Commun. Math. Phys. 102, 521–535 (1986)

Communications in
Mathematical
Physics

On a C^*-Algebra Approach to Phase Transition in the Two-Dimensional Ising Model. II

D. E. Evans[1] and J. T. Lewis[2]*

1 Mathematics Institute, University of Warwick, Coventry CV4 7AL, England

2 School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

Abstract. We investigate the states ϕ_β on the C^*-algebra of Pauli spins on a one-dimensional lattice (infinitely extended in both directions) which give rise to the thermodynamic limit of the Gibbs ensemble in the two-dimensional Ising model (with nearest neighbour interaction) at inverse temperature β. We show that if β_c is the known inverse critical temperature, then there exists a family $\{\nu_\beta : \beta \neq \beta_c\}$ of automorphisms of the Pauli algebra such that

$$\phi_\beta = \begin{cases} \phi_0 \circ \nu_\beta, & 0 \leqq \beta < \beta_c \\ \phi_\infty \circ \nu_\beta, & \beta > \beta_c . \end{cases}$$

1. Introduction

We consider the Ising Hamiltonian on a two-dimensional lattice, infinitely extended in all directions, with nearest neighbour interactions and zero field. Thus the problem is classically set in the commutative C^*-algebra $C(\mathscr{P}) = \bigotimes_{\mathbb{Z}^2} \mathbb{C}^2$ of all continuous functions on the configuration space $\mathscr{P} = \{\pm 1\}^{\mathbb{Z}^2}$. The transfer matrix method allows us to transform the model to a non-commutative algebra $\mathscr{A}^P = \bigotimes_{\mathbb{Z}} M_2$ in one dimension less. More precisely, for each inverse temperature β, suppose $\langle \cdot \rangle_\beta$ is the equilibrium state for the classical system obtained as the thermodynamic limit of the Gibbs ensembles on the configuration space $\mathscr{P}$ using free boundary conditions. Then there is for each β, a map $F \to F_\beta$ from the local observables in $C(P)$ into the Pauli or quantum algebra $\mathscr{A}^P$ such that $\langle F \rangle_\beta = \phi_\beta(F_\beta)$. Thus any classical correlation or expectation value can be computed using a knowledge of the Pauli algebra alone. The main result of [3] was the following:

* Partially supported by the Science and Engineering Research Council

Theorem 1. *The cyclic representation of $\mathscr{A}^P$ associated with ϕ_β is irreducible for $0 \leqq \beta \leqq \beta_c$, whilst it is reducible, with two-dimensional centre (for the weak closure) if $\beta > \beta_c$.*

Here β_c is the same as the (inverse) critical temperature given by Onsager [14].

We now improve on this, at least for $\beta \neq \beta_c$, to show:

Theorem 2. *There exist automorphisms $\{\nu_\beta : \beta \neq \beta_c\}$ of $\mathscr{A}^P$ such that*

$$\phi_\beta = \phi_0 \circ \nu_\beta, \quad 0 \leqq \beta < \beta_c \tag{1.1}$$

$$\phi_\beta = \phi_\infty \circ \nu_\beta, \quad \beta > \beta_c \tag{1.2}$$

In particular, since ϕ_0 and ϕ_∞ can be given explicitly, we give a simple proof of Theorem 1, independent of [3].

We make use of the crossed product C^*-algebra $\hat{\mathscr{A}}$ introduced by Araki [2] and described below. The algebra $\hat{\mathscr{A}}$ is generated by the Fermi algebra $\mathscr{A}^F$ and a self-adjoint element $T: \hat{\mathscr{A}} = \mathscr{A}^F_+ + T\mathscr{A}^F_+ + \mathscr{A}^F_- + T\mathscr{A}^F_-$, where $\mathscr{A}^F_+$ and $\mathscr{A}^F_-$ are the even and odd parts of $\mathscr{A}^F$, respectively. The important facts are that the Pauli algebra $\mathscr{A}^P$ sits in $\hat{\mathscr{A}}$ as $\mathscr{A}^P = \mathscr{A}^F_+ + T\mathscr{A}^F_-$, the state ϕ_β on $\mathscr{A}^P$ extends to a state $\hat{\phi}_\beta$ on $\hat{\mathscr{A}}$ whose restriction to $\mathscr{A}^F = \mathscr{A}^F_+ + A^F_-$ is the Fock state ω_β. As pointed out in [11, 21], the state ω_β is connected to the infinite temperature state ω_0 by a Bogoliubov automorphism γ_β: $\omega_\beta = \omega \circ \gamma_\beta$. In this paper we remark that the restriction of γ_∞ to $\mathscr{A}^F_+$ is the Kramers–Wannier automorphism, and γ_∞ relates ω_0 to the zero-temperature state $\omega_\infty : \omega_0 = \omega_\infty \circ \gamma_\infty^{-1}$. Our principal result is that $\{\gamma_\beta|_{\mathscr{A}_+} : 0 \leqq \beta < \beta_c\}$ and $\{\gamma_\infty^{-1}\gamma_\beta|_{\mathscr{A}_+} : \beta > \beta_c\}$ extend to automorphisms $\{\nu_\beta : \beta \neq \beta_c\}$ of $\mathscr{A}^P$, such that

$$\phi_\beta = \begin{cases} \phi_0 \circ \nu_\beta, & 0 \leqq \beta < \beta_c \\ \phi_\infty \circ \nu_\beta, & \beta > \beta_c. \end{cases}$$

Theorem 1 (for $\beta \neq \beta_c$) then follows from an examination of the explicit expressions for ϕ_0 and ϕ_∞.

2. The C^*-algebraic Formulation

We consider the two-dimensional Ising model with the Hamiltonian

$$H^{LM}(\xi) = -\left(\sum_{i=-L}^{L-1} \sum_{j=-M}^{M} J_1 \xi_{ij} \xi_{i+1,j} + \sum_{i=-L}^{L} \sum_{j=-M}^{M-1} J_2 \xi_{ij} \xi_{i,j+1} \right), \tag{2.1}$$

where $\xi_{ij} = \pm 1$ is the classical spin at the lattice site $(i,j) \in \mathbb{Z}^2$, and J_1 and J_2 are positive constants. Then the Gibbs ensemble average is given by

$$\langle F \rangle_{LM} = Z_{LM}^{-1} \sum_{\xi} F(\xi) \exp(-\beta H^{LM}(\xi)), \tag{2.2}$$

$$Z_{LM} = \sum_{\xi} \exp(-\beta H^{LM}(\xi)),$$

where the sum is over all configurations $\xi_{ij} = \pm 1$, $\beta \geqq 0$ and F, a local observable, is a function of ξ_{ij}, for $|i| \leqq l$, $|j| \leqq m$, and some $l \leqq L$, $m \leqq M$. The transfer matrix method [10] allows us to compute the expectation values $\langle \cdot \rangle_{LM}$ in terms of a state

