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Ground States of the XY-Model

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Abstract. Ground states of the XY-model on infinite one-dimensional lattice, specified by the Hamiltonian

$$-J[\sum\{(1+\gamma)\sigma_x^{(j)}\sigma_x^{(j)}+(1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)}\}+2\lambda\sum\sigma_z^{(j)}]$$

with real parameters $J\neq 0$, γ and λ, are all determined. The model has a unique ground state for $|\lambda|\geqq 1$, as well as for $\gamma=0$, $|\lambda|<1$; it has two pure ground states (with a broken symmetry relative to the $180°$ rotation of all spins around the z-axis) for $|\lambda|<1$, $\gamma\neq 0$, except for the known Ising case of $\lambda=0$, $|\gamma|=1$, for which there are two additional irreducible representations (soliton sectors) with infinitely many vectors giving rise to ground states.

The ergodic property of ground states under the time evolution is proved for the uniqueness region of parameters, while it is shown to fail (even if the pure ground states are considered) in the case of non-uniqueness region of parameters.

1. Main Results

We study ground states of the XY-model in the external transverse field on one-dimensional lattice (infinitely extended in two directions). Physical observables of the model are Pauli spins

$$\sigma_\alpha^{(j)} \quad (\alpha=x,y,z)$$

on each lattice site $j\in\mathbb{Z}$ ($[\sigma_\alpha^{(j)},\sigma_\beta^{(k)}]=0$ for $j\neq k$), which generates a UHF algebra $\mathfrak{A}$. The local Hamiltonian for an interval $[a,b]$ $(a<b)$ is

$$H(a,b)=-J\left[\sum_{j=a}^{b-1}\left\{\{(1+\gamma)\sigma_x^{(j)}\sigma_x^{(j+1)}+(1-\gamma)\sigma_y^{(j)}\sigma_y^{(j+1)}\}+2\lambda\sum_{j=a}^{b}\sigma_z^{(j)}\right.\right], \quad (1.1)$$

where J,γ (asymmetry of x and y), λ ($-2J\lambda$ being the strength of the external field) are real parameters and we assume* $J>0$.

* The sign of the first summation in $H(a,b)$ can be inverted by $180°$ rotation of σ spins around the z-axis at every other site (for example at all odd sites) and the sign of the last summation can be inverted by the $180°$ rotation of all σ-spins around the x-axis, for example. Therefore the case of $J<0$ can be reduced to the case of $J>0$ under consideration

The time evolution of an observable $A \in \mathfrak{A}$ is defined by

$$\alpha_t(A) \equiv \lim_{N \to \infty} e^{itH(-N,N)} A e^{-itH(-N,N)}, \tag{1.2}$$

where the limit is known to exist. (See Theorem 6.2.4 and 6.2.6 in [6].) Its generator

$$\delta(A) = (d/dt)\alpha_t(A)|_{t=0} \tag{1.3}$$

(defined whenever $\alpha_t(A)$ is norm differentiable) has as its core the subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ consisting of all polynomials of σ's (strictly local observables) and, for any $A \in \mathfrak{A}_0$,

$$\delta(A) = \dot{A} \equiv i[H(-N,N), A] \tag{1.4}$$

for sufficiently large N for which the right-hand side is independent of N. A ground state φ is then characterized by

$$-i\varphi(A^*\delta(A)) \geqq 0, \quad A \in \mathfrak{A}_0. \tag{1.5}$$

(The Definition 5.3.18 of [6].) It follows from the reality of the left-hand side of (1.5) that $\varphi(\delta(B)) = 0$ for $B = A^*A$ and hence for any $B \in \mathfrak{A}_0$, which then implies

$$\varphi(\alpha_t(A)) = \varphi(A) \quad (A \in \mathfrak{A}). \tag{1.6}$$

Theorem 1. (1) *The number of external ground states (which are necessarily pure) is as follows:*

(α) 1 *if* $|\lambda| \geqq 1$ *or if* $\gamma = 0$, $|\lambda| < 1$.
(β) 2 *if* $|\lambda| < 1$, $\gamma \neq 0$ *and* $(\lambda, \gamma) \neq (0, \pm 1)$.
(γ) ∞ *if* $(\lambda, \gamma) = (0, \pm 1)$.

(2) *For the case* (γ), *which is the Ising model, there are 2 extremal ground states which are the continuation from the region* (β) *and 2 additional irreducible representations of* $\mathfrak{A}$ *in which any vector in an infinite dimensional subspace (of the representation space) gives rise to an extremal ground state.*

Remark. Existence of two ground states in the region (β) has been previously indicated by a study of correlation functions [5, 9]. Results for the case (γ) have been obtained in Example 6.2.56 of [6].

For an extremal ground state φ, consider the locally perturbed state

$$\varphi_B(A) \equiv \varphi(B^*AB)/\varphi(B^*B) \tag{1.7}$$

for $B \in \mathfrak{A}$. We say that φ has the ergodic property under the time evolution if

$$\lim_{t \to \infty} \varphi_B(\alpha_t(A)) = \varphi(A) \tag{1.8}$$

holds for all $A, B \in \mathfrak{A}$.

Theorem 2. *The unique ground state in the case* (α) *has the ergodic property* (1.8) *under the time evolution, while any ground state in the cases* (β) *and* (γ) *fails to have such a property.*

2. Jordan–Wigner Transformation

We use the method developed in [4, 3]. We enlarge the algebra $\mathfrak{A}$ to a larger algebra

$\hat{\mathfrak{A}}$, adding a new element T having the following property:

$$T^2 = 1, \quad T^* = T, \quad TA = \Theta_-(A)T \quad (A \in \mathfrak{A}), \tag{2.1}$$

where Θ_- is an (involutive) automorphism of $\mathfrak{A}$ given by

$$\Theta_-(A) = \lim_{N\to\infty} \left(\prod_{j=0}^{-N} \sigma_z^{(j)} \right) A \left(\prod_{j=0}^{-N} \sigma_z^{(i)} \right) \tag{2.2}$$

(180° rotation of spins around the z-axis on the left half ($j \leqq 0$) of the lattice). More concretely,

$$\Theta_-(\sigma_x^{(j)}) = \begin{cases} \sigma_x^{(j)} \\ -\sigma_x^{(j)} \end{cases}, \quad \Theta_-(\sigma_y^{(j)}) = \begin{cases} \sigma_y^{(j)} & \text{if } j \geqq 1, \\ -\sigma_y^{(j)} & \text{if } j \leqq 0, \end{cases}$$

$$\Theta_-(\sigma_z^{(i)}) = \sigma_z^{(j)} \quad (\forall j).$$

$\hat{\mathfrak{A}}$ is the crossed product of $\mathfrak{A}$ by the group Z_2 (integers modulo 2) via its action $n \in Z_2 \to \Theta_-^n$ and is decomposed as a direct sum:

$$\hat{\mathfrak{A}} = \mathfrak{A} + \mathfrak{A}T. \tag{2.3}$$

We extend Θ_- to an automorphism of $\hat{\mathfrak{A}}$ by defining

$$\Theta_-(A_1 + A_2T) = \Theta_-(A_1) + \Theta_-(A_2)T \quad (A_1, A_2 \in \mathfrak{A}). \tag{2.4}$$

Within $\hat{\mathfrak{A}}$, we introduce creation and annihilation operators by the following Jordan–Wigner transformation [8], where T plays the role of a product of $\sigma_z^{(j)}$ from $j = 0$ to $-\infty$.

$$c_j^* = TS_j(\sigma_x^{(j)} + i\sigma_y^{(j)})/2, \tag{2.5a}$$

$$c_j = TS_j(\sigma_x^{(j)} - i\sigma_y^{(j)})/2, \tag{2.5b}$$

$$S_j = \begin{cases} \sigma_z^{(1)} \cdots \sigma_z^{(j-1)} & \text{if } j \geqq 2, \\ 1 & \text{if } j = 1, \\ \sigma_z^{(0)} \cdots \sigma_z^{(j)} & \text{if } j \leqq 0. \end{cases} \tag{2.6}$$

They satisfy the canonical anticommutation relations:

$$[c_j, c_k]_+ = [c_j^*, c_k^*]_+ = 0, \tag{2.7a}$$

$$[c_j, c_k^*]_+ = \delta_{ik} \mathbb{1} \tag{2.7b}$$

and generate a C^*-subalgebra of $\hat{\mathfrak{A}}$, which we denote $\mathfrak{A}^{\mathrm{CAR}}$. We have

$$\Theta_-(c_j^*) = \begin{cases} c_j^* \\ -c_j^* \end{cases}, \quad \Theta_-(c_j) = \begin{cases} c_j & \text{if } j \geqq 1, \\ -c_j & \text{if } j \leqq 0. \end{cases} \tag{2.8}$$

Let Θ be an automorphism of $\hat{\mathfrak{A}}$, uniquely determined by

$$\Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)}, \quad \Theta(\sigma_y^{(j)}) = -\sigma_y^{(j)}, \quad \Theta(\sigma_z^{(j)}) = \sigma_z^{(j)}, \tag{2.9}$$

$$\Theta(T) = T. \tag{2.10}$$

(The 180° rotation of all spins around the z-axis.) Then

$$\Theta(c_j^*) = -c_j^*, \quad \Theta(c_j) = -c_j. \tag{2.11}$$

Both subalgebras $\mathfrak{A}$ and $\mathfrak{A}^{\mathrm{CAR}}$ of $\hat{\mathfrak{A}}$ are Θ-invariant as sets and are split into sum of even and odd parts:

$$A = A_+ + A_-, \quad A_\pm \equiv (A \pm \Theta(A))/2, \tag{2.12}$$

$$\mathfrak{A} = \mathfrak{A}_+ + \mathfrak{A}_-, \quad \mathfrak{A}_\pm \equiv \{A \in \mathfrak{A}; \Theta(A) = \pm A\}, \tag{2.13}$$

$$\mathfrak{A}^{\mathrm{CAR}} = \mathfrak{A}^{\mathrm{CAR}}_+ + \mathfrak{A}^{\mathrm{CAR}}_-, \quad \mathfrak{A}^{\mathrm{CAR}}_\pm \equiv \{A \in \mathfrak{A}^{\mathrm{CAR}}; \Theta(A) = \pm A\}. \tag{2.14}$$

Two algebras are related by

$$\mathfrak{A}_+ = \mathfrak{A}^{\mathrm{CAR}}_+, \quad \mathfrak{A}_- = \mathfrak{A}^{\mathrm{CAR}}_- T. \tag{2.15}$$

Let $\tilde{\Theta}_-$ be an automorphism of $\hat{\mathfrak{A}}$, called the dual action in the language of the crossed product, defined by

$$\tilde{\Theta}_-(A_1 + A_2 T) = A_1 - A_2 T \quad (A_1, A_2 \in \mathfrak{A}). \tag{2.16}$$

According to even-odd properties under mutually commuting involutive automorphisms Θ and $\tilde{\Theta}_-$, $\hat{\mathfrak{A}}$ is decomposed into 4 pieces:

$$\hat{\mathfrak{A}} = \mathfrak{A}^{\mathrm{CAR}}_+ + \mathfrak{A}^{\mathrm{CAR}}_- + \mathfrak{A}^{\mathrm{CAR}}_+ T + \mathfrak{A}^{\mathrm{CAR}}_- T, \tag{2.17}$$

in which

$$\mathfrak{A} = \mathfrak{A}^{\mathrm{CAR}}_+ + \mathfrak{A}^{\mathrm{CAR}}_- T. \tag{2.18}$$

The local Hamiltonian (1.1) belongs to $\mathfrak{A}_+ = \mathfrak{A}^{\mathrm{CAR}}_+$ and is expressed in terms of c's by

$$H(a,b) = 2J\left[\sum_{j=a}^{b-1} \{(c_j^* c_{j+1} + c_{j+1}^* c_j) + \gamma(c_j^* c_{j+1}^* + c_{j+1} c_j)\} - \lambda \sum_{j=a}^{b} (2c_j^* c_j - 1)\right]. \tag{2.19}$$

The limit (1.2) exists and defines the time translation automorphism α_t for elements of $\hat{\mathfrak{A}}$:

$$\alpha_t(A_1 + A_2 T) = \alpha_t(A_1) + \alpha_t(A_2) V_t T, \tag{2.20}$$

$$\begin{aligned} V_t &= \lim_{N \to \infty} e^{itH(-N,N)} T e^{-itH(-N,N)} T \\ &= \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \alpha_{t_1}(\Delta) \cdots \alpha_{t_n}(\Delta), \end{aligned} \tag{2.21}$$

$$\Delta = H(-N,N) - \Theta_-(H(-N,N)) = -4J\{(1+\gamma)\sigma_x^{(0)}\sigma_0^{(1)} + (1-\gamma)\sigma_y^{(0)}\sigma_y^{(1)}\}. \tag{2.22}$$

The merit of the Jordan–Wigner transform is to enable us to write α_t on $\mathfrak{A}^{\mathrm{CAR}}$ in a compact form. Let

$$c^*(f) = \sum_{j \in \mathbb{Z}} c_j^* f_j, \quad c(f) = \sum_{j \in \mathbb{Z}} c_j f_j, \tag{2.23}$$

where $f = (f_j) \in l_2(\mathbb{Z})$. Further, let

$$B(h) = c^*(f) + c(g), \quad h = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.24}$$

The B's satisfy (and are characterized by)

$$[B(h_1)^*, B(h_2)] = (h_1, h_2), \quad B(h)^* = B(\Gamma h), \tag{2.25}$$

where, for $h_k = \begin{pmatrix} f_k \\ g_k \end{pmatrix}$ $(k = 1, 2)$, we denote

$$(h_1, h_2) = (f_1, f_2) + (g_1, g_2), \tag{2.26}$$

$$(f_1, f_2) = \sum_{j \in \mathbb{Z}} \bar{f}_{1j} f_{2j}, \tag{2.27}$$

$$\Gamma \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \bar{g} \\ \bar{f} \end{pmatrix}. \tag{2.28}$$

Then

$$\alpha_t(B(h)) = B(e^{2JiKt} h), \tag{2.29}$$

$$K = \begin{pmatrix} U + U^* - 2\lambda & \gamma(U - U^*) \\ -\gamma(U - U^*) & -(U + U^* - 2\lambda) \end{pmatrix} \tag{2.30}$$

$$(Uf)_j = f_{j+1}, \quad (U^* f)_j = f_{j-1}. \tag{2.31}$$

3. Main Points in the Proof of Theorem 1

Ground states of $(\mathfrak{A}^{\mathrm{CAR}}, \alpha_t)$ as well as those of $(\mathfrak{A}_+^{\mathrm{CAR}}, \alpha_t)$ can be determined by the following theorems proved in Sect. 9. (For an application we need only Theorem 3(1) and 4(1α), plus 4(3) for the Ising case.)

Theorem 3. *Let $\alpha_t(B(h)) = B(e^{iLt} h)$ where $L^* = L$, $\Gamma L = -L\Gamma$.*

(1) *There exists a unique ground state for $(\mathfrak{A}^{\mathrm{CAR}}, \alpha_t)$ if and only if* 0 *is not an eigenvalue of L. The unique ground state is the Fock state φ_{E_+} for which $B(h)$, $E_+ h = h$, is a creation operator and $B(h)$, $E_- h = h$, is an annihilation operator, $E_\pm$ being the spectral projection of L for* $(0, \infty)$ *and for* $(-\infty, 0)$, *respectively. More explicitly*

$$\varphi(B(h_1) \cdots B(h_{2n+1})) = 0, \tag{3.1}$$

$$\varphi(B(h_1) \cdots B(h_{2n})) = \sum \operatorname{sign} P \prod_{j=1}^{n} (\Gamma h_{P(2j-1)}, E_+ h_{P(2j)}), \tag{3.2}$$

where the sum is over all pairing, i.e. *over all permutations P satisfying*

$$P(2j-1) < P(2j) \quad (\forall j), \quad P(1) < P(3) < \cdots < P(2n-1).$$

(2) *If the eigenprojection E_0 of L for the eigenvalue* 0 *is not zero, then all ground states are given by*

$$\varphi(A_1 A_0) = \varphi_{E_+}(A_1) \varphi_0(A_0) \tag{3.3}$$

where A_0 belongs to the C^*-subalgebra $\mathfrak{A}_{E_0}^{\mathrm{CAR}}$ of $\mathfrak{A}^{\mathrm{CAR}}$ generated by $B(h)$ with h satisfying $E_0 h = h$, A_1 belongs to the C^*-subalgebra $\mathfrak{A}_{1-E_0}^{\mathrm{CAR}}$ of $\mathfrak{A}^{\mathrm{CAR}}$ generated by $B(h)$ with h satisfying $E_0 h = 0$, φ_{E_+} is the Fock state of $\mathfrak{A}_{1-E_0}^{\mathrm{CAR}}$ specified by the spectral *projection* E_+ *of L for* $(0,\infty)$ *as described in case (1) and* φ_0 *is an arbitrary state of* $\mathfrak{A}^{\mathrm{CAR}}_{E_0}$.

Theorem 4. *Consider the same situation as Theorem 3.*

(1) *There exists a unique ground state for* $(\mathfrak{A}^{\mathrm{CAR}}_+, \alpha_t)$ *if and only if one of the following (mutually exclusive) conditions* (α) *and* (β) *is satisfied:*

(α) $E_0 = 0$ *and the infimum of the positive part of the spectrum of L is not an eigenvalue of L.*

(β) $\dim E_0 = 1$.

In both cases, the unique ground state is the restriction of any ground state of $(\mathfrak{A}^{\mathrm{CAR}}, \alpha_t)$. *More explicitly*

$$\varphi(A_1 + A_2 B(h)) = \varphi_{E_+}(A_1) \tag{3.4}$$

for $A_1, A_2 \in \mathfrak{A}^{\mathrm{CAR}}_{1-E_0}$, $\Theta(A_1) = A_1$, $\Theta(A_2) = -A_2$ and $E_0 h = h$.

(2) *If* $\dim E_0 > 1$, *the set of all ground states of* $(\mathfrak{A}^{\mathrm{CAR}}_+, \alpha_t)$ *coincides with the set of restrictions of all ground states of* $(\mathfrak{A}^{\mathrm{CAR}}, \alpha_t)$ *to* $\mathfrak{A}^{\mathrm{CAR}}_+$.

