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The Chern Classes of Sobolev Connections

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Abstract. Assume F is the curvature (field) of a connection (potential) on $\mathbf{R}^4$ with finite L^2 norm $\left(\int_{\mathbf{R}^4} |F|^2 dx < \infty\right)$. We show the chern number $c_2 = 1/8\pi^2 \int_{\mathbf{R}^4} F \wedge F$ (topological quantum number) is an integer. This generalizes previous results which showed that the integrality holds for F satisfying the Yang–Mills equations. We actually prove the natural general result in all even dimensions larger than 2.

0. Introduction

All solutions of the Yang–Mills equations on $\mathbf{R}^4$ with finite action actually arise from connections defined on $\mathbf{R}^4 \cup (\infty) = S^4$ [1, 2]. This implies that the chern numbers of these connections are the chern numbers of a bundle over S^4, and hence are integers. It seems to be a question of general interest whether this result holds for arbitrary connections on $\mathbf{R}^4$ with finite energy [3]. Schlafley showed this is indeed true if the curvature or field $|F|$ has growth at most $(r^2 \log r)^{-1}$ [4]. We prove that finite energy $\int_{\mathbf{R}^4} |F|^2 dx$ is sufficient. We prove general n-dimensional results. We assume throughout the paper that G is a compact Lie group with bi-invariant metric and $\mathfrak{g}$ is the Lie algebra for G.

Theorem. *Let $A_j \in L^{n/2}_{1,\mathrm{loc}}(\mathbf{R}^n, \mathfrak{g})$, $j = 1, 2, \ldots, n > 2$ and let $F = F_A = dA + A \wedge A$ be the curvature of the connection $d + A$. If n is even, $n \neq 2$, and $\int_{\mathbf{R}^n} |F|^{n/2} dx < \infty$, then the chern number arising from a representation $\rho: G \to \mathrm{SU}(N)$ is integral.*

The proof is somewhat lengthy, and could be shortened considerably for the case A_j smooth. However, it seemed worthwhile to treat the most general case, $A_j \in L^{n/2}_{1,\mathrm{loc}}$, for the purpose of completeness. The various technical theorems we use to handle non-smooth A_j have interesting features and possible applications elsewhere. The main idea of the proof is to choose a good gauge near (∞). This relies on an earlier theorem on the existence of good (Coulomb) gauges [5]. The idea for the proof arose from conversations with L. M. Sibner about the removable singularities theorem in dimension 3 [6].

1. Sobolev connections and Chern Number

Let P be a principal bundle over a compact Riemannian n-manifold M with structure group G, Lie algebra $\mathfrak{g}$. Recall that the class of smooth connections is an affine space, described by choosing any base connection D_0 and setting

$$\mathscr{A}(P) = \{D = D_0 + A\colon A \in C^\infty(T^*M \otimes \operatorname{Ad} P)\}.$$

We shall use the bundles $T^*M \otimes \operatorname{Ad} P = \eta_1(P)$ and $T^*M \wedge T^*M \otimes \operatorname{Ad} P = \eta_2(P)$ of Lie algebra-valued one and two forms, so often we use the notation $\eta_1(P)$ and $\eta_2(P)$. The space of Sobolev connections is

$$\mathscr{A}_k^p(P) = \{D_0 + A\colon A \in L_k^p(\eta_1(P))\}.$$

Here $L_k^p(\eta)$ denotes the Banach space of sections of η with partials up through order k in L^p. There is an equivalent local description. If $\mathscr{U}_\alpha$ is a cover of M and $\rho_\alpha\colon P|\mathscr{U}_\alpha \sim G \times \mathscr{U}_\alpha$ are smooth local trivializations, then

$$D|\mathscr{U}_\alpha \sim d + A_\alpha,$$

where $A_\alpha \in L_k^p(\mathscr{U}_\alpha, \mathbf{R}^n \otimes \mathfrak{g})$. It is a well-known fact that $\mathscr{A}(P)$ is dense in $\mathscr{A}_k^p(P)$.

