Hyperbolicity, Sinks and Measure in One Dimensional Dynamics

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Abstract. Let f be a C^2 map of the circle or the interval and let $\Sigma(f)$ denote the complement of the basins of attraction of the attracting periodic orbits. We prove that $\Sigma(f)$ is a hyperbolic expanding set if (and obviously only if) every periodic point is hyperbolic and $\Sigma(f)$ doesn't contain the critical point. This is the real one dimensional version of Fatou's hyperbolicity criteria for holomorphic endomorphisms of the Riemann sphere. We also explore other applications of the techniques used for the result above, proving, for instance, that for every C^2 immersion f of the circle (i.e. a map of the circle onto itself without critical points), either its Julia set has measure zero or it is the whole circle and then f is ergodic, i.e. positively invariant Borel sets have zero or full measure.

Introduction

The subject of this paper is the dynamics of C^2 maps of the circle or the interval, on regions bounded away from the critical points. The aspects of the dynamics that we shall consider, and the corresponding results that we shall prove, can be summarized as follows:

Hyperbolicity—If the map is not topologically equivalent to an irrational rotation of the circle, every compact invariant set not containing critical points, sinks or non-hyperbolic periodic points is hyperbolic.

Stability—Structural stability is generic in the space of C^r immersions of the circle and is characterized by the hyperbolicity of the non-wandering set.

Ergodicity—Transitive C^2 immersions of the circle are ergodic, i.e. every invariant Borel set has either zero or full Lebesgue measure.

Measure—If Γ is a compact invariant set with empty interior not containing critical points, then either the Lebesgue measure of Γ is zero or there exists an interval U that is mapped diffeomorphically into itself by some power of the map and such that $\Gamma \cap U$ has positive Lebesgue measure.

Sinks—For every compact set K that doesn't contain critical points, the periods of the sinks or non-hyperbolic periodic orbits contained in K are bounded.

Before entering into the precise statements of these results, let us recall the definition and basic properties of the concepts they involve.

Let N denote either the circle $S^1 = \{z \in \mathbb{C} \,||z| = 1\}$ or the interval $[0,1]$. Let $\mathrm{End}^r(N)$ be the space of C^r maps $f: N \circlearrowleft$ (endomorphisms) endowed with the C^r topology. As usual we say that $x \in N$ is a *periodic point* of $f \in \mathrm{End}^r(N)$ if $f^n(x) = x$ for some $n \geq 1$. In this case we say that it is *hyperbolic* if $|(f^n)'(x)| \neq 1$, a *sink* if $|(f^n)'(x)| < 1$ and a *source* if $|(f^n)'(x)| > 1$. The *basin* of a sink x is defined as the set of points y such that $\lim_{n \to +\infty} d(f^n)(x), f^n(y)) = 0$. It is an open set containing x. We say that a set $\Lambda \subset N$ is an *invariant* set of $f \in \mathrm{End}^r(N)$ if $f(\Lambda) \subset \Lambda$. If there exists a neighborhood U of Λ such that $\Lambda = \bigcap_n f^n(U)$, we say that Λ is *isolated* and that U is an *isolating block* Λ. A *hyperbolic set* of $f \in \mathrm{End}^r(N)$ is a compact invariant set Λ such that there exists constants $K > 0$ and $\lambda > 1$ satisfying $|(f^n)'(x)| \geq K\lambda^n$ for all $x \in \Lambda$, $n \geq 0$. Hyperbolic sets have remarkable properties, the most outstanding being its stability and the reducibility of its dynamics to certain simpler models.

What follows is a short glossary of these properties:

I) If N is a hyperbolic set of $f \in \mathrm{End}^r(N)$, then $N = S^1$, and f *is topologically equivalent to a map* $f_d: S^1 \circlearrowleft$ given by $f_d(z) = z^d$, where d is an integer $\neq 1$ or -1. Recall that topologically equivalent means that there exists a homeomorphism $h: S^1 \circlearrowleft$ such that $hf = f_d h$.

