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Integrability of Two Interacting N-Dimensional Rigid Bodies

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Abstract. A new class of integrable Euler equations on the Lie algebra so$(2n)$ describing two n-dimensional interacting rigid bodies is found. A Lax representation of equations of motion which depends on a spectral parameter is given and complete integrability is proved. The double hamiltonian structure and the Lax representation of the general flow is discussed.

1. Introduction

The Euler equations on the SO(n) Lie group, which describe the rotation of a free n-dimensional rigid body about a fixed point, have the following set of n quadratic, mutually commuting, integrals of motion

$$K_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{\ell_{ij}^2}{\alpha_i - \alpha_j} \quad (i = 1, \ldots, n), \tag{1.1}$$

where ℓ_{ij} are the angular momentum dynamical variables and α_j, $j = 1, \ldots, n$ are real parameters. Integrals of the form (1.1) have been for the first time considered by Uhlenbeck (see [1]) for the motion of a mass point on a unit sphere under the influence of a harmonic potential. But they play a special role in the motion of an n-dimensional rigid body, since the Manakov [2] integrable system corresponds to the hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n}{}' \beta_i K_i = \frac{1}{2} \sum_{i<j} \frac{\beta_i - \beta_j}{\alpha_i - \alpha_j} \ell_{ij}^2, \tag{1.2}$$

where β_j are real parameters and the summation is taken over all pairs $i<j$.

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For two n-dimensional interacting rigid bodies described by dynamical variables ℓ_{ij}, m_{ij}, $i,j=1,\ldots,n$, little is known about integrable cases. Here we consider the following extension of the Uhlenbeck integrals (1.1),

$$K_i=\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(\ell_{ij}+m_{ij})^2}{\alpha_i-\alpha_j}+\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(\ell_{ij}-m_{ij})^2}{\alpha_i+\alpha_j}, \tag{1.3}$$

which are also quadratic in dynamical variables and mutually commute, as may be shown by direct calculations. Therefore the hamiltonian

$$\begin{aligned} H&=\frac{1}{2}\sum_{i=1}^{n}\beta_i K_i\\ &=\frac{1}{2}\sum_{i<j}{}'\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}(\ell_{ij}+m_{ij})^2+\frac{1}{2}\sum_{i<j}{}'\frac{\beta_i+\beta_j}{\alpha_i+\alpha_j}(\ell_{ij}-m_{ij})^2 \end{aligned} \tag{1.4}$$

becomes a natural candidate for an integrable case of two interacting rigid bodies.

Indeed, for the Hamilton's equations of motion generated by (1.4), we have found a Lax representation which depends on a spectral parameter λ and reduces to the Manakov [2] representation when $\ell_{ij}=m_{ij}$. It turns out that the eigenvalues of the Lax matrix $L(\lambda)$ corresponding to different values of the spectral parameter commute, which implies that all integrals of motion derived from the Lax matrix commute, too.

Actually, there are more than necessary integrals of motion coming from $\mathrm{Tr}(L(\lambda))^{2k}$, $k=1,\ldots,n$ and, because of commutativity, not all of them can be functionally independent; however, among them there are $\frac{1}{2}n(n-1)-\left[\frac{n}{2}\right]$ functionally independent expressions ($[x]$ is the integer part of x) so that the hamiltonian (1.4) is completely integrable.

In Sect. 2 we construct a Lax representation of the flow generated by (1.4) and explain the relationship between the quadratic integrals of motion (1.3) and the eigenvalues of the Lax matrix $L(\lambda)$. Section 3 is concerned with the proof of complete integrability of the hamiltonian (1.4) on the symplectic manifold defined by constant values of all Casimir functions constructed from ℓ_{ij} and m_{ij}. In Sect. 4 we give a Lax representation of the higher flows generated by $\mathrm{Tr}(L(\lambda))^{2k}$, and exhibit a double hamiltonian structure of this hierarchy of flows which helps to prove, in a simpler way, the commutativity of all integrals of motion associated with $L(\lambda)$. Finally, in Sect. 5 some open questions are briefly discussed.

2. Lax Representation

2.1. The Hamilton's equations of motion generated by (1.4) have the form

$$\dot{\ell}_{ij}=\{\ell_{ij},H\}, \qquad \dot{m}_{ij}=\{m_{ij},H\}, \tag{2.1}$$

where dot denotes time derivative and $\{\cdot,\cdot\}$ is the Poisson bracket defined by the commutation relations

$$\{\ell_{pq},\ell_{rs}\}=\delta_{ps}\ell_{rq}+\delta_{pr}\ell_{qs}+\delta_{qs}\ell_{pr}+\delta_{qr}\ell_{sp}, \tag{2.2}$$

$$\{m_{pq},m_{rs}\}=\delta_{ps}m_{rq}+\delta_{pr}m_{qs}+\delta_{qs}m_{pr}+\delta_{qr}m_{sp}, \tag{2.3}$$

of the generators of two $\mathrm{so}(n)$ Lie algebras. Such a Poisson structure is degenerated, and in order to investigate integrability of (1.4), we have to restrict it to an invariant manifold defined by constant values of Casimir functions of $\mathrm{so}(n)$ (both for ℓ_{ij} and m_{ij}) on which $\{\cdot,\cdot\}$ is not degenerated. But in order to find a Lax representation it is, for a while, more convenient to work with the Poisson bracket defined by (2.2) and (2.3).