The curvature of a smooth connection D is a smooth section $F(D)$ of the bundle $\eta_2(P)$ of Lie algebra-valued two forms. In local coordinates $F|\mathscr{U}_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$. From the local description, we easily see the following lemma:

Lemma 1.1. *The curvature map $F\colon \mathscr{A} \to C^\infty(\eta_2(P))$ extends to a smooth (in fact quadratic) map $F\colon \mathscr{A}_1^p \cap \mathscr{A}_0^{2p} \to L_0^p(\eta_2(P))$. If $2p \geqq n = \dim M$, then $\mathscr{A}_1^p \subset \mathscr{A}_0^{2p}$ and $F\colon \mathscr{A}_1^p \to L_0^p(\eta_2(P))$.*

Now let $\Lambda^n(M) = \Lambda^n$ be the bundle of n-forms over M. A smooth bundle map $\Psi\colon \eta \to \Lambda^n$ is said to be a homogeneous polynomial of degree q if $\Psi_x(\eta_x) = h_x(\eta_x \otimes \eta_x \otimes \cdots \otimes \eta_x)$, where $h\colon (\underset{q}{\otimes}\,\eta) \to \Lambda^n$ is linear.

This thinking leads us to the elementary observation:

Proposition 1.2. *Let $\Psi\colon \eta_2(P) \to \Lambda_n$ be a homogeneous polynomial map of degree p. Then the induced map $\bar{\Psi}\colon \mathscr{A} \to \mathbf{R}$ given by*

$$\bar{\Psi}(D) = \int_M \Psi(F(D))$$

extends to a smooth map $\Psi\colon \mathscr{A}_1^p(P) \cap \mathscr{A}^{2p}(P) \to \mathbf{R}$.

Proof. Note Ψ factors into a composition

$$\mathscr{A}_1^p(P) \cap \mathscr{A}^{2p}(P) \xrightarrow{F} L_0^p(\eta_2) \xrightarrow{\Psi} L_0^1(\Lambda^n) \xrightarrow{\int} \mathbf{R}.$$

Each piece is smooth. The map $\bar{\Psi}$ itself is polynomial. If Ψ is merely homogeneous of order p, $(\Psi_x(c\eta_x) = c^p \Psi_x(\eta_x),\ c > 0)$, then $\bar{\Psi}$ is C^k for k any integer less than p.

Corollary 1.3. *If $\bar{\Psi}$ takes on a constant value γ on $\mathscr{A}(P)$, then it has the value γ on every connection in $\mathscr{A}_1^p(P) \cap \mathscr{A}_0^{2p}(P)$.*

Proof. This follows from the density of $\mathscr{A}(P)$ in $\mathscr{A}_1^p(P) \cap \mathscr{A}_0^{2p}(P)$ and the continuity of Ψ. For $m < n/2$, the following can be compared to Sedlacek's result [7].

Corollary 1.4. *If $\rho: G \to \mathrm{SU}(N)$ is a representation, and ω a smooth n-2m form on M, then the characteristic class cupped with ω and evaluated on $[M]$, $c_m(\omega)$, which is a multiple of*

$$\int_M (\operatorname{tr} \rho(F)^m) \wedge \omega$$

is constant on $\mathscr{A}_1^m \cap \mathscr{A}^{2m}$. In particular, if n is even and $m = n/2$,

$$\int_M \operatorname{tr} \rho(F)^{n/2}$$

is a fixed integer times the appropriate dimensional constant on $\mathscr{A}_1^{n/2}$.

Proof. Here Ψ is given by $(\operatorname{tr} \rho(F)^m) \wedge \omega$, which is polynomial. Since $L_1^{n/2} \subset L^n$, we may replace $\mathscr{A}_1^{n/2} \cap \mathscr{A}^n$ by $\mathscr{A}_1^{n/2}$.