II) If $\Lambda \subset N - \partial N$ is an isolated hyperbolic set of $f \in \mathrm{End}^r(N)$ and $\Lambda \neq N$, then $f|\Lambda$ is topologically equivalent to a subshift of finite type. A subshift of finite type is a map $\partial: \Sigma^+(A) \circlearrowleft$, where A is a $m \times m$ matrix whose entries a (i,j) are 0 or 1; $\Sigma^+(A)$ is the space of the sequences $\theta: \mathbb{Z}^+ \to \{1,\dots,m\}$ that satisfy the transition law a $(\theta(n), \theta(n+1)) = 1$ for all $n \geq 0$, endowed with the pointwise convergence topology, and $\sigma: \Sigma^+(A) \circlearrowleft$ is defined by $\sigma(\theta)(n) = \theta(n+1)$.

III) If $\Lambda \subset N - \partial N$ is an isolated hyperbolic set of $f \in \mathrm{End}^1(N)$, then for every isolating block U of Λ, there exists a neighborhood $\mathcal{U}$ of f such that if $g \in \mathcal{U}, f|\Lambda$ is topologically equivalent to $g/\bigcap_n g^n(U)$.

IV) For every hyperbolic set $\Lambda \subset N - \partial N$ of $f \in \mathrm{End}^r(N)$ and every neighborhood V of Λ there exists an isolated hyperbolic contained in V and containing Λ.

Property (I) is the one dimensional case of a result of Shub [6]. Property (II) was proved by Jacobson [1] and Nitecki [5] (for a certain Λ, but their techniques with minor modifications prove (II)). Property (III) is nowadays a minor application of the stability theory of hyperbolic sets, but proofs can be found in [5]. (IV) is folklorical. We shall give a simple proof of it in the Appendix.

In [1], Jacobson introduced the set $\Sigma(f)$ of an endomorphism $f \in \mathrm{End}^r(N)$ defining it as the complement of the union of the basins of the sinks and, attempting to characterize the C^r structurally stable endormorphisms (i.e. those that are topoligically equivalent to every C^r nearby endomorphism), he considered the class of maps $f \in \mathrm{End}^r(N)$ such that $\Sigma(f)$ is hyperbolic and contained in $N - \partial N$. This class will be denoted $\mathcal{S}^r(N)$. He proved the following results:

V) $\mathcal{S}^1(N)$ is open and dense in $\mathrm{End}^1(N)$.

VI) Every $f \in \mathcal{S}^1(N)$ is Σ-stable, i.e. if $g \in \mathrm{End}^1(N)$ is C^1 near to f, $g/\Sigma(g)$ is topologically equivalent to $f/\Sigma(f)$.

VII) If $f \in \mathscr{S}^1(N)$ and $\Sigma(f) = N$, then $N = S^1$ and $f:S^1 \supset$ is topologically equivalent to a map $f_d:S^1 \supset$ (defined as in property (I), of which this property is an immediate corollary).

VIII) If $f \in \mathscr{S}^1(N)$ and $\Sigma(f) \neq N$, then $f/\Sigma(f)$ is topologically equivalent to a finite type subshift. This is a corollary of (II).

IX) If $f \in \mathscr{S}^1(N)$ and has not critical points, then f is C^1-structurally stable.

X) If $f \in \mathscr{S}^2(N)$ and satisfies:

a) Every critical point is non-degenerate, i.e. $f'(x) \Rightarrow f''(x) \neq 0$.

b) If $\tilde{C}(f)$ denotes the set of critical points of f, plus, in the case $N = [0, 1]$, the points 0 and 1, then $f''(x) \neq f^m(y)$ for all x and y in $\tilde{C}(f)$ and every $n \geq 0, m \geq 1$.

c) If $N = [0, 1]$, 0 and 1 are not critical points of f, then f is C^2-structurally stable.

These properties pose two questions:

Problem I. Is $\mathscr{S}^r(N)$ dense in End$^r(N)$ for $r \geq 2$?

Problem II. Is $\Sigma(f)$ hyperbolic for every C^r-structurally stable $f \in$ End$^r(N)$?

An affirmative answer to Problem II would yield a complete characterization of structural stability, namely that $f \in$ End$^1(N)$ is structurally stable if and only if $C(f) = \phi$ and $f \in \mathscr{S}^1(N)$, and that $f \in$ End$^r(N)(r \geq 2)$ is C^r-structurally stable if and only if $f \in \mathscr{S}^r(N)$ and satisfies (a), (b) and (c). We shall give positive answers to both questions when restricted to the space of C^r immersions (that is an open subset of End$^r(N)$) with $r \geq 2$. This will be based on the following hyperbolicity criteria that also has an independent intrinsic interest.