(3) *If* $E_0 = 0$ *and the infimum e of the positive part of the spectrum of L is an eigenvalue of L with the eigenprojection E, then an extremal ground state* φ *of* $(\mathfrak{A}^{\mathrm{CAR}}_+, \alpha_t)$ *is either the restriction of* φ_{E_+} *to* $\mathfrak{A}^{\mathrm{CAR}}_+$ *or the (Fock) state* φ_h *defined by*

$$\varphi_h(A) = \varphi_{E_+}(B(h)^* A B(h))/(h,h) = \varphi_{E_+ - P(h) + \Gamma P(h)\Gamma}(A), \tag{3.5}$$

where h is any vector satisfying $E_e h = h$ *and* $P(h)$ *is the orthogonal projection onto the one-dimensional space spanned by h. The cyclic representations associated with* φ_h *are all equivalent and are disjoint from the cyclic representation associated with the restriction of* φ_{E_+} *to* $\mathfrak{A}^{\mathrm{CAR}}_+$. *An arbitrary ground state of* $(\mathfrak{A}^{\mathrm{CAR}}, \alpha_t)$ *is of the following form:*

$$\varphi = \alpha_0 \varphi_{E_+} + \sum_{i=1}^{\infty} \alpha_i \varphi_{h_i}, \tag{3.6}$$

where $\alpha_j \geqq 0$ $(j = 0, \ldots)$, $\sum \alpha_j = 1$ *and* h_j *are mutually orthogonal.*

The following lemma (proved in Sect. 7(i)) shows that $E_0 = 0$ for all cases of our interest and $E_e = 0$ for almost all cases, except for the Ising model case $(\lambda, \gamma) = (0, \pm 1)$.

Lemma 3.1. (1) *If* $(\lambda, \gamma) \neq (0, \pm 1)$, *K has an absolutely continuous spectrum.*

(2) *If* $(\lambda, \gamma) = (0, \pm 1)$, *then* $\mathrm{Sp}\, K = \{2, -2\}$, i.e. $K/2$ *is selfadjoint unitary.*

We can now discuss a ground state φ of $(\mathfrak{A}, \alpha_t)$. By a version of the definition of a ground state, requiring (1.5) for all A in the domain of δ (within respective algebras under considerations), the restriction of φ to $\mathfrak{A}_+ = \mathfrak{A}^{\mathrm{CAR}}_+$ is immediately seen to be a ground state of $(\mathfrak{A}^{\mathrm{CAR}}_+, \alpha_t)$, and hence is given by Theorem 4. If φ is Θ-invariant (i.e. $\varphi(\Theta(A)) = \varphi(A)$ for all $A \in \mathfrak{A}$) in addition, then φ is completely determined by its restriction to $\mathfrak{A}_+$. Thus we obtain the following key intermediate result:

Proposition 3.2. (1) *If* $(\lambda, \gamma) \neq (0, \pm 1)$, *then a* Θ-*invariant ground state of* $(\mathfrak{A}, \alpha_t)$ *is unique and is given by*

$$\bar{\varphi}_{E_+}(A_+ + A_-) = \varphi_{E_+}(A_+), \quad A_\pm \in \mathfrak{A}_\pm. \tag{3.7}$$

(2) *If* $(\lambda, \gamma) = (0, \pm 1)$, *then all* Θ*-invariant ground states of* $(\mathfrak{A}, \alpha_t)$ *are given by*

$$\bar{\varphi}(A_+ + A_-) = \alpha_0 \varphi_{E_+}(A_+) + \sum_j \alpha_j \varphi_{h_i}(A_+), \tag{3.8}$$

where $\alpha_j \geqq 0\, (j = 0, 1, \ldots), \sum \alpha_j = 1$ *and* $h_j = E_{4J} h_j$ *are mutually orthogonal.*

Previous argument already shows that Θ-invariant ground states must be of the form given by (3.7) and (3.8). In converse direction, the existence of ground states of $(\mathfrak{A}, \alpha_t)$ is known (for example, Proposition 5.3.23 and 6.2.44(1) in [6]) and if φ is a ground state, then

$$\bar{\varphi}(A) \equiv \{\varphi(A) + \varphi(\Theta(A))\}/2, \quad A \in \mathfrak{A}, \tag{3.9}$$

is a Θ-invariant ground state due to $\Theta\alpha_t = \alpha_t \Theta$. Therefore (3.7) must be a ground state in case (1). On the other hand, a ground state of a smaller system might not have an extension to a ground state of a larger system. Thus it requires an additional proof to see that all states of the form (3.8) are ground states of $(\mathfrak{A}, \alpha_t)$. This is provided by Theorem 6 of Sect. 8 along with Lemma 4.5(3) and Corollary 4.4(1).

The final step of the proof is to find all possible decompositions of a Θ-invariant ground state $\bar{\varphi}$. Any ground state φ must be obtained by a decomposition of the form (3.9). On the other hand, the set of all ground states of a C^*-dynamical system is known to be a face in the set of all states (Proposition 5.3.39 in [61]) and hence any decomposition of $\bar{\varphi}$ yields ground states and any extremal ground state is a pure state.

Our result on decomposition of $\bar{\varphi}$, which implies Theorem 1, is as follows:

Proposition 3.3. (1) *If either* $|\lambda| \geqq 1$ *or* $|\lambda| < 1$, $\gamma = 0$, *then the unique* Θ*-invariant ground state* $\bar{\varphi}_{E_+}$ *is pure and is the unique ground state of* $(\mathfrak{A}, \alpha_t)$.

(2) *If* $|\lambda| < 1$, $\gamma \neq 0$ *and* $(\lambda, \gamma) \neq (0, \pm 1)$, *then the unique* Θ*-invariant ground state* $\bar{\varphi}_{E_+}$ *is an average of two pure states* $\varphi_\pm$, *which exhaust extremal ground states of* $(\mathfrak{A}, \alpha_t)$. *The cyclic representations of* $\mathfrak{A}$ *associated with* $\varphi_\pm$ *are mutually disjoint and* Θ *interchanges* φ_+ *and* φ_-: $\varphi_\pm(\Theta(A)) = \varphi_\mp(A)$, $A \in \mathfrak{A}$.

(3) *For* $(\lambda, \gamma) = (0, \pm 1)$, $\bar{\varphi}_{E_+}$ *and* $\bar{\varphi}_h$ *are averages of two pure states* $\varphi_\pm$ *and* $\varphi_{h\pm}$, *respectively, which exhaust all extremal ground states of* $(\mathfrak{A}, \alpha_t)$. *Cyclic representations of* $\mathfrak{A}$ *associated with* $\varphi_\pm$ *are mutually disjoint and disjoint from those associated with* $\varphi_{h\pm}$. *Cyclic representation of* $\mathfrak{A}$ *associated with* $\varphi_{h\pm}$ *for different* h *are all equivalent among* φ_{h+} *and among* φ_{h-}, *but are disjoint between* φ_{h+} *and* φ_{h-}.

We now describe main points in the proof of this Proposition.

4. Main Points in the Proof of Proposition 3.3

Let $\hat{\varphi}_{E_+}$ be the $(\Theta\tilde{\Theta}_-)$-invariant extension of φ_{E_+} to $\hat{\mathfrak{A}}$:

$$\hat{\varphi}_{E_+}(A_1 + A_2 T) = \varphi_{E_+}(A_1) \quad (A_1, A_2 \in \mathfrak{A}^{\mathrm{CAR}}). \tag{4.1}$$

Let $(\hat{\pi}, \hat{\mathfrak{H}})$ be the cyclic representation of $\hat{\mathfrak{A}}$ associated with $\hat{\varphi}_{E_+}$ and $\hat{\Omega} \in \hat{\mathfrak{H}}$ be the cyclic vector giving the state $\hat{\varphi}_{E_+}$. By Θ and $\tilde{\Theta}_-$ invariance of $\hat{\varphi}_{E_+}$, we have the following orthogonal decomposition:

$$\hat{\mathfrak{H}} = \mathfrak{H}_{11} + \mathfrak{H}_{12} + \mathfrak{H}_{21} + \mathfrak{H}_{22}, \tag{4.2}$$

$$\mathfrak{H}_{11} = (\hat{\pi}(\mathfrak{A}_+^{\text{CAR}})\hat{\Omega})^-, \quad \mathfrak{H}_{12} = (\hat{\pi}(\mathfrak{A}_-^{\text{CAR}})\hat{\Omega})^-, \tag{4.3}$$

$$\mathfrak{H}_{21} = (\hat{\pi}(\mathfrak{A}_+^{\text{CAR}})\hat{\pi}(T)\hat{\Omega})^-, \quad \mathfrak{H}_{22} = (\hat{\pi}(\mathfrak{A}_-^{\text{CAR}})\hat{\pi}(T)\hat{\Omega})^-. \tag{4.4}$$

The cyclic representation of $\mathfrak{A}$ associated with $\bar{\varphi}_{E_+}$ is the restriction of $\hat{\pi}(\mathfrak{A})$ to $\mathfrak{H}_{11} + \mathfrak{H}_{22}$, while that associated with $\bar{\varphi}_h$ is the restriction of $\hat{\pi}(\mathfrak{A})$ to $\mathfrak{H}_{12} + \mathfrak{H}_{21}$. (For cyclicity in the latter, note that $\hat{\pi}(B(h))^*\hat{\pi}(B(h))\hat{\Omega} = \|h\|^2\hat{\Omega}$ for $h = E_e h(= E_+ h)$.) Our problem is to find an irreducible decomposition of these representations.

We apply the following lemma to obtain the desired conclusion. Proof will be given in Sect. 8.

Lemma 4.1. *Let ω be a state of C^*-algebra $\mathfrak{A}$ and is invariant under an involutive automorphism Θ of $\mathfrak{A}$. Let π be a representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$ with a cyclic unit vector Ω giving rise to the given state $\omega(A) = (\Omega, \pi(A)\Omega)$. (The GNS-triplet.) Let $\mathfrak{A}_\pm$ be Θ-even and Θ-odd parts of $\mathfrak{A}$. Let $\pi_\pm$ be the restriction of the representation π of $\mathfrak{A}_+$ to subspaces $\mathfrak{H}_\pm \equiv \pi(\mathfrak{A}_\pm)\Omega$. Assume that π_+ is irreducible. Then π_- is irreducible too and the following hold.*

(1) *The representation π of $\mathfrak{A}$ is irreducible if and only if π_+ and π_- are disjoint.*

(2) *If π_+ and π_- are not disjoint, then π is a direct sum of mutually disjoint 2 irreducible representations of $\mathfrak{A}$ and ω is an average of 2 pure states $\omega_\pm$ which give rise to mutually disjoint representations and interchanged by Θ: $\omega_\pm(\Theta(A)) = \omega_\mp(A)$.*

Let the restriction of $\hat{\pi}(\mathfrak{A}_+^{\text{CAR}})(= \hat{\pi}(\mathfrak{A}_+))$ to $\mathfrak{H}_{ij}$ be π_{ij}. We have to compare π_{11} with π_{22} (and π_{12} with π_{21} in the Ising case), in order to apply the above lemma to our problem.

The representation $\hat{\pi}$ of $\mathfrak{A}^{\text{CAR}}$ restricted to $\mathfrak{H}_{11} + \mathfrak{H}_{12}$ is the cyclic representation π_{E_+} associated with the pure (Fock) state φ_{E_+} (any Fock state φ_E is known to be pure as it is uniquely defined by $\varphi_E(B(Eh)B(Eh)^*) = 0$ for all h) and the same restricted to $\mathfrak{H}_{21} + \mathfrak{H}_{22}$ is the cyclic representation $\pi_{\theta_- E_+ \theta_-}$ associated with the pure (Fock) state

$$\varphi_{\theta_- E_+ \theta_-}(A) = \varphi_{E_+}(\Theta_-(A)) = (\hat{\pi}(T)\Omega, \quad \hat{\pi}(A)\hat{\pi}(T)\Omega) \tag{4.5}$$

of $A \in \mathfrak{A}^{\text{CAR}}$, where $\Theta_-(B(h)) = B(\theta_-(h))$,

$$\theta_-\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \theta_- f \\ \theta_- g \end{pmatrix}, \quad (\theta_- f)_j = \begin{cases} f_j & \text{if } j \geqq 1, \\ -f_j & \text{if } j \leqq 0. \end{cases} \tag{4.6}$$

Since these Fock states are Θ-invariant, all representations π_{ij} of $\mathfrak{A}_+$ are irreducible (by Lemma 8.1. (i) in Sect. 8) and Lemma 4.1 (1) implies

$$\pi_{11} \not\sim \pi_{12}, \quad \pi_{21} \not\sim \pi_{22}. \tag{4.7}$$

Since π_{E_+} is the cyclic representation of $\mathfrak{A}^{\text{CAR}}$ associated with the Θ-invariant extension to $\mathfrak{A}^{\text{CAR}}$ of the state of $\mathfrak{A}_+ = \mathfrak{A}_+^{\text{CAR}}$ given by a vector in $\mathfrak{H}_{11}$ or $\mathfrak{H}_{12}$, and since exactly the same situation prevails with $\pi_{\theta_- E_+ \theta_-}$, $\mathfrak{H}_{21}$ and $\mathfrak{H}_{22}$, we see that

$$\pi_{E_+} \sim \pi_{\theta_- E_+ \theta_-} \quad \text{if and only if} \quad \pi_{11} \sim \pi_{21} \quad \text{or} \quad \pi_{11} \sim \pi_{22}. \tag{4.8}$$

where the "only if" part follows from (4.7). Taking (4.7) and (4.8) together we obtain the following result:

Lemma 4.2. (1) $\pi_{11} \sim \pi_{22}$ *if and only if*

$$\pi_{E_+} \sim \pi_{\theta_- E_+ \theta_-} \quad \textit{and} \quad \pi_{11} \not\sim \pi_{21}. \tag{4.9}$$

(2) $\pi_{12} \sim \pi_{21}$ *if and only if (4.9) holds.*

To apply this result to our problem, we can use the following known criterion ((1) by Theorem 1 of [1], (2) by Theorem 4 of [3]).

Proposition 4.3. (1) *Two Fock states* φ_{E_1} *and* φ_{E_2} *of* $\mathfrak{A}^{\text{CAR}}$ *give rise to equivalent representations if and only if* $E_1 - E_2$ *is in the Hilbert–Schmidt class.*

(2) *Restrictions of Fock states* φ_{E_1} *and* φ_{E_2} *of* $\mathfrak{A}^{\text{CAR}}$ *to* $\mathfrak{A}_+^{\text{CAR}}$ *give rise to equivalent representations if and only if* $E_1 - E_2$ *is in the Hilbert–Schmidt class and* $\dim(E_1 \wedge (1-E_2))$ *is even.*

Note that $\dim(E_1 \wedge (1-E_2))$ is finite if $E_1 - E_2$ is in the Hilbert–Schmidt class and is equal to $\dim((1-E_1) \wedge E_2)$ due to $\Gamma E_i \Gamma = 1 - E_i$.

Corollary 4.4. (1) $\pi_{11} \sim \pi_{22}$ *if and only if*

$$\|E_+ - \theta_- E_+ \theta_-\|_{\text{HS}} < \infty \quad \textit{and} \quad \dim(\theta_- E_+ \theta_- \wedge (1-E_+)) \textit{ is odd.} \tag{4.10}$$

(2) $\pi_{12} \sim \pi_{21}$ *if and only if (4.10) holds.*

Quantities in (4.10) are computed in Sect. 7. The result is summarized as follows.

Lemma 4.5. (1) *If either* $|\lambda| = 1$, $\gamma \neq 0$ *or* $|\lambda| < 1$, $\gamma = 0$, *then*

$$\|E_+ - \theta_- E_+ \theta_-\|_{\text{HS}} = \infty. \tag{4.11}$$

(2) *If either* $|\lambda| > 1$ *or* $(\lambda, \gamma) = (\pm 1, 0)$, *then*

$$\|E_+ - \theta_- E_+ \theta_-\|_{\text{HS}} < \infty, \quad \dim(\theta_- E_+ \theta_- \wedge (1-E_+)) = \textit{even}. \tag{4.12}$$

(3) *If* $|\lambda| < 1$, $\gamma \neq 0$, *then*

$$\|E_+ - \theta_- E_+ \theta_-\|_{\text{HS}} < \infty, \quad \dim(\theta_- E_+ \theta_- \wedge (1-E_+)) = \textit{odd}. \tag{4.13}$$

To finish up the proof of Proposition 3.3, (1) in that Proposition follows from Lemma 4.5 (1), (2), the "only if" part of Corollary 4.4 (1), the "if" part of Lemma 4.1 (1) and Proposition 3.2 (1). (2) in Proposition 3.3 follows from Lemma 4.5 (3), the "if" part of Corollary 4.4 (1), Lemma 4.1 (2) and Proposition 3.2 (1).

As for (3) of Proposition 3.3, the statement about decomposition of $\bar{\varphi}_{E_+}$ and $\bar{\varphi}_h$ follows from Lemma 4.5 (3), Corollary 4.4 (1) and (2) and Lemma 4.1 (2).

Since the state (3.8) of $\mathfrak{A}$ can be given by a density matrix on $\hat{\mathfrak{H}}$, only 4 irreducible representations, two each by decomposition of $\hat{\pi}(\mathfrak{A})$ restricted to $\mathfrak{H}_{11} + \mathfrak{H}_{12}$ and to $\mathfrak{H}_{12} + \mathfrak{H}_{21}$, can appear in the cyclic representation associated with (3.8). Since representations of $\mathfrak{A}_+$ on $\mathfrak{H}_{11} + \mathfrak{H}_{22}$ and on $\mathfrak{H}_{12} + \mathfrak{H}_{21}$ are disjoint in the present case, two irreducible representations arising from $\mathfrak{H}_{11} + \mathfrak{H}_{22}$, which correspond to $\varphi_\pm$ and are mutually disjoint by Lemma 4.1 (2), must be both disjoint from each of two irreducible representations arising from $\mathfrak{H}_{12} + \mathfrak{H}_{21}$, which corresponds to $\bar{\varphi}_{h\pm}$ and are mutually disjoint. The label $\pm$ of $\bar{\varphi}_{h\pm}$ can be so chosen that the same sign of $\pm$ gives rise to one of two irreducible representations arising from $\mathfrak{H}_{12} + \mathfrak{H}_{21}$, common for all h. Thus the statement about equivalence and disjointness of representations in (3) of the proportion follows.