Corollary 1.5. *Let $\mathbf{S}: S^{2m} - \{\infty\} \to \mathbf{R}^n \cdot \mathbf{S}^{-1}: \mathbf{R}^{2m} \to S^{2m} - \{0\}$ be the usual stereographic projection and its inverse. Let $D \in \mathscr{A}_1^m$ be a connection in a principal bundle P on S^{2m}, and let the connection $d + A$ on $\mathbf{R}^{2m}$*

$$d + A = (\mathbf{S}^{-1})^* D$$

be obtained by pull-back of D. Then the chern number

$$c_m = \left(\frac{1}{2\pi i}\right)^m \frac{(-1)^{m+1}}{m} \int_{\mathbf{R}^{2m}} \operatorname{tr}(\rho(F)^m)$$

is integral, where $F = F(d + A) = dA + A \wedge A$.

Proof. Due to the way chern number is defined, on S^n

$$c_m = \left(\frac{1}{2\pi i}\right)^m \frac{(-1)^{m+1}}{m} \int_{S^{2m}} \operatorname{tr}(\rho(F(D))^m), \tag{3.1}$$

which is integral by the previous corollary.

2. Construction of the Bundles on $\mathbf{S}^n$

Recall that if M' is a non-compact manifold, by $L_{k,\mathrm{loc}}^p(M')$ we mean the Frechet space of functions whose restriction to any compact domain $\bar{\mathscr{U}} \subset M'$ lie in $L_k^p(\mathscr{U})$. Let $B' = \{x \in \mathbf{R}^n, 0 < |x| \leqq 1\}$. In Sect. 4, we obtain the following result as Corollary 4.6.

Theorem 2.1. *There exists $\varepsilon = \varepsilon(G, n) > 0$ such that if $D \in \mathscr{A}_{1,\mathrm{loc}}^{n/2}(B')$ and*

$$\int_{0<|x|\leqq 1} |F(D)|^{n/2} dx = \lim_{r \to 0} \int_{r \leqq |x| \leqq 1} |F(D)|^{n/2} dx < \varepsilon,$$

then there exists $s \in L_{2,\mathrm{loc}}^{n/2}(B^n, G)$ such that in the gauge

$$s^{-1} D s = d + s^{-1} ds + s^{-1} A s = d + \tilde{A},$$

we have $A \in L_1^{n/2}(B, \mathbf{R}^n \otimes \mathrm{g})$. Also

$$\| A \|_{L_1^{n/2}(B)} \leqq c(n, G) \| F(D) \|_{L^{n/2}(B)}.$$

Moreover, s may be chosen smooth on $\{x{:}0 < \rho \leqq |x| \leqq 1 - \rho < 1\}$.

We now describe how to pass from this technical result to our main result. We state the result for $\mathbf{R}^n$, although it, of course, applies to any conformal equivalent of a compact manifold with a finite number of points omitted.

Corollary 2.2. *Let* $A \in L^{n/2}_{1,\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^n \otimes \mathfrak{g})$, *and* $\int_{\mathbf{R}^n} |F_A|^{n/2} dx < \infty$. *Then the connection* $D = d + A$ *is gauge equivalent to the pull-back via the inverse stereographic projection of a connection* $D \in \mathscr{A}^{n/2}_1(P)$, *for some smooth principle bundle on* S^n.

Proof. Since $\int_{\mathbf{R}^n} |F_A|^{n/2} dx < \infty$, we can choose a $K < \infty$ such that $\int_{|x| \geqq K} |F_A|^{n/2} dx < \varepsilon$. Cover S^n with two coordinate charts, $\mathscr{U}_1 = \mathbf{S}^{-1}(x{:}|x| \leqq 4K)$ and $\mathscr{U}_2 = \mathbf{S}^{-1}(x{:}|x| \geqq K) \cup \{\infty\}$. Parametrize $B = \mathscr{U}_2$, $B' = \mathbf{S}^{-1}(x{:}|x| \geqq K)$ conformally in the obvious way. Now apply Theorem 2.1 to the connection $d + A$, which is now regarded as a connection $d + A'$ on B'. By conformal invariance.

$$\int_{|x| \geqq K} |F_A|^2 dx = \int_{|x| \leqq 1} |F_{A'}|^2 dx < \varepsilon.$$