φ_β^{LM} on the Pauli spin algebra $\mathscr{A}_M^P$ generated by the spin matrices $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$ on sites i where $|i| \leqq M$. Then $\mathscr{A}_M^P \simeq \bigotimes_{-M}^{M} M_2$, and we adopt the convention that

$$\sigma_x^{(i)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y^{(i)} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_z^{(i)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can identify a function of $\xi = (\xi_j)$, $\xi_j = \pm 1$, $\xi' = (\xi'_j)$, $\xi'_j = \pm 1$ with a $2^{2M+1} \times 2^{2M+1}$ matrix, and hence with an element of $\mathscr{A}_M^P$. If we define

$$(T_M)_{\xi,\xi'} = \exp\left\{\frac{K_2}{2} \sum_{j=-M}^{M} (\xi_j \xi_{j+1} + \xi'_j \xi'_{j+1}) + K_1 \sum_{j=-M}^{M} \xi_j \xi'_j\right\}, \tag{2.3}$$

then under the above identifications

$$T_M = (2 \sinh 2K_1)^{M+1/2} V^{1/2} W V^{1/2}, \tag{2.4}$$

if

$$V = \exp\left\{K_2 \sum_{j=-M}^{M-1} \sigma_x^{(j)} \sigma_x^{(j+1)}\right\}, \tag{2.5}$$

$$W = \exp\left\{K_1^* \sum_{j=-M}^{M} \sigma_z^{(j)}\right\}, \tag{2.6}$$

and

$$K_j = \beta J_j, \quad j = 1, 2, \quad K_1^* = \tfrac{1}{2} \log(\coth K_1). \tag{2.7}$$

If

$$\Omega_M(\xi) = \exp\left\{\frac{K_2}{2} \sum_{j=-M}^{M-1} \xi_j \xi_{j+1}\right\},$$

then $Z_{LM} = \|(T_M)^L \Omega_M\|^2$, and $\langle F \rangle_{LM} = \varphi_\beta^{LM}(F_{\beta M})$ for some $F_{\beta M} \in \mathscr{A}_M^P$, if φ^{LM} is the vector state $\langle T_M^L \Omega_M, \cdot\, T_M^L \Omega_M \rangle Z_{LM}^{-1}$ on $\mathscr{A}_M^P$. If $K_1^* < \infty$, then by the Perron Frobenius theorem, T_M has a unique unit vector $\Omega^M = \Omega^M(\xi)$, $\Omega^M(\xi) > 0$ belonging to the largest eigenvalue. Then as $L \to \infty$:

$$\lim_{L \to \infty} \langle F \rangle_{LM} = \langle \Omega^M, F_{\beta M} \Omega^M \rangle.$$

Then if $\mathscr{A}^P$ denotes the Pauli algebra generated by the spin matrices $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$ for $i \in \mathbb{Z}$, so that $\mathscr{A}^P = \lim_{M \to \infty} \mathscr{A}_M^P$, we have

$$\lim_{M \to \infty} \lim_{L \to \infty} \langle F \rangle_{LM} = \varphi_\beta(F_\beta),$$

where $F_\beta = \lim_{M \to \infty} F_{\beta M}$ and $\varphi_\beta = \lim_{M \to \infty} \langle \Omega^M, \cdot\, \Omega^M \rangle$ is a state on $\mathscr{A}^P$.

Following [17, 21, 11, 12, 2, 3] the states φ_β on the Pauli algebra $\mathscr{A}^P$ are best studied by introducing a Fermion algebra $\mathscr{A}^F$ generated by annihilation and creation operators c_i and c_i^*, $i \in \mathbb{Z}$, satisfying the canonical anticommutation relations:

$$[c_i, c_j]_+ = [c_i^*, c_j^*]_+ = 0, \quad [c_i, c_j^*]_+ = \delta_{ij} 1. \tag{2.8}$$

We adopt the self dual formalism of [3], so that $\mathscr{A}^F$ is generated by the range of a linear map B on $l_2 \oplus l_2$ given by

$$B(h) = \sum_{-\infty}^{\infty} (c_j^* f_j + c_j g_j), \quad h = \begin{pmatrix} f \\ g \end{pmatrix} \quad f = (f_j) \quad g = (g_j). \tag{2.9}$$

Here the convergence of (2.9) is in norm, and B satisfies

$$[B(h_1)^*, B(h_2)]_+ = \langle h_1, h_2 \rangle 1, \quad B(h)^* = B(\Gamma h),$$

where

$$\Gamma \begin{pmatrix} f \\ g^* \end{pmatrix} = \begin{pmatrix} g \\ f^* \end{pmatrix}$$

A unitary U on $l_2 \oplus l_2$ commuting with Γ gives rise to an automorphism $\tau(U)$ of $\mathscr{A}^F$ by

$$\tau(U) B(h) = B(Uh), \tag{2.10}$$

and is called a Bogoliubov automorphism. A basis projection is a projection E on $l_2 \oplus l_2$ such that

$$\Gamma E \Gamma = 1 - E.$$

Any basis projection E gives rise to a unique state ω on $\mathscr{A}^F$ such that $\omega(B(f)B(f)^*) = 0, f \in E(l_2 \oplus l_2)$. We write $\omega = \omega_E$. Then ω_E is called a Fock state, is irreducible, and satisfies

$$\omega_E(B(f)^* B(g)) = \langle f, E_g \rangle, \quad f, g \in l_2 \oplus l_2.$$

We define a unitary u_- on l_2 by

$$(u_- f)_j = \begin{cases} f_j & j \geqq 1 \\ -f_j & j \leqq 0 \end{cases}, \tag{2.11}$$

and $\theta_- = u_- \oplus u_-$ on $l_2 \oplus l_2$. The corresponding Bogoliubov automorphism $\tau(\theta_-)$ is denoted by Θ_- so that

$$\Theta_- c_j = \begin{cases} c_j & j \geqq 1 \\ -c_j & j \leqq 0 \end{cases}. \tag{2.12}$$

We construct the crossed product C^*-algebra

$$\hat{\mathscr{A}} = \mathscr{A}^F X_{\Theta_-} \mathbb{Z}_2,$$

which is generated by $\mathscr{A}^F$ and a self adjoint unitary T in $\hat{\mathscr{A}}$ satisfying $TaT = \Theta_-(a)$, $a \in \mathscr{A}^F$. The Pauli spin algebra $\mathscr{A}^P$ is identified with a C^*-subalgebra of $\hat{\mathscr{A}}$ generated by

$$\sigma_z^{(j)} = 2c_j^* c_j - 1 \tag{2.13}$$

$$\sigma_x^{(j)} = TS_j\,(c_j + c_j^*), \quad \sigma_y^{(j)} = TS_j i\,(c_j - c_j^*), \tag{2.14}$$

$$S_j = \begin{cases} \prod_{k=1}^{j-1} \sigma_z^{(k)} & \text{if } j > 1, \\ 1 & \text{if } j = 1, \\ \prod_{k=0}^{j} \sigma_z^{(k)} & \text{if } j < 1. \end{cases} \tag{2.15}$$