The right-hand side of (2.1) is quadratic in ℓ_{ij}, m_{ij}, and for such systems both matrices in a Lax representation are usually linear in dynamical variables. The most natural building blocks are the $n\times n$ matrices ℓ_{ij}, m_{ij}, and it takes a bit of experimenting to find the correct ansatz for $2n\times 2n$ matrices,

$$L(\lambda)=\left(\begin{array}{c|c}\ell & i\lambda a\\ \hline i\lambda a & m\end{array}\right), \tag{2.4}$$

$$M(\lambda)=\left(\begin{array}{c|c}\tilde{\ell} & i\lambda b\\ \hline i\lambda b & \tilde{m}\end{array}\right), \tag{2.5}$$

in order to make Eqs. (2.1) equivalent to

$$\dot{L}=[M,L], \tag{2.6}$$

where $[\cdot,\cdot]$ denotes the usual matrix commutator.

The ingredients of (2.4), (2.5) are defined as $(\ell)_{ij}=\ell_{ij}$, $(m)_{ij}=m_{ij}$, $(\tilde{\ell})_{ij}=a_{ij}\ell_{ij}+b_{ij}m_{ij}$, $(\tilde{m})_{ij}=c_{ij}\ell_{ij}+d_{ij}m_{ij}$, $a=\mathrm{diag}(\alpha_1,\ldots,\alpha_n)$, $b=\mathrm{diag}(\beta_1,\ldots,\beta_n)$. From comparison of (2.6) with (2.1) it follows that

$$a_{ij}=d_{ij}=\frac{\alpha_i\beta_i-\alpha_j\beta_j}{\alpha_i^2-\alpha_j^2};\qquad b_{ij}=c_{ij}=\frac{\alpha_j\beta_i-\alpha_i\beta_j}{\alpha_i^2-\alpha_j^2}.$$

By performing a similarity transformation $S^{-1}LS$, $S^{-1}MS$ with $S=\left(\begin{array}{c|c}I & I\\ \hline I & -I\end{array}\right)$, we get another representation for Lax matrices, namely:

$$L'(\lambda)=\frac{1}{2}\left(\begin{array}{c|c}\ell+m & \ell-m\\ \hline \ell-m & \ell+m\end{array}\right)+i\lambda\left(\begin{array}{c|c}a & 0\\ \hline 0 & -a\end{array}\right):=L_0+i\lambda A, \tag{2.7}$$

$$M'(\lambda)=\frac{1}{2}\left(\begin{array}{c|c}\tilde{\ell}+\tilde{m} & \tilde{\ell}-\tilde{m}\\ \hline \tilde{\ell}-\tilde{m} & \tilde{\ell}+\tilde{m}\end{array}\right)+i\lambda\left(\begin{array}{c|c}b & 0\\ \hline 0 & -b\end{array}\right):=M_0+i\lambda B, \tag{2.8}$$

with a more transparent dependence on the spectral parameter. We will assume that λ is real to make $L(\lambda)$ skew-adjoint. Note that

$$(\tilde{\ell}+\tilde{m})_{ij}=\frac{\beta_i-\beta_j}{\alpha_i-\alpha_j}(\ell+m)_{ij};\qquad(\tilde{\ell}-\tilde{m})_{ij}=\frac{\beta_i+\beta_j}{\alpha_i+\alpha_j}(\ell-m)_{ij}, \tag{2.9}$$

and $H=-(1/2)\,\mathrm{tr}\,L_0M_0$.

2.2. To this representation we can apply immediately Dubrovin's theorem [3], which states that any equation of the type $[A,\dot{V}]=[[B,V],[A,V]]$, where V is an arbitrary matrix with zero diagonal elements and A,B are arbitrary diagonal matrices, is solvable in terms of Riemann θ-functions: namely we can take

$V=\left(\begin{array}{c|c} u & v \\ \hline -v & -u \end{array}\right)$, where $u_{ij}=(\ell-m)_{ij}/[2(\alpha_i-\alpha_j)]$ for $i\neq j$, $u_{jj}=0$ for $j=1,\ldots,n$ and $v_{ij}=(\ell+m)_{ij}/[2(\alpha_i+\alpha_j)]$. So the system (1.4) is, in principle, integrable but the formal verification of the conditions of Liouville's theorem [4] is more involving. We will do this in the next section.

2.3. We show now the relationship between the starting integrals (1.3) and the eigenvalues of the Lax matrix (2.7) which, by (2.6), are integrals of motion, too. For this purpose we assume for the moment that our dynamical variables are defined as

$$\ell_{ij}:=q_ip_j-q_jp_i; \qquad m_{ij}:=\bar{q}_i\bar{p}_j-\bar{q}_j\bar{p}_i,$$

where of (q_i,p_i) and $(\bar{q}_i,\bar{p}_i)$, $i=1,\ldots,n$, we can think of as a set of $2n$ pairs of canonically conjugate variables. However, we shall not make use of the standard Poisson bracket relations $\{q_i,p_j\}=\delta_{ij}$ and $\{\bar{q}_i,\bar{p}_j\}=\delta_{ij}$.