Finally, to see that $\varphi_\pm$ and $\varphi_{h\pm}$ exhaust all pure ground states, any pure ground state φ belongs to one of 4 irreducible representations and $(\varphi(A) + \varphi(\Theta(A)))/2 \equiv \bar{\varphi}(A)$ is of the form (3.8), in which either (i) all $\alpha_j, j \neq 0$, are 0 or (ii) $\alpha_0 = 0$

(due to the difference in associated representations. In either case, a decomposition into two irreducible parts (one for the original φ and the other for its Θ-transform, which is always not equivalent to the original representation for each of the 2 representations under consideration) is unique and therefore φ must be one of $\alpha_0\varphi_\pm$ in case (i) and one of $\Sigma\alpha_j\varphi_{h_j\pm}$ in case (ii). In the latter, only one of α's should be non-zero in order that φ be pure and hence φ must be one of $\varphi_\pm$ and $\varphi_{h\pm}$. Q.E.D.

5. Concrete Decomposition of $\bar{\varphi}_{E_+}$ into $\varphi_\pm$

Except for the cases $|\lambda|=1, \gamma\neq 0$ and $|\lambda|<1, \gamma=0$, representations π_{E_+} and $\pi_{\theta_-E_+\theta_-}$ of $\mathfrak{A}^{\mathrm{CAR}}$ are equivalent (Proposition 4.3 (1) and Lemma 4.5 (2), (3)), and hence there exists a unitary $U^{\mathrm{CAR}}(\Theta_-)$ on the cyclic representation space $\mathfrak{H}_{11}+\mathfrak{H}_{12}$ of π_{E_+} satisfying $U^{\mathrm{CAR}}(\Theta_-)\pi_{E_+}(A)U^{\mathrm{CAR}}(\Theta_-)^*=\pi_{E_+}(\Theta_-(A))$. Since π_{E_+} is irreducible, it belongs to $\pi_{E_+}(\mathfrak{A}^{\mathrm{CAR}})''$ and, since π_{E_+} and $\pi_{\theta_-E_+\theta_-}$ are equivalent, it extends to a unitary operator $\hat{U}(\Theta_-)\in\hat{\pi}(\mathfrak{A}^{\mathrm{CAR}})''$.

Consider the case $|\lambda|<1, \gamma\neq 0$. Since the representation $\pi_{11}\circ\Theta_-$ of $\mathfrak{A}_+^{\mathrm{CAR}}$ is disjoint from π_{11} and is equivalent to π_{12}, and since $\pi_{12}\circ\Theta_-$ is disjoint from π_{12} and is equivalent to π_{11} ((4.8), Proposition 4.3 (2) and Lemma 4.5 (3)), $U^{\mathrm{CAR}}(\Theta_-)$ has to interchange $\mathfrak{H}_{11}$ and $\mathfrak{H}_{12}$ in this case.

Let $\hat{U}(\Theta)$ be a unitary operator which is 1 on $\mathfrak{H}_{11}+\mathfrak{H}_{21}$ and -1 on $\mathfrak{H}_{12}+\mathfrak{H}_{22}$. It implements Θ on $\hat{\mathfrak{A}}$ (hence on $\mathfrak{A}^{\mathrm{CAR}}$) and $\hat{U}(\Theta)$ anticommutes with $U^{\mathrm{CAR}}(\Theta_-)$ on $\mathfrak{H}_{11}+\mathfrak{H}_{12}$. Hence if $\pi_{E_+}(A_\alpha)$ is a net tending to $U^{\mathrm{CAR}}(\Theta_-)$ with $A_\alpha\in\mathfrak{A}^{\mathrm{CAR}}$, then

$$\pi_{E_+}(A_{\alpha-})=\{\pi_{E_+}(A_\alpha)-\hat{U}(\Theta)\pi_{E_+}(A_\alpha)\hat{U}(\Theta)\}/2\to U^{\mathrm{CAR}}(\Theta_-)$$

with $A_{\alpha-}\equiv(A_\alpha-\Theta(A_\alpha))/2\in\mathfrak{A}_-^{\mathrm{CAR}}$, and hence

$$U^{\mathrm{CAR}}(\Theta_-)\in\pi_{E_+}(\mathfrak{A}_-^{\mathrm{CAR}})^- \quad \hat{U}(\Theta_-)\in\hat{\pi}(\mathfrak{A}_-^{\mathrm{CAR}})^-, \tag{5.1}$$

where $-$ on the shoulder indicates closure. Since $U^{\mathrm{CAR}}(\Theta_-)^2$ commutes with $\pi_{E_+}(\mathfrak{A}^{\mathrm{CAR}})$ due to $\Theta^2=\mathrm{id}$, and since π_{E_+} is irreducible, $U^{\mathrm{CAR}}(\Theta_-)^2$ is a scalar operator, which can be made to be $\mathbb{1}$ by redefining $U^{\mathrm{CAR}}(\Theta_-)$ (through multiplication of a phase factor $e^{i\theta}$). Thus we assume

$$\hat{U}(\Theta_-)^2=1. \tag{5.2}$$

We have, for $A\in\mathfrak{A}^{\mathrm{CAR}}$,

$$\hat{U}(\Theta_-)\hat{\pi}(A)\hat{U}(\Theta_-)=\hat{\pi}(T)\hat{\pi}(A)\hat{\pi}(T). \tag{5.3}$$

By approximating $\hat{U}(\Theta_-)$ by $\hat{\pi}(A)$, $A\in\mathfrak{A}^{\mathrm{CAR}}$, we obtain

$$\hat{U}(\Theta_-)=\hat{\pi}(T)\hat{U}(\Theta_-)\hat{\pi}(T). \tag{5.4}$$

Thus $W\equiv\hat{U}(\Theta_-)\hat{\pi}(T)$ is a selfadjoint unitary element ($W\equiv W^*$, $W^2=1$) in the center of $\hat{\pi}(\hat{\mathfrak{A}})''$ (due to (5.3) and (5.4) together with the selfadjoint unitary property of $\hat{U}(\Theta_-)$ and $\hat{\pi}(T)$), belonging to $\hat{\pi}(\mathfrak{A}_-)^-$.

Proposition 5.1. *Assume $|\lambda|<1, \gamma\neq 0$. Let $\hat{U}(\Theta_-)$ and W be as above. Let*

$$\Omega_\pm=2^{-1/2}\quad(1\pm W)\hat{\Omega}, \tag{5.5}$$

$$\varphi_\pm(A)=(\Omega_\pm,\hat{\pi}(A)\Omega_\pm)\quad(A\in\mathfrak{A}). \tag{5.6}$$

(1) *$\varphi_\pm$ are pure states of $\mathfrak{A}$ and*

$$\bar{\varphi}_{E_+} = (\varphi_+ + \varphi_-)/2. \tag{5.7}$$

(2) *The cyclic representation $\pi_\pm$ associated with $\varphi_\pm$ can be realized on $\mathfrak{H}_{11}$ by*

$$\pi_\pm(A) = \hat{\pi}(A_+) \pm \hat{\pi}(A_-)W, \tag{5.8}$$

$$A_\pm = (A \pm \Theta(A))/2, \quad A \in \mathfrak{A}$$

$$\varphi_\pm(A) = (\hat{\Omega}, \pi_\pm(A)\hat{\Omega}). \tag{5.9}$$

In terms of $\mathfrak{A}^{\mathrm{CAR}}$

$$\hat{\pi}(A_-)W = \hat{\pi}(A_- T)\hat{U}(\Theta_-), \tag{5.10}$$

with $A_- T \in \mathfrak{A}_-^{\mathrm{CAR}}$ and $\hat{U}(\Theta_-) \in \hat{\pi}(\mathfrak{A}_-^{\mathrm{CAR}})^-$.

(3) *π_+ and π_- are disjoint.*

Proof. Let

$$\mathfrak{H}_\pm = (1 \pm W)(\mathfrak{H}_{11} + \mathfrak{H}_{22}). \tag{5.11}$$

Since W is in the center of $\hat{\pi}(\mathfrak{A})$ (in the center of $\hat{\pi}(\hat{\mathfrak{A}})$ and belonging to $\hat{\pi}(\mathfrak{A}_-)^-$), $\mathfrak{H}_\pm$ are invariant subspaces under $\hat{\pi}(\mathfrak{A})$. Since $\hat{\Omega}$ is cyclic for $\hat{\pi}(\mathfrak{A})$ on $\mathfrak{H}_{11} + \mathfrak{H}_{22}$, $\Omega_\pm$ are cyclic on $\mathfrak{H}_\pm$. Since $W = W^* = W^{-1}$, $\mathfrak{H}_+$ is orthogonal to $\mathfrak{H}_-$, and hence (5.7) holds.

Since $W \in \hat{\pi}(\mathfrak{A}_-)^-$, $(\hat{\Omega}, W\hat{\Omega}) = 0$ (by Θ-invariant of φ_{E_+}), and hence $\Omega_\pm$ are unit vectors and $\varphi_\pm$ are states. Furthermore, for $A \in \mathfrak{A}$,

$$(\Omega_\pm, \hat{\pi}(A)\Omega_\pm) = (\hat{\Omega}, \hat{\pi}(A)(1 \pm W)\hat{\Omega}) = (\hat{\Omega}, (\hat{\pi}(A_+) \pm \hat{\pi}(A_-)W)\hat{\Omega}) = (\hat{\Omega}, \pi_\pm(A)\hat{\Omega}),$$

where the first equality is due to $[\hat{\pi}(A), W] = 0$ and $(1 \pm W)^2 = 2(1 \pm W)$, the second equality is due to

$$(\hat{\Omega}, \hat{\pi}(A_-)\hat{\Omega}) = (\hat{\Omega}, \hat{\pi}(A_+)W\hat{\Omega}) = 0$$

$(\hat{U}(\Theta)\hat{\Omega} = \hat{\Omega}$ and $\hat{U}(\Theta)x\hat{U}(\Theta) = -x$ for $x = \hat{\pi}(A_-)$ and for $x = \hat{x}(A_+)W)$, and the third equality is the definition (5.8) for which the representation property is immediate because W commutes with $\hat{\pi}(A)$ and $W = W^* = W^{-1}$. Since $\pi_\pm(\mathfrak{A}_+) = \hat{\pi}(\mathfrak{A}_+)$ is irreducible on $\mathfrak{H}_{11}$, $\pi_\pm(\mathfrak{A})$ is irreducible there and $\varphi_\pm$ are pure states. This proves (1) and (2).

Let $A_\alpha \in \mathfrak{A}_-$ be such that $\hat{\pi}(A_\alpha) \to W$. Then $\pi_\pm(A_\alpha) = \pm\hat{\pi}(A_\alpha)W \to \pm W^2 = \pm 1$. Therefore the two representations $\pi_\pm$ can not be equivalent and hence are disjoint. Q.E.D.

Exactly the same decomposition can be made for $\bar{\varphi}_h$ in the case $(\lambda, \gamma) = (0, \pm 1)$.

6. Proof of Theorem 2

Case (γ). The local Hamiltonian is $-2J\Sigma\sigma_\alpha^{(j)}\sigma_\alpha^{(j+1)}$ with $\alpha = x$ or y according as $\gamma = \pm 1$. Therefore $\alpha_t(\sigma_\alpha^{(j)}) = \sigma_\alpha^{(j)}$ for all j. Since $\mathfrak{A}$ is simple (being a UHF algebra), a straightforward argument shows existence of $B \in \mathfrak{A}$ for any given state φ such that $\varphi_B(A)$ given by (1.7) differs from $\varphi(A)$ for $A = \sigma_\alpha^{(j)}$, and hence the ergodic property (1.8) fails for such A and B.

Case (α). Because K has an absolutely continuous spectrum

$$[B(h_1)^*, \alpha_t(B(h_2))]_+ = (h_1, e^{2JiKt}h_2)\mathbb{1} \to 0$$

as $t \to \pm\infty$ by the Riemann–Lebesgue Lemma. Therefore

$$\lim_{t\to\infty} \| [A_1, \alpha_t(A_2)]_\Theta \| = 0 \quad (A_1, A_2 \in \mathfrak{A}^{\mathrm{CAR}}), \tag{6.1}$$

where the Θ-graded commutator is defined by

$$[A_1, A_2]_\Theta = [A_{1+}, A_2] + [A_{1-}, A_{2+}] + [A_{1-}, A_{2-}]_+, \tag{6.2}$$
$$A_j = A_{j+} + A_{j-}, \quad A_{j\pm} \in \mathfrak{A}^{\mathrm{CAR}}_\pm \quad (j = 1, 2, \ldots).$$

Lemma 6.1. *In case* (α) *and* (β), *the following holds for* $A \in \mathfrak{A}^{\mathrm{CAR}}$:

$$\text{w-lim}\, \hat{\pi}(\alpha_t(A)) = \varphi_{E_+}(A)\mathbb{1}. \tag{6.3}$$

Proof. The restrictions of $\hat{\pi}(\mathfrak{A}^{\mathrm{CAR}})$ to $\mathfrak{H}_{11} + \mathfrak{H}_{12}$ and $\mathfrak{H}_{21} + \mathfrak{H}_{22}$ are both irreducible representations of $\mathfrak{A}^{\mathrm{CAR}}$ associated with Θ-invariant (Fock) states. Therefore the last half of Lemma 2 in [2] is applicable (separately on $\mathfrak{H}_{11} + \mathfrak{H}_{12}$ and on $\mathfrak{H}_{21} + \mathfrak{H}_{22}$) and we obtain (6.3) on $\mathfrak{H}_{11} + \mathfrak{H}_{12}$ and

$$\text{w-lim}\,(\hat{\pi}(\alpha_t(A)) - \varphi_{\theta_- E_+ \theta_-}(\alpha_t(A))\mathbb{1}) = 0 \tag{6.4}$$

on $\mathfrak{H}_{21} + \mathfrak{H}_{22}$.

Now $\varphi_{\theta_- E_+ \theta_-}(\alpha_t(A)) = \varphi_{F_t}(A)$ with

$$F_t = e^{-2JiKt}\theta_- E_+ \theta_- e^{2JiKt} = (e^{-2JiKt}\theta_- e^{2JiKt})E_+(e^{-2JiKt}\theta_- e^{2JiKt}).$$

By the next lemma, $(e^{-2JiKt}\theta_- e^{2JiKt})$ strongly tends to a selfadjoint unitary operator commuting with E_+ as $t \to \pm\infty$ (different limits for $t \to \infty$ and for $t \to -\infty$), and hence $\lim F_t = E_+$. Therefore

$$\lim_{t\to\infty} \varphi_{\theta_- E_+ \theta_-}(\alpha_t(A)) = \varphi_{E_+}(A), \tag{6.5}$$

and we obtain (6.3) on the whole space $\hat{\mathfrak{H}}$. Q.E.D.

Lemma 6.2. (1) *The following limit exists in cases* (α) *and* (β).

$$w_\pm = \lim_{t\to\pm\infty} \theta_- e^{2JiKt}\theta_- e^{-2JiKt}. \tag{6.6}$$

(2) *Both* $\theta_- w_\pm$ *are selfadjoint unitary and commute with spectral projections of* K *as well as with* Γ.

Remark. $e^{-2JiKt}\theta_- e^{2JiKt} \to \theta_- w_\pm$ as $t \to \mp\infty$ by (6.6).

Proof. Because $\theta_- U \theta_- - U$ is rank 1 (being 0 for all $f \in l_2(\mathbb{Z})$ with $f_1 = 0$), $\theta_- K \theta_- - K$ is at most rank 4. Furthermore, the spectrum of K and that of $\theta_- K \theta_-$ are both absolutely continuous by Lemma 3.1. (1). Hence

$$w_\pm = \lim_{t\to\pm\infty} e^{2Ji(\theta_- K \theta_-)t} e^{-2JiKt}, \tag{6.7}$$

which is the same as (6.6), exist and are both unitary by Theorem X.4.4 (and Theorem

X.3.5) in [7]. Equation (6.7) implies $[w_{\pm}, \Gamma] = 0$. Further

$$\theta_- w_{\pm} = \lim_{t\to\pm\infty} e^{2JiKt}\theta_- e^{-2JiKt}$$

shows $(\theta_- w_{\pm})^* = (\theta_- w_{\pm})$, $(\theta_- w_{\pm})^2 = 1$ and

$$e^{2JiKt}\theta_- w_{\pm} e^{-2JiKt} = \theta w_{\pm},$$

which implies the commutativity of $\theta_- w_{\pm}$ with the spectral projections of K. Q.E.D.

Let $\Xi_{\pm}$ be the (Bogoliubov) automorphism of $\mathfrak{A}^{\mathrm{CAR}}$ determined by

$$\Xi_{\pm}(B(h)) = B(w_{\pm} h), \tag{6.8}$$

which exists due to $[\Gamma, w_{\pm}] = 0$ and due to the unitarity of $w_{\pm}$. We have

$$\lim_{t\to\pm\infty} \alpha_t(T) A \alpha_t(T) = \lim \alpha_t \Theta_- \alpha_{-t}(A) = \Theta_- \Xi_{\pm}(A). \tag{6.9}$$

Properties of $\Theta_- w_{\pm}$ given by Lemma 6.2 imply

$$\Theta_- \Xi_{\pm} \alpha_t = \alpha_t \Theta_- \Xi_{\pm}, \quad (\Theta_- \Xi)^2 = \mathrm{id}, \tag{6.10}$$

$$\varphi_{E_+}(\Theta_- \Xi_{\pm}(A)) = \varphi_{E_+}(A). \tag{6.11}$$

We now concentrate on the case (α) for a while.

Lemma 6.3. *In case* (α) (i.e. $|\lambda| \geqq 1$ *or* $|\lambda| < 1$, $\gamma = 0$),

$$\text{w-}\lim_{t\to\pm\infty} \bar{\pi}_{E_+}(\alpha_t(A)) = \bar{\varphi}_{E_+}(A)\mathbb{1} \quad (A \in \mathfrak{A}), \tag{6.12}$$

where $\bar{\pi}_{E_+}$ is the cyclic representation of $\mathfrak{A}$ associated with $\bar{\varphi}_{E_+}$.

Proof. $\bar{\pi}_{E_+}$ can be identified with the restriction of $\hat{\pi}(\mathfrak{A})$ to $\mathfrak{H}_{11} + \mathfrak{H}_{22}$. By Lemma 6.1, (6.12) has already been shown for $A \in \mathfrak{A}_+ = \mathfrak{A}_+^{\mathrm{CAR}}$. It remains to show (6.12) for $A \in \mathfrak{A}_-$ $(= \mathfrak{A}_-^{\mathrm{CAR}} T)$.