We extend the automorphism Θ to $\hat{\mathscr{A}}$ by defining $\Theta(T) = T$, so that

$$\Theta(\sigma_z^{(j)}) = \sigma_z^{(j)}, \quad \Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)}, \quad \Theta(\sigma_y^{(j)}) = -\sigma_y^{(j)}. \tag{2.16}$$

Then Θ gives gradings to both $\mathscr{A}^F$ and $\mathscr{A}^P$, so that if $\mathscr{A}_\pm^F = \{x \in \mathscr{A}^F : \Theta(x) = \pm x\}$, then

$$\{x \in \mathscr{A}^P : \Theta x = x\} = \mathscr{A}_+^F, \quad \{x \in \mathscr{A}^P : \Theta x = -x\} = T\mathscr{A}_-^F,$$

and

$$\mathscr{A}^F = \mathscr{A}_+^F + \mathscr{A}_-^F, \quad \mathscr{A}^P = \mathscr{A}_+^F + T\mathscr{A}_-^F.$$

The state $\phi_\beta = \phi_\beta \circ \Theta$ on $\mathscr{A}^P$ gives rise to an even state $\omega_\beta = \omega_\beta \circ \Theta$ on $\mathscr{A}^F$ by

$$\omega_\beta(a + b) = \phi_\beta(a), \quad a \in \mathscr{A}_+^F, \quad b \in \mathscr{A}_-^F. \tag{2.17}$$

Then ω_β is a Fock state, whose basis projection E_β is described after taking Fourier series as follows on $L^2(\mathbb{T}) \oplus L^2(\mathbb{T})$. (No confusion should arise when we often identify, in the sequel, l^2 with $L^2(\mathbb{T})$ in this way).

First $\gamma(\theta) \geqq 0$ is determined by

$$\cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos\theta = \cosh\gamma(\theta), \tag{2.18}$$

and $\delta(\theta) = \Theta(\theta) - \theta$ is determined by

$$\cos\delta(\theta) = (\sinh\gamma(\theta))^{-1}(\cosh 2K_1^* \sinh 2K_2 - \sinh 2K_1^* \cosh 2K_2 \cos\theta) \tag{2.19}$$

$$\sin\delta(\theta) = (\sinh\gamma(\theta))^{-1} \sinh 2K_1^* \sin\theta. \tag{2.20}$$

Then if V_β is the self adjoint unitary

$$V_\beta(\theta) = \begin{pmatrix} \cos\Theta(\theta) & -i\sin\Theta(\theta) \\ i\sin\Theta(\theta) & -\cos\Theta(\theta) \end{pmatrix}, \tag{2.21}$$

E_β is the multiplication operator $(1 - V_\beta)/2$.

The states ϕ_0 and ϕ_∞ correspond to infinite and zero temperatures ($\beta = 0, \beta = \infty$ respectively) as follows. The region $\beta > \beta_c$ corresponds to $K_1^* < K_2$, and $\beta < \beta_c$ to $K_1^* > K_2$. As in [3], we will regard K_1^* and K_2 as independent parameters. Then $K_2 = 0$, $K_1^* > 0$ corresponds to $\beta = 0$, and $K_1^* = 0$, $K_2 > 0$ to $\beta = \infty$.

Case (*A*). $K_2 = 0$, $K_1^* > 0$, $(\beta = 0)$. Here $\gamma(\theta) = 2K_1^*$, $\delta(\theta) = \pi - \theta$, $\Theta(\theta) = \pi$,

$$V_0(\theta) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_0 = (1 - V_0)/2.$$

Then the even state ϕ_0 on $\mathscr{A}^P$ corresponding to the quasi-free state $\omega_0 = \omega_{E_0}$ on $\mathscr{A}^F$

as in (2.17) is the product state,

$$\phi_0 = \bigotimes_{-\infty}^{\infty} \omega_+ ,$$

where $\omega_+ = \langle z_+, \cdot z_+ \rangle$ is the vector state on M_2 given by $z_+ = 2^{-1/2}\begin{pmatrix}1\\1\end{pmatrix}$. Note that $\sigma_z = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ has eigenvectors z_+ and $z_- = 2^{-1/2}\begin{pmatrix}1\\-1\end{pmatrix}$ with eigenvalues 1 and -1 respectively. Thus the eigenspace of $W = \exp\left\{K_1^* \sum_{j=-M}^{M} \sigma_z^{(j)}\right\}$ corresponding to the largest eigenvalue is non-degenerate, and spanned by $\bigotimes_{-M}^{M} z_+$. The transfer matrix T_M in the case when $K_2 = 0$ is a scalar multiple of W (see (2.4)) and so the same applies to T_M.

Case (*B*). $K_1^* = 0$, $K_2 > 0$, $(\beta = \infty)$. Here $\gamma(\theta) = 2K_2$, $\delta(\theta) = 0$, $\Theta(\theta) = \theta$,

$$V_\infty(\theta) = \begin{pmatrix}\cos\theta & -i\sin\theta\\ i\sin\theta & -\cos\theta\end{pmatrix}, \quad E_\infty = (1 - V_\infty)/2.$$

The even state ϕ_∞ on $\mathscr{A}^P$ corresponding to the quasi-free state $\omega_\infty = \omega_{E_\infty}$ on $\mathscr{A}^F$ as in (2.17) is the state

$$\phi_\infty = \frac{1}{2}\left(\bigotimes_{-\infty}^{\infty} \mu_+ + \bigotimes_{-\infty}^{\infty} \mu_-\right)$$

where $\mu_\pm$ are the vector states $\langle x_\pm, \cdot x_\pm \rangle$ on M_2, if $x_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $x_- = \begin{pmatrix}0\\1\end{pmatrix}$. Note that $\sigma_x = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ has eigenvectors x_+ and x_- with eigenvalues 1 and -1 respectively. Thus the eigenspace of $V = \exp\left\{K_2 \sum_{j=-M}^{M-1} \sigma_x^{(j)} \sigma_x^{(j+1)}\right\}$ corresponding to the largest eigenvalue is doubly degenerate and spanned by $\bigotimes_{-M}^{M} x_+$ and $\bigotimes_{-M}^{M} x_-$ (corresponding to all spins up and all spins down respectively). The transfer matrix T_M in the case when $K_1^* = 0$ is a scalar multiple of V, and so the same applies to this T_M.

Note that ϕ_0 is clearly pure, whilst ϕ_∞ is clearly not. Moreover the cyclic representation of the state ϕ_∞ is a direct sum of two disjoint irreducible representations, and so has a two dimensional centre.