In terms of (q_i,p_i), $(\bar{q}_i,\bar{p}_i)$ we can write the Lax matrix (2.7) as a perturbation of rank 4,

$$L'(\lambda)=i\lambda A+Q\otimes P-P\otimes Q+\bar{Q}\otimes\bar{P}-\bar{P}\otimes\bar{Q}, \tag{2.10}$$

where

$$Q=2^{-1/2}(q_1,\ldots,q_n;\,q_1,\ldots,q_n), \qquad P=2^{-1/2}(p_1,\ldots,p_n;\,p_1,\ldots,p_n),$$
$$\bar{Q}=2^{-1/2}(\bar{q}_1,\ldots,\bar{q}_n;\,-\bar{q}_1,\ldots,-\bar{q}_n), \qquad \bar{P}=2^{-1/2}(\bar{p}_1,\ldots,\bar{p}_n;\,-\bar{p}_1,\ldots,-\bar{p}_n).$$

Then to the matrix (2.10) we can apply a Weinstein-Aronszajn formula [5] which says that

$$\frac{\det(Iz-L'(\lambda))}{\det(Iz-i\lambda A)}=\det(I-W_z)$$

if $L'(\lambda)$ has the general form $L'(\lambda)=i\lambda A+\sum_k x_k\otimes y_k$, and W_z is the matrix defined as $(W_z)_{kl}=((Iz-i\lambda A)^{-1}x_k,y_\ell)$. I denotes the identity matrix and $(\cdot,\cdot)$ is a real scalar product. For $L(\lambda)$ given by (2.10) we may read off vectors x_k, y_k to find the 4×4 matrix

$$I-W_z$$
$$=\begin{bmatrix} 1-2\sum_j t_jp_jq_j; & -2\sum_j t_jq_j^2; & -i\lambda\sum_j \alpha_jt_j\bar{p}_jq_j; & -i\lambda\sum_j \alpha_jt_jq_j\bar{p}_j \\ 2\sum_j t_jp_j^2; & 1+2\sum_j t_jp_jq_j; & i\lambda\sum_j \alpha_jt_j\bar{p}_jp_j; & i\lambda\sum_j \alpha_jt_jp_j\bar{q}_j \\ -i\lambda\sum_j \alpha_jt_jp_j\bar{q}_j; & -i\lambda\sum_j \alpha_jt_j\bar{q}_jq_j; & 1-2\sum_j t_j\bar{q}_j\bar{p}_j; & -2\sum_j t_j\bar{q}_j^2 \\ i\lambda\sum_j \alpha_jt_jp_j\bar{p}_j; & i\lambda\sum_j \alpha_jt_j\bar{p}_jq_j; & 2\sum_j t_j\bar{p}_j^2; & 1+2\sum_j t_j\bar{p}_j\bar{q}_j \end{bmatrix},$$

where $t_j=(z^2+\lambda^2\alpha_j^2)^{-1}$. The calculation of $\det(I-W_z)$ is somewhat involving, but at the end we obtain

$$\begin{aligned}
&\frac{\det(I_z-L'(\lambda))}{\det(I_z-i\lambda A)}\\
&=\left(1-\frac{z^2}{2}\sum_j\sum_{k\neq j}\frac{\ell_{jk}^2}{(z^2+\lambda^2\alpha_j^2)(z^2+\lambda^2\alpha_k^2)}\right)\left(1-\frac{z^2}{2}\sum_j\sum_{k\neq j}\frac{m_{jk}^2}{(z^2+\lambda^2\alpha_j^2)(z^2+\lambda^2\alpha_k^2)}\right)\\
&\quad+\lambda^2\sum_j\sum_{k\neq j}\frac{\alpha_j\alpha_k\ell_{jk}m_{jk}}{(z^2+\lambda^2\alpha_j^2)(z^2+\lambda^2\alpha_k^2)}+\lambda^4\left(\sum_j\sum_{k\neq j}\frac{\alpha_j\alpha_k\ell_{jk}m_{jk}}{(z^2+\lambda^2\alpha_j^2)(z^2+\lambda^2\alpha_k^2)}\right)^2\\
&\quad+z^2\lambda^2\sum_j\sum_k\sum_{r\neq j,k}\sum_{s\neq j,k}\frac{\alpha_j\alpha_k\ell_{jr}\ell_{kr}m_{js}m_{ks}}{(z^2+\lambda^2\alpha_j^2)(z^2+\lambda^2\alpha_k^2)(z^2+\lambda^2\alpha_r^2)(z^2+\lambda^2\alpha_s^2)},
\end{aligned}\tag{2.11}$$

which is completely expressed in terms of the dynamical variables ℓ_{ij}, m_i^{jj}. Therefore, we can forget about variables (q_i,p_i), $(\bar q_i,\bar p_i)$, which were used here as a vehicle to obtain formula (2.11), and consider just (2.11). The simple change of variables $\zeta=z/\lambda$ gives