The overlap between two charts is in the coordinates in $\mathscr{U}_2 = B$ (letting $\tilde{\mathscr{U}}_2 = \{|x| \leqq \frac{1}{2}\}$.

$$\tilde{\mathscr{U}}_2 \cap \mathscr{U}_1 = \{x \in B'{:}\tfrac{1}{4} \leqq |x| \leqq \tfrac{1}{2}\}.$$

The map $s{:}B' \to G$ restricts to the overlap function $g_{12}{:}\tilde{\mathscr{U}}_2 \cap \mathscr{U}_1 \to G$, and can be chosen smooth in this range $\{x{:}\frac{1}{4} \leqq |x| \leqq \frac{1}{2}\}$ $(\rho = \frac{1}{4})$. This map $g_{12} = s$ describes the principal bundle P on S^n. Since we started with $A|\mathscr{U}_1$ in $L^{n/2}_1$, and by coordinate change we obtain in $\tilde{\mathscr{U}}_2$

$$\tilde{A} = s^{-1} ds + s^{-1}(A'|\tilde{\mathscr{U}}_2)s,$$

in $L^{n/2}_1$, we have a local description of a connection in $\mathscr{A}^{n/2}_1(P)$.

Corollary 2.3. *If* $A \in L^{n/2}_{1,\mathrm{loc}}(\mathbf{R}^n, \mathbf{R}^n \otimes \mathfrak{g})$, $(n = 2m)$ *and* $\lim_{r \to \infty} \int_{|x| \leqq r} |F_A|^{n/2} dx < \infty$, *then*

$$c_m = \left(\frac{1}{2\pi i}\right)^m \frac{(-1)^{m+1}}{m} \int_{\mathbf{R}^n} \operatorname{tr}(\rho(F))^m dx$$

is an integer.

Proof. By the preceding corollary, the connection $d + A$ is obtained via a pull-back from an $\mathscr{A}^{n/2}_1(P)$ connection on S^n. By Corollary 1.5, $c_{n/2}$ is integral.

3. A Density Theorem

In Sect. 4, we obtain a map u: $B' \to G$, $u \in L^{n/2}_{2,\mathrm{loc}}(B', G)$. Now u is not necessarily continuous (it is if $u \in L^p_{2,\mathrm{loc}}(B', G)$ for any $p > n/2$). We wish to approximate u by a map which is smooth in an annullus $0 < \rho \leqq |x| \leqq 1 - \rho < 1$. To do this we mimic the proof of the approximation of L^2_1 maps from surfaces by Schoen and Uhlenbeck [8]. The slight technical difference is that we wish to keep the approximation fixed near the boundary.

For the following, G is any compact manifold isometrically immersed in $\mathbf{R}^k$ (i.e., the Lie group structure is irrelevant). In our case, we consider $\rho: G \subset SU(N) \subset \mathbf{C}^N \times \mathbf{C}^N = \mathbf{R}^k$ $(k = 2N^2)$. Let $\mathcal{O}_\delta$ be the set of points at distance δ from G. If δ is sufficiently small, the nearest point projection from $\mathbf{R}^k$ to G is well-defined and smooth on $\mathcal{O}_\delta$. Call this $\Pi: \mathcal{O}_\delta \to G$. Given any domain $\Omega \subset \mathbf{R}^n$, define

$$L_k^p(\Omega, G) = \{u \in L_k^p(\Omega, \mathbf{R}^k) : u(x) \in G \quad \text{a.e.}\}.$$

Let $\Omega_h = \{x \in \Omega : \operatorname{dist}(x, \mathbf{R}^n - \Omega) \leqq h\}$. Also, let φ be any positive, smooth bump function with compact support in the unit ball, $\int\limits_{\mathbf{R}^n} \varphi(y)dy = 1$. Given $u \in L_k^p(\Omega, \mathbf{R}^k)$, $x \in \Omega_n$

$$u^h(x) = \int\limits_{|x-y| \leqq h} u(x+hy)\varphi(y)dy. \tag{3.2}$$

It is well-known that u^h is smooth on Ω_h. However, $u^h(x) \notin G$.