Let Z be a weak accumulation point of $\bar{\pi}_{E_+}(\alpha_t(A))$ as $t \to +\infty$. For any $B \in \mathfrak{A}_+^{\mathrm{CAR}} = \mathfrak{A}_+$, we have

$$Z \bar{\pi}_{E_+}(B) = \bar{\pi}_{E_+}(\Theta_- \Xi_+(B)) Z \tag{6.13}$$

by (6.9) and (6.1), where Θ graded commutator is an ordinary commutator for $B \in \mathfrak{A}_+ = \mathfrak{A}_+^{\mathrm{CAR}}$.

Since $\bar{\varphi}_{E_+} = \varphi_{E_+}$ on $\mathfrak{A}_+ = \mathfrak{A}_+^{\mathrm{CAR}}$ is invariant under $\Theta_- \Xi_+$, $\bar{\pi}_{E_+}(\Theta_- \Xi_+(B))$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{11}$ is equivalent to $\bar{\pi}_{E_+}(B)$, $B \in \mathfrak{A}_+$, on $\mathfrak{H}_{11}$, which is disjoint from $\bar{\pi}_{E_+}(B)$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{22}$ in the present case by Lemma 4.5 (1), (2) and Corollary 4.4 (1). Also $\bar{\pi}_{E_+}(\Theta_- \Xi_+(B))$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{22}$ is disjoint from $\bar{\pi}_{E_+}(\Theta_- \Xi_+(B))$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{11}$ by Lemma 4.5 (1) and Corollary 4.4 (1) (mapped by automorphism $\Theta_- \Xi_+$), and the latter is equivalent to $\pi_{E_+}(B)$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{11}$.

Since Z is in the weak closure of $\bar{\pi}_{E_+}(\mathfrak{A}_-)$, it maps $\mathfrak{H}_{11}$ into $\mathfrak{H}_{22}$, and $\mathfrak{H}_{22}$ into $\mathfrak{H}_{11}$. Thus Z from $\mathfrak{H}_{11}$ into $\mathfrak{H}_{22}$ intertwine $\bar{\pi}_{E_+}(B)$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{11}$ with $\bar{\pi}_{E_+}(\Theta_- \Xi_+(B))$. $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{22}$ and Z from $\mathfrak{H}_{22}$ into $\mathfrak{H}_{11}$ intertwine $\bar{\pi}_{E_+}(B)$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{22}$ with $\bar{\pi}_{E_+}(\Theta_- \Xi_+(B))$, $B \in \mathfrak{A}_+$ on $\mathfrak{H}_{11}$. From disjointness proved above, if follows that

$Z=0$. Thus

$$\underset{t\to+\infty}{\text{w-lim}}\,\bar\pi_{E_+}(\alpha_t(A))=0\quad(A\in\mathfrak{A}_-).\tag{6.14}$$

The same argument works for $t\to-\infty$.

By Θ-invariance of $\bar\varphi_{E_+}$, $\bar\varphi_{E_+}(A)=0$ for $A\in\mathfrak{A}_-$ and we have proved (6.12). Q.E.D.

Case (β). Lemmas 6.1 and 6.2 are still valid. Due to the invariance $\bar\varphi_{E_+}(\Theta_-\Xi_\pm(A))=\bar\varphi_{E_+}(A)$, $A\in\mathfrak{A}_+=\mathfrak{A}_+^{\mathrm{CAR}}$, there exists a unitary operator $V_\pm$ on $\mathfrak{H}_{11}$ which satisfies

$$V_\pm\hat\Omega=\hat\Omega,\quad V_\pm\pi_{11}(A)V_\pm^*=\pi_{11}(\Theta_-\Xi_\pm(A))\tag{6.15}$$

for $A\in\mathfrak{A}_+$. Since π_{11} is an irreducible representation of $\mathfrak{A}_+$, $V_\pm\in\pi_{11}(\mathfrak{A}_+)''$. Since π_{11} and π_{22} are equivalent in the present case, we have $V_\pm\in\bar\pi_{E_+}(\mathfrak{A}_+)''$ defined on $\mathfrak{H}_{11}+\mathfrak{H}_{22}$ and satisfies

$$V_\pm\bar\pi_{E_+}(A)V_\pm^*=\bar\pi_{E_+}(\Theta_-\Xi_\pm(A))\quad(A\in\mathfrak{A}_+).\tag{6.16}$$

Since $(\Theta_-\Xi_\pm)^2=\mathrm{id}$ by (6.10), $(V_\pm)^2$ commutes with all $\bar\pi_{E_+}(A)$, $A\in\mathfrak{A}_+$ and must be in the center of $\bar\pi_{E_+}(\mathfrak{A}_+)''$. Since $\bar\pi_{E_+}(A_+)''$ is isomorphic to $\pi_{11}(\mathfrak{A}_+)''$ in the present case, and since $\pi_{11}(\mathfrak{A}_+)''$ is irreducible on $\mathfrak{H}_{11}$, $\bar\pi_{E_+}(\mathfrak{A}_+)''$ is a factor and $(V_\pm)^2$ are multiples of identity. $V_\pm\hat\Omega=\hat\Omega$ given by (6.15) then implies

$$(V_\pm)^2=1.\tag{6.17}$$

Proposition 6.4. *In case* (β) (i.e. $|\lambda|<1$, $\gamma\neq0$, $(\lambda,\gamma)\neq(0,\pm1)$),

$$\underset{t\to\pm\infty}{\text{w-lim}}\,\bar\pi_{E_+}(\alpha_t(A))=\begin{cases}\bar\varphi_{E_+}(A)\mathbb{1} & \text{if } A\in\mathfrak{A}_+, & (6.18)\\ \varphi_+(A)V_\pm W & \text{if } A\in\mathfrak{A}_-. & (6.19)\end{cases}$$

Remark. $\varphi_+(A)=-\varphi_-(A)$ for $A\in\mathfrak{A}_-$, while $\varphi_+(A)=\varphi_-(A)=\bar\varphi_{E_+}(A)$ for $A\in\mathfrak{A}_+$.

Proof. Equation (6.18) follows from (6.3), as $\bar\pi_{E_+}$ is the restriction of $\hat\pi$ and $\mathfrak{A}_+=\mathfrak{A}_+^{\mathrm{CAR}}$. Let Z be a weak accumulation point of $\bar\pi_{E_+}(\alpha_t(A))$ with $A\in\mathfrak{A}_-$ as $t\to+\infty$. For any $B\in\mathfrak{A}_+=\mathfrak{A}_+^{\mathrm{CAR}}$, we obtain (6.13) again by (6.9) and (6.1), which hold also for the case (β). Since $(\Theta_-\Xi_\pm)^2=\mathrm{id}$ by (6.10), and since W introduced in Sect. 5 is in the center of $\bar\pi_{E_+}(\mathfrak{A})''$, we obtain

$$[ZV_+W,\bar\pi_{E_+}(B)]=0\tag{6.20}$$

for all $B\in\mathfrak{A}_+$. Furthermore, $Z\in\bar\pi_{E_+}(\mathfrak{A}_-)^-$, $V_+\in\bar\pi_{E_+}(\mathfrak{A}_+)''$ and $W\in\bar\pi_{E_+}(\mathfrak{A}_-)^-$ (where we restrict W to $\mathfrak{H}_{11}+\mathfrak{H}_{22}$). Therefore ZV_+W must be in the center of $\bar\pi_{E_+}(\mathfrak{A}_+)''$. In the present case $\bar\pi_{E_+}(\mathfrak{A}_+)''$ is a factor as we have seen above, and ZV_+W is a multiple of identify. By $W^2=1$ and (6.17), we obtain

$$Z=aWV_+.\tag{6.21}$$

The number a can be found by computing

$$a=(\hat\Omega,ZW\hat\Omega)=\lim_{t\to+\infty}(\hat\Omega,\bar\pi_{E_+}(\alpha_t(A))W\hat\Omega)=\varphi_+(\alpha_t(A))=\varphi_+(A)(=-\varphi_-(A)),\tag{6.22}$$

where we have used $V_+\hat{\Omega}=\hat{\Omega}$ ((6.15)) in the first equality, (5.8) and (5.9) in the third equality and the α_t-invariance of the ground state φ_+ (a general property of any ground state) in the fourth equality. This proves (6.19) for $t\to+\infty$. The same argument holds for $t\to-\infty$. Q.E.D.

Consider a ground state $\varphi_1(A)=(\Omega_1,\bar{\pi}_{E_+}(A)\Omega_1)$. For $\Omega_1=\Omega_\pm$ and $\varphi_1=\varphi_\pm$, we have

$$(\Omega_1,\varphi_+(A)W\Omega_1)=\varphi_1(A), \tag{6.23}$$

and hence the same holds for any ground state $\varphi_1=\alpha\varphi_+ +(1-\alpha)\varphi_-$ $(0\leqq\alpha\leqq 1)$ with $\Omega_1=\alpha^{1/2}\Omega_+ +(1-\alpha)^{1/2}\Omega_-$. (Note the orthogonality of invariant subspaces $\mathfrak{H}_\pm$ of $\mathfrak{H}_{11}+\mathfrak{H}_{22}$.) Since $\bar{\pi}_{E_+}(\mathfrak{A}_+)''$ is a factor, if $V_+\in\bar{\pi}_{E_+}(\mathfrak{A}_+)''$ is not an identity operator, there exists $B\in\mathfrak{A}$ such that

$$(\bar{\pi}_{E_+}(B)\Omega_1,W\bar{\pi}_{E_+}(B)\Omega_1)\neq(\bar{\pi}_{E_+}(B)\Omega_1,V_+W\bar{\pi}_{E_+}(B)\Omega_1),$$

and (1.8) would fail for any φ_1 and $t\to+\infty$. Similarly, if V_- is not an identity, (1.8) would fail for any φ_1 and $t\to+\infty$.

We now want to see that $V_\pm\neq 1$. Due to (6.16), it is enough to see that $\Theta_-\Xi_\pm$ are not an identity, or equivalently it is enough to have the following lemma, proved in Sect. 7(v). Q.E.D.

Lemma 6.5. *In case* (β), $\theta_-w_+\neq 1$, $\theta_-w_-\neq 1$.

7. Fourier Analysis on Test Function Space

(i) *Proof of Lemma 3.1.* We use the Fourier transform

$$\tilde{f}(\theta)=\sum_{n\in\mathbb{Z}}e^{in\theta}f_n,\quad f_n=(2\pi)^{-1}\int e^{-in\theta}\tilde{f}(\theta)d\theta \tag{7.1}$$

for $f=(f_n)\in l_2(\mathbb{Z})$ and $\tilde{f}\in L^2([0,2\pi])$ to analyze the operator K. Due to $(Uf)^\sim(\theta)=e^{-i\theta}\tilde{f}(\theta)$, the operator K is represented as

$$(Kh)^\sim(\theta)=K(\theta)\tilde{h}(\theta), \tag{7.2}$$

$$K(\theta)=2\begin{pmatrix}\cos\theta-\lambda & -i\gamma\sin\theta\\ i\gamma\sin\theta & -(\cos\theta-\lambda)\end{pmatrix}. \tag{7.3}$$

The eigenvalues of the 2×2 matrix $K(\theta)$ are $\pm\mu(\theta)$,

$$\mu(\theta)=2[(\cos\theta-\lambda)^2+\gamma^2\sin^2\theta]^{1/2}, \tag{7.4}$$

which is an algebraic function of $\cos\theta$ and is not a constant except for the case $(\lambda,\gamma)=(0,\pm 1)$, when (7.4) is 2. This proves Lemma 3.1.

(ii) *Computation of* E_+. E_+ is the multiplication of

$$E_+(\theta)=(K(\theta)+\mu(\theta))/(2\mu(\theta)). \tag{7.5}$$

In some special cases, it takes a simpler form.

If $\gamma=0$, $\lambda\geqq 1$, then

$$E_+=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{7.6}$$

If $\gamma=0$, $\lambda \leqq -1$, then

$$E_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.7}$$

If $\gamma=0$, $|\lambda|<1$, then

$$E_+(\theta) = (1-\chi_\lambda(\theta))\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \chi_\lambda(\theta)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.8}$$

where $\chi_\lambda(\theta)=1$ if $\cos\theta>\lambda$ and $\chi_\lambda(\theta)=0$ if $\cos\theta \leqq \lambda$.

(iii) *Computation of Hilbert-Schmidt Norm.* We have to estimate

$$\| E_+ - \theta_- E_+ \theta_- \|_{\mathrm{HS}}^2 = 2\| E_- \theta_- E_+ \|_{\mathrm{HS}}^2 = 8\| E_- q E_+ \|_{\mathrm{HS}}^2, \tag{7.9}$$

where $E_- = (1-E_+)$ and $q=(1+\theta_-)/2$. (See (5.2) in [3].) The operator q is given by

$$q\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} qf \\ qg \end{pmatrix}, \quad (qf)_j = \begin{cases} f_j & \text{if } j \geqq 1 \\ 0 & \text{if } j \leqq 0. \end{cases} \tag{7.10}$$

Lemma 7.1.

$$\| E_- q E_+ \|_{\mathrm{HS}}^2 = \lim_{\varepsilon \to +0} (2\pi)^{-2} \int_\varepsilon |1-e^{i(\theta_1-\theta_2)}|^{-2} \mathrm{tr}(E_+(\theta_1)E_-(\theta_2))d\theta_1 d\theta_2, \tag{7.11}$$

where the suffix ε to the integral denotes the integration over $\theta_1, \theta_2 \in [0, 2\pi]$ excluding the region $|\theta_1-\theta_2|<\varepsilon$ and $||\theta_1-\theta_2|-2\pi|<\varepsilon$.

Proof. Let $q_\eta(\eta>0)$ be a bounded operator given by

$$q_\eta\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} q_\eta & f \\ q_\eta & g \end{pmatrix}, \quad (q_\eta f)_j = \begin{cases} e^{-\eta j} f_j & \text{if } j \geqq 1, \\ 0 & \text{if } j \leqq 0. \end{cases} \tag{7.12a}$$

Then $q_\eta \to q$ strongly as $\eta \to +0$. We have

$$\begin{aligned} (q_\eta f)\tilde{}(\theta) &= \sum_{n=1}^{\infty} \int e^{[i(\theta-\theta')-\eta]n} \tilde{f}(\theta')d\theta'/(2\pi) \\ &= \int (e^{-i(\theta-\theta')+\eta}-1)^{-1} \tilde{f}(\theta')d\theta'/2\pi). \end{aligned} \tag{7.12b}$$

Hence

$$\begin{aligned} \| E_- q E_+ \|_{\mathrm{HS}}^2 &\leqq \inf \lim_{\eta \to +0} \| E_- q_\eta E_+ \|_{\mathrm{HS}}^2 \\ &= \lim_{\eta \to +0} (2\pi)^{-2} \int |1-e^{i(\theta_1-\theta_2)+\eta}|^{-2} \mathrm{tr}(E_+(\theta_1)E_-(\theta_2))d\theta_1 d\theta_2, \end{aligned} \tag{7.13}$$

where the right-hand side is a monotone increasing limit due to

$$\mathrm{tr}(E_+(\theta_1)E_-(\theta_2)) = \mathrm{tr}(E_-(\theta_2)E_+(\theta_1)E_-(\theta_2)) \geqq 0. \tag{7.14}$$

Let $E(\Delta_1)$ be the projection operator defined by

$$(E(\Delta_1)h)\tilde{}(\theta) = \chi(\theta, \Delta_1)\tilde{h}(\theta), \tag{7.15}$$

where $\chi(\theta, \Delta_1)$ is the characteristic function for the subset Δ_1 of $[0, 2\pi]$. Let

$$d(\Delta_1, \Delta_2) = \sup\{|\theta_1 - \theta_2| \quad \text{and} \quad ||\theta_1 - \theta_2| - 2\pi|, \quad \theta_1 \in \Delta_1, \quad \theta_2 \in \Delta_2\}. \tag{7.16}$$

If $d(\Delta_1, \Delta_2) > 0$, then $E(\Delta_1)qE(\Delta_2) = \lim E(\Delta_1)q_\eta\ E(\Delta_2)$ has a bounded integral kernel.

$$(E(\Delta_1)qE(\Delta_2)f)\tilde{}(\theta) = \chi(\theta, \Delta_1)(2\pi)^{-1} \int_{\Delta_2} (e^{-i(\theta-\theta')} - 1)^{-1} \tilde{f}(\theta')d\theta'. \tag{7.17}$$

Let $\Delta(n,j) = [3^{-n}(j-1)2\pi,\ 3^{-n}j2\pi]\ (j = 1, \ldots, 3^n)$. Then

$$\| E_- qE_+ \|_{\mathrm{HS}}^2 \geqq \lim_{n\to\infty} \sum_{k,j}' \| E(\Delta(n,k))E_- qE_+ E(\Delta(n,j)) \|_{\mathrm{HS}}^2, \tag{7.18}$$

where $\sum'$ is the summation over all k, j satisfying $d(\Delta(n,k), \Delta(n,j)) > 0$, and the limit in n is monotone increasing. The right-hand side can be computed by using (7.17), and we obtain

$$\| E_- qE_+ \|_{\mathrm{HS}}^2 \geqq \lim_{n\to\infty} \sum_{k,j}' (2\pi)^{-2} \int_{k,j} |1 - e^{i(\theta_1 - \theta_2)}|^{-2} \operatorname{tr}(E_+(\theta_1)E_-(\theta_2)) d\theta_1 d\theta_2, \tag{7.19}$$

where the suffix k, j indicates the integral over $\Delta(n,k) \times \Delta(n,j)$. Since the integrand is positive by (7.14) and the domains of integration $\sum_{k,j}' \int_{k,j}$ and $\int_\varepsilon$ are cofinal in the limit $n \to \infty$ and $\varepsilon \to 0$ (i.e. the former contains the latter for large n for any fixed ε, and the latter contains the former for small ε for any fixed n), the right-hand side of (7.19) coincides with the right-hand side of (7.11).

Finally, the integrand of (7.13) is positive and monotone increasing as $\eta \to 0$ and tends to the integrand of (7.11). Hence the right-hand side of (7.13) does not exceed the right-hand side of (7.11) or that of (7.19). Then (7.13) and (7.19) prove (7.11). Q.E.D.