$$\begin{aligned}
&\frac{\det(I_z-L(\lambda))}{\det(I_z-i\lambda A)}\\
&=\left(1-\frac{1}{2}\zeta^2\lambda^{-2}\sum_j\sum_{k\neq j}\frac{\ell_{jk}^2}{(\zeta^2+\alpha_j^2)(\zeta^2+\alpha_k^2)}\right)\\
&\quad\cdot\left(1-\frac{1}{2}\zeta^2\lambda^{-2}\sum_j\sum_{k\neq j}\frac{m_{jk}^2}{(\zeta^2+\alpha_j^2)(\zeta^2+\alpha_k^2)}\right)\\
&\quad+\lambda^{-2}\sum_j\sum_{k\neq j}\frac{\alpha_j\alpha_k\ell_{jk}m_{jk}}{(\zeta^2+\alpha_j^2)(\zeta^2+\alpha_k^2)}+\lambda^{-4}\left(\sum_j\sum_{k\neq j}\frac{\alpha_j\alpha_k\ell_{jk}m_{jk}}{(\zeta^2+\alpha_j^2)(\zeta^2+\alpha_k^2)}\right)^2\\
&\quad+\zeta^2\lambda^{-4}\sum_j\sum_k\sum_{r\neq j,k}\sum_{s\neq j,k}\frac{\alpha_j\alpha_k\ell_{jr}\ell_{kr}m_{js}m_{ks}}{(\zeta^2+\alpha_j^2)(\zeta^2+\alpha_k^2)(\zeta^2+\alpha_r^2)(\zeta^2+\alpha_s^2)},
\end{aligned}$$

from which we infer that there are no poles of order higher than first. The coefficients of λ^{-2} of the residues at the simple poles $\pm i\alpha_j$ read

$$\pm i/4\sum_{\substack{k=1\\k\neq j}}^{n}\frac{(\ell_{jk}+m_{jk})^2}{\alpha_j-\alpha_k}+\frac{(\ell_{jk}-m_{jk})^2}{\alpha_j+\alpha_k}=\pm\frac{i}{4}k_j,$$

which are proportional to integrals (1.3) and are quadratic in dynamical variables. The coefficients of λ^{-4} of the residues at $\pm i\alpha_j$ yield another set of n integrals of motion which are quartic in dynamical variables. The formulas are so messy that we shall not give them here. What seems worthwhile to point out is that the matrix $L(\lambda)$ considered as a function of dynamical variables (q_i,p_i), $(\bar q_i,\bar p_i)$ is probably the first example in which by using the Weinstein-Aronszajn formula we get an additional set of n integrals which are quartic in momenta $p_i,\bar p_i$ (see also [6, 7]).

3. Integrability

3.1. As is known [4] the Poisson structure defined by (2.2, 2.3) is degenerated. Indeed, all Casimir functions constructed from ℓ_{ij} and m_{ij} variables, which are generated by $\mathrm{Tr}\, L_0^{2k}$, $k=1,\ldots,n$, belong to the kernel of the Poisson bracket and give rise to trivial dynamics. So it is necessary to restrict the hamiltonian system (1.4) on the invariant manifold defined by constant values of Casimir functions. On the so defined manifold of even dimension, $\{\cdot,\cdot\}$ is nondegenerate and thus Liouville's theorem can be applied to prove integrability of (1.4).

In our case of two interacting rigid bodies we have $n(n-1)$ dynamical variables, $2\left[\frac{n}{2}\right]$ Casimir functions, so that the number of degrees of freedom is $d=\frac{n(n-1)}{2}-\left[\frac{n}{2}\right]$. Hence we need d commuting and functionally independent integrals of motion. For counting the number of integrals coming from the expansion of

$$\mathrm{Tr}(L_0+i\lambda A)^{2k}:=\mathrm{Tr}\left(\sum_{r=0}^{2k} L_{2k,r}(i\lambda)^r\right):=\sum_{r=0}^{2k} I_{2k,r}(i\lambda)^r \tag{3.1}$$

into powers of $(i\lambda)$ [$L_{2k,r}$ denotes the matrix coefficient of $(i\lambda)^r$], we note that traces of odd powers of $L(\lambda)$ vanish [which is more obvious from (2.4)] and we obtain nontrivial expressions only at even powers of λ in (3.1), whose coefficients contain an even number of factors L_0 and A. The $\mathrm{Tr}(L_0^{2k})$, $k=1,\ldots,n$ are Casimir functions, and thus the number of nontrivial integrals of motion arising from (3.1) is $\frac{1}{2}n(n-1)$, i.e. higher than necessary. Hence they have to be functionally dependent because they all commute as we shall show below, and the Poisson bracket (2.2), (2.3) is nondegenerated on the manifold of constant values of Casimir functions.