Lemma 3.1. *There exists $\varepsilon_0 = \varepsilon_0(n, G)$ such that if $\int\limits_{B_h(x)} |du|^n(y)dy \leqq \varepsilon \leqq \varepsilon_0$, then the mollified function u^h has the property* $\operatorname{dist}(u^h(x), G) < K\varepsilon^{1/n}$.

Proof. The condition given in Schoen and Uhlenbeck [9.3.2] is $h^{-n+2} \int\limits_{B_h(x)} |du|^2 dy \leqq c\varepsilon^{1/2}$, which is implied by our assumption and the Hölder inequality.

Theorem 3.2. *Let $u \in L_1^n(\Omega, G)$, and Ω be a compact domain in $\mathbf{R}^n$ with smooth boundary. Then given $\mu > 0$, $d > 0$ there exists $\tilde{u} \in L_1^n(\Omega, G)$, ,$\tilde{u} = u$ on $\overline{\Omega - \Omega_d}$, $\tilde{u}|\Omega_{2d} \in C^\infty(\Omega_{2d}, G)$ and $\|\tilde{u} - u\|_{L^n(\Omega, \mathbf{R}^k)} \leqq \mu$. If $\mu \in L_2^{n/2}(\Omega, G)$, we may find $\tilde{u}$ with $\|\tilde{u} - u\|_{L_2^{n/2}(\Omega, \mathbf{R}^k)} \leqq \mu$.*

Proof. For $x \in \Omega_d$, we have that $\varepsilon(x, h) = \int\limits_{|x-y| \leqq h} |du|^n dx$ is a continuous family (in h) of continuous functions on Ω_d decreasing to 0. Therefore $\varepsilon(n) = \max \varepsilon(x, h) \to 0$. So for h sufficiently small, $\operatorname{dist}(u^h(x), G) \leqq K(\varepsilon(h))^{1/n} \to 0$. We observe that the proof that $\lim\limits_{h \to 0} u_h^* = u$ in $L_1^n(\Omega)$ is exactly as in the classical case $f \equiv 1$, once we show that the linear map $v \to v_h^*$ on L_1^n satisfies $\|v_h^*\|_{L_1^n(\Omega)} \leqq \bar{K} \|v\|_{L_1^n(\Omega)}$. But

$$\begin{aligned} \|v_h^*\|_{L^n(\Omega)}^n &\leqq \int \varphi(y) \int |v(x - hf(x)y)|^n dx dy \\ &\leqq \int \varphi(y) K \int |v(z)|^n dz dy \\ &= K \|v\|_{L^n(\Omega)}^n. \end{aligned} \tag{3.3}$$

Here $K = \max\limits_{y,x,h \leqq d} \left(\det \dfrac{\partial(x - hf(x)y)}{\partial x} \right)^{-1}$. By differentiating (3.2), we obtain

$$|d(v_h^*)(x)| \leqq ||dv|_h^*(x)| + h \max |df| (|v_h^*(x)|).$$

From this we get (using (3.3) again on dv and $|v|$)

$$\|d(v_h^*)\|_{L^n(\Omega)} \leqq K^{1/n} \|dv\|_{L^n(\Omega)} + K^{1/n} h \max |df| \; \|v\|_{L^n(\Omega)}.$$

Finally, we show $\| u_h^* - \tilde{u}_h \|_{L_1^n(\Omega)} \to 0$, (which completes the proof). We have already

$$\max_{x \in \Omega} |\tilde{u}_h(x) - u_h^*(x)| \leqq K_2(\varepsilon(n))^{1/n} \to 0$$

from Lemma 3.1. By the chain rule

$$\begin{aligned} |d\tilde{u}_h(x) - du_h^*(x)| &\leqq |d\Pi(u_h^*(x)) - I|\,|du_h^*(x)| \\ &\leqq K_3(\varepsilon(h))^{1/n} |du_h^*(x)|. \end{aligned}$$

Integrating both inequalities completes the proof. The proof for $L_2^{n/2}(\Omega)$ simply involves one more differentiation.