(iv) *Proof of Lemma 4.5.*

(I) *The case of* $|\lambda| > 1$ *or* $|\lambda| < 1, \gamma \neq 0$. Then $\mu_+(\theta) \neq 0$ for all values of θ, and hence $E_\pm(\theta)$ are holomorphic in θ (and with period 2π). Hence $\operatorname{tr}(E_+(\theta_1)E_-(\theta_2))$ has 0 at $\theta_1 = \theta_2$ and $|\theta_1 - \theta_2| = 2\pi$. By the positivity (7.14), the degree of 0 cannot be 1. Furthermore

$$|1 - e^{i(\theta_1 - \theta_2)}|^{-2} = (4 \sin^2[(\theta_1 - \theta_2)/2])^{-1}$$

is holomorphic for $\theta_1 \neq \theta_2$ and has a pole of degree 2 at $\theta_1 = \theta_2 \bmod 2\pi$. Therefore the integrand of (7.11) is holomorphic over whole values of θ_1, θ_2 and (7.11) is obviously finite. Namely,

$$\| E_+ - \theta_- E_+ \theta_- \|_{\mathrm{HS}} < \infty \tag{7.20}$$

in this case.

We also see that E_+ is holomorphic in the real parameters λ and γ. Hence the Z_2-index (the even-odd property of $\dim(\theta_- E_+ \theta_- \cap (1 - E_+))$ is constant in each of 4 connected components of this region by Theorem 3 of [3].

For $|\lambda| \geqq 1$, $\gamma = 0$, we have $E_+(\theta)$ independent of θ by (7.6) and (7.7). Hence

$\theta_- E_+ \theta_- = E_+$ and

$$\dim(\theta_- E_+ \theta_- \wedge (1 - E_+)) = 0.$$

Together with (7.20), this shows Lemma 4.5(2) for the case of $|\lambda| > 1$.

For $|\lambda| < 1$, $\gamma \neq 0$, we compute the Z_2-index at $(\lambda, \gamma) = (0, \pm 1)$ after making a unitary transform:

$$\dim(\theta_- E_+ \theta_- \wedge E_-) = \dim(\theta_- v E_+ v \theta_- \wedge v E_- v), \tag{7.21}$$

where

$$v = 2^{-1/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{7.22}$$

is a selfadjoint unitary commuting with θ_-. We have

$$v E_\pm v = \frac{1}{2} \left\{ 1 \pm \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} \right\}. \tag{7.23}$$

We want to determine the number of linearly independent solutions of the following simultaneous equations:

$$v E_+ v \theta_- h = h, \quad v E_- v h = h. \tag{7.24}$$

From the second equation with $h = \begin{pmatrix} f \\ -g \end{pmatrix}$, we obtain

$$f = U^* g, \quad g = U f. \tag{7.25}$$

From the first equation, we obtain

$$\theta_- f = -U^* \theta_- g, \quad \theta_- g = -U \theta_- f. \tag{7.26}$$

Thus we obtain

$$(U^* + \theta_- U^* \theta_-) g = 0, \quad (U + \theta_- U \theta_-) f = 0.$$

Namely, $g_n = 0$ except for $n = 0$ and $f_n = 0$ except for $n = 1$. Furthermore, $f_1 = g_0$ by (7.25). Conversely, this solution

$$f_n = \delta_{n1}, \quad g_n = \delta_{n0} \tag{7.27}$$

satisfies (7.25) and (7.26) and hence (7.24). Thus

$$\dim(\theta_- E_+ \theta_- \wedge E_-) = 1 \tag{7.28}$$

for $(\lambda, \gamma) = (0, 1)$.

In the case of $(\lambda, \gamma) = (0, -1)$, $E_\pm$ are transposed matrix of $E_\pm$ for the case $(\lambda, \gamma) = (0, 1)$. Hence the roles of U and U^*, and hence the roles of f and g are interchanged in the above calculation and we obtain the same conclusion. Thus we have proved Lemma 4.5 (3).

(II) *The case of* $(\lambda, \gamma) = (\pm 1, 0)$. As already indicated above we have E_+ independent of θ and $E_+ = \theta_- E_+ \theta_-$. Hence $E_+ - \theta_- E_+ \theta_- = 0$ and $\theta_- E_+ \theta_- \wedge (1 - E_+) = 0$. This proves Lemma 4.5 (2) for this case.

(III) *The case of* $|\lambda| < 1, \gamma = 0$. We have

$$\operatorname{tr}(E_+(\theta_1)E_-(\theta_2)) = \chi_\lambda(\theta_1) + \chi_\lambda(\theta_2) - 2\chi_\lambda(\theta_1)\chi_\lambda(\theta_2)$$
$$= \begin{cases} 1 & \text{if} (\cos\theta_1 - \lambda)(\cos\theta_2 - \lambda) < 0 \\ 0 & \text{otherwise} \end{cases}$$

(modulo a set of measure 0). The integration domain contains the region $0 < \theta_1 < \theta_0$, $\theta_0 < \theta_2 < \pi$, where $0 < \theta_0 < \pi$, $\cos\theta_0 = \lambda$. Then

$$\int_0^{\theta_0} d\theta_1 \int_{\theta_0}^{\pi} d\theta_2 |1 - e^{i(\theta_1 - \theta_2)}|^{-2} = \infty$$

because the integrand is bounded below by

$$|1 - e^{i(\theta_1 - \theta_2)}|^{-2} = (4\sin^2[(\theta_1 - \theta_2)/2])^{-1} \geqq (\theta_1 - \theta_2)^{-2}. \tag{7.29}$$

This proves Lemma 4.5 (1) in the present case.

(IV) *The case of* $\lambda = 1$, $\gamma \neq 0$. In this case, $\mu(\theta)$ vanishes at $\theta = 0$ and 2π. We use the fact

$$\lim_{\theta \to 0} E_+(\theta) = \lim_{\theta \to 2\pi - 0} E_-(\theta) = \tfrac{1}{2}\begin{pmatrix} 1 & -i\gamma/|\gamma| \\ i\gamma/|\gamma| & 1 \end{pmatrix}, \tag{7.30}$$

and hence

$$\lim_{\theta_1 \to +0, \theta_2 \to 2\pi - 0} \operatorname{tr}(E_+(\theta_1)E_-(\theta_2)) = 1.$$

Thus, there exists $\delta \in (0, \pi)$ such that, for $0 < \theta < \delta$ and $2\pi - \delta < \delta_2 < 2\pi$,

$$\operatorname{tr}(E_+(\theta_1)E_-(\theta_2)) \geqq 1/2. \tag{7.31}$$

Since the integrand of (7.11) is positive due to (7.14), (7.11) is bounded below by the integration over $0 < \theta_1 < \delta$ and $2\pi - \delta < \theta_2 < \delta_2$, which is estimated below, using (7.31) and a variation of (7.29):

$$(8\pi^2)^{-1} \int_0^{\delta} d\theta_1 \int_{2\pi - \delta}^{\pi} d\theta_2 (\theta_1 - \theta_2 + 2\pi)^{-2} = (8\pi^2)^{-1} \int_0^{\delta} d\theta_1 \int_0^{\delta} d\theta_2 (\theta_1 + \theta_2)^{-2} = \infty. \tag{7.32}$$

This proves Lemma 4.5 (1) in the present case.

(V) *The case of* $\lambda = -1, \gamma \neq 0$. In this case $\mu(\theta)$ vanishes at $\theta = \pi$ and we have

$$\lim_{\theta \to \pi - 0} E_+(\theta) = \lim_{\theta \to \pi + 0} E_-(\theta) = \frac{1}{2}\begin{pmatrix} 1 & -i\gamma/|\gamma| \\ i\gamma/|\gamma| & 1 \end{pmatrix}. \tag{7.33}$$

We can proceed as in the case of (IV) and the contribution from

$$\pi - \delta < \theta_1 < \pi, \quad \pi < \theta_2 < \pi + \delta,$$

for example, becomes infinite. This completes proof of Lemma 4.5.

(V) *Non-triviality* of $\theta_- w_\pm$.

We compute

$$(h_1, \theta_- w_\pm h_2) = \lim_{t \to \pm\infty} (h_1, e^{2JiKt}(2q - 1)e^{-2JiKt}h_2)$$
$$= 2 \lim_{t \to \pm\infty} \lim_{\eta \to +0} (2\pi)^{-2} \int (\tilde{h}_1(\theta_1), e^{2JiK(\theta_1)t} e^{-2JiK(\theta_2)t} h_2(\theta_2))$$
$$\cdot (e^{-i(\theta_1 - \theta_2) + \eta} - 1)^{-1} d\theta_1 d\theta_2 - (h_1, h_2), \tag{7.34}$$

where the first equality uses the definition (6.6) of $w_\pm$ and $\theta_- = (2q-1)$, and the second equality uses (7.12a, b). The following explicit formula confirms that $\theta_- w_\pm$ is not a multiple of the identity operator.

Proposition 7.2. *If* $|\lambda| < 1, \gamma \neq 0$ *and* $(\lambda, \gamma) \neq (0, \pm 1)$,

$$(h_1, \theta_- w_\pm h_2) = \pm (h_1, (E_+ - E_-) S_{\lambda,\gamma} h_2), \tag{7.35}$$

where $E_\pm$ *are the spectral projections of* K *for* $(0, \infty)$ *and* $(-\infty, 0)$, *respectively, and* $S_{\lambda,\gamma}$ *is defined by*

$$(S_{\lambda,\gamma} h)\tilde{}(\theta) = \begin{cases} h(\theta) & if \quad k(\theta) > 0, \\ -h(\theta) & if \quad k(\theta) < 0. \end{cases} \tag{7.36}$$

$$k(\theta) = \sin\theta \{(1-\gamma^2)\cos\theta - \lambda\}. \tag{7.37}$$

Computation. Since (7.35) is bounded by $\|h_1\| \|h_2\|$, it is enough to prove it for a dense set of h_1 and h_2. We propose to prove (7.35) for h's which vanish in a neighbourhood of $\theta = 2\pi$ and are bounded. We decompose

$$(e^{-i(\theta_1-\theta_2)+\eta} - 1)^{-1} = Q_1 + Q_2 + Q_3, \tag{7.38}$$

$$\begin{aligned} Q_1 &= (-i(\theta_1 - \theta_2) + \eta)^{-1} \eta (e^\eta - 1)^{-1}, \\ Q_2 &= g(-i(\theta_1 - \theta_2), \eta) - g(-i(\theta_1 - \theta_2), 0), \\ Q_3 &= g(-i(\theta_1 - \theta_2), 0) = \{(e^{-i(\theta_1-\theta_2)} - 1)^{-1} - i(\theta_1 - \theta_2)^{-1}\}, \\ g(\theta, \eta) &= (e^{\theta+\eta} - 1)^{-1} - (\theta + \eta)^{-1} \eta (e^\eta - 1)^{-1} \\ &= (\theta + \eta)^{-1} (g_1(\theta + \eta) - g_1(\eta)), \\ g_1(\theta) &= \theta/(e^\theta - 1). \end{aligned}$$

We start with the third term Q_3 which gives rise to a contribution of the form

$$(h_1, e^{2JiKt} q_3 e^{-2JiKt} h_2), \tag{7.39}$$

where q_3 is an operator with the kernel Q_3 for $0 \leqq \theta_j \leqq 2\pi - \varepsilon$ and 0 kernel for other values of θ's where $\varepsilon > 0$ is chosen so that $[0, 2\pi - \varepsilon]$ contains the support of $\tilde{h}_j$, $j = 1, 2$. Since q_3 is a Hilbert–Schmidt operator (having a bounded kernel), it can be uniformly approximated by finite rank operators, for which evaluation of (7.39) reduces to that of $(h_1, e^{2JiKt} h_a)$ $(h_b, e^{-2JiKt} h_2)$ for finite number of h_a, h_b. Because K has an absolutely continuous spectrum, $(h_1, e^{2JiKt} h_a) \to 0$ as $t \to \pm\infty$ by the Riemann–Lebesgue Lemma, and hence (7.39), which is independent of η, tends to 0 as $t \to \pm\infty$.

We now come to the second term Q_2. Since $g_1(\theta)$ is holomorphic for $|\mathrm{Im}\,\theta| < 2\pi$, $\theta^{-1}(g_1(\theta + \eta) - g_1(\eta))$ is uniformly bounded for real η $(|\eta| \leqq \eta_0)$ and pure imaginary θ satisfying $|\theta| \leqq 2\pi - \varepsilon$ for some $\varepsilon > 0$. Since $|\theta/(\theta + \eta)| < 1$ for pure imaginary θ, Q_2 is uniformly bounded for $|\theta_j| \leqq 2\pi - \varepsilon$, $|\eta| \leqq \eta_0$. Furthermore, for $|\theta| \geqq \delta$, for any given $\delta > 0$, $g(\theta, \eta)$ for pure imaginary θ and real η is continuous in (θ, η), and hence Q_2 tends to 0 as $\eta \to 0$ uniformly in θ_j in the region $2\pi - \varepsilon \geqq \theta_j \geqq \delta$. Since h_j are bounded by our choice and δ can be arbitrarily small, the contribution of Q_2 to the integral in (7.34) tends to 0 as $\eta \to +0$.

We are now left with Q_1 which gives rise to

$$(h_1, e^{2JiKt}Q_1 e^{-2JiKt}h_2) = \lim_{\eta\to+0} (2\pi)^{-2} i \int (\tilde{h}_1(\theta_1), \quad e^{2JiK(\theta_1)t} e^{-2JiK(\theta_2)t} \tilde{h}_2(\theta_2))$$
$$\times (\theta_1 - \theta_2 + i\eta)^{-1} d\theta_1 d\theta_2, \tag{7.40}$$

where we have omitted $\eta(e^\eta - 1)^{-1}$ which tends to 1 as $\eta \to 0$. We substitute

$$e^{2JiK(\theta_1)t} e^{-2JiK(\theta_2)t} = \sum_{\sigma,\tau} E_\sigma(\theta_1) E_\tau(\theta_2) \exp 2Jit(\sigma\mu(\theta_1) - \tau\mu(\theta_2)) \tag{7.41}$$

into (7.40) where $\sigma = \pm$, $\tau = \pm$ and $\mu(\theta)$ is given by (7.4).

In the term with opposite signs for σ and τ in (7.41), we may replace $E_\tau(\theta_2)$ by $E_\tau(\theta_2) - E_\tau(\theta_1)$ due to orthogonality of E_+ and E_-. Since $E_\tau(\theta)$ is holomorphic in θ in the present case, $(E_\tau(\theta_2) - E_\tau(\theta_1))/(\theta_1 - \theta_2)$ is a uniformly bounded kernel and our previous argument for Q_3 term is applicable with the conclusion that such a term gives a vanishing contribution in the limit of $t \to \pm\infty$.

For the remaining term, we apply the following (more or less known) lemma, for which we give a proof in the Appendix for completeness sake, and we obtain (7.36) by noticing that the sign of $-\mu'(\theta)$ is the same as that of $k(\theta)$.

Lemma 7.3. *Let F be a piecewise C^2 function, which is nowhere constant and $f_j (j = 1, 2)$ be L_2 functions, all in one real variable. Then*

$$\lim_{t\to\pm\infty} \lim_{\delta\to+0} \int dx_1 dx_2 \overline{f_1(x_1)} f_2(x_2)(x_1 - x_2 + \sigma i\delta)^{-1} \exp i(F(x_2) - F(x_1))t$$
$$= -2\pi i\sigma \int dx \overline{f_1(x)} f_2(x) \chi(\pm\sigma F'(x)), \tag{7.42}$$

where $\sigma = \pm 1$, and χ is the characteristic function of $(0, \infty)$.

(In our application, $\sigma = +1$, $F(x) = -\mu(x)$ and $(2\chi(\pm F'(\theta)) - 1)\tilde{h}(\theta) = \pm (S_{\lambda,\gamma}h)\tilde{}(\theta)$. Note that $(h_1, h_2) = (2\pi)^{-1} \int (\tilde{h}_1(\theta), \tilde{h}_2(\theta)) d\theta$.)

8. Ground States of the Even Part of an Algebra—Proof of Lemma 4.1

Before going into proof of Lemma 4.1, we discuss its significance, in terms of Theorem 5 below which follows easily from Lemma 4.1. For a ground state φ of a C^*-dynamical system $(\mathfrak{A}, \alpha_t)$, the cyclic representation π_φ of $\mathfrak{A}$ on the Hilbert space $\mathfrak{H}_\varphi = \overline{\pi_\varphi(\mathfrak{A})\Omega_\varphi}$ with $\Omega_\varphi \in \mathfrak{H}_\varphi$ giving rise to $\varphi(A) = (\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi)$ allows a continuous one-parameter group of unitaries $U_\varphi(t)$ implementing α_t

$$U_\varphi(t)\pi_\varphi(A)U_\varphi(t)^* = \pi_\varphi(\alpha_t(A)), \tag{8.1}$$

$$U_\varphi(t)\Omega_\varphi = \Omega_\varphi, \tag{8.2}$$

and with a positive generator

$$U_\varphi(t) = \exp ith_\varphi, \quad h_\varphi \geqq 0. \tag{8.3}$$

Any cyclic representation π_φ associated with a ground state will be called a ground state representation.

Let Θ be an involutive automorphism of $\mathfrak{A}$ ($\Theta^2 = \text{id}$) and $\mathfrak{A}_+$ be the Θ-fixed-point subalgebra of $\mathfrak{A}$ (consisting of all $A \in \mathfrak{A}$ such that $\Theta(A) = A$). We assume $\alpha_t \Theta = \Theta \alpha_t$. Then $\mathfrak{A}_+$ is α_t-invariant as a set and $(\mathfrak{A}_+, \alpha_t)$ is another system. Since the restriction of a ground state φ of $(\mathfrak{A}, \alpha_t)$ to $\mathfrak{A}_+$ is a ground state of $(\mathfrak{A}_+, \alpha_t)$, a ground state representation π of $\mathfrak{A}$ "contains" a ground state representation π_1 of $\mathfrak{A}_+$ in the sense that π restricted to $\mathfrak{A}_+$ contains π_1 as a subrepresentation. If φ is an extremal ground state, then φ is pure (because the set of all ground states is a face of the set of states) and the associated representation is irreducible. The following theorem describes all possibilities for irreducible ground state representations of $(\mathfrak{A}, \alpha_t)$ and irreducible ground state representations of $(\mathfrak{A}_+, \alpha_t)$ contained in the former.