3.2. In order to prove that the integrals $I_{2k,r}$ are in involution we show first that the Poisson bracket of the eigenvalues of the $L(\lambda)$ matrix (2.4) corresponding to two different values of the real spectral parameter, say λ and λ' vanish. Let $\mu(\lambda)$, $v(\lambda')$ be two eigenvalues

$$L(\lambda)\psi=\mu(\lambda)\psi\,,\qquad \psi=(\psi_1,\ldots,\psi_n,\psi_{n+1},\ldots,\psi_{2n}):=(\psi_1^{(1)},\ldots,\psi_n^{(1)};\psi_1^{(2)},\ldots,\psi_n^{(2)})\,,$$

$$L(\lambda')\varphi=v(\lambda')\varphi\,,\qquad \varphi=(\varphi_1,\ldots,\varphi_n,\varphi_{n+1},\ldots,\varphi_{2n}):=(\varphi_1^{(1)},\ldots,\varphi_n^{(1)};\varphi_1^{(2)},\ldots,\varphi_n^{(2)})\,,$$

with the corresponding normalized $\langle\psi,\psi\rangle=\langle\varphi,\varphi\rangle=1$ eigenvectors ψ and φ. The symbol $\langle\cdot,\cdot\rangle$ denotes here complex scalar product of $2n$-dimensional vectors. The eigenvalues μ, v are purely imaginary quantities because the matrix L is skew-adjoint. They can be expressed as

$$\mu=\langle\psi,L\psi\rangle\,;\qquad v=\langle\varphi,L\varphi\rangle\,.$$

Taking into account the bilinearity of $\langle\cdot,\cdot\rangle$, the normalization of eigenvectors, the differentiation property of $\{\cdot,\cdot\}$ and the commutation rules (2.2, 2.3), we get:

$$
\begin{aligned}
&\{\mu(\lambda), v(\lambda')\} \\
&\quad = \{\langle\psi, L(\lambda)\psi\rangle, v(\lambda')\} = (\psi, \{L(\lambda), v(\lambda')\}\psi) \\
&\quad = \sum_{R=1}^{2n} \sum_{S=1}^{2n} \bar{\psi}_R\{L_{RS}(\lambda), v(\lambda')\}\psi_S = \sum_R \sum_S \bar{\psi}_R\langle\varphi, \{L_{RS}(\lambda), L(\lambda')\}\varphi\rangle\psi_S \\
&\quad = \sum_{R,S} \sum_{T,W} \bar{\psi}_R\varphi_T\{L_{RS}(\lambda), L_{TW}(\lambda')\}\varphi_W\psi_S \\
&\quad = \sum_{rst} \bar{\psi}_r\varphi_r\bar{\varphi}_t\ell_{ts}\psi_s + \sum_{rsw} \bar{\psi}_r\bar{\varphi}_r\psi_s\ell_{sw}\varphi_w + \sum_{rst} \psi_s\varphi_s\bar{\psi}_r\ell_{rt}\bar{\varphi}_t \\
&\qquad + \sum_{rsw} \bar{\varphi}_s\psi_s\varphi_w\ell_{wr}\bar{\psi}_r + \sum_{\varrho,\sigma,\tau} \bar{\psi}_\varrho\varphi_\varrho\bar{\varphi}_\tau m_{\tau-n,\sigma-n}\psi_\sigma \\
&\qquad + \sum_{\varrho\sigma\omega} \bar{\psi}_\varrho\bar{\varphi}_\varrho\psi_\sigma m_{\sigma-n,\omega-n}\varphi_\omega + \sum_{\varrho\sigma\tau} \psi_\sigma\varphi_\sigma\bar{\psi}_\varrho m_{\varrho-n,\tau-n}\bar{\varphi}_\tau \\
&\qquad + \sum_{\varrho\sigma\omega} \bar{\varphi}_\sigma\psi_\sigma\varphi_\omega m_{\omega-n,\varrho-n}\bar{\psi}_\varrho\,,
\end{aligned}
\tag{3.2}
$$

where the summation over big latin indices runs from 1 to $2n$, over the corresponding small latin indices runs from 1 to n and over the greek indices runs from $n+1$ to $2n$. A bar over ψ_t and other components denotes complex conjugation. Next we can eliminate all ℓ_{ts} and $m_{\tau-n,\varrho-n}$ from (3.2) by the use of the eigenequations

$$\sum_s \ell_{ts}\psi_s^{(1)} = \mu(\lambda)\psi_t^{(1)} - i\lambda\alpha_t\psi_t^{(2)}\,, \tag{3.3}$$

$$\sum_w \ell_{sw}\varphi_w^{(1)} = v(\lambda')\varphi_s^{(1)} - i\lambda'\alpha_s\varphi_s^{(2)}\,, \tag{3.4}$$

$$\sum_\sigma m_{\tau-n,\sigma-n}\psi_{\sigma-n}^{(2)} = \mu(\lambda)\psi_{\tau-n}^{(2)} - i\lambda\alpha_{\tau-n}\psi_{\tau-n}^{(1)}\,, \tag{3.5}$$

$$\sum_\omega m_{\sigma-n,\omega-n}\varphi_{\omega-n}^{(2)} = v(\lambda')\varphi_{\sigma-n}^{(2)} - i\lambda'\alpha_{\sigma-n}\varphi_{\sigma-n}^{(1)}\,, \tag{3.6}$$

and their complex conjugate expressions (note that $\alpha_{\tau-n}=\alpha_t$ if $\tau=t+n$). Then all coefficients of μ and v cancel by themselves and the remaining terms are