Theorem A. *If $f \in$ End$^2(N)$ and $\Lambda \subset N$ is a compact invariant set that doesn't contain critical points, sinks or non-hyperbolic periodic points, then either $\Lambda = N = S^1$ and f is topologically equivalent to an irrational rotation or Λ is a hyperbolic set.*

Corollary I. *If all the periodic points of $f \in$ End$^2(N)$ are hyperbolic and $\Sigma(f)$ doesn't contain critical points, then either $N = S^1$ and f is topologically equivalent to an irrational rotation or $\Sigma(f)$ is a hyperbolic set.*

Corollary II. *For all $r \geq 1$, every C^r immersion $f:S^1 \supset$ can be approximated in the C^r topology by an immersion $g:S^1 \supset$ such that $\Sigma(g)$ is a hyperbolic set.*

Corollary III. *For all $r \geq 2$, a C^r immersion $f:S^1 \supset$ is C^r structurally stable if and only if $\Sigma(f)$ is hyperbolic.*

Corollary I is an easy consequence of Theorem A. Corollary II also follows easily from Theorem A recalling that for all $r \geq 1$, the hyperbolicity of all the periodic points is a generic property in End$^r(N)$. Corollary III follows from Theorem A, property (IX) and the fact that all the periodic points of a C^r-structurally stable endomorphism are hyperbolic.

We were not able to solve Problems I and II, but using Theorem A we can reduce its solution to a problem related to links of critical points. To define this notion first recall that given $f \in$ End$^r(N)$, the α-limit set of a point $x \in N$ is defined by:

$$\alpha(x) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} f^{-m}(\{x\})}.$$

Denote $\tilde{C}(f)$ the set of critical points of f plus, in the case $N = [0, 1]$, the points 0 and 1. A *link* of f is a pair (x, y) of points in $\tilde{C}(f)$ such that y is in the forward orbit of x. A *fake link* is a pair (x, y) of points in $\tilde{C}(f)$ such that it is not a link and $x \in \alpha(y)$. Denote $v(f)$ the number of links of f.

Conjecture. If all the critical points of $f \in \text{End}^r(N)$ $(r \geq 2)$ are non-degenerate and all its periodic points are hyperbolic, then, if f has a fake link, there exists $g \in \text{End}^r(N)$ arbitrarily near to f in the C^r topology and satisfying $v(g) > v(f)$.

In other words, using the fake link a new link is created by a small perturbation without destroying those already existing. To see how this conjecture implies an affirmative answer to Problems I and II we need the following approximation theorem.

Theorem B. *Suppose that* $f \in \text{End}^r(N)$, $r \geq 2$, *satisfies:*

a) *all the periodic points of* f *are hyperbolic,*
b) *all the critical points are non-degenerate,*
c) f *has no fake links.*

Then f *can be approximated in the* C^r *topology by an endomorphism* $g \in \mathscr{S}^r(N)$.

Now let $\mathscr{S}_0^r(N)$ be the set of maps $f \in \text{End}^r(N)$ such that all its periodic points are hyperbolic and all its critical points are nondegenerate. If $f_0 \in \mathscr{S}_0^r(N)$ take an open neighborhood $\mathscr{U}$ of f_0 such that the number of critical points of every $f \in \mathscr{U}$ is the same, say N. Then $v(f) \leq (N + 2)^2$ for all $f \in \mathscr{U}$. Take $f_1 \in \mathscr{U}$ such that $v(f_1) \geq v(f)$ for all $f \in \mathscr{U}$. It is easy to see that we can construct an endomorphism $f_2 \in \mathscr{U}$ such that all its periodic points are hyperbolic and having the same links of f_1. Then $v(f_2) = v(f_1)$. Suppose that f_2 has a fake link. If the conjecture is true there exists $f_3 \in \mathscr{U}$ with $v(f_3) > v(f_2)$. But then $v(f_3) > v(f_1)$ contradicting the way we choose f_1. Therefore f_2 has no fake links. By Theorem B we can approximate f_2 by $g \in \mathscr{U}$ such that $g \in \mathscr{S}^r(N)$. Since $\mathscr{U}$ is arbitrary, we have proved that, if the conjecture is true, every element of $\mathscr{S}_0^r(N)$ can be approximated by an element in $\mathscr{S}^r(N)$. Since $\mathscr{S}_0^r(N)$ is obviously dense in $\text{End}^r(N)$ this yields an affirmative answer to Problem I. Therefore, if $f \in \text{End}^r(N)$ is C^r-structurally stable $(r \geq 2)$ we can approximate it by $g \in \mathscr{S}^r(N)$, and because of the structural stability of f, g and f are topologically equivalent. Moreover, it is easy to verify that if $r \geq 2$ all the critical points of a C^r-structurally stable endomorphism are non-degenerate and that a topological equivalence transforms non-degenerate critical points in critical points. Then, the fact that $\Sigma(g)$ contains no critical points implies that $\Sigma(f)$ contains no critical points. Since all the periodic points of f are hyperbolic (because f is structurally stable) it follows from Corollary I that $\Sigma(f)$ is hyperbolic.

