Scaling of Mandelbrot Sets Generated by Critical Point Preperiodicity

J.-P. Eckmann[*] and H. Epstein

Institut des Hautes Etudes Scientifiques, 35, route de Chartres, F-91440 Bures-sur-Yvette, France

Abstract. Let $z \to f_\mu(z)$ be a complex holomorphic function depending holomorphically on the complex parameter μ. If, for $\mu = 0$, a critical point of f_0 falls after a finite number of steps onto an unstable fixed point of f_0, then, in the parameter space, near 0, an infinity of more and more accurate copies of the Mandelbrot set appears. We compute their scaling properties.

In several numerical experiments, the dynamics of rational maps has been studied as the coefficients of the map vary. It has been observed [M, DH, S, CGS, C] that remarkably precise copies of the standard Mandelbrot set $\mathcal{M}$ appear in the corresponding plots. In their recent work [DH], Douady and Hubbard explain this phenomenon by the local occurrence of "polynomial-like maps of degree 2." In this paper, we elaborate on this explanation by showing that the mechanism of "quadratification by large order iteration" [G, EEW] produces infinite sequences of copies of $\mathcal{M}$, obeying simple scaling properties.

We describe now the setting of a pre-periodic critical point, but immediately restrict our discussion to the case of period one. We consider a complex holomorphic function $(z, \mu) \to f_\mu(z)$ over $H \times D_M \subset \mathbb{C}^2$, where H is a domain in $\mathbb{C}$ and $D_M = \{\zeta \in \mathbb{C} \,|\, |\zeta| < M\}$, $M > 0$. We denote by f_μ the function $z \to f_\mu(z)$, and $D^{\alpha\beta}f_\mu = \partial_z^\alpha \partial_\mu^\beta f_\mu$ [1], $f_\mu' = \partial_z f_\mu$, etc. We make the following assumptions:

A1: f_0 has a non-degenerate critical point c.

A2: f_0 has an unstable fixed point u, $f_0'(u) = \gamma$, $|\gamma| > 1$. (The restriction of f_0 to a sufficiently small neighbourhood of u has a unique inverse which we denote f_0^{-1}.)

A3: (critical point preperiodicity) $f_0^Q(c) = u$,

$$(f_0^Q)''(c) \neq 0, \quad \text{for some} \quad Q \geq 1.$$

A4: There is a sequence $\{x_n\}$ of points in H accumulating at u such that

$$f_0(x_{n+1}) = x_n, f_0'(x_n) \neq 0, \quad \text{for} \quad n = 1, 2, \ldots,$$

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[*] On leave from the University of Geneva

[1] By abuse of notation, we write $D^{\alpha\beta}f_0$ instead of $D^{\alpha\beta}f_\mu|_{\mu=0}$

and $x_0 = c$. Furthermore, for sufficiently large n one has $x_n = f_0^{-1}(x_{n-1})$, with f_0^{-1} as in A2.

If M above is chosen sufficiently small, then f_μ will possess a critical point c_μ and an unstable fixed point u_μ for $\mu \in D_M$, depending analytically on μ and such that $c_0 = c$, $u_0 = u$. We further assume

A5: $\partial_\mu [f_\mu^Q(c_\mu) - u_\mu]|_{\mu=0} \neq 0$.

By the implicit function theorem, reducing M and H somewhat, we may change coordinates in such a way that $c_\mu = c$, $u_\mu = u$ for all $\mu \in D_M$ and that A1–A5 still hold.

Our result is a scaling relation for p-fold iterated functions.

