_1 = 1$, $N_0 = N-2$ (D I):

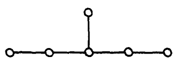


Note that blocks of black roots having zero length are explicitly allowed: the neighbouring white roots are then directly connected

Table 4. Painted Dynkin diagrams for the exceptional groups. Beneath each diagram, the corresponding semisimple part $\mathfrak{k}'$ of the stability algebra $\mathfrak{k}$ is given. The diagrams are ordered according to the number $r' = \operatorname{rank} \mathfrak{k}'$ of black roots, and $\mathfrak{k} = \mathrm{u}(1) \oplus \ldots \oplus \mathrm{u}(1) \oplus \mathfrak{k}'$ with $\hat{r}$ copies of $\mathrm{u}(1)$, where $\hat{r} = \operatorname{rank} \mathfrak{g} - \operatorname{rank} \mathfrak{k}'$ is the number of white roots

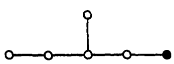
E_6:

$r' = 0$:



$\{0\}$

$r' = 1$:

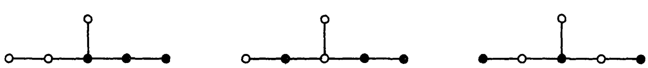


su(2)

$r' = 2$:

su(3)   su(2) ⊕ su(2)

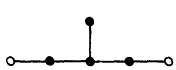
$r' = 3$:



su(4)   su(2) ⊕ su(3)   su(2) ⊕ su(2) ⊕ su(2)

$r' = 4$:

su(5)   su(2) ⊕ su(4)   su(3) ⊕ su(3)   su(2) ⊕ su(2) ⊕ su(3)



so(8)

$r' = 5$:

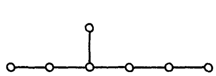
su(6)   so(10)   su(2) ⊕ su(5)   su(2) ⊕ su(3) ⊕ su(3)

↓ (from so(10))

Hermitian symmetric
(EIII)

E_7:

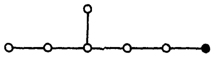
$r' = 0$:



$\{0\}$

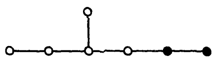
Table 4 (continued)

$r' = 1$:

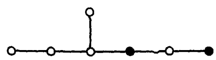


su(2)

$r' = 2$:



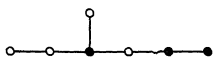
su(3)



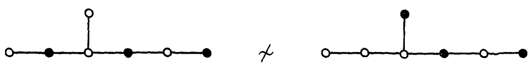
su(2) ⊕ su(2)

$r' = 3$:

su(4)

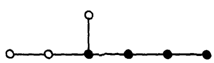


su(2) ⊕ su(3)

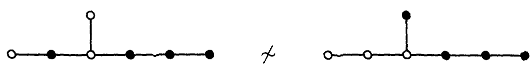


su(2) ⊕ su(2) ⊕ su(2)

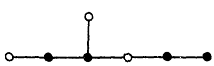
$r' = 4$:



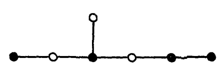
su(5)



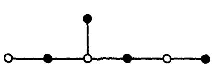
su(2) ⊕ su(4)



su(3) ⊕ su(3)



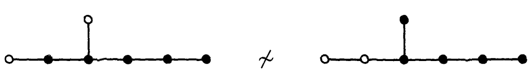
su(2) ⊕ su(2) ⊕ su(3)



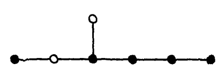
su(2) ⊕ su(2) ⊕ su(2) ⊕ su(2)

so(8)

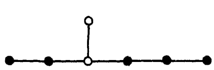
$r' = 5$:



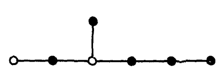
su(6)



su(2) ⊕ su(5)

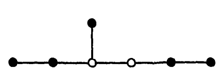


su(3) ⊕ su(4)

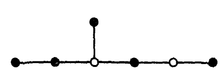


su(2) ⊕ su(2) ⊕ su(4)

su(2) ⊕ so(8)



su(2) ⊕ su(3) ⊕ su(3)