This approach is obviously motivated by Peixoto's proof [6] of the density of Morse–Smale vectorfields in two dimensional compact orientable manifolds, where the role of the critical points is played by the saddles and saddle connections correspond to links. The phenomena corresponding to fake links is a separatrix of a saddle accumulating in a saddle. In Peixoto's proof there is a step, corresponding to Theorem B, where it is proved that a Kupka–Smale vectorfield without fake links can be approximated by a Morse–Smale vectorfield. But the crucial point is to show that when a separatrix of a saddle accumulates in another saddle then a new saddle

connection can be created without destroying those already existing. However this step doesn't work in the non-orientable case because of the reversing of orientation of certain Poincaré maps. The same problem appears here, but instead of being produced by the non-orientability of the manifold its cause is the existence of critical points.

Now let us consider the problem of whether there exists a one dimensional version of the infinitely many sinks phenomena found by Newhouse for diffeomorphisms in dimension ≥ 2. To produce examples of endomorphisms $f:N \supset$ with infinitely many sinks is trivial. For instance one can easily construct C^∞ diffeomorphisms $f:[0,1] \supset$ having infinitely many sinks. However the following theorem states that the only way to produce C^2 endomorphisms with infinitely many sinks bounded away from the critical points is by inserting intervals where some power of f acts as a diffeomorphism with infinitely many sinks. To give the precise statement of this property, let us first introduce a definition. Given if $\in End^r(N)$ we shall say that two periodic orbits γ_1 and γ_2 are homologous if they have the same period n and there exist $x_1 \in \gamma_1$ and $x_2 \in \gamma_2$ and an interval J with x_1 and $x_2 \in \gamma_2$ as endpoints such that f^n/J maps J diffeomorphically onto itself. Obviously this is an equivalence relation.

Theorem C. *If* $f \in End^2(N)$, *every compact set that doesn't contain critical points contains only finitely many non-homologous orbits of sinks or non-hyperbolic periodic points.*

Corollary I. *If* $f \in End^2(N)$ *and K is a compact set not containing critical points, then the periods of the sinks or non-hyperbolic periodic points whose orbit is contained in K are bounded.*

Corollary II. *If* $f:N \supset$ *is real analytic and K is a compact set not containing critical points then the set of sinks or non-hyperbolic periodic points whose orbits are contained in K is finite.*

Both Corollaries follow immediately from Theorem C. Another interesting application is the finiteness of the set of periodic plateaus of an immersion. Let us recall the basic dynamical properties of immersions. Given a C^1 endomorphism $f:S^1 \supset$ with degree $d \neq 1$ or -1 there exists a continuous map $h:S^1 \supset$ of degree 1 such that $hf = f_d h$, where $f_d:S_1 \supset$ is defined by $f_d(z) = z^d$. Moreover, if f is an immersion, h is monotone, i.e. for every $z \in S^1$, $h^{-1}(\{z\})$ is either a unique point or an interval $[a,b]$ with $a \neq b$. In the last case we say that (a,b) is a *plateau* of f. Denote $J(f)$ the complement of the union of the plateaux of f. Using the map h it is easy to check the following properties:

XI) Two plateaux are either disjoint or coincide.

XII) f maps diffeomorphically plateaux onto plateaux.

XIII) Every plateau U is either periodic (i.e. $f^N(U) = U$ for some $N \geq 1$), eventually periodic (i.e. $f^m(U)$ is periodic for some $m \geq 1$) or wandering (i.e. $f^n(U) \cap f^m(U) = l$ for all $n \geq 1$).