For more details on the C^*-formulation of the two-dimensional Ising model, we refer to [17, 21, 11, 12, 5, 3].

3. The Kramers–Wannier Automorphism Revisited

The even algebra $\mathscr{A}_+$ is generated by

$$\sigma_z^{(j)} = 2c_j^* c_j - 1, \tag{3.1}$$

and

$$\sigma_x^{(j)}\sigma_x^{(j+1)} = (c_j - c_j^*)(c_{j+1} + c_{j+1}^*). \tag{3.2}$$

Define an automorphism κ of $\mathscr{A}_+$ by

$$\kappa(\sigma_z^{(j)}) = \sigma_x^{(j)}\sigma_x^{(j+1)}, \tag{3.3}$$

$$\kappa(\sigma_x^{(j)}\sigma_x^{(j+1)}) = \sigma_z^{(j+1)}. \tag{3.4}$$

This automorphism relates high and low temperatures (cf. (2.4–6)) and is essentially the mechanism by which Kramers and Wannier [10] located the critical point of the classical two-dimensional Ising model, assuming only one critical point existed. See also [14, p. 123].) Note that κ^2 is the restriction of the shift on $\mathscr{A}^P = \bigotimes_{-\infty}^{\infty} M_2$ to $\mathscr{A}_+$, but we will see in Corollary 4.3 that κ does not extend to an automorphism of $\mathscr{A}^P$. However κ does extend to an automorphism of $\mathscr{A}^F$:

Let U be the shift on l_2:

$$(Uf)_k = f_{k+1}, \quad f = (f_k) \in l_2, \tag{3.5}$$

identified with multiplication by $e^{-i\theta}$ on $L^2(\mathbb{T})$. Let

$$W = i/2\begin{pmatrix} 1 - U^* & 1 + U^* \\ -1 - U^* & U^* - 1 \end{pmatrix}. \tag{3.6}$$

Note that

$$W^2 = \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix},$$

so that $\tau(W)^2 = \tau(W^2)$ is the Bogoliubov automorphism on the CAR algebra induced by the shift, or $\tau^2(c_j) = c_{j+1}$.

Lemma 3.1. *The restriction of the Bogoliubov automorphism $\tau(W)$ from $\mathscr{A}^F$* to $\mathscr{A}^F_+$ *is κ.*

Proof. We have if $\tau = \tau(W)$

$$\tau(c_j^*) = \frac{i}{2}(c_j^* - c_{j+1}^* - c_j - c_{j+1})$$

$$\tau(c_j) = \frac{i}{2}(c_j^* + c_{j+1}^* - c_j + c_{j+1}).$$

Then

$$\tau(c_j - c_j^*) = i(c_{j+1}^* + c_{j+1})$$

$$\tau(c_j + c_j^*) = i(c_j^* - c_j)$$

Hence

$$\begin{aligned}\tau((c_j - c_j^*)(c_{j+1} + c_{j+1}^*)) \\ = -(c_{j+1}^* + c_{j+1})(c_{j+1}^* - c_{j+1}) \\ = 2c_{j+1}^* c_{j+1} - 1.\end{aligned}$$

Since $\tau^2(c_j) = c_{j+1}$, we see

$$\tau(2c_j^* c_j - 1) = (c_j - c_j^*)(c_{j+1} + c_{j+1}^*).$$

We now extend the Kramers–Wannier automorphism κ to $\mathscr{A}^F$ by putting $\kappa = \tau(W)$. Also note that

$$W^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W = \frac{1}{2} \begin{pmatrix} -(U+U^*) & U^* - U \\ U - U^* & U + U^* \end{pmatrix}. \tag{3.7}$$

This means that κ takes the infinite temperature state ω_0 to the zero temperature state ω_∞:

$$\omega_0 \circ \kappa = \omega_\infty, \tag{3.8}$$

as one would expect from (3.1–2) and (2.4–6).

We now define

$$U_\beta = e^{-i\Theta} \tag{3.9}$$

where Θ is as defined in (2.18–20), and

$$W_\beta = \frac{i}{2} \begin{pmatrix} 1 - U_\beta^* & 1 + U_\beta^* \\ -(1 + U_\beta^*) & U_\beta^* - 1 \end{pmatrix} \tag{3.10}$$

Then

$$W_\beta^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_\beta = \tfrac{1}{2} \begin{pmatrix} -(U_\beta + U_\beta^*) & U_\beta^* - U_\beta \\ U_\beta - U_\beta^* & U_\beta + U_\beta^* \end{pmatrix} = -V_\beta.$$

This means that if $\gamma_\beta = \tau(W_\beta)$, the Bogoliubov automorphism induced by W_β, then

$$\omega_0 \circ \gamma_\beta = \omega_\beta, \tag{3.11}$$

and

$$\omega_\infty \circ \delta_\beta = \omega_\beta, \tag{3.12}$$

if $\delta_\beta = \kappa^{-1}\gamma_\beta = \tau(W^* W_\beta)$. We will show that $\{\gamma_\beta|_{\mathscr{A}_+} : 0 \leqq \beta < \beta_c\}$ and $\{\delta_\beta|_{\mathscr{A}_+} : \beta > \beta_c\}$ extend to automorphisms $\{\nu_\beta : \beta \neq \beta_c\}$ of $\mathscr{A}^P$ such that

$$\left.\begin{aligned} \phi_0 \circ \nu_\beta &= \phi_\beta, \quad 0 \leqq \beta < \beta_c, \\ \phi_\infty \circ \nu_\beta &= \phi_\beta, \quad \beta > \beta_c. \end{aligned}\right\} \tag{3.13}$$

Remark 3.2. The Kramers–Wannier transformation on the even subalgebra of the Pauli algebra also has an analogue on a certain subalgebra of the UHF algebra $\mathscr{F}_q = \bigotimes_{\infty}^{\infty} M_q$ which is relevant for the high temperature–low temperature duality in the q-state Potts model, and has also recently appeared in work on the index of subfactors and entropy [8, 9, 16].