Theorem 5. (1) *The correspondence between irreducible ground state representations of* $(\mathfrak{A}, \alpha_t)$ *and those of* $(\mathfrak{A}_+, \alpha_t)$ *by containment up to unitary equivalence is either* (i) *one-to-two,* (ii) *one-to-one, or* (iii) *two-to-one, depending on individual ground state representation.*

(2) *For a given pure ground state* φ *of* $(\mathbf{A}, \alpha_t)$, (iii) *occurs if and only if* φ *and* $\varphi \circ \Theta$ *give rise to non-equivalent representations.*

(3) *If* φ *and* $\varphi \circ \Theta$ *give rise to equivalent irreducible representations, then there exists a* Θ*-invariant pure ground state* φ_0 *giving rise to the same representation.* (i) *occurs if and only if the infimum of the spectrum of the restriction of* h_φ *to* $\pi_{\varphi_0}(\mathfrak{A}_-)\Omega_{\varphi_0}$ *belongs to its point spectrum.*

(4) *The restriction of a pure ground state* φ *of* $(\mathfrak{A}, \alpha_t)$ *to* $\mathfrak{A}_+$ *is not pure if and only if* $\varphi \neq \varphi \circ \Theta$ *and* φ *and* $\varphi \circ \Theta$ *give rise to equivalent representations. This can happen only in case* (i).

(5) *If* ψ *is a pure ground state of* $(\mathfrak{A}_+, \alpha_t)$, *and the associated cyclic representation of* $\mathfrak{A}_+$ *is contained in a ground state representation of* $(\mathfrak{A}, \alpha_t)$, (iii) *occurs if and only if the* Θ*-invariant extension* $\bar{\psi}$ *of* ψ *to* $\mathfrak{A}$ *defined by*

$$\bar{\psi}(A_+ + A_-) = \psi(A_+), \quad A_\pm \in \mathfrak{A}_\pm \tag{8.4}$$

is not pure.

For the proof of this theorem, we need the following lemma, which we prove first.

Lemma 8.1. *Let* φ *be a* Θ*-invariant state of* $\mathfrak{A}$, $\mathfrak{H}, \pi, \Omega$ *be the* GNS*-triplet associated with* φ, $\mathfrak{H}_\pm = \overline{\pi(\mathfrak{A}_\pm)\Omega}$ *and* $\pi_\pm$ *be the representation* π *of* $\mathfrak{A}_+$ *restricted to invariant subspaces* $\mathfrak{H}_\pm$. *Then both* π_+ *and* π_- *are irreducible if one of the following three conditions are satisfied.* (i) φ *is pure,* (ii) $2\varphi = \psi + \psi \circ \Theta$ *with* ψ *pure and not equivalent to* $\psi \circ \Theta$, (iii) π_+ *is irreducible.*

Proof. (i) Let φ_+ be the restriction of φ to $\mathfrak{A}_+$. If $\lambda\varphi_+ \geqq \varphi_1$ for $\lambda > 0$ and for a state φ_1 of $\mathfrak{A}_+$, then for Θ-invariant extensions $\varphi = \bar{\varphi}_+$ and $\bar{\varphi}_1$ of φ_+ and φ_1 we have

$$\begin{aligned} \bar{\varphi}_1((A_+ + A_-)^*(A_+ + A_-)) &= \varphi_1(A_+^* A_+) + \varphi_1(A_-^* A_-) \leqq \lambda\varphi_+(A_+^* A_+) \\ &\quad + \lambda\varphi_+(A_-^* A_-) = \lambda\varphi((A_+ + A_-)^*(A_+ + A_-)). \end{aligned} \tag{8.5}$$

Since φ is assumed to be pure, $\bar{\varphi}_1 = \varphi$ and $\varphi_1 = \varphi_+$. This shows that φ_+ is pure, π_+ is irreducible and we are in case (iii).

(ii) The same computation shows that $\bar{\varphi}_1 \leqq \lambda(\psi + \psi \circ \Theta)/2$. Since ψ is assumed to be pure, $\psi \circ \Theta$ is also pure. Since ψ and $\psi \circ \Theta$ are assumed to be not equivalent, we conclude $\bar{\varphi}_1 = \mu\psi + (1-\mu)\psi\circ\Theta$ for some $\mu\in[0,1]$. Since $\bar{\varphi}_1$ is Θ-invariant, we have $\mu = 1/2$, and hence $\bar{\varphi}_1 = \varphi$ and $\varphi_1 = \varphi_+$. Again π_+ is irreducible.

(iii) Let $\Psi\in\mathfrak{H}_-$, $\Psi\neq 0$. We consider two possibilities (a) $\pi(\mathfrak{A}_-)\Psi = 0$, (b) there exists $A_-\in\mathfrak{A}_-$ such that $\pi(A_-)\Psi\neq 0$. In case (a), Ψ is orthogonal to $\pi(\mathfrak{A}_-)\Omega$ (because $\pi(\mathfrak{A}_-)^* = \pi(\mathfrak{A}_-)$). It is also orthogonal to $\pi(\mathfrak{A}_+)\Omega\subset\mathfrak{H}_+$ because $\Psi\in\mathfrak{H}_-\perp\mathfrak{H}_+$. Hence Ψ is orthogonal to $\pi(\mathfrak{A})\Omega$. Since Ω is cyclic, $\Psi = 0$. Thus case (a) does not occur. Since π_+ is irreducible, $\overline{\pi(\mathfrak{A}_+)\pi(\mathfrak{A}_-)\Psi} = \mathfrak{H}_+\ni\Omega$ in case (b). Hence $\overline{\pi(\mathfrak{A}_-)\pi(\mathfrak{A}_+)\pi(\mathfrak{A}_-)\Psi}$ contains $\mathfrak{H}_- = \overline{\pi(\mathfrak{A}_-)\Omega}$. Since $\pi(\mathfrak{A}_+)\supset \pi(\mathfrak{A}_-)\pi(\mathfrak{A}_+)\pi(\mathfrak{A}_-)$, Ψ is cyclic for $\pi_-(\mathfrak{A}_+)$ (in $\mathfrak{H}_-$). Since every non-zero $\Psi\in\mathfrak{H}_-$ is cyclic for $\pi_-(\mathfrak{A}_+)$, π_- is irreducible. Q.E.D.

Proof of Theorem 5. (I) First consider the case where φ is a Θ-invariant pure ground state of $\mathfrak{A}$. Then the restriction φ_+ of φ to $\mathfrak{A}_+$ is a pure ground state of $\mathfrak{A}$, with π_+ and π_- both irreducible by Lemma 8.1 (i). If there is an eigenvector $\Psi\in\mathfrak{H}_-$ (in the notation of Lemma 8.1) of h_φ belonging to an eigenvalue which is the infimum of the spectrum of the restriction of h_φ to $\mathfrak{H}_-$, then $(\Psi, \pi_-(A)\Psi)$, $A\in\mathfrak{A}_+$ is a ground state and $\pi_\pm$ are both ground state representations.

Since π_- is irreducible, $U_\varphi(t)$ on $\mathfrak{H}_-$ satisfying $U_\varphi(t)\pi_-(A)U_\varphi(t)^* = \pi_-(\alpha_t(A))$ for all $A\in\mathfrak{A}_+$ is unique up to multiplication of numbers $e^{i\alpha t}$, and hence the eigenspace belonging to the eigenvalue at the infimum of the spectrum of the generator of such $U_\varphi(t)$ on $\mathfrak{H}_-$ does not depend on the ground state (of $\mathfrak{A}_+$) from which this representation might be constructed. Hence if the infimum in question is not an eigenvalue of h_φ, then π_- is not a ground state representation. By Lemma 4.1 (2), π_+ and π_- are disjoint, and hence the cyclic representation associated with φ contains either 2 or 1 ground state representations depending on the condition about the spectrum of h_φ discussed above.

Conversely, if an irreducible (ground state) representation π_1 of $\mathfrak{A}$ on a space $\mathfrak{H}_1$ contains either π_+ or π_-, then there exists a vector $\Psi_1\in\mathfrak{H}_1$ and Ψ either in $\mathfrak{H}_+$ or $\mathfrak{H}_-$ (representation spaces of π_+ and π_-) such that $\psi_1(A) = (\Psi_1, \pi_1(A)\Psi_1)$ and $\psi(A) = (\Psi, \pi(A)\Psi)$ coincides for all $A\in\mathfrak{A}_+$. By construction, ψ is Θ-invariant, and hence $(\psi_1 + \psi_1\circ\Theta)/2 = \psi$. Since π is irreducible in the present case, $\psi_1 = \psi$. The cyclic representation associated with ψ_1 is π_1 and hence π_1 and π are equivalent. This shows that an irreducible representation π_1 containing π_+ or π_- is unique and the correspondence is either one-to-two or one-to-one.

(II) Consider the case where φ is a pure ground state and is not equivalent to $\varphi\circ\Theta$. We apply Lemma 8.1 (ii) to $\bar{\varphi} = (\varphi + \varphi\circ\Theta)/2$ to see that the restriction of φ to $\mathfrak{A}_+$ is a pure state, and hence gives rise to an irreducible ground state representation π_+ of $\mathfrak{A}_+$. By Lemma 4.1, π_+ and π_- are equivalent and φ, being pure, must coincide with ω_+ or ω_- given in that lemma. Since Θ commutes with α_t, if one of $\omega_\pm$ (which coincides with φ) is a ground state, then the other is also a ground state and both representations (which are disjoint) contain π_+.

Conversely, if an irreducible (ground state) representation contains π_+ and π_-, it must be a subrepresentation of the cyclic representation associated with $\bar{\varphi}$ by the same argument as in the case (I), and hence it must coincide with the cyclic representation associated with either φ or $\varphi\circ\Theta$ (ω_+ or ω_- in the notation of Lemma 4.1). Thus the correspondence is two-to-one in this case.

(III) Finally we consider the case where φ is a pure ground state of $(\mathfrak{A}, \alpha_t)$, $\varphi \neq \varphi \circ \Theta$ and φ equivalent to $\varphi \circ \Theta$. Let $\mathfrak{H}_\varphi$, π_φ, Ω_φ be the GNS triplet associated with φ. Since φ and $\varphi \circ \Theta$ give rise to an equivalent representation, there exists a unitary operator $U(\Theta)$ on $\mathfrak{H}_\varphi$ such that $U(\Theta)\pi_\varphi(A)U(\Theta)^* = \pi_\varphi(\Theta(A))$. Since π_φ is irreducible, $U(\Theta)^2$ is a multiple of identity due to $\Theta^2 = \mathrm{id}$, and we can redefine $U(\Theta)$ such that $U(\Theta)^2 = \mathbb{1}$, which we now assume.

Since Θ commutes with α_t, $U(\Theta)U_\varphi(t)U(\Theta)U_\varphi(t)^*$ must be a multiple of the identity, say $c(t)\mathbb{1}$. We then have

$$U_\varphi(t)U(\Theta)U_\varphi(t)^{-1} = c(t)U(\Theta).$$

By taking square, we have $c(t)^2 = 1$ and hence $c(t) = \pm 1$. Since $c(t)$ is continuous in t, we have $c(t) = 1$. Hence $U(\Theta)$ commutes with $U_\varphi(t)$. The vector $U(\Theta)\Omega_\varphi$ is not proportional to Ω_φ as it gives rise to the state $\varphi \circ \Theta \neq \varphi$. Let

$$\Omega_0 = \| \Omega_\varphi + U(\Theta)\Omega_\varphi \|^{-1}(\Omega_\varphi + U(\Theta)\Omega_\varphi).$$

Since h_φ commutes with $U(\Theta)$, $h_\varphi \Omega_0 = 0$ and Ω_0 gives rise to a ground state $\varphi_0(A) = (\Omega_0, \pi_\varphi(A)\Omega_0)$ which is Θ-invariant due to $U(\Theta)\Omega_0 = \Omega_0$. Since π is irreducible, φ_0 is a pure state giving rise to π_φ. We are now in the situation (I) as far as representations are concerned.

We now use the notation $\mathfrak{H}$, π, Ω, $\mathfrak{H}_\pm$ relative to φ_0. Then φ is a vector state by some $\Phi \in \mathfrak{H}$, not belonging to either $\mathfrak{H}_+$ or $\mathfrak{H}_-$ because $\varphi \neq \varphi \circ \Theta$. It is decomposed as

$$\Phi = \Phi_- + \Phi_-, \quad \Phi_\pm = (1 + U(\Theta))\Phi/2 \in \mathfrak{H}_\pm,$$

where $\Phi_\pm \neq 0$ give rise to pure ground states of $\mathfrak{A}$ (due to $h_\varphi \Phi_\pm = 0$ which follows from $[h_\varphi, U(\Theta)] = 0$). Denoting their restriction to $\mathfrak{A}_+$ by $\psi_\pm$, the restriction φ_+ of φ to $\mathfrak{A}_+$ is decomposed as

$$\varphi_+ = \| \Phi_+ \|^2 \psi_+ + \| \Phi_- \|^2 \psi_-, \quad \psi_\pm(A) = \| \Phi_\pm \|^{-2}(\Phi_\pm, \pi_\pm(A)\Phi_\pm).$$

Since π_+ and π_- are disjoint, $\psi_+ \neq \psi_-$ and φ_+ is not pure. Q.E.D.

Proof of Lemma 4.1. The irreducibility of π_- is already proved by Lemma 8.1. (iii).

(I) Assume that π_+ and π_- are disjoint. Let $C \in \pi(\mathfrak{A})'$. Since $C \in \pi(\mathfrak{A}_+)'$, C has to leave $\mathfrak{H}_\pm$ invariant because $\pi_\pm$ are disjoint and must be multiples of identity $c_\pm \mathbb{1}_\pm$ on each of $\mathfrak{H}_\pm$ (due to the irreducibility of $\pi_\pm$). Since $\pi(\mathfrak{A}_-)$ bridges $\mathfrak{H}_+$ and $\mathfrak{H}_-$ (cyclicity of Ω), $c_+ = c_-$ and C is a multiple of identity. Therefore π is irreducible.

(II) Assume that π_+ and π_- are equivalent. Then there exists a unitary map u from $\mathfrak{H}_+$ to $\mathfrak{H}_-$ satisfying $u\pi_+(A) = \pi_-(A)u$ for all $A \in \mathfrak{A}$.

Claim. There exists a unitary U in the weak closure of $\pi(\mathfrak{A}_-)$ such that $U\Psi = u\Psi$ for all $\Psi \in \mathfrak{H}_+$.

Claim is proved as follows. If $\pi(\mathfrak{A}_-) = 0$, we have $\mathfrak{H}_- = 0$, which contradicts with $\mathfrak{H}_+ \ni \Omega \neq 0$ and the equivalence. Thus we have $\pi(A) \neq 0$ for some $A \in \mathfrak{A}_-$. Let $\{e_i\}$ be a complete orthonormal set in $\mathfrak{H}_+$, u_{ij} be the matrix unit of $\pi(\mathfrak{A}_+)''$ relative to $\{e_i\}$. (We use the isomorphism of $\pi(\mathfrak{A}_+)''$ with $\pi_+(\mathfrak{A}_+)'' = \mathscr{B}(\mathfrak{H}_+)$ due to the equivalence of π_+ and π_-.) For some i and k, $(ue_k, \pi(A)e_i) \neq 0$. Let $U_i = \lambda u_{ik}\pi(A)u_{ii}$, where λ is adjusted so that $U_i^* U_i = u_{ii}$. (Since u_{ii} is a minimal projection in $\pi(\mathfrak{A}_+)''$, $U_i^* U_i$ is proportional to u_{ii} in general.) Then $U_i^* U_i = u_{ii}$ and $U_i e_j = \delta_{ij} u e_i$. Since $U_j \equiv u_{ji} U_i u_{ij}$ has the initial and final projection u_{jj}, which are mutually orthogonal for different j, the sum $U=\sum_j U_j$ converges and belongs to the weak closure $\pi(\mathfrak{A}_-)$. By definition, $U\Psi=u\Psi$ for $\Psi=e_j$ for any j and hence for all $\Psi\in\mathfrak{H}$.

Since $(\Phi, U^*U\Psi)=(\Phi,\Psi)$ for all $\Phi,\Psi\in\mathfrak{H}_+$ and $U^*U\in\pi(\mathfrak{A}_+)''$, $U^*U=1$ by the isomorphism of $\pi(\mathfrak{A}_+)''$ and $\pi_+(\mathfrak{A}_+)''$. Since U maps $\mathfrak{H}_\pm$ into $\mathfrak{H}_\mp$, $U^*=u^*$ on $\mathfrak{H}_-$, and we have $(\Phi, UU^*\Psi)=(\Phi,\Psi)$ for all $\Phi,\Psi\in\mathfrak{H}_-$. By the isomorphism of $\pi(\mathfrak{A}_+)''$ and $\pi_-(\mathfrak{A}_+)''$, we have $UU^*=1$. Therefore U is a unitary element of the closure of $\pi(\mathfrak{A}_-)$. This completes proof of Claim.

From the property of u, we have $U\pi(A)\Psi=u\pi(A)\Psi=\pi(A)u\Psi=\pi(A)U\Psi$ for all $A\in\mathfrak{A}_+$ and $\Psi\in\mathfrak{H}_+$. Therefore $U^*\pi(A)U-\pi(A)=0$ on $\mathfrak{H}_+$. Since $U^*\pi(A)U-\pi(A)\in\pi(\mathfrak{A}_+)''$ for $A\in\mathfrak{A}_+$, we have $U^*\pi(A)U-\pi(A)=0$ for all $A\in\mathfrak{A}_+$. Therefore $[U,x]=0$ for all $x\in\pi(\mathfrak{A}_+)''$. If $A\in\mathfrak{A}_-$, $\pi(A)U^*\in\pi(\mathfrak{A}_+)''$ and $\pi(A)=(\pi(A)U^*)U$. Hence $[U,\pi(A)]=0$ for $A\in\mathfrak{A}_-$. We have now proved that U is in the center of $\pi(\mathfrak{A})''$.

Since U belongs to the closure of $\pi(\mathfrak{A}_-)$, $U^2\in\pi(\mathfrak{A}_+)''$. Since $\pi(\mathfrak{A}_+)''$ is a factor, U^2 must be multiple of identity. By redefining U by a multiplication of a number $e^{i\theta}$, we may assume that $U^2=1$.