XIV) If f is C^2, every plateau is periodic or eventually periodic.

XV) $x \in J(f)$ if and only if for every neighborhood W of x there exists $n \geq 0$ such that $f^n(U) = S^1$. This implies that $F/J(f)$ is transitive.

XVI) $J(f)$ is either a Cantor set or coincides with S^1. In this case f is topologically equivalent to $z \mapsto z^a$.

XVII) $J(f)$ contains a dense subset of sources.

Property (XIV) is not a corollary of the existence of the semiconjugacy h but follows easily from an adaptation of Denjoy's theorem (see Lemma I.4 below).

Each periodic plateau is mapped by h in a periodic point of f_d with the same period as the plateau. Moreover different plateaux are mapped in different points. Therefore if an immersion $f: S^1 \supset$ has infinitely many periodic plateaux, the periods of these periodic plateaux is an unbounded set. But it is clear that every periodic plateau contains a sink or a non-hyperbolic periodic points. Then, by Corollary I of Theorem C, there can be only finitely many periodic plateaux. We have thus proved:

Corollary III. *The set of periodic plateaux of a C^2 immersion $f: S^1 \supset$ is finite*

The question of the Lebesgue measure of compact invariant sets not containing critical points has an answer similar to that of the finiteness of the set of sinks. Every diffeomorphism of the interval has compact invariant sets with positive measure, but the next theorem shows that this is essentially the only way of producing examples of a compact invariant set without critical points and with positive measure.

Theorem D. *If $f \in \mathrm{End}^2(N)$ and $\Lambda \underset{\neq}{\subseteq} N$ is a compact invariant set not containing critical points, then, either the Lebesgue measure of Λ is a zero of there exist an interval $J \subset N$ and an integer $n \geq 1$ such that $f^n(J) \subset J$, f^n/J has no critical points and $J \cap \Lambda$ has positive Lebesgue measure.*

Corollary I. *If $f : S^1 \supset$ is a C^2 immersion, either $J(f) = S^1$ or the Lebesgue measure of $J(f)$ is zero.*

Now observe that if all the periodic points of a C^2 immersion are hyperbolic, then by Theorem A, f is expanding, i.e. there exists $N \geq 1$ such that $|(f^N)'(x)| > 1$ for all $x \in S^1$. This class of transformations have a very developed ergodic theory. For instance, they have a unique f-invariant probability absolutely continuous with respect to the Lebesgue measure and this f-invariant probability is ergodic and its Radon–Nikodym derivative with respect to the Lebesgue measure is a Holder continuous positive function. For simple reasons this result is false if there exists a non-hyperbolic periodic point x_0, even if for some $N \geq 1$ the inequality $|(f^N)'(x)| > 1$ holds for all $x \neq x_0$. However, immersions $f : S^1 \supset$ with $J(f) = S^1$ are ergodic:

Theorem E. *Every C^2 immersion $f : S^1 \supset$ with $J(f) = S^1$ is ergodic, i.e., every invariant Borel set has either zero or full Lebesgue measure.*

The proof of these theorems, with the exception of Theorem B, will be based on Lemma I.3, stated in Sect. I and proved in Sect. II. Essentially it states that given a compact invariant set $\Lambda \subset N$ of a C^2 endomorphism, then if Λ doesn't contain critical points and $\bigcap_{n \geq 0} f^n(\Lambda)$ is not a union of periodic orbits, there exists an open interval J having non-empty intersection with $\bigcap_{n \geq 0} f^n(\Lambda)$, where the backward dynamics of $f/\bigcap_{n \geq 0} f^n(\Lambda)$ is reasonably hyperbolic. More precisely, if $a \in J \cap (\bigcap_{n \geq 0} f^n(\Lambda))$

and $a = a_0, a_1, \ldots$, are points in $\bigcap_{n \geq 0} f^n(\Lambda)$ satisfying $f(a_{n+1}) = a_n$ for all $n \geq 0$, there exists maps $\varphi_n : J \to N, n = 1, 2, \ldots$ such that:

$$\varphi_n(a) = a_n,$$

$$f^n \varphi_n(x) = x \quad \text{for all } x \in J$$

for all $n \geq 1$, and also satisfying the following three properties:

a) $\varphi_n(J) \cap J = \phi \quad \text{or} \quad \varphi_n(J) \subset J$,

b) $\lim\limits_{n \to +\infty} |\varphi_n(y)| = 0 \quad \text{for all } y \in J$,

c) There exists $K > 0$ such that

$$\frac{1}{K} \leq \frac{|\varphi_n'(z)|}{|\varphi_n'(y)|} \leq K$$

for all $n \geq 1, z \in J, y \in J$.