Note that this would be the first step in a proof that an $L_2^{n/2}$ bundle is "equivalent" to a smooth bundle. See Sedlacek [7] for a situation where such a theorem might be useful.

4. Coulomb Gauges

The theorem we wish to extend is the following theorem, proved in [5].

Theorem 4.1. *There exists $\varepsilon_0 = \varepsilon(n, G) > 0$ and $K_0 = K_0(n, G)$ such that if $A \in L_1^{n/2}(B^n, \mathbf{R}^n \times \mathfrak{g})$ and $\int_{B^n} |F_A|^{n/2} dx < \varepsilon$, then there exists $s \in L_2^{n/2}(B^n, G)$ such that $\tilde{A} = s^{-1} ds + s^{-1} A s$ satisfies*

a) $d^* A = 0$

b) $\int_{B^n} (|d\tilde{A}|^{n/2} + |\tilde{A}|^{n/2}) dx \leqq K_1 \int_{B^n} |F_A|^{n/2} dx.$

We use this above theorem and the following compactness theorem where $m = n - 1$ [5].

Theorem 4.2. *If $2p > m$, $M = M^m$ is a compact manifold, and D_i is a sequence of connections with $\int_M |F(D_i)|^p dx < b$, then a subsequence of the D_i is gauge equivalent to a sequence D_i which converges weakly in the space of L_1^p connections.*

The general techniques give an immediate corollary, which is what we shall use.

Corollary 4.3. *If $2p > m$ and $M = M^m$ a compact manifold, then there exists an $\varepsilon_1(p, M, G) > 0$ such that if D is a connection with $\int_M |F(D)|^p dx < \varepsilon_1$, then there exists a flat connection d on M such that D is gauge equivalent to D' with*

$$\| d - D' \|_{L_1^p(M)}^p \leqq K_1 \int_M |F(D)|^p d\mu.$$

Proof. By the weak compactness theorem we may assume that for ε_1 small, D' is weakly close to a flat connection and close in L_0^q for $m < q < 2p$. We may then find a Coulomb gauge for D' by solving

$$d^*(g^{-1} dg + g^{-1}(d - D')g) = 0$$

in the space $g \in L_1^q(M,G) \subset C^0(M,G)$ by means of the implicit function theorem. In the new gauge $D' = d + A$, where A is small in L_0^q, $q > m$, $d^*A = 0$, and it is easily seen that the L_1^p norm of A is bounded by a multiple of $\int_M |F(D)|^p d\mu$ by the usual techniques of standard elliptic theory.

We now prove our main extension lemma.

Lemma 4.4. *There exists* $\varepsilon_2 = \varepsilon_2(n, G)$ *such that if* $A \in L_1^{n/2}(\{x: \rho_1 \leqq x \leqq \rho_2\}, \mathbf{R}^n \times \mathfrak{g})$ *and* $\rho_1^{-1} \int_{|x|=\rho_1} |F_A|^{n/2} d\mu = \varepsilon < \varepsilon_2$, *then A is gauge equivalent to a connection* $\tilde{A}$ *which extends to a connection (again called* $\tilde{A}$*),* $\tilde{A} \in L_1^{n/2}(|x| \leqq \rho_2, \mathbf{R}^n \times \mathfrak{g})$ *and* $\int_{|x| \leqq \rho_1} |F_A|^{n/2} dx \leqq K_2 \varepsilon$.

Proof. By conformal invariance, we may assume $\rho_1 = 1$, $\rho_2 = \rho > 1$. Let $i(x) = x$ imbed $S^{n-1} \subset \mathbf{R}^n$ and let $i^*A = A^*$ be the pull-back Sobolev connection in a bundle over S^{n-1}. $\int_{S^{n-1}} |F_{A^*}|^{n/2} * 1 = \int_{S^{n-1}} |i^* F_A|^{n/2} * 1 \leqq \varepsilon_2$. By Corollary 4.3, with $p = n/2$, $m = n-1$, $M = S^{n-1}$, if ε_2 is sufficiently small $d + A^*$ is gauge equivalent to $d + A^{**}$, where

$$\|A^{**}\|_{L_1^{n/2}(S^{n-1})}^{n/2} \leqq K_1 \int_{S^{n-1}} |F_A|^{n/2} * 1 \leqq K_1 \varepsilon_2 .$$

Since C^∞ is dense in $L_1^{n/2}(S^{n-1})$ we may assume the gauge transformation s is smooth. Extend it to all of the annulus by $s(x) = s\left(\frac{x}{|x|}\right)$ and transform A via it. In the new gauge i^*A, the tangential part of A is small on S^{n-1}.