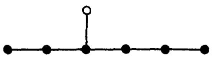


su(2) ⊕ su(2) ⊕ su(2) ⊕ su(3)

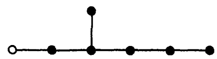
so(10)

Table 4 (continued)

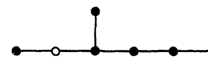
$r'=6$:



su(7)

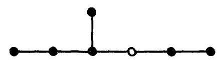


so(12)

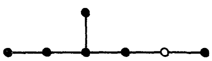


su(2) ⊕ su(6)

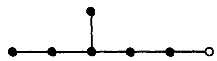
su(2) ⊕ su(3) ⊕ su(4)



su(3) ⊕ su(5)



su(2) ⊕ so(10)

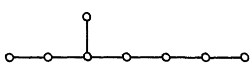


$\mathfrak{e}_6$

Hermitian symmetric
(EVII)

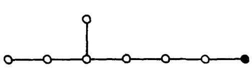
E_8:

$r'=0$:



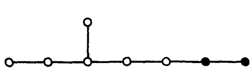
{0}

$r'=1$:

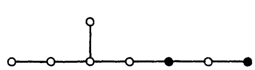


su(2)

$r'=2$:

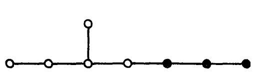


su(3)

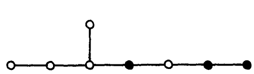


su(2) ⊕ su(2)

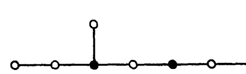
$r'=3$:



su(4)



su(2) ⊕ su(3)

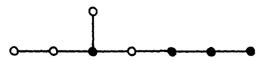


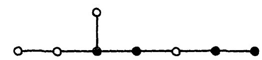
su(2) ⊕ su(2) ⊕ su(2)

Table 4 (continued)

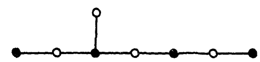
$r'=4$:


su(5)


su(2) ⊕ su(4)


su(3) ⊕ su(3)

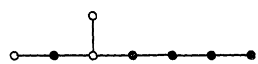

su(2) ⊕ su(2) ⊕ su(3)

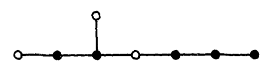

su(2) ⊕ su(2) ⊕ su(2) ⊕ su(2)

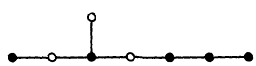

so(8)

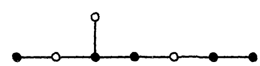
$r'=5$:

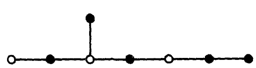

su(6)

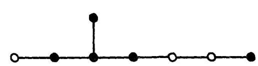

su(2) ⊕ su(5)

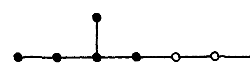

su(3) ⊕ su(4)


su(2) ⊕ su(2) ⊕ su(4)

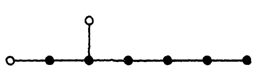

su(2) ⊕ su(3) ⊕ su(3)

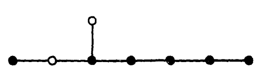

su(2) ⊕ su(2) ⊕ su(2) ⊕ su(3)

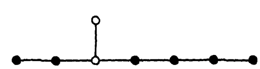

su(2) ⊕ so(8)

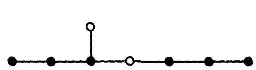

so(10)

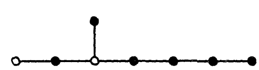
$r'=6$:


su(7)


su(2) ⊕ su(6)


su(3) ⊕ su(5)

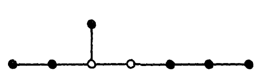

su(4) ⊕ su(4)

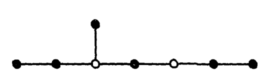

su(2) ⊕ su(2) ⊕ su(5)


su(3) ⊕ so(8)


su(2) ⊕ so(10)


so(12)


su(2) ⊕ su(3) ⊕ su(4)


su(2) ⊕ su(2) ⊕ su(3) ⊕ su(3)

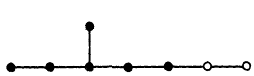
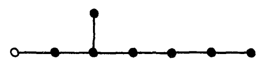