The reader familiar with the ergodic theory of hyperbolic sets will recognize in (a), (b), (c) an analog of the properties on which this theory is founded. In fact, once the existence of J satisfying all these properties is proved, the proofs of Theorems A, C, D and E requires little or very standard work. Theorem B is a corollary of Theorem A and a folklorical extension of Denjoy's theorem (Lemma I.4). It has no intrinsic value. Its interest may reside in motivating research on the link conjecture. This conjecture has a Closing Lemma flavour and as such it can prompt the remark that it can be harder than the problems it attempts to solve. However one must not forget that in the interval, Closing Lemma problems have been successfully handled (L. S. Young [7], I. Malta [4]) and that what Jacobson does in his paper is (essentially) to solve the conjecture exploiting the fact that the points involved are critical.

We suggest to the readers to (at least in the first lecture) follow the proofs, reasoning in the case of f being an immersion of S^1. By doing this they will reach the core of the proofs faster and avoid tedious technicalities.

A natural question is whether Theorems C and D survive without the hypothesis that keeps the critical points away. A concise simple question that exposes the lack of good general techniques to analyse the dynamics near critical points is the following.

Problem III. Does there exist a real analytic endomorphism $f : N \circlearrowleft$ with infinitely many sinks?

I. Proof of the Theorems

To prove the hyperbolicity of a set, instead of directly finding the constants K and λ required by the definition, it is easier to check the formally weaker condition required by the next lemma (whose very easy proof is left to the reader). We shall use the following notation: If $f \in \mathrm{End}^1(N)$ and Λ is a subset of N, we denote $\mathscr{S}(\Lambda, x)$ the set of sequences $\theta : \mathbb{Z}^+ \to \Lambda$ such that $\theta(0) = x$ and $f(\theta(n)) = \theta(n-1)$ for all $n \geq 1$.

Lemma I.1. *If $f \in \mathrm{End}^1(N)$ and Λ is a compact invariant set such that*

$$\lim_{n \to +\infty} |(f^n)'(\theta(n))| = +\infty \tag{1}$$

for all $x \in \Lambda$ and $\theta \in \mathscr{S}(\Lambda, x)$, then Λ is hyperbolic.

To verify the hypothesis of this lemma we shall use the concept of coherent sequence of branches. If $J \subset N$ is an open interval we say that $\varphi : J \to N$ is a branch of f^{-n}/J if φ is C^1 and $f^n \varphi(x) = x$ for all $x \in J$. A *coherent sequence of branches* is a couple $(J, \{\varphi_n\})$, where J is an open interval and $\varphi_n : J \to N$ is a branch of f^{-n}/J, $n = 1, 2, 3, \ldots$, satisfying $f \varphi_{n+1} = \varphi_n$ for all $n \geq 1$.

Given $f \in \mathrm{End}^1(N)$ and a compact invariant set $\Lambda \subset N$, we say that an open interval J is *adapted* to Λ if there exists $\delta > 0$ such that for every $x \in J \cap \Lambda$ and $\theta \in \mathscr{S}(\Lambda, x)$ there exists a coherent sequence $(J, \{\varphi_n\})$ satisfying:

a) $\varphi_n(x) = \theta(n)$,

b) $d(\varphi_n(J), \ C(f)) > \delta$,

c) $\varphi_n(J) \subset J$ or $\varphi_n(J) \cap J = \phi$,

for all $n \geq 1$. Let us say that a coherent sequence $(J, \{\varphi_n\})$ is *associated* to Λ if there exists $x \in \Lambda \cap J$ such that $\varphi_n(x) \in \Lambda$ for all $n \geq 1$.