To describe this, let $\{E_{ij}: i, j = 1, \ldots, q\}$ be matrix units for M_q, and then let

$$f = \sum_{i,j=1}^{q} E_{ij}/q, \quad g = \sum_{i=1}^{q} E_{ii} \otimes E_{ii}$$

be rank one and rank q–projections in M_q and $M_q \otimes M_q$ respectively. Then define a doubly infinite sequence $\{e_i\}_{-\infty}^{\infty}$ of projections in $\mathscr{F}_q$ by

$$e_{2i-1} = \cdots 1 \otimes 1 \otimes f \otimes 1 \otimes 1 \ldots, \quad i^{th}\text{ position}$$

$$e_{2i} = \cdots 1 \otimes 1 \otimes g \otimes 1 \otimes 1 \ldots, \quad i\text{-}(i+1)\text{ positions}$$

and let $\mathscr{A}_q$ be the C^*-algebra generated by $\{e_i\}_{-\infty}^{\infty}$. Thus if $q=2$,

$$e_{2i-1} = (\sigma_z^{(i)} + 1)/2, \quad e_{2i} = (\sigma_x^{(i)} \sigma_x^{(i+1)} + 1)/2.$$

and so $\mathscr{A}_2$ is the even part of the Pauli algebra. The projections $\{e_i\}$ satisfy the relations

$$e_i e_j = e_j e_i, \quad |i-j| \geqq 2, \tag{3.14}$$

$$e_i e_{i\pm 1} e_i = \frac{1}{q} e_i, \tag{3.15}$$

$$\operatorname{tr} x\, e_i = \frac{1}{q} \operatorname{tr} x, \text{ if } x \in C^*\text{-algebra generated by } \{e_j\}_{-\infty}^{i-1}, \tag{3.16}$$

where tr is the trace on $\mathscr{F}_q$.

The local transfer matrix in the q-state Potts model can be written [4, 22], up to a scalar, as $X^{1/2} Y X^{1/2}$, where $X = \exp 2K_2 \Sigma e_{2i}$, $Y = \exp 2K_1^* \Sigma e_{2i-1}$, and $K_j = \beta J_j$, $j=1, 2$, $(e^{2K_1^*} - 1)(e^{2K_1} - 1) = q$.

As in the Ising model, where $q=2$, the automorphism κ_q: $e_i \to e_{i+1}$ of $\mathscr{F}_q$ can be used to locate the critical temperature (see e.g. [4]). Families of projections satisfying (3.14–16) and the automorphisms κ_q have recently occurred in the work of Jones [8, 9] on index of subfactors and braid groups, and Pimsner and Popa [16] on index and entropy of subfactors.

4. Extendibility of Automorphisms

We consider the problem of deciding when an automorphism of the even algebra $\mathscr{A}_+$ extends to an automorphism of the Pauli algebra $\mathscr{A}^P$.

Let $\mathscr{C}$ be a graded unital C^*-algebra, i.e. $\mathscr{C}$ is equipped with an automorphism Θ such that $\Theta^2 = 1$, and we define the even and odd parts of $\mathscr{C}$ by

$$\mathscr{C}_{\pm} = \{x \in \mathscr{C} : \Theta(x) = \pm x\},$$

respectively. We say that an automorphism v of $\mathscr{C}$ is graded if $v\mathscr{C}_{\pm} \subseteq \mathscr{C}_{\pm}$. An inner automorphism of $\mathscr{C}$ is said to be even (respectively odd) if it is implemented by an even (respectively odd) unitary.

Note that if $\mathscr{C}$ is simple, then a graded inner automorphism v on $\mathscr{C}$ is always either even or odd. For then, if $v = \operatorname{Ad}(u)$, $u \in \mathscr{C}$, we have $v = \Theta v \Theta$, since v is graded, and so $\operatorname{Ad} \Theta(v) = \operatorname{Ad}(v)$ on $\mathscr{C}$. Since $\mathscr{C}$ is simple, this implies $\Theta(v) = \lambda v$ for some $\lambda \in \mathbb{T}$. Letting $v = a + b$, where a, b are even and odd respectively, we see that $a - b = \lambda(a+b)$, or $a(1-\lambda) = b(1+\lambda) = 0$. Hence either $b=0$, $\lambda = 1$ and v is even, or $a = 0$, $\lambda = -1$, and v is odd.

We need something stronger than this:

Lemma 4.1. *Let u be a self adjoint unitary in a graded C^*-algebra such that $\mathscr{C}_+$ is simple and*

$$u\mathscr{C}_+u = \mathscr{C}_+.$$

Then u is either odd or even.

Proof. Let $u = a + b$, where a, b are even and odd respectively. We have to show that either a or b is zero. Now a, b are self adjoint and $(a+b)x(a+b) \in \mathscr{C}_+$, for all $x \in \mathscr{C}_+$. This means

$$axb + bxa = 0, \quad \text{for all } x \in \mathscr{C}_+. \tag{4.1}$$

In particular $ab + ba = 0$, and since u is unitary we have

$$a^2 + b^2 = 1. \tag{4.2}$$

From (4.1) with $x = a$ we get

$$a^2b + ba^2 = 0. \tag{4.3}$$

Then using (4.2) we have $(1 + b^2)b + b(1 - b^2) = 0$, or

$$b = b^3. \tag{4.4}$$

Then $(ab)^*ab = ba^2b = b(1 - b^2)b = 0$, using (4.4), hence $ab = 0 = ba$. But then using (4.1), $b(axb + bxa) = 0$, for all $x \in \mathscr{C}_+$ implies that $b^2xa = 0$ for all $x \in \mathscr{C}_+$, or $(1 - a^2)xa = 0$ for all $x \in \mathscr{C}_+$. But $\mathscr{C}_+$ is simple and so either $a^2 = 1$ or $a = 0$, i.e. by (3.2) either $b = 0$ or $a = 0$.

We now consider the following general situation. Let $\mathscr{A}$ be a unital C^*-algebra, with α, β two commuting automorphisms such that $\alpha^2 = \beta^2 = 1$, and a unitary element U satisfying $\alpha(U) = -U$, $U^2 = 1$, $\beta(U) = U$.

Let $\hat{\mathscr{A}}$ be the crossed product of $\mathscr{A}$ by the β-action of $\mathbb{Z}_2$ which is generated by $\mathscr{A}$ and a $T \in \hat{\mathscr{A}}$ satisfying $T^2 = 1$, $T^* = T$, $Ta = \beta(a)T$, $a \in \hat{\mathscr{A}}$. We grade $\mathscr{A}$ by α so that $\mathscr{A}_\pm = \{x \in \mathscr{A} : \alpha(x) = \pm x\}$, and let $\mathscr{B} = \mathscr{A}_+ + T\mathscr{A}_-$, which is a C^*-subalgebra of $\hat{\mathscr{A}}$. Extend α, β to $\hat{\mathscr{A}}$ by

$$\hat{\alpha}(a + Tb) = \alpha(a) + T\alpha(b),$$

$$\hat{\beta}(a + Tb) = \beta(a) + T\beta(b), \quad a, b \in \mathscr{A}.$$

We grade $\hat{\mathscr{A}}$, $\mathscr{B}$ by $\hat{\alpha}$ and $\tilde{\alpha} = \hat{\alpha}|_{\mathscr{B}}$ respectively, so that $\mathscr{B}_+ = \mathscr{A}_+$, $\mathscr{B}_- = T\mathscr{A}_-$.

If v is a graded automorphism of $\mathscr{A}$, we now give a criterion when $v|\mathscr{A}_+$ extends to an automorphism of $\mathscr{B}$. We will then apply these criteria to the case $\mathscr{A} = \mathscr{A}^F$, $\alpha = \Theta$, $\beta = \Theta_-$, $\mathscr{B} = \mathscr{A}^P$, $U = c_i + c_i^*$ for any $i \geqq 1$, and v a quasi-free automorphism of $\mathscr{A}^F$.