$$
\begin{aligned}
&-i\lambda\sum_r \bar{\psi}_r\varphi_r(\bar{\varphi}_1^{(1)}, a\psi^{(2)}) - i\lambda'\sum_r \bar{\psi}_r\bar{\varphi}_r(\psi^{(1)}, a\varphi^{(2)}) + i\lambda'\sum_s \psi_s\varphi_s(\bar{\psi}^{(1)}, a\bar{\varphi}^{(2)}) \\
&\quad + i\lambda\sum_s \bar{\varphi}_s\psi_s(\varphi^{(1)}, a\bar{\psi}^{(2)}) - i\lambda\sum_\varrho \bar{\psi}_\varrho\varphi_\varrho(\bar{\varphi}^{(2)}, a\psi^{(1)}) \\
&\quad - i\lambda'\sum_\varrho \bar{\psi}_\varrho\bar{\varphi}_\varrho(\psi^{(2)}, a\varphi^{(1)}) + i\lambda'\sum_\sigma \psi_\sigma\varphi_\sigma(\bar{\psi}^{(2)}, a\bar{\varphi}^{(1)}) + i\lambda\sum_\sigma \bar{\varphi}_\sigma\psi_\sigma(\varphi^{(2)}, a\bar{\psi}^{(1)})\,,
\end{aligned}
\tag{3.7}
$$

where the factors of type $(\bar{\varphi}^{(1)}, a\psi^{(2)})$ denote, for a moment, the real scalar product of $(\bar{\varphi}_1^{(1)}, \ldots, \bar{\varphi}_n^{(1)})$ with $(\alpha_1\psi_1^{(2)}, \ldots, \alpha_n\psi_n^{(2)})$. Further we can eliminate terms like $\sum_r \bar{\psi}_r\varphi_r$, etc., by making use of the identities

$$
\begin{aligned}
(\mu-v)\sum_t \bar{\varphi}_t\psi_t &= i\lambda(\bar{\varphi}^{(1)}, a\psi^{(2)}) - i\lambda'(\psi^{(1)}, a\bar{\varphi}^{(2)})\,, \\
(\mu+v)\sum_t \varphi_t\psi_t &= i\lambda(\varphi^{(1)}, a\psi^{(2)}) + i\lambda'(\psi^{(1)}, a\varphi^{(2)})\,, \\
(\mu-v)\sum_\tau \bar{\varphi}_\tau\psi_\tau &= i\lambda(\bar{\varphi}^{(2)}, a\psi^{(1)}) - i\lambda'(\psi^{(2)}, a\bar{\varphi}^{(1)})\,, \\
(\mu+v)\sum_\tau \varphi_\tau\psi_\tau &= i\lambda(\varphi^{(2)}, a\psi^{(1)}) + i\lambda'(\psi^{(2)}, a\varphi^{(1)})\,,
\end{aligned}
$$

the first two of which simply follow from the skew-symmetry of $\ell_{ts}=-\ell_{st}$ by multiplying (3.3) by $\bar{\varphi}_t^{(1)}$, (3.4) by $\psi_s^{(1)}$ and summing over indices t and s. The other two follow in a similar way from (3.5) and (3.6). After substitution of these identities we can immediately see that the coefficients of $(\mu-\nu)^{-1}$ and $(\mu+\nu)^{-1}$ vanish independently and thus $\{\mu(\lambda), \nu(\lambda')\}=0$.

Now, since the eigenvalues of $L(\lambda)$ and $L(\lambda')$ commute, we can conclude that the coefficients $K_{2k}(\lambda)$ and $K_{2\ell}(\lambda')$ of the expansion of the characteristic polynomial into powers of μ^2,

$$\prod_{j=1}^{2n} (\mu-\mu_j(\lambda))=\det(I\mu-L(\lambda))=\sum_{k=0}^{n} K_{2n-2k}(\lambda)\mu^{2k},$$

are in involution for any two different values λ, λ' of the spectral parameter. Thus also all the coefficients $K_{2k,2r}$ of

$$K_{2k}(\lambda)=\sum_{r=1}^{k} K_{2k,2r}\lambda^{2r}$$

mutually commute, and thus the same holds for $I_{2k,2r}$ since, by the Newton identities, we can express $(K_{2k}(\lambda))$ in terms of $(I_{2k}(\lambda))$.

The proof of functional independence of d integrals of motion of type $I_{2k,2r}$ goes essentially along the lines of the work [8] and we will not repeat the arguments used there.

4. Higher Flows and Double Hamiltonian Structure

The higher flows of the system (1.4) are those generated by the integrals $I_{2k,2r}$ in the expansion (3.1), which are of higher (than second) order in dynamical variables. They are all integrable since we have proved that integrals $I_{2k,2r}$ commute, and $d=\frac{n(n-1)}{2}-\left[\frac{n}{2}\right]$ of them are independent.

More interesting is the fact that for each flow generated by $I_{2k,2r}$ we can construct a Lax representation, and the hierarchy of vectorfields generated by $I_{2k,2r}$ is endowed with a double hamiltonian structure as it is the case for soliton equations. This double hamiltonian structure may be used for giving another simpler proof that all $I_{2k,2r}$ are in involution.