Lemma I.2. *If $f \in \mathrm{End}^2(N)$ is not topologically equivalent to an irrational rotation and $\Lambda \subset N - \partial N$ is a compact invariant set not containing critical points, then, for all non-periodic points $x \in \bigcap_{n \geq 0} f^n(\Lambda)$ there exists an interval $J \supset \{x\}$ adapted to Λ.*

This lemma and the next one will be proved in Sect. II. Given $f \in \mathrm{End}^1(N)$, a compact invariant set Λ and an interval J adapted to Λ, we say that a map $\psi : J \supset$ is a *return map* of Λ if there exist $m \geq 1$ such that ψ is a branch of f^{-m}/J satisfying $f^j(\psi(J)) \cap J = \phi$ for all $0 < j < m$ and there exists $x \in J \cap (\bigcap_{n \geq 0} f^n(\Lambda))$ such that $\psi(x) \in \bigcap_{n \geq 0} f^n(\Lambda)$. It is clear that $f^m(\psi(J)) = J$. Denote $F(\Lambda, J)$ the set of return maps $\psi : J \supset$ of Λ. The next lemma is the fundamental step of the proof of the theorems.

Lemma I.3. *If $f \in \mathrm{End}^2(N)$ is not topologically equivalent to an irrational rotation and $\Lambda \subset N - \partial N$ is a compact invariant set not containing critical points, then, if $\bigcap_{n \geq 0} f^n(\Lambda)$ contains non-periodic points, there exists an interval J adapted to Λ and constants $K_1 > 0, 0 < \lambda < 1$ such that every coherent sequence $(J, \{\varphi_n\})$ associated to Λ satisfies*

$$\sum_{m=1}^{\infty} |\varphi'_m(x)| \leq K_1, \tag{2}$$

$$|\varphi'_n(x)| \leq K_1 |\varphi'_n(y)|, \tag{3}$$

for all $x \in J$, $y \in J$, $n \geq 1$, and

$$|\psi'(x)| \leq \lambda \tag{4}$$

for all $x \in J$, $\psi \in F(J, \Lambda)$.

Now let us prove Theorem A. First observe that *given $f \in \mathrm{End}^r(N)$ and a compact invariant set Λ, we can assume without loss of generality that $\Lambda \subset N - \partial N$. In fact, $\Lambda \cap \partial N \neq \phi$ means that $N = [0, 1]$ and Λ contains either 0 or 1. Then we can extend f to a C^r map g of a bigger interval, say $[-1, 2]$, and now Λ will be a compact invariant set of $g \in \mathrm{End}^r([-1, 2])$ such that $\Lambda \subset [-1, 2] - \partial [-1, 2]$. Now we work with g and $N = [-1, 2]$ (where we can apply I.2 and I.3) and the conclusion on Λ will hold also for

Λ as an invariant set of $f : [0, 1] \supset$. For this reason, *in the proofs of* A, B, C *and* D, *we shall assume that the compact invariant sets that we shall handle, are contained in $N - \partial N$.*

To prove Theorem A suppose by contradiction that f is not topologically equivalent to an irrational rotation and that Λ is not hyperbolic. Denote $\mathscr{F}$ the family of non-hyperbolic compact invariant subsets of Λ. Order $\mathscr{F}$ by inclusion. $\mathscr{F} \neq \phi$ because $\Lambda \in \mathscr{F}$. Moreover, if $\mathscr{F}_0 \subset \mathscr{F}$ is a totally ordered subfamily, the set $\cap \{ \Gamma | \Gamma \in \mathscr{F}_0 \}$ belongs to $\mathscr{F}_0$ because otherwise it would be hyperbolic and then it would have a compact neighborhood U whose maximal invariant $\bigcap_n f^n(U)$ is hyperbolic. But $\Gamma \subset U$ for some $\Gamma \in \mathscr{F}_0$, and then $\Gamma \subset \bigcap_n f^n(U)$, thus implying that Γ is hyperbolic and contradicting $\Gamma \in \mathscr{F}_0 \subset \mathscr{F}$. Therefore, by Zorn's Lemma, there exists a minimal $\Lambda_0 \in \mathscr{F}_0$. Observe that $\bigcap_{n \geq 0} f^n(\Lambda_0)$ is again non-hyperbolic (because if it were, Λ_0 would be also hyperbolic; this follows easily from Lemma I.1). Then the minimality of Λ_0 implies $\Lambda_0 = \bigcap_{n \geq 0} f^n(\Lambda_0)$. Hence $f(\Lambda_0) = \Lambda_0$. Moreover Λ_0 can't be a union of periodic orbits because every periodic orbit is hyperbolic and then Λ_0 would be hyperbolic (a compact union of hyperbolic sets is hyperbolic, this is another simple corollary of I.1). Then we can apply Lemma I.3. Let J be the interval given by I.3. Take any $\theta \in \mathscr{S}(\Lambda_0, x)$, $x \in \Lambda_0$. Let Γ be the limit set of θ (i.e. the set of points p such that $\liminf_{n \to +\infty} d(\theta(n), p) = 0$). Γ is compact and invariant. If $\Gamma \neq \Lambda_0$ then Γ is hyperbolic by the minimality of Λ in $\mathscr{F}$. Using the hyperbolicity of Γ it is easy to prove that