Theorem 4.2. *Let v be a graded automorphism of $\mathscr{A}$, where $\mathscr{A}_+$ is simple. If $v|_{\mathscr{A}_+}$ extends to an automorphism $\tilde{v}$ of $\mathscr{B}$, then $\tilde{v}$ must be graded.*

Proof. Let $\sigma = TU \in T\mathscr{A}_-$, so that σ is a self adjoint unitary in $\mathscr{B}$, and $\mathscr{B} = \mathscr{A}_+ + \sigma\mathscr{A}_+$. If $v|_{\mathscr{A}_+}$ extend to an automorphism $\tilde{v}$ of $\mathscr{B}$, then v, $\mathrm{Ad}(\sigma)$ leave $\mathscr{A}_+$ invariant and

$$v\mathrm{Ad}(\sigma)v^{-1} = \mathrm{Ad}(\tilde{v}(\sigma)) \quad \text{on} \quad \mathscr{A}_+.$$

Hence by Lemma 4.1, $\tilde{v}(\sigma)$ is either odd or even. If $\tilde{v}(\sigma)$ is odd, then $\tilde{v}$ is graded, but if $\tilde{v}(\sigma)$ is even, then $\tilde{v}(\mathscr{B}) \subset \mathscr{B}_+$ which is impossible as $\tilde{v}$ is an automorphism.

Corollary 4.3. *The Kramers–Wannier automorphism $\kappa:\mathscr{A}_+ \to \mathscr{A}_+$ does not extend to an automorphism of $\mathscr{A}^P$.*

Proof. Suppose κ extends to a graded automorphism $\tilde{\kappa}$ of $\mathscr{A}^P$. Then $\phi_0 \circ \kappa = \phi_\infty$ on $\mathscr{A}_+$ means that $\phi_0 \circ \tilde{\kappa} = \phi_\infty$ on $\mathscr{A}^P$, since ϕ_0 and ϕ_∞ are even states. But this is impossible as ϕ_0 and ϕ_∞ are pure, impure respectively by Sect. 2 or [3].

Note that since κ extends to an automorphism of $\mathscr{A}^F$, it follows from Corollary 4.3 that the Jordan–Wigner transformation which identifies $\mathscr{A}^P_+$ with $\mathscr{A}^F_+$ in (3.1) and (3.2) cannot be extended to an isomorphism between $\mathscr{A}^P$ and $\mathscr{A}^F$ (although $\mathscr{A}^P$ and $\mathscr{A}^F$ are isomorphic C^*-algebras).

If v is a graded automorphism of $\mathscr{A}$, which extends to an automorphism $\hat{v}$ of $\hat{\mathscr{A}}$, then

$$\beta v \beta v^{-1}(x) = T\hat{v}(T)x\hat{v}(T)T \quad \text{for all} \quad x \in \mathscr{A}.$$

In particular, if $\hat{v}$ is graded, then $T\hat{v}(T)$ is in $\hat{\mathscr{A}}_+$. Note that by the argument of Theorem 4.2, if $\hat{\mathscr{A}}_+$ is simple, then $\hat{v}$ must be graded. In the converse direction we have:

Theorem 4.4. *Let v be a graded automorphism of $\mathscr{A}$, where $\mathscr{A}_+$ is simple, and $\beta v \beta v^{-1}$ is an inner even automorphism of $\mathscr{A}$. Then v extends to a graded automorphism of $\hat{\mathscr{A}}$, leaving $\mathscr{B}$ invariant, and given by*

$$\hat{v}(a + Tb) = v(a) + Tvv(b), \quad a, b \in \mathscr{A}. \tag{4.5}$$

where v is a unitary in $\mathscr{A}_+$ such that

$$v\beta(v) = 1, \quad \beta v \beta v^{-1} = \mathrm{Ad}(v) \text{ on } \mathscr{A}$$

Proof. Suppose $\beta v \beta v^{-1} = \mathrm{Ad}(v)$, for some v unitary in $\mathscr{A}_+$. If $\gamma = \beta v \beta v^{-1}$, we have $\gamma\beta\gamma\beta = 1$. Therefore for $x \in \mathscr{A}$:

$$x = \gamma\beta\gamma\beta(x) = v\beta(v)x\beta(v)^*v^*.$$

But $\mathscr{A}_+$ is simple and so we must have $v\beta(v) \in \mathbb{C}$. By rotating v we may assume $v\beta(v) = 1$. Define $\hat{v}:\hat{\mathscr{A}} \to \hat{\mathscr{A}}$ by (4.5). We use $v\beta(x) = Tvv(x)v^*T$, $x \in \mathscr{A}$, and $TvT = v^*$ to check that $\hat{v}$ is an automorphism: For a, $b \in \mathscr{A}$,

$$\begin{aligned}\hat{v}[(a + Tb)^*] &= \hat{v}[a^* + T\beta b^*] = v(a^*) + Tvv\beta(b^*)\\ &= v(a)^* + v(b^*)v^*T = [\hat{v}(a + Tb)]^*.\end{aligned}$$

Moreover

$$\begin{aligned}\hat{v}(a_1 + Tb_1)(a_2 + Tb_2) &= \hat{v}(a_1a_2 + \beta(b_1)b_2 + Tb_1a_2 + T\beta(a_1)b_2)\\ &= v(a_1a_2) + v(\beta(b_1)b_2) + Tvv(b_1a_2) + Tvv(\beta(a_1)b_2)\\ &= v(a_1)v(a_2) + Tvv(b_1)Tvv(b_2) + Tvv(b_1)v(a_2) + v(a_1)Tvv(b_2)\\ &= [v(a_1) + Tvv(b_1)][v(a_2) + Tvv(b_2)]\\ &= \hat{v}(a_1 + Tb_1)\hat{v}(a_2 + Tb_2).\end{aligned}$$

Thus $\hat{v}$ is an automorphism, and if v is in $\mathscr{A}_+$, it is clear that $\hat{v}$ is graded and leaves $\mathscr{B}$ invariant.