Theorem 1. *There are non-zero constants K_1, K_2, K_3, and K such that for all sufficiently large p one has:*

1. *There is a number μ_p for which $f_{\mu_p}^p(c) = c$ and which satisfies*

$$|\mu_p - K_1 \gamma^{-p}| \leq K|\gamma|^{-2p}.$$

2. *The quantity $\lambda_p \equiv 2/(f_{\mu_p}^p)''(c)$ satisfies*

$$|\lambda_p - K_2 \gamma^{-p}| \leq K|\gamma|^{-3p/2}.$$

3. *The quantity $\delta_p \equiv \lambda_p^{-1} \partial_\mu f_\mu^p(c)|_{\mu=\mu_p}$ satisfies*

$$|\delta_p - K_3 \gamma^{2p}| \leq K|\gamma|^{7p/4}.$$

4. *The function*

$$F_a(z; p) \equiv \lambda_p^{-1}(f_{\mu_p + a/\delta_p}^p(\lambda_p z + c) - c) \tag{1}$$

satisfies

$$|F_a(\zeta; p) - \zeta^2 - a| \leq K|\gamma|^{-p/4}$$

for all $|\zeta| < |\gamma|^{p/16}/2K_2$ and $|a| < |\gamma|^{p/8}$.

For numerical calculations, it may be useful to give the constants K_1, K_2, K_3. One has

$$K_1 = \gamma^Q \lim_{n \to \infty} [\gamma^n (x_n - u)]/\partial_\mu f_0^Q(c), \tag{2}$$

$$K_2 = 2\gamma^Q \Big/ \Big(\lim_{n \to \infty} [(f_0^n)'(x_n)\gamma^{-n}] \cdot (f_0^Q)''(c) \Big), \tag{3}$$

$$K_3 = 2\partial_\mu f_0^Q(c)/(K_2^2(f_0^Q)''(c)). \tag{4}$$

As a result of the theorem, the sequence of functions $(\zeta, a) \to F_a(\zeta; p)$ tends, as $p \to \infty$, uniformly on every compact set to $(\zeta, a) \to P_a(\zeta) \equiv \zeta^2 + a$. For any $A \geq 4$, $|a| \leq A$, we have $P_a^{-1}(D_{A^2}) \subset D_{2A}$. Hence, for $r > 2A$ and sufficiently large p, we have $F_a^{-1}(D_{A^2}; p) \subset D_r$ and the triple $(D_{A^2}, F_a^{-1}(D_{A^2}; p), F_a(\cdot; p))$ is a "polynomial-like map of degree 2" in the sense of Douady and Hubbard [DH]. Recall that the Mandelbrot set $\mathscr{M}$ of the family $\{P_a\}$ is given by

$$\mathscr{M} = \bigcap_{n=1}^{\infty} g_n^{-1}(D_{A^2}), \quad \text{where} \quad g_n(a) = P_a^n(0).$$

Clearly, the local Mandelbrot set $\mathcal{M}_p$ of the family $\{F_a(\cdot; p)\}$, similarly defined, converges to $\mathcal{M}$ as $p \to \infty$. In particular,

$$\mathcal{M}^c \subset \bigcup_{R \geq 1} \bigcap_{p \geq R} \mathcal{M}_p^c. \tag{5}$$

On the other hand, $\mathcal{M}$ contains the open set $\mathcal{M}' = \{a \,|\, P_a \text{ has an attractive cycle}\}$ [2]. If $a \in \mathcal{M}'$, then, for sufficiently large p, $F_a(\cdot; p)$ has an attractive cycle of the same period as P_a which, by the results of [DH], attracts the critical point 0. Thus, if we define $\mathcal{M}_p'$ in analogy with $\mathcal{M}'$, we find

$$\mathcal{M}' \subset \bigcup_{R \geq 1} \bigcap_{p \geq R} \mathcal{M}_p'. \tag{6}$$

Theorem 1, combined with (5), (6) shows the asserted accumulation of Mandelbrot sets near $\mu = 0$, and their scale and position.

Before we start with the proof of the theorem proper, we restate A1–A5 and some of their consequences as H1–H4 below.

There are positive constants $L, M, U < 1$, $R(= 8 + 16L|\gamma|/(1 - |\gamma|^{-3/5}))$, such that:

H1: For $\mu \in D_M$, f_μ has a non-degenerate critical point at c.