Let $U(\Theta)$ be a unitary selfadjoint operator being 1 on $\mathfrak{H}_+$ and -1 on $\mathfrak{H}_-$. Then $U(\Theta)\pi(A)U(\Theta)=\pi(\Theta(A))$ and $U(\Theta)UU(\Theta)=-U$.

Let $\Omega_\pm=2^{-1/2}(1\pm U)\Omega$, $\mathfrak{H}^\pm=\overline{\pi(\mathfrak{A})\Omega_\pm}$. Since U is in the center of $\pi(\mathfrak{A})''$, $\mathfrak{H}^+$ and $\mathfrak{H}^-$ are orthogonal due to $(1+U)^*(1-U)=0$. Since $\Omega\in\mathfrak{H}_+$ and $U\Omega\in\mathfrak{H}_-$, $\Omega_\pm\neq 0$. Since $\pi(\mathfrak{A}_+)''$ is a type I factor with multiplicity 2, and since $\mathfrak{H}^\pm$ are both invariant non-trivial subspaces, $\pi(\mathfrak{A}_+)$ restricted to $\mathfrak{H}^\pm$ are both irreducible.

Let $\omega_+(A)=(\Omega_\pm,\pi(A)\Omega_\pm)$. They are pure states of $\mathfrak{A}$ (as well as of $\mathfrak{A}_+$), being vector states in an irreducible representation. Since $U(\Theta)\Omega_\pm=\Omega_\mp$, we have $\omega_\pm\circ\Theta=\omega_\mp$. By a direct computation, we see that $\omega=(\omega_+ +\omega_-)/2$. (Note that the vector states by Ω and $U\Omega$ both are ω, as U is in the center of $\pi(\mathfrak{A})''$.) Q.E.D.

By construction, we have

$$\omega_\pm(A_+ + A_-)=\omega(A_+)\pm\bar{\omega}(A_-U)\quad (A_\pm\in\mathfrak{A}_\pm). \tag{8.6}$$

($\bar{\omega}$ is the continuous extension of ω to $\pi(\mathfrak{A})''$.) Therefore, the GNS triplet for $\omega_\pm$ can be given by $\mathfrak{H}_+$, the following representation $\rho_\pm$ of $\mathfrak{A}$ (restricted to $\mathfrak{H}_+$) and Ω:

$$\rho_\pm(A_+ + A_-)=\pi(A_+)\pm\pi(A_-)U\quad (A_\pm\in\mathfrak{A}_\pm). \tag{8.7}$$

Note that $\rho_\pm(A)\in\pi(\mathfrak{A}_+)''$ leaves $\mathfrak{H}_+$ invariant. Thus $\pi(\mathfrak{A}_+)$ is already irreducible in the cyclic representation space of $\mathfrak{A}$ associated with $\omega_\pm$.

In the case (2) of Lemma 4.1, a ground state of $\mathfrak{A}_+$ extends to a ground state of $\mathfrak{A}$:

Theorem 6. *The Θ-invariant extension $\bar{\omega}$ of a pure ground state ω of $\mathfrak{A}_+$ to $\mathfrak{A}$ is a ground state of $\mathfrak{A}$ if $\bar{\omega}$ is not pure.*

Proof. We are in the situation (5) in Theorem 5. Since ω is α_t-invariant (being a ground state) and Θ commutes with α_t, $\bar{\omega}$ is also α_t-invariant. By Lemma 4.1, $\bar{\omega}$ is an average of two non-equivalent pure states $\omega_\pm$ of $\mathfrak{A}$. The α_t-invariance of $\bar{\omega}$ implies that $\bar{\omega}$ is also an average of $\omega_\pm\circ\alpha_t$, which are pure, $\omega_\pm\circ\alpha_t$ must coincide with ω_+ or ω_-. By continuity in t, we have $\omega_\pm\circ\alpha_t=\omega_\pm$. Hence both ω_+ and ω_- are α_t-invariant. There exists one-parameter group of unitaries $U_\pm(t)$ in the GNS representation space of $\omega_{\pm}$ implementing α_t. By using (8.7) along with $\mathfrak{H}_+, \Omega$ as the GNS triplets for $\omega_{\pm}$, $U_{\pm}(t)$ are already determined by $U_{\pm}(t)\pi(A_+)\Omega = \pi(\alpha_t(A_+))\Omega$ for $A_+ \in \mathfrak{A}_+$ and coincide with $U_\omega(t)$ for the ground state ω of $\mathfrak{A}_+$. Hence their generators are non-negative and $\omega_{\pm}$ are pure ground states of $\mathfrak{A}$. Therefore $\bar{\omega}$ is also a ground state of $\mathfrak{A}$. Q.E.D.

9. Ground States of Quasifree Motion—Proof of Theorems 3 and 4

Proof of Theorem 3. (Case I) 0 is not an eigenvalue of L.

We use the following characterization of a ground state φ of a C^*-algebra $\mathfrak{A}$ (Theorem 5.3.19 in [6]):

$$\varphi(A(f)^*A(f)) = 0, \tag{9.1}$$

$$A(f) = \int \alpha_t(A)f(t)dt, \tag{9.2}$$

for all $A \in \mathfrak{A}$ and for all f of the form

$$f(t) = (2\pi)^{-1} \int e^{-itp}\tilde{f}(p)dp \tag{9.3}$$

with C^∞ function $\tilde{f}$ of a compact support contained in the open interval $(-\infty, 0)$.

Applying the above characterization to $A = B(h)$, we obtain

$$\varphi(B(h(f))^*B(h(f))) = 0,$$

$$h(f) = \int e^{iLt}hf(t)dt = \tilde{f}(L)h.$$

Hence, by norm continuity of $B(h)$ as a function of h,

$$\varphi(B(h)^*B(h)) = 0 \tag{9.4}$$

for all h satisfying $E_+h = 0$. (We are using $E_0 = 0$.) This already fixes φ to the Fock state φ_{E_+}. It is easy to check that φ_{E_+} is a ground state of $(\mathfrak{A}^{\mathrm{CAR}}, \alpha_t)$. (Cf. case (a) of proof of Theorem 4 below.)

(Case II) $E_0 \neq 0$.

We split the Hilbert space of h into images of E_0 and $1 - E_0$, denoting them $\mathscr{L}_0$ and $\mathscr{L}_1$. Since $\Gamma L = -L\Gamma$, $[\Gamma, E_0] = 0$ and these spaces are Γ-invariant. Let $\mathfrak{A}_0^{\mathrm{CAR}}$ and $\mathfrak{A}_1^{\mathrm{CAR}}$ be the subalgebra of $\mathfrak{A}^{\mathrm{CAR}}$ generated by $B(h)$ with $h \in \mathscr{L}_0$ and $h \in \mathscr{L}_1$, respectively. By the Γ-invariance of $\mathscr{L}_0$ and $\mathscr{L}_1$, they are $*$-algebras. Furthermore, linear combinations of $A_1A_0 (A_0 \in \mathfrak{A}_0, A_1 \in \mathfrak{A}_1)$ are dense in $\mathfrak{A}^{\mathrm{CAR}}$.

As in case I, we have $\varphi(B(h)^*B(h)) = 0$ whenever $E_-h = h$. By Schwarz' inequality, $\varphi(B(h)^*A) = \varphi(AB(h)) = 0$ for any $A \in \mathfrak{A}^{\mathrm{CAR}}$ whenever $E_-h = h$. Since $\Gamma L = -L\Gamma, \Gamma E_+ = E_-\Gamma$. If $h \in \mathscr{L}_1$, then $h = h_+ + h_-$ with $E_{\pm}h_{\pm} = h_{\pm}$ and $B(h) = B(\Gamma h_+)^* + B(h_-)$ with both Γh_+ and h_- in $E_-\mathscr{L}_1$. Hence linear combination of the identity operator and

$$B(h_1)^* \dots B(h_m)^*B(h_{m+1}) \dots B(h_{m+n}), \tag{9.5}$$

for various choices of m, n and $h_1 \dots h_{m+n}$ from $E_-\mathscr{L}_1$, are dense in $\mathfrak{A}_1^{\mathrm{CAR}}$. For $m + n \neq 0$,

$$\varphi(B(h_1)^* \dots B(h_m)^*B(h_{m+1}) \dots B(h_{m+n})A_0) = 0 \tag{9.6}$$

for any $A_0 \in \mathfrak{A}_0^{\mathrm{CAR}}$ because of $B(h_1)^*$ if $m \neq 0$ and because of $B(h_{m+n})\tilde{A}_0 = \Theta(A_0)B(h_{m+n})$ if $n \neq 0$. Since the Fock state φ_{E_+} on $\mathfrak{A}_1^{\mathrm{CAR}}$ also vanishes on (9.5), if $n+m \neq 0$, we have

$$\varphi(A_1 A_0) = \varphi_{E_0}(A_1)\varphi(A_0). \tag{9.7}$$

Thus we have proved the necessity of (3.3).

We now show that (3.3) defines a ground state by explicitly constructing the GNS triplet for such φ. Let $\mathfrak{H}_1, \pi_1, \Omega_1$ be the GNS triplet for the state φ_{E_+} of $\mathfrak{A}_1^{\mathrm{CAR}}$ and $\mathfrak{H}_0 \pi_0, \Omega_0$ be the GNS triplet for the state φ_0 of $\mathfrak{A}_0^{\mathrm{CAR}}$. Let $U_1(\Theta)$ be the selfadjoint unitary operator in $\mathfrak{H}_1$ satisfying

$$U_1(\Theta)\pi_1(A)\Omega_1 = \pi_1(\Theta(A))\Omega_1. \tag{9.8}$$

Let us consider $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_0$ and $\Omega = \Omega_1 \otimes \Omega_0$. Let $h_0 \in \mathfrak{L}_0$, $h_1 \in \mathfrak{L}_1$,

$$\pi(B(h_0 + h_1)) = \pi_1(B(h_1)) \otimes \mathbb{1} + U_1(\Theta) \otimes \pi_0(B(h_0)). \tag{9.9}$$

It is then immediately checked that (9.9) satisfies

$$\pi(B(h))^* = \pi(B(\Gamma h)), \tag{9.10}$$

$$[\pi(B(h_a))^*, \quad \pi(B(h_b))]_+ = (h_a, h_b)\mathbb{1}, \tag{9.11}$$

and hence it generates a representation of $\mathfrak{A}^{\mathrm{CAR}}$ on $\mathfrak{H}$.

From the definition and $U_1(\Theta)\Omega_1 = \Omega_1$, it follows that

$$\begin{aligned}(\Omega, \pi(A_1 A_0)\Omega) &= (\Omega_1, \pi_1(A_1)\Omega_1)(\Omega_0, \pi_0(A_0)\Omega_0) \\ &= \varphi_{E_+}(A_1)\varphi_0(A_0)\end{aligned}$$

for any $A_1 \in \mathfrak{A}_1^{\mathrm{CAR}}$ and $A_0 \in \mathfrak{A}_0^{\mathrm{CAR}}$. Since π_1 is irreducible, $U_1(\Theta) \otimes \mathbb{1} \in \pi(\mathfrak{A}_1)''$. Hence $\mathbb{1} \otimes \pi_0(\mathfrak{A}_0) \subset \pi(\mathfrak{A})''$. Since Ω_1 and Ω_0 are cyclic, $\Psi_1 \otimes \Psi_0$ with any $\Psi_1 \in \mathfrak{H}_1$ and $\Psi_0 \in \mathfrak{H}_0$ is in the closure of $\pi(\mathfrak{A}^{\mathrm{CAR}})''\Omega \supset \pi_1(\mathfrak{A}_1^{\mathrm{CAR}})\Omega_1 \otimes \pi_0(\mathfrak{A}_0^{\mathrm{CAR}})\Omega_0$. Hence Ω is a cyclic vector and $(\mathfrak{H}, \pi, \Omega)$ is the GNS triplet for φ of the form (3.3). (It also shows that φ is a state.)

It remains to show that φ is a ground state. Let

$$\begin{aligned}U_1(t)\pi_1(A)\Omega_1 &= \pi_1(\alpha_t(A))\Omega_1 \quad (A \in \mathfrak{A}_1^{\mathrm{CAR}}), \\ U_1(t) &= \exp ith_1.\end{aligned} \tag{9.12}$$

Since $\mathfrak{A}_0^{\mathrm{CAR}}$ is generated by $B(h)$ with $E_0 h = h$ (i.e. $e^{iLt}h = h$), $\alpha_t(A_0) = A_0$ for any $A_0 \in \mathfrak{A}_0^{\mathrm{CAR}}$. Furthermore $\alpha_t \Theta = \Theta \alpha_t$, which implies $[U_1(t), U_1(\Theta)] = 0$. Therefore $U(t) = U_1(t) \otimes \mathbb{1}$ satisfies

$$U(t)\pi(A)\Omega = \pi(\alpha_t(A))\Omega, \tag{9.13}$$

$$U(t) = \exp i(h_1 \otimes \mathbb{1})t, \quad h_1 \otimes \mathbb{1} \geqq 0. \tag{9.14}$$

Therefore φ is a ground state. Since $\mathfrak{A}_0^{\mathrm{CAR}}$ is non-trivial, there are more than one φ_0 and hence more than one ground state of $\mathfrak{A}$. Q.E.D.

Proof of Theorem 4. The first and major step in the proof is to show that the Θ-invariant extension $\bar{\psi}$ of a pure ground state ψ of $\mathfrak{A}_+^{\mathrm{CAR}}$ to $\mathfrak{A}^{\mathrm{CAR}}$ always gives rise to a ground state representation of $\mathfrak{A}^{\mathrm{CAR}}$:

Let $\mathfrak{H}$, π, Ω be the GNS triplet for $\bar{\psi}$. We use an elementary result about spectral subspaces. For any closed subset Δ of the real line, let $\mathfrak{A}^{\mathrm{CAR}}(\Delta)$ be the set of $A \in \mathfrak{A}^{\mathrm{CAR}}$ such that

$$\int \alpha_t(A) f(t) dt = 0 \tag{9.15}$$

whenever the support of

$$\int e^{itp} f(t) dt \equiv \tilde{f}(p) \tag{9.16}$$

is disjoint from Δ. If $A_i \in \mathfrak{A}^{\mathrm{CAR}}(\Delta_i)$ $(i=1,2)$, then $A_1 A_2 \in \mathfrak{A}^{\mathrm{CAR}}(\Delta)$ for $\Delta = \overline{\Delta_1 + \Delta_2}$ (the closure of the set of all $x_1 + x_2$ with $x_1 \in \Delta_1$, $x_2 \in \Delta_2$). Let h_1 and h_2 be such that their L-spectral supports are contained in $(-\infty, -\varepsilon]$ and $(-\infty, 0]$, respectively. Then $B(h_1) \in \mathfrak{A}^{\mathrm{CAR}}((-\infty, -\varepsilon])$, $B(h_2) \in \mathfrak{A}^{\mathrm{CAR}}((-\infty, 0])$ and

$$B(h_2)B(h_1) \in \mathfrak{A}_+^{\mathrm{CAR}}((-\infty, -\varepsilon]). \tag{9.17}$$

We use now the characterization of a ground state ψ of $\mathfrak{A}_+^{\mathrm{CAR}}$ asserting that $\psi(A^*A) = 0$ for any $\varepsilon > 0$ and any $A \in \mathfrak{A}_+^{\mathrm{CAR}}((-\infty, -\varepsilon])$. We immediately obtain

$$\pi(B(h_2))\pi(B(h_1))\Omega = 0. \tag{9.18}$$

We now have the following two alternative possibilities:
Case (I).

$$\pi(B(h_1))\Omega = 0 \tag{9.19}$$

for all $\varepsilon > 0$ and for all h_1. Then $\pi(B(h))\Omega = 0$ whenever $E_- h = h$ by continuity. By the same argument as the proof of Theorem 2 in case II, we prove the formula (9.7) for $\bar{\psi}(A_1 A_0)$ and show that $\bar{\psi}$ is a ground state of $\mathfrak{A}^{\mathrm{CAR}}$.

Case (II). There exists h_1 such that $\Psi \equiv \pi(B(h_1))\Omega \neq 0$ and

$$\Psi(B(h_2))\Psi = 0 \tag{9.20}$$

for all h_2 satisfying $E_- h_2 = h_2$. Hence $\psi(A) = (\Psi, \pi(A)\Psi)/(\Psi, \Psi)$ is a ground state of $\mathfrak{A}^{\mathrm{CAR}}$. Let E' be the projection operator onto the subspace $\overline{\pi(\mathfrak{A}^{\mathrm{CAR}})\Psi}$. By $\pi(\mathfrak{A}^{\mathrm{CAR}})$ invariance of the subspace, $E' \in \pi(\mathfrak{A}^{\mathrm{CAR}})'$. Since $(\Omega, \pi(B(h_1))^* \Psi) = \|\Psi\|^2 \neq 0$, we see that $E'\Omega \neq 0$.

Since $\pi(B(h_1))E'\Omega = E'\pi(B(h_1))\Omega = E'\Psi = \Psi$, $E'\Omega$ is cyclic in $\overline{\pi(\mathfrak{A})\Psi}$ and hence the cyclic representation of the state

$$\varphi(A) = (E'\Omega, \pi(A)E'\Omega)/\|E'\Omega\|^2$$

of $\mathfrak{A}^{\mathrm{CAR}}$ is a ground state representation.

By Lemma 4.1, $\bar{\psi}$ is either pure or $\bar{\psi} = (\omega_+ + \omega_-)/2$ with $\omega_\pm$ pure and giving rise to mutually disjoint irreducible representations π^+ and π^-. In the former case $\varphi \leqq \|E'\Omega\|^{-2}\bar{\psi}$ implies $\varphi = \bar{\psi}$ and π is a ground state representation. In the latter case $E' \neq 0$ can be either projection on one of inequivalent irreducible subspaces or 1. Hence either $\varphi = \omega_\pm$ or $\varphi = \bar{\psi}$. The case $\varphi = \bar{\psi}$ is the same as before. If $\varphi = \omega_+$, then π^+ is a ground state representation. Since Θ commutes with α_t and $\omega_- = \omega_+ \circ \Theta$, π^- must be also a ground state representation (equivalent to $\pi^+ \circ \Theta$). Hence $\bar{\psi} = (\omega_+ + \omega_-)/2$ is also a ground state representation. This completes the first step of the proof.

We now consider different cases for the ground state representation of $\mathfrak{A}^{\mathrm{CAR}}$ to discuss the ground state representation of $\mathfrak{A}_+$, contained in the former.