5. The Main Results

We now apply the criterion of the previous section for extending automorphisms from the even algebra $\mathscr{A}_+^P$ to the Pauli algebra $\mathscr{A}^P$ to deduce:

Theorem 5.1. *The (Bogoliubov) automorphisms* $\{\tau(W^*W_\beta)|_{\mathscr{A}_+}:\beta>\beta_c\}$ *and* $\{\tau(W_\beta)|_{\mathscr{A}_+}:0\leqq\beta<\beta_c\}$ *extend to graded automorphisms* $\{v_\beta:\beta\neq\beta_c\}$ *of the Pauli algebra* $\mathscr{A}^P$ *such that*

$$\phi_0\circ v_\beta=\phi_\beta,\quad 0\leqq\beta<\beta_c, \tag{5.1}$$

$$\phi_\infty\circ v_\beta=\phi_\beta,\quad \text{for } \beta>\beta_c. \tag{5.2}$$

The (Bogoliubov) automorphisms

$$\{\tau(W^*W_\beta)|_{\mathscr{A}_+}:0\leqq\beta<\beta_c\}\quad \textit{and}\quad \{\tau(W_\beta)|_{\mathscr{A}_+}:\beta>\beta_c\}$$

do not extend to automorphisms of the Pauli algebra.

First we need some lemmas.

Lemma 5.2. *The operators*

$$\delta-u_-\delta u_-,\quad \textit{for}\quad \beta>\beta_c, \tag{5.3}$$

and

$$\Theta-u_-\Theta u_-,\quad \textit{for}\quad 0\leqq\beta<\beta_c, \tag{5.4}$$

are trace class.

Proof. If $z_i=\tanh K_i=e^{-2K_i^*}$, and $z_i^*=\tanh K_i^*=e^{-2K_i}$, then (2.18), (2.19) and (2.20) can be solved (see e.g. [13]) to get

$$e^{2i\delta(\omega)}=\frac{(1-z_2z_1^*e^{i\omega})(1-z_1^*e^{-i\omega}/z_2)}{(1-z_2z_1^*e^{-i\omega})(1-z_1^*e^{i\omega}/z_2)}. \tag{5.5}$$

Then for $\beta>\beta_c$ (i.e. $z_1^*<z_2<1$), the Fourier coefficients $\{k_n\}$ of $i\delta$ are given by

$$k_n=\frac{1}{2n}[(z_1^*/z_2)^n-(z_2z_1^*)^n]=-k_{-n},\quad n>0$$

([13, Eq. (75)]). Let $p_\pm=(1\pm u_-)/2$, then $\delta-u_-\delta u_-=2(p_+\delta p_-+p_-\delta p_+)$. Since δ is real, it is enough to show that $p_+\delta p_-$ is trace class if $\beta>\beta_c$. Let $\{e^{-ik\omega}:k=0,1,2,\ldots\}$ and $\{e^{ik\omega}:k=1,\ldots\}$ be complete orthonormal bases for p_-l_2 and p_+l_2 respectively. Then the matrix of $ip_+\delta p_-$ with respect to these bases is

$$\{k_{r+s+1}:r,s=0,1,2,\ldots\}.$$

Thus with this identification of p_-l_2 and p_+l_2 with $l_2(\mathbb{N})=l_2^+$, and if $\chi_\lambda=\{\lambda^i\}_{i=0}^\infty\in l_2^+$, for $0\leqq\lambda<1$, we have

$$ip_+\delta p_-=\{k_{r+s+1}:r,s\}=\frac{1}{2}\int_{z_2z_1^*}^{z_1^*/z_2}|\chi_\lambda\rangle\langle\chi_\lambda|d\lambda, \tag{5.6}$$

which is trace class for $0 \leqq z_1^* < z_2$. We have from (5.5) that

$$e^{2i\Theta(\omega)} = \frac{(1 - z_2 z_1^* e^{i\omega})(1 - z_2 e^{i\omega}/z_1^*)}{(1 - z_2 z_1^* e^{-i\omega})(1 - z_2 e^{-i\omega}/z_1^*)},$$

so that $i\Theta$ has Fourier coefficients $\{h_n\}$ given by

$$h_n = \frac{-1}{2n}[(z_2/z_1^*)^n + (z_2 z_1^*)^n] = -h_{-n}, \quad n > 0.$$

Thus

$$ip_+ \Theta p_- = -\frac{1}{2}\left[\int_0^{z_2/z_1^*} |\chi_\lambda\rangle\langle\chi_\lambda| d\lambda + \int_0^{z_2 z_1^*} |\chi_\lambda\rangle\langle\chi_\lambda| d\lambda\right], \tag{5.7}$$

which is trace class for $0 \leqq z_2 < z_1^*$ or $0 \leqq \beta < \beta_c$.

Lemma 5.4. *The operators*

$$W^* W_\beta - \theta_- W^* W_\beta \theta_- \tag{5.8}$$

and

$$W_\beta - \theta_- W_\beta \theta_- \tag{5.9}$$

are trace class for $\beta \neq \beta_c$.

Proof. We have

$$W^* W_\beta = \frac{1}{2}\begin{pmatrix} 1 + UU_\beta^* & 1 - UU_\beta^* \\ 1 - UU_\beta^* & 1 + UU_\beta^* \end{pmatrix}. \tag{5.10}$$

Thus $W^* W_\beta - \theta_- W^* W_\beta \theta_-$ being trace class is equivalent to $UU_\beta^* - u_- UU_\beta^* u_-$ being trace class. But $UU_\beta^* = e^{i\delta}$ as $\Theta(\theta) - \theta = \delta(\theta)$, and

$$e^{i\delta} - u_- e^{i\delta} u_- = i\int_0^1 u_- e^{i(1-s)\delta} u_- [\delta - u_- \delta u_-] e^{is\delta} ds. \tag{5.11}$$

Thus by (5.3) in Lemma 5.2 we see that

$$W^* W_\beta - \theta_- W^* W_\beta \theta_-, \quad \beta_c < \beta \leqq \infty \tag{5.12}$$

are trace class. Since $U_\beta^* = e^{i\Theta}$, we see in a similar manner using (5.4) of Lemma 5.2 that

$$W_\beta - \theta_- W_\beta \theta_-, \quad 0 \leqq \beta < \beta_c \tag{5.13}$$

are trace class. Now $W_\infty = W$, and a direct computation shows that

$$W^* - \theta_- W^* \theta_- \tag{5.14}$$

is trace class. The lemma now follows from the identity:

$$W^* W_\beta - \theta_- W^* W_\beta \theta_- = W^*(W_\beta - \theta_- W_\beta \theta_-) + (W^* - \theta_- W^* \theta_-)\theta_- W_\beta \theta_-, \tag{5.15}$$

and the operators in (5.12), (5.13) and (5.14) being trace class.

Remark 5.5. It follows from [3, Sect. 6] that the operators in (5.9) (and hence in (5.8) using (5.15)) are Hilbert–Schmidt, but this is not enough for our approach here. Note that it also follows from [3, Sect. 6] that $W^*W_{\beta_c} - \theta_- W^*W_{\beta_c}\theta_-$ and $W_{\beta_c} - \theta_- W_{\beta_c}\theta_-$ are not Hilbert Schmidt.