H2: For all $\mu \in D_M$, the restriction of f_μ to $u + D_U$ has a unique inverse f_μ^{-1}, which satisfies for all $z \in u + D_U$:
 1. $f_\mu^{-1}(z) \in u + D_U$, $|(f_\mu^{-1})'(z)| \leq |\gamma|^{-4/5}$.
 2. $|f_\mu^{-1}(z) - u - \gamma^{-1}(z - u)| \leq L|z - u|(|z - u| + |\mu|)$.
 3. $|(f_\mu'(z) - \gamma| \leq L(|z - u| + |\mu|)$. In particular, $|f_\mu'(z)| \leq |\gamma|^{5/4}$.
 4. $|\partial_\mu f_\mu(z)| \leq L|z - u|$.

H3: There is a $P \geq 1$ and a holomorphic function $\mu \to z_\mu$ from D_M to $u + D_{U/R}$, such that for all $\mu \in D_M$ one has
 1. $f_\mu^P(z_\mu) = c$.
 2. $|D^{\alpha\beta} f_\mu^P(z) - D^{\alpha\beta} f_0^P(z_0)| \leq L(|z - z_0| + |\mu|)$, $\forall \alpha + \beta \leq 3$, $\forall z \in u + D_U$.
 3. $|(f_\mu^P)'(z_\mu)| > L^{-1}$, $|\partial_\mu f_0^P(z_0)| \leq L$.
 4. $|z_\mu - u| \geq L^{-1}$.
 5. $x_P = z_0$, $x_{P+n} = f_0^{-n}(x_P)$ for $n \geq 0$, $x_{P-n} = f_0^n(x_P)$ for $0 \leq n \leq P$.

H4: There is a $C > 0$ such that for every $\mu \in D_M$, one has
 1. $|D^{\alpha\beta} f_\mu^Q(z) - D^{\alpha\beta} f_0^Q(c)| \leq L(|z - c| + |\mu|)$, $\forall \alpha + \beta \leq 3$, $\forall z \in c + D_C$.
 2. $f_0^Q(c) = u$, $|\partial_\mu f_\mu^Q(c)| > L^{-1}$.
 3. $|(f_\mu^Q)''(c)| > L^{-1}$.

Proof of Theorem 1.1 (Well-known: see e.g. [G], [DH]). By H4.2, we see that for sufficiently small M, $\{f_\mu^Q(c) \,|\, |\mu| = M/2\}$ is a contour around u. By H2.2 and H3, $f_\mu^{-s}(z_\mu)$ lies inside this contour for sufficiently large s. Thus, by Rouché's theorem, there is a unique μ in $D_{M/2}$ for which $f_\mu^Q(c) = f_\mu^{-s}(z_\mu)$, i.e. $f_\mu^{s+P+Q}(c) = c$. Thus we have found μ_{s+P+Q}. For a quantitative treatment, we define, for all $(z, \mu) \in (u + D_U) \times D_M$,

$$h_\mu(z) = \lim_{n \to \infty} \gamma_\mu^n(f_\mu^{-n}(z) - u), \tag{7}$$

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2   We have no result concerning $\mathcal{M} \backslash \mathcal{M}'$, the so-called "composantes farfelues" [DH, MSS, ST]

where $\gamma_\mu = f'_\mu(u)$. By H2.2, the above limit exists uniformly in z, μ and satisfies

$$h_\mu(u) = 0, h'_\mu(u) = 1, h_\mu(f_\mu^{-1}(z)) = \gamma_\mu^{-1} h_\mu(z).$$ (8)

We can now rewrite $f_\mu^{s+P+Q}(c) = c$ as

$$\gamma_\mu^{-s} = h_\mu(f_\mu^Q(c))/h_\mu(z_\mu) \equiv v(\mu).$$

Since $z_0 \neq u$ by H3.4, we see from H3.2 and (6) that v is defined and $v'(0) \neq 0$. Using again H2.2, we see that

$$\mu = \gamma^{-s}(1 + \mu s k_s(\mu))/v'(0),$$

with $k_s(\mu)$ uniformly bounded for large s and $\mu \in D_{M/2}$. Theorem 1.1 follows by the contraction mapping theorem. A little algebra shows (2) in the form

$$K_1 = \gamma^Q h_0(x_0)/\partial_\mu f_0^Q(c), \ h_0(x_0) \equiv \gamma^P h_0(x_P).$$ (2')