Let the normal part of A, $A_r \in L_1^{n/2}(S^{n-1}, \mathfrak{g})$, be approximated (to within $(\varepsilon)^{2/n}$) by a smooth C^∞ section g_r. Define a second smooth gauge transformation on $1 \leqq |x| \leqq \rho$ by $g(1, \theta) = I$, $\partial g / \partial r\,(1, \theta) = -g_r$ and $g(r, \theta)$ is geodesic in G as a function of r. Gauge transform again. Now we have the norm in this gauge

$$\|A | S^{n-1}\|_{L_1^{n/2}(S^{n-1}, \mathbf{R}^n \times \mathfrak{g})}^{n/2} \leqq K_2' \varepsilon .$$

Extend (in this gauge) by setting

$$A(x) = f(x) A\left(\frac{x}{|x|}\right)$$

for $f(x)$ a smooth function, $f(x) = 0$ for $|x| \leqq 1/2$ and $f(x) = 1$ for $|x| \geqq 3/4$. Now

$$\begin{aligned} \int_{|x| \leqq 1} |F_A|^{n/2} dx &\leqq C_1 \|A | B^n\|_{L_1^{n/2}(B^n)}^{n/2} \\ &\leqq C_1 C_2 \|A | S^{n-1}\|_{L_1^{n/2}(S^{n-1})}^{n/2} \leqq C_1 C_2 K_2' \varepsilon . \end{aligned}$$

This completes the proof, with $K_2 = C_1 C_2 K_2'$. The main technical difficulty is next solved.

Theorem 4.5. *There exists an $\varepsilon_3>0$ and $K_3<\infty$ such that if $A\in L^{n/2}_{1,\mathrm{loc}}(B',\mathbf{R}^n\times \mathfrak{g})$ and $\int_{B^n}|F_A|^{n/2}dx<\varepsilon_3$, then there exists a gauge transformation $g\in L^{n/2}_{2,\mathrm{loc}}(B',G)$ such that the gauge transformed connection*

$$\tilde{A}=g^{-1}dg+g^{-1}Ag\in L^{n/2}_1(B',\mathbf{R}^n\times \mathfrak{g}).$$

Moreover $d^*\tilde{A}=0$ and $\|\tilde{A}\|^{n/2}_{L^{n/2}_1(B^n)}\leqq K_3\int_{B^n}|F_{\tilde{A}}|^{n/2}dx$.

Proof. Choose a sequence of radii $\rho_i\to 0$ such that $A|\{|x|=\rho_i\}\in L^{n/2}_1$ and $\rho_i\int_{|x|=\rho_i}|F_A|^{n/2}dx=\varepsilon_i\to 0$. Let A_i be the extension of A_i gauge equivalent to $A|\{x:\rho_i\leqq|x|\leqq 1\}$ given by Lemma 4.4. We see immediately that

$$\int_{|x|\leqq 1}|F_{A_i}|^{n/2}dx\leqq\int_{|x|\leqq\rho_i}|F_{A_i}|^{n/2}dx+\int_{|x|\leqq 1}|F_A|^{n/2}dx\leqq K_2\varepsilon_i+\varepsilon_3.$$

Apply Theorem 4.1, which gives a gauge equivalent $\tilde{A}_i$ which can be estimated by $\|\tilde{A}_i\|^{n/2}_{L^{n/2}_1(B_n)}\leqq K_0(K_2\varepsilon_i+\varepsilon_3)$. Here $d^*\tilde{A}_i=0$.