$\mathfrak{e}_6$

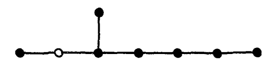
Table 4 (continued)

$r'=7$:

su(8)

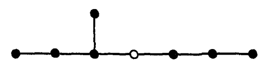


so(14)

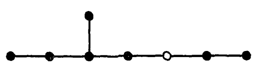


su(2) ⊕ su(7)

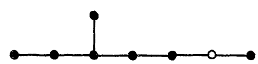
su(2) ⊕ su(3) ⊕ su(5)



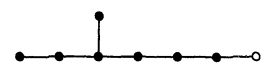
su(4) ⊕ su(5)



su(3) ⊕ so(10)



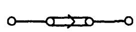
su(2) ⊕ e_6



e_7

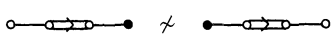
F_4:

$r'=0$:



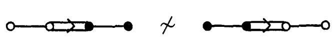
{0}

$r'=1$:

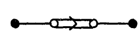


su(2)

$r'=2$:



su(3)

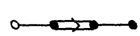


su(2) ⊕ su(2)

so(5) ≅ sp(2)

$r'=3$:

su(2) ⊕ su(3)



sp(3)

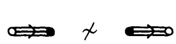
so(7)

G_2:

$r'=0$:

{0}

$r'=1$:



su(2)

Table 5. Matrix form of 2δ and $2\hat{\delta}$ for the classical series A, B, D

A-Series: $\mathfrak{g}=\mathrm{su}(N)$ with $N=r+1$

$\mathfrak{k}=\mathrm{s}(\mathrm{u}(N_1)\oplus\ldots\oplus\mathrm{u}(N_p))$ with $\sum_{j=1}^{p} N_j=N$ (cf. Table 3).

In terms of the standard Cartan subalgebra of $\mathrm{su}(N)$ consisting of all diagonal matrices, $2\delta=\mathrm{diag}(2\delta_1,\ldots,2\delta_N)$ and $2\hat{\delta}=\mathrm{diag}(2\hat{\delta}_1,\ldots,2\hat{\delta}_N)$, where for $1\leqq k\leqq N$

$$2\delta_k=i(N-2k+1)\,,$$

$$2\hat{\delta}_k=i\left(N-N_l-2\sum_{j=1}^{l-1} N_j\right) \quad \text{if} \quad \sum_{j=1}^{l-1} N_j<k\leqq \sum_{j=1}^{l} N_j, \qquad 1\leqq l\leqq p\,.$$

B-Series: $\mathfrak{g}=\mathrm{so}(N)$ with $N=2r+1$ odd

D-Series: $\mathfrak{g}=\mathrm{so}(N)$ with $N=2r$ even

$\mathfrak{k}=\mathrm{u}(N_1)\oplus\ldots\oplus\mathrm{u}(N_p)\oplus\mathrm{so}(N_0)$ with $N_0+2\sum_{j=1}^{p} N_j=N$ (cf. Table 3).

In terms of the standard Cartan subalgebra of $\mathrm{so}(N)$ consisting of all block diagonal matrices with antisymmetric (2×2)-blocks, of the form

$$D(\lambda_1,\ldots,\lambda_r)=\begin{pmatrix} 0 & \lambda_1 & & & & \\ -\lambda_1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & \lambda_r & \\ & & & -\lambda_r & 0 & \\ & & & & & 0 \end{pmatrix} \quad \text{if} \quad N=2r+1\,,$$

$$D(\lambda_1,\ldots,\lambda_r)=\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & \lambda_r \\ & & & -\lambda_r & 0 \end{pmatrix} \quad \text{if} \quad N=2r\,,$$

$2\delta=D(2\delta_1,\ldots,2\delta_r)$ and $2\hat{\delta}=D(2\hat{\delta}_1,\ldots,2\hat{\delta}_r)$, where for $1\leqq k\leqq r$,