As a preparation for the proof of Theorem 1.2–1.4, we study the deviation of a function φ from a quadratic polynomial by considering ([G, EEW]) the magnitude of $\varphi''/\varphi' - 1/z$. We therefore set, for large p,

$$\Delta_\mu(z;p) = \frac{(f_\mu^p)''(z)}{(f_\mu^p)'(z)} - \frac{(f_\mu^Q)''(z)}{(f_\mu^Q)'(z)}.$$

Lemma 2. *There is a constant K such that for sufficiently large p and all*

$$z \in c + D_{|\gamma|^{-3p/4}}, \ \mu \in \mu_p + D_{|\gamma|^{-3p/2}},$$ (9)

the inequality

$$|\Delta_\mu(z;p)| < K|(f_\mu^Q)'(z)||\gamma|^p$$ (10)

holds.

Proof. We set $s = p - P - Q$. Let $|\zeta - u| < 2U/R < U < 1$, $|\mu| < 2K_1|\gamma|^{-p}$. By H2, $|f_\mu^{-n}(\zeta) - u| \leq |\gamma|^{-4n/5}|\zeta - u|$. By induction, using H2.2, for all $n = 0, 1, \ldots, s$,

$$|f_\mu^{-n}(\zeta) - u| \leq |\gamma|^{-n}|\zeta - u| \left[1 + L|\gamma|(|\zeta - u| + 2K_1|\gamma|^{-p/5}) \frac{1 - |\gamma|^{-3n/5}}{1 - |\gamma|^{-3/5}} \right].$$

For sufficiently large p, this gives, by the definition of R,

$$|f_\mu^{-n}(\zeta) - u| < 2|\gamma|^{-n} \frac{U}{R} [1 + 2L|\gamma|/(1 - |\gamma|^{-3/5})] \leq |\gamma|^{-n} U/4.$$

We now fix z and μ satisfying (9) and denote, for $0 \leq n \leq s$,

$$w = f_\mu^Q(z), \hat{w} = f_{\mu_p}^Q(c), \hat{w}_n = f_{\mu_p}^n(\hat{w}) = f_{\mu_p}^{n-s}(z_{\mu_p}).$$

By the above estimate and Theorem 1.1,

$$|\hat{w}_n - u| < |\gamma|^{n-s} U/4.$$ (11)

By (9), $|w - \hat{w}| \leq 2L|\gamma|^{-3p/2}$ and $|w - u| < U/2$. An induction on n, $1 \leq n \leq s$, shows that $w_n \equiv f_\mu^n(w)$ is defined, and

$$|w_n - \hat{w}_n| \leq 2L|\mu - \mu_p| + |\gamma|^{5/4}|w_{n-1} - \hat{w}_{n-1}|$$
$$\leq 2L|\gamma|^{5n/4 - 3p/2}(1 - |\gamma|^{-5(n+1)/4})/(1 - |\gamma|^{-5/4}).$$

This implies $|w_n - u| < U/2$, $|w_s - u| < 2U/R$, for sufficiently large p, since $|\hat{w}_s - u| < U/R$. The first estimate, with $\zeta = w_s$, now gives:

$$|w_n - u| < |\gamma|^{n-s} U/4, \qquad 0 \le n \le s. \tag{12}$$

We can now turn to

$$\Delta_\mu(z; p) = (f_\mu^Q)'(z) \left\{ \frac{(f_\mu^P)''(w_s)}{(f_\mu^P)'(w_s)} \cdot (f^s)'(w) + \frac{(f^s)''(w)}{(f^s)'(w)} \right\}. \tag{13}$$