First we may choose a weakly convergent subsequence $\tilde{A}_i-\tilde{A}\in L^{n/2}_1(B^n,\mathbf{R}^n\times\mathfrak{g})$. The weak convergence implies $d^*\tilde{A}=0$, and the inequality

$$\|A\|^{n/2}_{L^{n/2}_1(B^n)}\leqq K_0\varepsilon_3.$$

Let $\mathscr{C}_i=\{x:\rho_i\leqq|x|\leqq 1\}$ and $\tilde{g}_i\in L^{n/2}_1(\mathscr{C}_i,G)$ be the composition of the several gauge transformations constructed by Lemma 4.4 and Theorem 4.5 relating $A|\mathscr{C}_i$ and $\tilde{A}_i$. Then

$$\tilde{A}_i=\tilde{g}_i^{-1}\circ d\tilde{g}_i+\tilde{g}_i^{-1}A\tilde{g}_i.$$

If we fix j, this equation holds on $\mathscr{C}_j$ for $i\geqq j$. Since $A|\mathscr{C}_j\in L^{n/2}_1(\mathscr{C}_j,\mathbf{R}^n\times\mathfrak{g})$ and on $\mathscr{C}_j$, $i\geqq j$, we have for almost all x

$$\begin{aligned}|d\tilde{g}_i|&=|\tilde{g}_i\tilde{A}_i-A\tilde{g}_i|\leqq|\tilde{A}_i|+|A|,\\ |d^2\tilde{g}_i|&=|\tilde{g}_id\tilde{A}_i-dA\tilde{g}_i+d\tilde{g}_i\tilde{A}_i-\tilde{A}dg_i|\\ &\leqq|d\tilde{A}_i|+|dA|+|d\tilde{g}_i||\tilde{A}_i|+|A||d\tilde{g}_i|.\end{aligned}$$

From the first inequality and the Sobolev theorem we find for $i\geqq j$

$$\begin{aligned}\|\tilde{g}_i\|_{L^n_1(\mathscr{C}_j)}&\leqq\|\tilde{A}_i\|_{L^n(B^n)}+\|A|\mathscr{C}_j\|_{L^n(\mathscr{C}_j)}+\ell\\ &\leqq c^j(\|\tilde{A}_i\|_{L^{n/2}_1(B^n)}+\|A|\mathscr{C}_j\|_{L^{n/2}_1(\mathscr{C}_j)})+\ell.\end{aligned}$$

From this estimate, the second almost everywhere inequality and the Holder inequality we obtain

$$\|\tilde{g}_i\|_{L^{n/2}_2(\mathscr{C}_j)}\leqq\|\tilde{A}_i\|_{L^{n/2}_1(B^n)}+\|A|\mathscr{C}_j\|_{L^{n/2}_1(B^n)}+\|\tilde{g}_i\|_{L^n_1(\mathscr{C}_j)}(\|\tilde{A}_i\|_{L^n(B^n)}+\|A|\mathscr{C}_j\|_{L^n(\mathscr{C}_j)}).$$

By a diagonal argument, we extract a subsequence which converges weakly to a limit g in $L^{n/2}_2(\mathscr{C}_j,G)$ for all j. This produces $g\in L^{n/2}_{2,\mathrm{loc}}(B^n,G)$. Since the weak limit implies almost everywhere convergence for $\tilde{A}_i$ and $\tilde{g}_i$ we obtain

$$\tilde{A}=g^{-1}dg+g^{-1}Ag.$$

This completes the proof.

We obtain Theorem 2.1 as a simple corollary of this theorem and Theorem 3.2.

Corollary 4.6. In the statement of Theorem 4.5, if we relax the condition that $d^*\tilde{A}=0$, we may assume $g|\{x:\rho \leqq |x| \leqq 1-\rho\} \in C^\infty$.

Proof. Let $\Omega = \{x:\rho/2 \leqq |x| \leqq 1\}$ and $d=\rho/4$. We may approximate $g \in L_2^{n/2}(\Omega, G)$ as closely as we like by a function which is smooth on Ω_d. If the approximation is close enough, the inequalities remain valid, although $d^*\tilde{A}$ will no longer be zero in the new gauge.

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