First let us recall that our matrix L_0 is an element of the Lie algebra so$(2n)$ with the natural Cartan-Killing scalar product defined as

$$(x,y)=-\tfrac{1}{2}\mathrm{Tr}(xy)=g_{ij,\ell k}x^{ij}y^{\ell k} \tag{4.1}$$

for $x, y \in \mathrm{so}(2n)$, $x=x^{ij}e_{ij}$, $y=y^{k\ell}e_{k\ell}$, where e_{ij} is a basis of so$(2n)$ and the summation convention over repeated indices is used. On the right-hand side of (4.1) we have a coordinate description of the scalar product with $g_{ij,\ell k}=\delta_{ij,k\ell}$, as may be easily found by the use of the basis $(e_{ij})_{rs}=\delta_{ir}\delta_{js}-\delta_{is}\delta_{jr}$. The gradient of a function $f(x)$ of elements of the simple Lie algebra so $(2n)$ is again an element of the same Lie algebra and in the coordinate description it reads

$$\nabla f(x)=e_\alpha\partial^\alpha f=e_\alpha g^{\alpha\beta}\frac{\partial f}{\partial x^\beta}, \tag{4.2}$$

where α labels the elements of a basis of $\mathrm{so}(2n)$ and $g^{\alpha\beta}$ is the inverse of the metric tensor $g_{\alpha\beta}$, which here is just the identity. In particular, if $f(x)=\mathrm{Tr}(x^k)$, where $x\in\mathrm{so}(2n)$, we get $\nabla f(x)=kx^{k-1}$.

The standard Poisson bracket of two functions on a Lie algebra is defined as

$$\{f(x), h(x)\}=(x, [\nabla f, \nabla h])=x^\alpha g_{\alpha\beta}c^\beta_{\gamma\delta}\partial^\gamma f\partial^\delta h\,, \tag{4.3}$$

where $c^\beta_{\gamma\delta}$ are the structure constants of the Lie algebra: $[e_\gamma, e_\delta]=c^\beta_{\gamma\delta}e_\beta$, and, due to the definition of the Cartan-Killing scalar product, we have the identities

$$(x, [\nabla f, \nabla h])=(\nabla h, [x, \nabla f])=(\nabla f, [\nabla h, x])\,. \tag{4.4}$$

It may be verified that for $\mathrm{so}(n)$ Lie algebra the Poisson bracket (4.3) agrees with (2.2) and the hamiltonian vectorfield of $h(x)$ is just equal to

$$\{x, h\}=[x, \nabla h]\,. \tag{4.5}$$

In the case of the hamiltonian $(1/4k)I_{2k,2r}$, we get $\nabla(1/4k)I_{2k,2r}=L_{2k-1,2r}$, which is the matrix multiplying $(i\lambda)^{+2r-1}$ in the expansion of $\mathrm{tr}(L_0+i\lambda A)^{2k-1}$. So the equations of motion read

$$\frac{d}{ds}L_0=\left\{L_0, \frac{1}{4k}I_{2k,2r}\right\}=[L_0, L_{2k-1,2r}]\,, \tag{4.6}$$

where s denotes the time conjugate to $(1/4k)I_{2k,2r}$. Equation (4.6) has a Lax form, which, however, not involving a spectral parameter, is not very useful since $\mathrm{Tr}\,L_0^{2k}$, $k=1, \dots, n$ give rise to trivial dynamics. However, we can introduce a spectral parameter into Eq. (4.6) by making use of the identity

$$\begin{aligned}0&=[L_0+i\lambda A, (L_0+i\lambda A)^r]=[L_0, (L_0+i\lambda A)^r]+i\lambda[A, (L_0+i\lambda A)^r]\\ &=\left[L_0, \sum_{s=0}^{r} L_{rs}(i\lambda)^s\right]+i\lambda\left[A, \sum_{s=0}^{r} L_{rs}(i\lambda)^s\right],\end{aligned}$$

which implies

$$[L_0, L_{r,s+1}]+[A, L_{r,s}]=0 \qquad (s=0, \dots, r-1)\,, \tag{4.7}$$

since $[A, L_{r,r}]=[A, A^r]=0$ is satisfied automatically. Then it is easy to see that due to (4.7),

$$\begin{aligned}\frac{d}{ds}(L_0+i\lambda A)&=[L_0+i\lambda A, L_{2k-1,2r}+i\lambda L_{2k-1,2r+1}+\dots\\ &\quad+(i\lambda)^{2(k-r-1)}L_{2k-1,2k-2}+(i\lambda)^{2k-2r-1}L_{2k-1,2k-1}]\\ &=[L_0+i\lambda A, ((i\lambda)^{-2r}(L+i\lambda A)^{2k-1})_+]\,,\end{aligned}$$

where $(\cdot)_+$ means that we have to take only the positive part of the power expansion in the bracket. For a general hamiltonian $\frac{1}{4k}\mathrm{Tr}(L_0+i\lambda A)^{2k}$, we have

$$\frac{d}{ds}(L_0+i\lambda A)=\left[L_0+i\lambda A, \sum_{s=0}^{2k-1}(s+1)L_{2k-1,s}(i\lambda)^s\right].$$