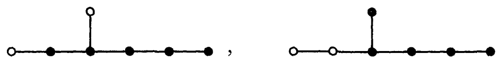
$$2\delta_k=N-2k\,,$$

$$2\hat{\delta}_k=\begin{cases} N-N_l-2\sum_{j=1}^{l} N_j-1 & \text{if} \quad \sum_{j=1}^{l-1} N_j<k\leqq \sum_{j=1}^{l} N_j, \quad 1\leqq l\leqq p \\ 0 & \text{otherwise} \end{cases}\,.$$

One of these occurs for the D-Series with $r_0=3$ or $r_0=2$, where the diagrams given in Table 3 under Case 1 and Case 2 have the same stability algebra $\mathfrak{k}=\mathrm{u}(1)\oplus\ldots\oplus\mathrm{u}(1)\oplus\mathfrak{k}'$ ($p\,\mathrm{u}(1)$-summands), with

$$\mathfrak{k}'=\mathrm{su}(N_1)\oplus\ldots\oplus\mathrm{su}(N_p)\oplus\mathfrak{k}''\,, \quad \begin{array}{ll} \mathfrak{k}''=\mathrm{so}(6)=\mathrm{su}(4) & \text{for} \quad r_0=3\,, \\ \mathfrak{k}''=\mathrm{so}(4)=\mathrm{su}(2)\oplus\mathrm{su}(2) & \text{for} \quad r_0=2\,, \end{array} \tag{5.20}$$

but are inequivalent. The other exceptions occur for E_7, where there are three pairs of diagrams, also mentioned in Dynkin's list [15, p. 149] that have the same stability algebra, namely



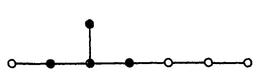
$\mathfrak{k} = u(1) \oplus u(1) \oplus \mathfrak{k}'$, $\mathfrak{k}' = su(6)$ (5.21)

$\mathfrak{k} = u(1) \oplus u(1) \oplus u(1) \oplus \mathfrak{k}'$, $\mathfrak{k}' = su(2) \oplus su(4)$ (5.22)

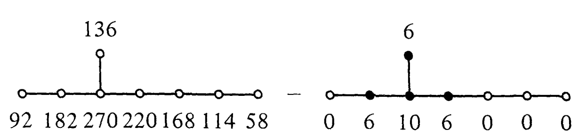
$\mathfrak{k} = u(1) \oplus u(1) \oplus u(1) \oplus u(1) \oplus \mathfrak{k}'$, $\mathfrak{k}' = su(2) \oplus su(2) \oplus su(2)$ (5.23)

but they are once again inequivalent. [This can be proved, e.g., as follows. First, for each of the two diagrams in (5.21) or (5.22) or (5.23), one writes the vectors in the centre of the corresponding stability algebra $\mathfrak{k}$ as explicit linear combinations (with the appropriate number, 2 or 3 or 4, of parameters) of the vectors iH_α (cf. (3.14)), where α runs through the simple roots for E_7. Using this, one can decide which of these vectors belong to the root system Δ_7 for E_7, and what is their number. Now if the two diagrams in question were equivalent, there would exist a (necessarily inner) automorphism of Δ_7 taking the centre of $\mathfrak{k}$ for the first diagram to the centre of $\mathfrak{k}$ for the second diagram, so these numbers would have to be the same for both diagrams. But in all these cases, it turns out that they are not.]

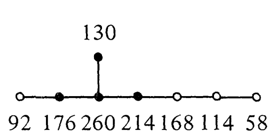
To conclude, we would like to point out that the vector $\hat{\delta}$ which, as explained in Sect. 4, gives rise to the Ricci tensor and, hence, the Einstein-Kähler metric on $M = G/K$, can easily be computed from Table 2 by subtracting the sum of positive roots for the semisimple part $\mathfrak{k}'$ of the stability algebra $\mathfrak{k}$ from the sum of positive roots for the symmetry algebra $\mathfrak{g}$. As an example, consider the painted Dynkin diagram for $\mathfrak{g} = \mathfrak{e}_8$, $\mathfrak{k}' = so(8)$:



Then take the difference



to obtain the coefficient of $2\hat{\delta}$ for this case:



In Table 5, $2\hat{\delta}$ is written down in matrix form for the classical Lie algebras su(N) and so(N), using the standard simple roots [5, pp. 462–465].

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