Case (*a*). $E_0=0$. In this case φ_{E_+} is the unique ground state of $(\mathfrak{A}^{\mathrm{CAR}}, \alpha_t)$ and contains mutually disjoint representations π_+ and π_- of $\mathfrak{A}_+^{\mathrm{CAR}}$. π_+ is the cyclic representation associated with the restriction of φ_{E_+} to $\mathfrak{A}_+^{\mathrm{CAR}}$, which is a ground state of $(\mathfrak{A}_+^{\mathrm{CAR}}, \alpha_t)$.

In order to see whether π_- is a ground state representation or not, we study the spectrum of h_φ for $\varphi=\varphi_{E_+}$ on $\mathfrak{H}_-$. As is known,

$$\mathfrak{H}_+ = \mathbb{C}\Omega \oplus \sum_{n=1}^{\infty\oplus} \mathrm{Asym}_{2n} \mathscr{L}_+^{\otimes 2n} ,$$

$$\mathfrak{H}_- = \sum_{n=0}^{\infty\oplus} \mathrm{Asym}_{2n+1} \mathscr{L}_+^{\otimes(2n+1)},$$

where $\mathscr{L}_+$ is the Hilbert space consisting of all E_+h and Asym_n is the projection onto the totally antisymmetric part of n-fold tensor product of copies of $\mathscr{L}_+$:

$$\mathrm{Asym}_n = (n!)^{-1} \sum \mathrm{sign}\,(P)\pi_n(P), \quad \pi_n(P)(h_{P(1)}\otimes \ldots \otimes h_{P(n)}) = h_1 \otimes \ldots \otimes h_n,$$

where the sum is over all permutations P of $(1,\ldots,n)$. We have

$$U_\varphi(t)\,\Psi = (e^{iLt})^{\otimes(2n+1)}\,\Psi, \tag{9.21}$$

$$h_\varphi \Psi = \{L\otimes 1 \otimes \cdots \otimes 1 + 1 \otimes L \otimes \cdots \otimes 1 + \cdots + 1\otimes 1 \otimes \cdots \otimes L\}\,\Psi, \tag{9.22}$$

for $\Psi \in \mathrm{Asym}_{2n+1}\mathscr{L}_+^{\otimes(2n+1)}$. Thus the spectrum of h_φ on the n^{th} summand of $\mathfrak{H}_-$ is the sum of $(2n+1)$ copies of positive spectrum of L and the point spectrum of h_φ on $\mathfrak{H}_-$ is the sum of $(2n+1)$ copies of the positive point spectrum of L. (Note that L in (9.22) is acting on $\mathscr{L}_+$.) The infimum of the spectrum of h_φ restricted to each $\mathrm{Asym}_{2n+1}\mathscr{L}_+^{\otimes(2n+1)}$ (which is $U_\varphi(t)$ invariant) is $(2n+1)$ times the infimum of the positive spectrum of L. Hence the infimum of the spectrum of h_φ restricted to $\mathscr{H}_-$ is the infimum of the positive spectrum of L and it is an eigenvalue of h_φ if and only if the infimum e of the positive spectrum of L is a point spectrum of L. (Then it is non-zero because $E_0=0$.) By Theorem 5(2) and (3), this proves the uniqueness of the ground state representation in case (α).

Furthermore, similar computation as above shows that the eigenspace of h_φ belonging to 0 is one-dimensional, and hence the restriction of φ_{E_+} is the unique ground state of $\mathfrak{A}_+^{\mathrm{CAR}}$ if $E_0=0$. This then completes the proof of uniqueness of ground states of $\mathfrak{A}_+^{\mathrm{CAR}}$ in case (α).

If $e>0$ is an eigenvalue of L and the infimum of the positive spectrum of L, then any pure ground state giving rise to the representation π_- (still $E_0=0$) is a vector state by an eigenvector Ψ of h_φ belonging to e, which implies $\Psi=\pi(B(h_e))\Omega$ with $Lh_e=eh_e$. In this case $\pi(B(h))\Psi=0$ if $E_-h=h$ and $(h,\Gamma h_e)=0$. It vanishes also for $h=h_e$. Therefore $\pi(B(h))\Psi=0$ holds whenever $E'h=0$ for $E'=E_+-P(h_e)+\Gamma P(h_e)\Gamma$. Thus the vector state by ψ is the Fock state $\varphi_{E'}$. Note that $\Gamma E'\Gamma=1-E'$. This completes the proof of (3).

Case (*b*) $\dim E_0=1$. ($=$ Case (β).). In this case $\mathfrak{A}_0^{\mathrm{CAR}}$ is commutative and two-dimensional, spanned by 1 and $B(h_0)$ with $E_0h_0=h_0$. Since $[\Gamma,E_0]=0$, $\Gamma h_0=ch_0$ with $|c|=1$ due to $\Gamma^2=1$. The constant c can be reduced to 1 by redefinition of h_0

with a suitable phase factor $e^{i\theta}$. We also normalize $(h_0, h_0) = 2$. Then $B(h_0)$ is self-adjoint unitary due to $B(h_0)^* = B(\Gamma h_0) = B(h_0)$ and $B(h_0)^2 = 2^{-1}[B(h_0), B(h_0)]_+ = 1$. There exist two pure states of $\mathfrak{A}_0^{\mathrm{CAR}}$ characterized by $\varphi_\pm(B(h_0)) = \pm 1$. Since $\varphi_\pm \Theta = \varphi_\mp$, we have exactly two pure ground states for $\mathfrak{A}^{\mathrm{CAR}}$ given by (3.3) with $\varphi_0 = \varphi_\pm$. The two pure states are disjoint ($\pi(B(h_0)) = \pm 1$) and are mapped to each other by Θ. Hence we have the unique Θ-invariant ground state of $\mathfrak{A}^{\mathrm{CAR}}$ and the unique ground state representation of $\mathfrak{A}_+^{\mathrm{CAR}}$ (contained in the former) by Lemma 4.1, for example. The restriction of two pure ground states of $\mathfrak{A}^{\mathrm{CAR}}$ to $\mathfrak{A}_+^{\mathrm{CAR}}$ coincides and gives the unique ground state of $\mathfrak{A}_+^{\mathrm{CAR}}$. This is because the cyclic representation space associated with either one of pure ground states of $\mathfrak{A}^{\mathrm{CAR}}$ is the Fock space for φ_{E_+} of $\mathfrak{A}_1^{\mathrm{CAR}}$, which is generated by $B(h)$, $E_0 h = 0$, and can be identified with the cyclic representation space of the ground state of $\mathfrak{A}_+^{\mathrm{CAR}}$ due to $\pi(B(h_0)) = \pm 1$. Then the uniqueness follows from the fact that the eigenspace of h_φ with eigenvalue 0 is one-dimensional and spanned by Ω. This completes the analysis of case (β).

Case(*c*) $\dim E_0 > 1$. (= Case (2).). The fixed point algebra $\mathfrak{A}_0^{\mathrm{CAR}}$ contains elements in $\mathfrak{A}_-^{\mathrm{CAR}}$, and hence any ground state of $\mathfrak{A}_+^{\mathrm{CAR}}$ giving rise to the π_- representation must also extend to a ground state of $\mathfrak{A}^{\mathrm{CAR}}$ (via the representation π). Note that the definition of h_φ for an irreducible representation π_- does not depend on how the representation is constructed and a ground state of $\mathfrak{A}_+^{\mathrm{CAR}}$ giving rise to the π_- representation must be a vector state by $\Psi \in \mathfrak{H}_-$ satisfying $h_\varphi \Psi = 0$. Therefore all ground states of $\mathfrak{A}_+^{\mathrm{CAR}}$ are obtained as the restriction of a ground state of $\mathfrak{A}^{\mathrm{CAR}}$ to $\mathfrak{A}_+^{\mathrm{CAR}}$. Since $\mathfrak{A}_0^{\mathrm{CAR}} \cap \mathfrak{A}_+^{\mathrm{CAR}}$ is nontrivial (which is not true for the case (β)), there exist different φ_0's which have different restrictions on $\mathfrak{A}_0^{\mathrm{CAR}} \cap \mathfrak{A}_+^{\mathrm{CAR}}$. (Any state of the latter can be extended to a state of $\mathfrak{A}_0^{\mathrm{CAR}}$.) Hence the ground state of $\mathfrak{A}_+^{\mathrm{CAR}}$ is not unique. Q.E.D.

Appendix

Proof of Lemma 7.3. Consider the Fourier transform

$$\tilde{f}(p) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ipx} f(x)dx. \tag{A.1}$$

Define

$$(U_t f)(x) = e^{iF(x)t} f(x), \tag{A.2}$$

$$(q_{\delta\pm} f)^{\sim}(p) = \begin{cases} e^{-|p|\delta} \tilde{f}(p) & \text{if} \quad \pm p \geqq 0, \\ 0 & \text{if} \quad \pm p < 0. \end{cases} \tag{A.3}$$

In the Hilbert space of L_2 functions f with inner product

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(x)^* f_2(x) dx = \int_{-\infty}^{\infty} \tilde{f}_1(p)^* \tilde{f}_2(p) dp, \tag{A.4}$$

we have, for $\delta > 0$ and $\sigma = \pm$

$$\begin{aligned} &\int dx_1 dx_2 \overline{f_1(x_1)} f_2(x_2)(x_1 - x_2 + \sigma i\delta)^{-1} \exp i(F(x_2) - F(x_1))t \\ &\quad = -2\sigma i\pi (U_t f_1, q_{\delta\sigma} U_t f_2). \end{aligned} \tag{A.5}$$

Hence the limit $\delta \to +0$ is given by $q_{o\sigma}$ which has norm 1. Thus (7.42) is uniformly bounded by $2\pi \| f_1 \| \| f_2 \|$, and it is enough to prove it for a dense set of f's.

Since F is piecewise C^2 and nowhere constant, there exists a countable family of mutually disjoint open finite intervals $I_j = (a_j, b_j)$ $(j = 1, 2, \dots)$ with dense union such that F is C^2 and monotone on each I_j. Let $I_{nj} = [a_j + 1/n, b_j - 1/n]$ ($[\alpha, \beta]$ is taken to be an empty set if $\alpha > \beta$), $\Delta_n = \bigcup_{j=1}^{n} I_{nj}$, χ_n be the characteristic function for the set Δ_n, and P_n be the multiplication of χ_n on f as an operator. Then $\lim P_n = 1$ and $[P_n, U_t] = 0$.

We propose to prove (A.4) for a dense set of f's having the following properties:

(i) The support of f is contained in Δ_n for some n.

(ii) On each I_{nj}, f is of the following form

$$f(x) = \tau F'(x) g(\tau F(x)), \tag{A.6}$$

where $\tau = \pm$ is the sign of $F'(x)$ for $x \in I_{nj}$ (where $F(x)$ is monotone) and g is an entire function (polynomial, for example). It is then enough to prove (7.42) when the integral is over $(x_1, x_2) \in I_{nj} \times I_{nk}$.

If $j \neq k$, then $(x_1 - x_2 + \sigma i \delta)^{-1}$ is uniformly bounded. By the same argument as the case of (7.34), the contribution from $I_{nj} \times I_{nk}$ tends to 0 as $t \to \pm\infty$. Hence we have only to consider the integral of x_1 and x_2 over the same finite interval (a, b) for some $a < b$.

By a change of variable from x_j to $y_j = \tau F(x_j)$, the left-hand side of (7.42) (apart from limits)

$$\int_{\alpha}^{\beta} dy_1 \int_{\alpha}^{\beta} dy_2 \bar{g}_1(y_1) g_2(y_2) (G(y_1) - G(y_2) + \sigma i \delta)^{-1} \exp \tau i (y_2 - y_1) t, \tag{A.7}$$

where $\alpha = \tau F(a) < \beta = \tau F(b)$, g_j is g of (A.6) for $f = f_j$ $(j = 1, 2)$, $\bar{g}(y) = \overline{g(\bar{y})}$ is also an entire function, $G(y_j) = x_j$ is monotone increasing with $G'(y) = (\tau F'(x))^{-1}$.

We now consider the following decomposition similar to (7.38).

$$(G(y_1) - G(y_2) + \sigma i \delta)^{-1} = R_1 + R_2 + R_3, \tag{A.8}$$

$$R_1 = G'(y_1)^{-1} (y_1 - y_2 + \sigma i \delta G'(y_1)^{-1})^{-1}, \tag{A.9}$$

$$R_2 = (y_1 - y_2 + \sigma i \delta G'(y_1)^{-1})^{-1} \{ [(y_1 - y_2)/(G(y_1) - G(y_2))] - G'(y_1)^{-1} \}, \tag{A.10}$$

$$\begin{aligned} R_3 = (y_1 - y_2 + \sigma i \delta G'(y_1)^{-1})^{-1} \{ & [(y_1 - y_2 + \sigma i \delta G'(y_1)^{-1})/(G(y_1) - G(y_2) + \sigma i \delta)] \\ & - [(y_1 - y_2)/(G(y_1) - G(y_2))] \}. \end{aligned} \tag{A.11}$$

We discuss the limit as $\delta \to 0$ and then $t \to \pm\infty$ for each term of (A.8).

We start with R_3. Denoting $z = (y_1 - y_2)$, $G' = G'(y_1)$, $R = -z^{-2}[G(y_1 - z) - G(y_1) - (-z) G'(y_1)]$ we obtain

$$R_3 = \delta i \sigma z (zG' + \sigma i \delta)^{-2} G'^{-1} R (1 + z^2 (G'z + \sigma i \delta)^{-1} R)^{-1} (1 + z G'^{-1} R)^{-1}. \tag{A.12}$$

From (A.11), R_3 is uniformly bounded and uniformly tends to 0 as $\delta \to 0$ if $|z| \geqq \varepsilon$ for any fixed $\varepsilon > 0$. Since F is C^2 and $\tau F'$ is strictly positive, G is C^2 and R is uniformly bounded. Therefore (A.12) shows that R_3 is uniformly bounded for $|z| \leqq \varepsilon$ due to

$$|\sigma i \delta z G' (zG' + \sigma i \delta)^{-2}| \leqq 1.$$

Therefore the contribution from R_3 vanishes in the limit of $\delta \to 0$.

We now come to R_2. With the same notation as above,

$$R_2 = -zRG'^{-1}(zG' + \sigma i\delta)^{-1}(1 + zG'^{-1}R)^{-1}. \tag{A.13}$$

Since G is monotone, R_2 is uniformly bounded for $|y_1 - y_2| \geqq \varepsilon$ for any $\varepsilon > 0$ fixed by (A.10) and R_2 is uniformly bounded for $|y_1 - y_2| \leqq \varepsilon$ for sufficiently small $\varepsilon > 0$ by (A.13). (Again $|z(zG' + \sigma i\delta)G'| < 1$.) Furthermore R_2 tends to

$$R_{20} = -G'^{-2}R(1 + zG'^{-1}R)^{-1} \tag{A.14}$$

pointwise except at $z = 0$ and uniformly for $|z| \geqq \varepsilon$ for any $\varepsilon > 0$. Therefore the contribution from R_2 in the limit of $\delta \to +0$ is given by an integral with uniformly bounded kernel, and hence tends to 0 as $t \to \pm\infty$ by the same reason as for (7.39).

Finally, the contribution from R_1 takes the following form

$$\begin{aligned}&\int_\alpha^\beta dy_1 \int_\alpha^\beta dy_2 \bar{g}_1(y_1) g_2(y_2) G'(y_1)^{-1}(y_1 - y_2 + \sigma i\delta G'(y_1)^{-1})^{-1} e^{-\tau izt}\\ &= \int_\alpha^\beta dy_1 \int_{y_1-\beta}^{y_1-\alpha} dz \bar{g}_1(y_1) g_2(y_1 - z) G'(y_1)^{-1}(z + \sigma i\delta G'(y_1)^{-1})^{-1} e^{-\tau izt}. \end{aligned}\tag{A.15}$$

We split $g_2(y_1 - z) = (g_2(y_1 - z) - g_2(y_1)) + g_2(y_1)$. Since g_2 is analytic,

$$R(\delta) \equiv (g_2(y_1 - z) - g_2(y_1))(z + \sigma i\delta G'(y_1)^{-1})^{-1}$$

is uniformly bounded and, in the limit of $\delta \to 0$, the integral is described in terms of the bounded kernel $R(0)$. By the same reasoning as before, this contribution tends to 0 as $t \to \pm\infty$. We are now left with

$$\int_\alpha^\beta dy_1 \bar{g}_1(y_1) g_2(y_1) G'(y_1)^{-1} \int_{y_1-\beta}^{y_1-\alpha} (z + \sigma i\delta G'(y_1)^{-1})^{-1} e^{-\tau izt} dz. \tag{A.16}$$

We deform the integration in complex z-plane to $\gamma_1 \cup \gamma_2 \cup \gamma_3$, where

$$\gamma_1 = \{y_1 - \beta - \sigma' ir; r \in [0, \varepsilon]\}, \tag{A.17}$$

$$\gamma_2 = \{r - \sigma' i\varepsilon; r \in [y_1 - \beta, y_1 - \alpha]\}, \tag{A.18}$$

$$\gamma_3 = \{y_1 - \alpha - \sigma' ir; r \in [0, \varepsilon]\}, \tag{A.19}$$

and σ' is the sign of τt. For sufficiently small δ, we obtain the contribution from the pole at $z = -\sigma i\delta G'(y_1)^{-1}$ if and only if $\sigma = \sigma'$, which is in the limit of $\delta \to 0$

$$-\sigma' 2\pi i \int_\alpha^\beta dy G'(y)^{-1} \bar{g}_1(y) g_2(y) = -\sigma 2\pi i \int_a^b \bar{f}_1(x) f_2(x) dx, \tag{A.20}$$

where we have used the change of integration variables $y = \tau F(x)$ along with (A.6) and $G'(y)^{-1} = \tau F'(x)$.

The contribution from γ_2 in the limit of $\delta \to 0$ tends to 0 as $t \to \pm\infty$ due to the exponential factor $e^{-\tau izt}$ (due to our choice of the sign σ'). The contribution from γ_1 and γ_3 tends to 0 as $\varepsilon \to 0$ because the integrand is integrable for $\delta = 0$.

Thus we obtain (A.20) if the sign of $F'(x)$ in (a, b) is the same as $\pm\sigma$ depending on whether t tends to $+\infty$ or $-\infty$ and 0 otherwise. This coincides with the right-hand side of (7.42) and Lemma 7.3 is proved.

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