The identity (4.7), which gives a relationship between the gradients of two subsequent hamiltonians $(1/4r)I_{r,s+1}$ and $(1/4r)I_{r,s+2}$, indicates the possibility of introducing [9] a second Poisson bracket for which $[A, L_{r,s}]$ will also be a vectorfield of $(1/4r)I_{r,s+1}$. Namely we can define

$$\{f(x), h(x)\}_A = (A, [\nabla f, \nabla h]), \tag{4.8}$$

and verify that $\{\cdot,\cdot\}_A$ is skew-symmetric and satisfies the Jacobi identity. In fact, the Jacobi identity is fulfilled here without making use of the Jacobi identity for a Lie algebra bracket: skew-symmetry of structure constants $c^{\alpha}_{\beta\gamma}$ is sufficient. For $\{\cdot,\cdot\}_A$ we also have the identities:

$$(A, [\nabla f, \nabla h]) = (\nabla h, [A, \nabla f]) = (\nabla f, [\nabla h, A]). \tag{4.9}$$

Now by the use of the second hamiltonian structure, or more precisely by the use of (4.7), (4.4), and (4.9), we may prove directly, in a standard way, commutativity of $I_{2k,2j}$ and $I_{2r,2s}$. Namely we have

$$\begin{aligned}
\left\{\frac{1}{4k} I_{2k,2j}, \frac{1}{4r} I_{2r,2s}\right\} &= (L_0, [L_{2k-1,2j}, L_{2r-1,2s}]) \\
&= (L_{2k-1,2j}, [L_{2r-1,2s}, L_0]) = (L_{2k-1,2j}[A, L_{2r-1,2s-1}]) \\
&= (L_{2r-1,2s-1}, [L_{2k-1,2j}, A]) = (L_{2r-1,2s-1}, [L_0, L_{2k-1,2j+1}]) \\
&= (L_0, [L_{2k-1,2j+1}, L_{2r-1,2s-1}]) = (L_0, [L_{2k-1,2j+2}, L_{2r-1,2s-2}]) \\
&= \left\{\frac{1}{4k} I_{2k,2j+2}, \frac{1}{4r} I_{2r,2s-2}\right\}.
\end{aligned} \tag{4.10}$$

By definition $I_{2k,2k} = \operatorname{Tr} A^{2k} = \text{const}$ and $I_{2k,0} = \operatorname{Tr} L_0^{2k}$, which commuted with all dynamical variables. Thus after subsequent applications of the identity (4.10) we may see that the Poisson bracket vanishes if either $2j + 2t$ $(t = 0, 1, \ldots)$ reaches $2k$ or $2s - 2t$ reaches 0.

5. Conclusions

In this paper we have found a Lax representation and have proved integrability of the equations of motion generated by the hamiltonian (1.4) and of the hierarchy of flow associated with (1.4). Equations generated by (1.4) are understood as describing the motion of two n-dimensional interacting rigid bodies since the hamiltonian (1.4) is different from the simple sum of two integrable hamiltonians of type (1.2) corresponding to variables ℓ_{ij} and m_{ij}, respectively. It is clear from the Lax representation (2.7), (2.8) that, because of (2.9), the integrable case studied here is closely related with the integrable Manakov case for $\mathrm{so}(2n)$ Lie algebra. It is, in fact, an example of nontrivial integrable reduction $(L_0)_{ij} = (L_0)_{i+n,j+n}$, $(L_0)_{i,j+n} = -(L_0)_{j,i+n}$, $i,j = 1, \ldots, n$, of the general, $L_0 \in \mathrm{so}(2n)$, Manakov case which suggests that it is worthwhile to study a general reduction problem for the Euler equations on Lie groups. In particular, it would be interesting to find other nontrivial reductions of the Manakov case and to have them classified.

Another question which naturally arises in connection with the study of the system (1.4) is whether there exist some values of the parameters α_j, β_j which correspond to real physical situations. In principle equations following from (1.4) have for $n=3$ the same underlying Lie algebra $\mathrm{so}(3)\times\mathrm{so}(3)=\mathrm{so}(4)$ as two interacting 3-dimensional rigid bodies or a single rigid body with an ellipsoidal cavity filled with an ideal incompressible fluid. Similarly for $n=4$ the underlying Lie algebra is

$$\mathrm{so}(4)\times\mathrm{so}(4)=\mathrm{so}(3)\times\mathrm{so}(3)\times\mathrm{so}(3)\times\mathrm{so}(3),$$

and the system (1.4) could be interpreted either as two interacting rigid bodies with an ellipsoidal cavity each or as one 3-dimensional rigid body with three ellipsoidal cavities filled with an ideal incompressible fluid. However, it is quite a nontrivial problem to investigate whether the physically admissible set of parameters has a nonvoid intersection with the set of parameters in (1.4). For instance, an important question is the physical meaning of the interaction term in the hamiltonian (1.4). At the moment these questions remain open.

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Note added in proof. After this research was accomplished, the authors realized that similar Hamiltonian systems had been independently investigated. In: Reiman, A.G.: The framework of affine Lie algebras. J. Sov. Math. **19**, 1507–1546 (1982).