By [1], a Bogoliubov automorphism $\tau(v)$ on $\mathscr{A}^F$ is inner if and only if one of the following conditions hold:

$$1 - v \text{ is trace class and } \det v = 1, \tag{5.16}$$

$$1 + v \text{ is trace class and } \det(-v) = -1. \tag{5.17}$$

An inspection of the proofs in [1] shows that if (5.16) holds then $\tau(v)$ is even, and if (5.17) holds then $\tau(v)$ is odd.

Also note that if a unitary v commutes with Γ, and $1 - v$ is trace class, then $\det(v) = \pm 1$ [1, p. 414]. Moreover the map $w \to \det(1 - w)$ is continuous on the trace class operators [20].

We apply these considerations to the unitaries

$$\theta_- W^* W_\beta \theta_- W_\beta^* W, \tag{5.18}$$

$$\theta_- W_\beta \theta_- W_\beta^* \tag{5.19}$$

for $\beta \neq \beta_c$.

Lemma 5.6.

$$\det(\theta_- W^* W_\beta \theta_- W_\beta^* W) = \begin{cases} 1 & \beta_c < \beta \leqq \infty \\ -1 & 0 \leqq \beta < \beta_c \end{cases}, \tag{5.20}$$

$$\det(\theta_- W_\beta \theta_- W_\beta^*) = \begin{cases} 1 & 0 \leqq \beta < \beta_c \\ -1 & \beta_c < \beta \leqq \infty \end{cases}. \tag{5.21}$$

Proof. By Lemma 5.4, we see that

$$1 - \theta_- W^* W_\beta \theta_- W_\beta^* W = (W^* W_\beta - \theta_- W^* W_\beta \theta_-) W_\beta^* W \tag{5.22}$$

and

$$1 - \theta_- W_\beta \theta_- W_\beta^* = (W_\beta - \theta_- W_\beta \theta_-) W_\beta^* \tag{5.23}$$

are trace class if $\beta \neq \beta_c$.

As in [3] we now treat K_1^* and K_2 (or z_1 and z_2^*) as independent parameters. From [3, p. 500] we see that W_β is norm continuous in the region $z_1^* \neq z_2$. Then we have from (5.10), (5.11) (5.6), (5.7), (5.22), (5.23) and (5.15) that $1 - \theta_- W^* W_\beta \theta_- W_\beta^* W$ and $1 - \theta_- W_\beta \theta_- W_\beta^*$ are continuous in z_1^* and z_2 in the trace class norm when $z_1^* \neq z_2$. Hence using continuity of the determinant, it is enough to compute the determinants in the cases $z_1^* = 0$, $z_2 > 0$, ($\beta = \infty$) and $z_2 = 0$, $z_1^* > 0$, ($\beta = 0$), which is an easy exercise.

Proof of Theorem 5.1. We now apply Theorem 4.4 to the automorphisms

$$v_\beta = \begin{cases} \tau(W^* W_\beta), & \beta > \beta_c \\ \tau(W_\beta), & 0 \leqq \beta < \beta_c \end{cases},$$

using Lemmas 5.5 and 5.6 and (5.16) to see that $v_\beta|_{\mathscr{A}_+}$ extend to graded automorphisms of $\mathscr{A}^P$ also denoted by v_β. Then (5.1) and (5.2) follow from (2.17), (3.11) and (3.12).

Finally, it is now clear using Corollary 4.3 that the automorphisms

$$\{\tau(W^*W_\beta)|_{\mathscr{A}_+}: 0 \leqq \beta < \beta_c\} \quad \text{and} \quad \{\tau(W_\beta)|_{\mathscr{A}_+}: \beta > \beta_c\}$$

do not extend to $\mathscr{A}^P$.

References

1. Araki, H.: On quasifree states of CAR and Bogoliubov automorphisms. Publ. Res. Inst. Math. Sci **6**, 385–442 (1970)
2. Araki, H.: On the XY-model on two-sided infinite chain. Publ. Res. Inst. Math. Sci. **20**, 277–296 (1984)
3. Araki, H., Evans, D. E.: On a C^*-algebra approach to phase transition in the two-dimensional Ising model. Commun. Math. Phys. **91**, 489–503 (1983)
4. Baxter, R. J.: Exactly solved models in statistical mechanics. London: Academic Press 1982
5. Evans, D. E., Lewis, J. T.: The spectrum of the transfer matrix in the C^*-algebra of the Ising model at high temperatures. Commun. Math. Phys. **92**, 309–327 (1984)
6. Kaufman, B.: Crystal statistics II. Phys. Rev. **76**, 1232–1243 (1949)
7. Kaufman, B., Onsager, L.: Crystal statistics III. Phys. Rev. **76**, 1244–1252 (1949)
8. Jones, V. F. R.: Index for subfactors. Invent. Math. **72**, 1–25 (1983)
9. Jones, V. F. R.: Braid groups, Hecke algebras and type II_1 factors. Proceedings Japan US Conference 1983. (to appear)
10. Kramers, H. A., Wannier, G. H.: Statistics of the two-dimensional ferromagnet. Part 1. Phys. Rev. **60**, 252–262 (1941)
11. Lewis, J. T., Sisson, P. N. M.: A C^*-algebra of the two-dimensional Ising model. Commun. Math. Phys. **44**, 279–292 (1975)
12. Lewis, J. T., Winnink, M.: The Ising model phase transition and the index of states on the Clifford algebra. Colloq. Math. Soc. János Bolyai **27**, Random fields. Hungary Esztergom: 1979
13. Montroll, E., Potts, R. B., Ward, J. C.: Correlations and spontaneous magnetisation of the two-dimensional Ising model. J. Math. Phys. **4**, 308–322 (1963)
14. Onsager, L.: Crystal statistics. I. Phys. Rev. **65**, 117–149 (1944)
15. Onsager, L.: Discussion remark. (Spontaneous magnetisation of the two-dimensional Ising model). Nuovo Cimento (Suppl) **6**, 261–262 (1949)
16. Pimsner, M., Popa, S.: Entropy and index for subfactors. Preprint INCREST 1983
17. Pirogov, S.: States associated with the two-dimensional Ising model. Theor. Math. Phys. **11**, (3), 614–617 (1972)
18. Schultz, T. D., Mattis, D. C., Lieb, E.: Two-dimensional Ising model as a solvable problem of many Fermions. Rev. Mod. Phys. **36**, 856–871 (1964)
19. Shale, D., Stinespring, W. F.: Spinor representations of infinite orthogonal groups. J. Math. Mech. **14**, 315–322 (1965)
20. Simon, B.: Trace ideals and their applications. London Math. Soc. Lecture Note Series **35**. Cambridge: Cambridge University Press 1979
21. Sisson, P. N. M.: A C^*-algebra of the Ising model. Ph.D. Thesis. Dublin University 1975
22. Temperley, H. N. V., Lieb, E. H.: Graph-theoretical problems and planar lattices. Proc. R. Soc. Lond. **A322**, 251–280 (1971)

Communicated by H. Araki

Received March 6, 1985