By H3, we have

$$|(f_\mu^P)''(\zeta)/(f_\mu^P)'(\zeta)| \le L^2, \qquad \forall \zeta \in u + D_U,$$

while H2 implies

$$|(f_\mu^j)'(w)| = \prod_{k=0}^{j-1} |f_\mu'(w_k)| \le |\gamma|^j \prod_{k=0}^{j-1} (1 + \mathcal{O}(|\gamma|^{-(k-s)/4})) \le \mathcal{O}(|\gamma|^j). \tag{14}$$

Since

$$\frac{(f_\mu^s)''(w)}{(f_\mu^s)'(w)} = \sum_{j=0}^{s-1} \frac{f_\mu''(w_j)}{f_\mu'(w_j)} (f^j)'(w), \tag{15}$$

we find

$$|(f_\mu^s)''(w)/(f_\mu^s)'(w)| \le \mathcal{O}(|\gamma|^s), \tag{16}$$

and Lemma 2 follows by combining (13)–(16).

For later use, we note that (12) describes accurately the position of w_n.

Proof of Theorem 1.2. By definition, we have $2\lambda_p^{-1} = (f_{\mu_p}^p)''(c)$, which we decompose as

$$(f_{\mu_p}^p)''(c) = (f_{\mu_p}^P)'(\hat{w}_s) \cdot (f_{\mu_p}^s)'(\hat{w}) \cdot (f_{\mu_p}^Q)''(c),$$

with the notations of the proof of Lemma 2. Using now (11) and H2 to control $(f_{\mu_p}^s)'$ and H4 to control $(f_{\mu_p}^s)'$ and $(f_{\mu_p}^Q)''$, one derives immediately Theorem 1.2, and a simple calculation leads to (3), in the form

$$K_2 = 2\gamma^Q h_0'(x_0)/(f_0^Q)''(c), \quad h_0'(x_0) \equiv \gamma^P h_0'(x_P)/(f_0^P)'(x_P). \tag{3'}$$

Note that (2') and (3') may be more convenient for numerical work than (2) and (3), since h_0 can be computed from its Taylor series.

We define

$$H_\mu(\zeta; p) = \lambda_p^{-1} f_\mu^P(\lambda_p \zeta + c). \tag{17}$$

Lemma 3. *For large p and for*

$$\zeta \in D_{|\gamma|^{p/4}/2K_2}, \qquad \mu \in \mu_p + D_{|\gamma|^{-3p/2}} \tag{18}$$

(where K_2 is defined by Theorem 1.2), one has

$$|H_\mu(\zeta; p) - H_\mu(0; p) - H_\mu''(0; p)\zeta^2/2| \le |H_\mu''(0; p)| \mathcal{O}(|\zeta|^3 |\gamma|^{-3p/4}). \tag{19}$$

Proof. Denote

$$G_\mu(\zeta; p) = H''_\mu(\zeta; p)/H'_\mu(\zeta; p) - 1/\zeta,$$

$$g(z, \mu) = (f^Q_\mu)''(z)/(f^Q_\mu)'(z) - 1/(z - c).$$

Then

$$G_\mu(\zeta; p) = \lambda_p \cdot \Delta_\mu(\lambda_p \zeta + c; p) + \lambda_p g(\lambda_p \zeta + c, \mu). \tag{20}$$

Our choice of domain, Eq. (18), assures that by Lemma 2, (20) is defined. Note that by H4, $|g(\lambda_p \zeta + c, \mu)| < 2L^2$, and hence by Lemma 2 and (17),

$$|G_\mu(\zeta; p)| \leq \mathcal{O}(|\gamma|^{-3p/4}) \quad \text{for} \quad \zeta, \mu \text{ as in (18)}.$$

Since $G_\mu(\zeta; p) = \partial_\zeta \log(H'_\mu(\zeta; p)/\zeta)$ and $H'_\mu(\zeta; p)/\zeta|_{\zeta=0} = H''_\mu(0; p)$, we can integrate, and we get

$$H'_\mu(\zeta; p) = \zeta H''_\mu(0; p) \exp \int_0^\zeta G_\mu(\eta; p) \, d\eta.$$

Hence,

$$|H'_\mu(\zeta; p) - \zeta H''_\mu(0; p)| \leq \mathcal{O}(|\zeta^2 H''_\mu(0; p)| |\gamma|^{-3p/4});$$

and the assertion follows by an additional integration.

