Commun. Math. Phys. 102, 517–519 (1985)

Communications in Mathematical Physics

Comments

On the Concept of Attractor: Correction and Remarks

John Milnor

Institute for Advanced Study, Princeton, NJ 08540, USA

The following consists of three unrelated comments on the author's paper [1].

1. Correction

Let f be a continuous map from a compact metric space X to itself, with n^{th} iterate denoted by f^n, and let $A \subset X$ be a closed non-vacuous subset with $f(A)=A$. Consider the following two properties of A.

(I) *For any sufficiently small neighborhood U of A, the intersection of the images $f^n(U)$ for $n \geqq 0$ is equal to A (compare Smale* [2, p. 786]*).*

(II) (Asymptotic stability) *For any sufficiently small neighborhood U, the successive images $f^n(U)$ converge to A, in the sense that for any neighborhood V there exists n_0 so that $f^n(U) \subset V$, for $n \geqq n_0$.*

In [1, Sect. 1] the author mistakenly described an example satisfying (I) but not (II). (The example was based on a remark of Besicovitch [3], which was corrected in a later paper [4].) In fact, (I) implies (II). The following proof is a minor modification of Hurley [8, Lemma 1.6], which demonstrates a corresponding statement for flows on a compact manifold. The proof shows also that (I) implies the existence of arbitrarily small neighborhoods $W \supset A$ with $f(W) \subset W$.

Proof that (I) *implies* (II). Let U be an open neighborhood which is small enough so that the intersection of the forward images of the closure $\bar{U}$ is equal to A. Let U_n be the open neighborhood consisting of all points x such that $f^i(x) \in U$ for $0 \leqq i \leqq n$. Thus $U = U_0 \supset U_1 \supset \ldots \supset A$ and $f(U_n) \subset U_{n-1}$. Hence the intersection W of the U_n satisfies $f(W) \subset W$. We will show that W is equal to U_n for n sufficiently large, and hence that W is an open set. Otherwise, for infinitely many integers n there must exist a point x_n which belongs to U_n but not U_{n+1}. Let $y_n = f^n(x_n) \in U$. Then we can choose some subsequence of these points y_n which converges to a point $y \in \bar{U}$. Since y_n belongs to the intersection of the sets $f^i(\bar{U})$ for $0 \leqq i \leqq n$, it follows that y belongs to the intersection of all of the $f^i(\bar{U})$, which is equal to A by hypothesis. But $f(y_n) \notin U$, hence $f(y) \notin U$, contradicting the hypothesis that $f(A) = A \subset U$. This proves that W is open. Hence the compact set $\bar{W} \subset \bar{U}$ is a neighborhood of A with $f(\bar{W}) \subset \bar{W}$. It follows easily from compactness that the successive images $\bar{W} \supset f(\bar{W}) \supset f^2(\bar{W}) \supset \ldots$ with intersection A actually converge to A in the sense described in (II). □

Here is an example. Let $U \subset X$ be any open set satisfying $f^n(\bar{U}) \subset U$ for some n, and suppose that the intersection $B = \bar{U} \cap f(\bar{U}) \cap \ldots \cap f^{n-1}(\bar{U})$ is non-vacuous. Then the intersection of the forward images of B is a non-vacuous, compact, f-invariant set which satisfies (I) and (II). The proof is not difficult.

In the context of a smooth flow, Conley [5] uses the term *attractor* for a compact, non-vacuous invariant set satisfying the analogues of conditions (I) and (II), while Auslander et al. [6] call such a set an [asymptotically] *stable attractor*. Note, however, that the word attractor is used in a quite different sense in [1]. Here is an example: Let f be the map of the unit square given by $f(x, y) = (1-x)(x, y)$. Then the edge $x=0$ is the unique compact invariant set satisfying (I) and (II). However, the omega limit set $\omega(x, y)$ is equal to the origin for almost every point (x, y) in the square, hence by the definitions of [1] the origin is the unique attractor.

2. Remark

In order to know that the concept of "minimal attractor," as defined in [1], is reasonable and useful, one would at least like an affirmative answer to the following. *For a C^k-generic map or flow on a compact manifold, is it true that almost every point belongs to the realm of attraction of some "minimal attractor"?* Using standard definitions, a related question would be the following: *Does the chain-recurrent set of a C^k-generic map or flow have at most a countable number of chain components?* Whenever this is true, one can at least say that almost every point belongs to the realm of attraction $\varrho(A)$ for some chain component A which is "attractive" in the sense that $\varrho(A)$ has strictly positive measure.

For some of the known generic and non-generic properties of maps or flows, see [7–12] below.

3. Addendum

The discussion of an attractor related to the solenoid in [1, Appendix 3] should have included a reference to Mayer and Roepstorff [13], which contains a detailed discussion of a similar example.

References

1. Milnor, J.: On the concept of attractor. Commun. Math. Phys. **99**, 177–195 (1985)
2. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–817 (1967)
3. Besicovitch, A.S.: A problem on topological transformation of the plane. Fundam. Math. **27**, 61–65 (1937) (compare [4])
4. Besicovitch, A.S.: A problem on topological transformations of the plane. II. Proc. Camb. Phil. Soc. **47**, 38–45 (1951)
5. Conley, C.: Isolated invariant sets and the Morse index. CBMS-NSF Reg. Conf. **38**, Am. Math. Soc. 1978
6. Auslander, J., Bhatia, N.P., Seibert, P.: Attractors in dynamical systems. Bol. Soc. Mat. Mex. **9**, 55–66 (1964)

7. Dobrynskiĭ, V.A., Sarkovskiĭ, A.N.: Genericity of dynamical systems almost all orbits of which are stable under sustained perturbations. Sov. Math. Dokl. **14**, 997–1005 (1973)
8. Hurley, M.: Attractors: persistence, and density of their basins. Trans. Am. Math. Soc. **269**, 247–271 (1982) (compare [9])
9. Hurley, M.: Bifurcation and chain recurrence. Ergodic Theory Dyn. Syst. **3**, 231–240 (1983)
10. Newhouse, S.: Lectures on dynamical systems, pp. 1–114 of "Dynamical Systems". Guckenheimer, Moser, Newhouse (ed.). Boston, Basel, Stuttgart: Birkhäuser 1980
11. Robinson, R.C., Williams, R.F.: Finite stability is not generic, pp. 451–462 of "Dynamical Systems". Peixoto, M. (ed.). New York: Academic Press 1973
12. Takens, F.: Tolerance stability, pp. 293–304 of "Dynamical Systems". Lecture Notes in Mathematics, Vol. 468, Manning, A. (ed.). Berlin, Heidelberg, New York: Springer 1975
13. Mayer, D., Roepstorff, G.: Strange attractors and asymptotic measures of discrete-time dissipative systems. J. Stat. Phys. **31**, 309–326 (1983)

Communicated by A. Jaffe

Received September 3, 1985