Proof of Theorem 1.3. We want to relate the parametrization of H_μ to that given by the quantity $H_\mu(0; p)$. For this purpose, we study $\partial_\mu f^p_\mu(z)$, with $z = \lambda_p \zeta + c$, and ζ, μ as in (18). We rewrite

$$\partial_\mu f^p_\mu(z) = (\partial_\mu f^P_\mu)(w_s) + (f^P_\mu)'(w_s) \cdot (\partial_\mu f^s_\mu)(w_0) + (f^P_\mu)'(w_s) \cdot (f^s_\mu)'(w_0) \partial_\mu f^Q_\mu(z), \tag{21}$$

with $w_s = f^{s+Q}_\mu(z)$. We shall see that the last term in (21) dominates, and we therefore start by majorizing the first two terms.

By H3, and our earlier bounds (12) on w_s, we have

$$|\partial_\mu f^P_\mu(w_s)| \leq C_1$$

with C_1 independent of p, s, μ, z. By H2, we may write

$$(\partial_\mu f^s_\mu)(w_0) = \sum_{n=0}^{s-1} (f^n_\mu)'(w_{s-n})(\partial_\mu f_\mu)(w_{s-n-1}).$$

Using (12) and H2.4, we see that

$$|\partial_\mu f^s_\mu(w_0)| \leq \sum_{n=0}^{s-1} C_2 |\gamma|^n |w_{s-n-1} - u| \leq p \cdot C_3 |\gamma|^n |\gamma|^{-n}.$$

Thus the second term in (21) is bounded by a constant multiple of p. Using H4.1, 2, H2.3, and H3.2, we see that

$$(f^P_\mu)'(w_s)(f^s_\mu)'(w_0) \partial_\mu f^Q_\mu(z) = ((f^P_0)'(z_0)/h'_0(z_0)) \gamma^{p-P-Q} \partial_\mu f^Q_0(c)(1 + R|\gamma|^{-p/4}), \tag{22}$$

with R bounded independently of p, s, μ, z. This asymptotic behavior applies now to $\partial_\mu f_\mu^p$ as well and Theorem 1.3 follows at once. Again, we leave (4) as an exercise.

Proof of Theorem 1.4. We have just seen that

$$\delta_p \equiv \lambda_p^{-1} \partial_\mu f_\mu^p(c)|_{\mu=\mu_p} = K_3 \gamma^{2p}(1 + \mathcal{O}(|\gamma|^{-p/4})).$$

Using the definition of H_μ, we get from Theorem 1.3, that

$$|\partial_\mu H_\mu(\zeta; p)| \leqq C_5 |\gamma|^{2p}, \quad \text{for} \quad \zeta, \mu \quad \text{as in (18)}.$$

By the Cauchy inequalities, this implies

$$|\partial_\mu H_\mu''(0; p)| \leqq 2C_5 |\gamma|^{2p}/|\gamma|^{p/2} = 2C_5 |\gamma|^{3p/2},$$

and hence

$$|H_\mu''(0; p) - 2| \leqq 2C_5 |\gamma|^{3p/2} |\mu - \mu_p|. \tag{23}$$

If we require now $|\mu - \mu_p| < |\gamma|^{-15p/8}$, then

$$|H_\mu(0; p) - \delta_p(\mu - \mu_p)| \leqq C_6 |\gamma|^{2p} |\mu - \mu_p|^2/(|\gamma|^{-3p/2} - |\gamma|^{-15p/8}) \leqq \mathcal{O}(|\gamma|^{-p/4}). \tag{24}$$

Combining (23), (24) with Lemma 3, we get Theorem 1.4.

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Note added in proof. We have learned from A. Douady that he and J. Hubbard have obtained the same results, which will appear in their paper in preparation.