$$\lim_{n \to +\infty} |(f^n)'(\theta(n))| = \infty. \tag{5}$$

If $\Gamma = \Lambda_0$ there exists n_0 such that $\theta(n_0) \in J \cap \Lambda_0$. Define $\hat{\theta} \in \mathscr{S}(\Lambda_0, \theta(n_0))$ by $\hat{\theta}(n) = \theta(n + n_0)$. Since J is adapted to Λ_0 there exists a coherent sequence $(J, \{\varphi_n\})$ satisfying $\varphi_n(\theta(n_0)) = \hat{\theta}(n)$ for all $n \geq 1$. Moreover (2) implies

$$\sum_{n=1}^{\infty} |\varphi_n'(\hat{\theta}(n))| < +\infty.$$

In particular $\lim_{n \to +\infty} |\varphi_n'(\hat{\theta}(n))| = 0$. Then $\lim_{n \to +\infty} |(f^n)'(\theta(n))| = \infty$ because $(f^n)'(\hat{\theta}(n)) = (\varphi_n'(\hat{\theta}(n)))^{-1}$. Hence:

$$\lim_{n \to +\infty} |(f^n)'(\hat{\theta}(n))| = \lim_{n \to +\infty} |(f^n)'(\hat{\theta}(n - n_0))|$$

$$= \lim_{n \to +\infty} |(f^{n_0})'(\theta(n_0))| \; |(f^{n-n_0})'(\hat{\theta}(n - n_0))|$$

$$= +\infty.$$

Therefore (5) holds for all $x \in \Lambda_0$ and $\theta \in \mathscr{S}(\Lambda_0, x)$ thus proving, by Lemma I.1, that Λ_0 is hyperbolic and contradicting $\Lambda_0 \in \mathscr{F}$.

To prove Theorem B we need one more lemma. It is a property essentially contained in Schwarz's proof of Denjoy's theorem. Given $f \in \mathrm{End}^1(N)$ we say that an open interval $J \subset N$ is a *d-interval* if f^n / J is injective and has no critical points for all $n \geq 1$. We say that a *d*-interval J is *eventually periodic* if there exist a *d*-interval J_1 and

integers $n \geq 0$, $m \geq 1$ satisfying

$$f^m(J_1) \subset J_1, \tag{6}$$

$$f^n(J) \subset J_1. \tag{7}$$

Lemma I.4. *If* $f \in \mathrm{End}^2(N)$ *is not topologically equilvalent to an irrational rotation, every d-interval J is either eventually periodic or satisfies*:

$$\liminf_{n \to +\infty} d(f^n(J); C(f)) = 0,$$

where $C(f)$ *is the set of critical points of* f.

Now suppose that $f \in \mathrm{End}^r(N)$ satisfies the hypothesis of Theorem B. Denote $\hat{C}(f)$ the set of critical points of f contained in $\Sigma(f)$. Order the points in $\hat{C}(f)$ by the relation $y \geq x$ if there exists $n \geq 1$ such that $y = f^n(x)$. This order has no cycles because a cycle would be a periodic critical point, hence a sink that cannot be contained in $\Sigma(f)$. Then there exist points $x_1, \ldots, x_m$ in $\hat{C}(f)$ such that every $x \in \hat{C}(F)$ satisfies $x \leq x_i$ for some $1 \leq i \leq m$ and if $x_i \leq x$ then $x = x_i$. Denote S_i the set of points $x \in \hat{C}(f)$ such that $x \leq x_i$. Let us say that a d-interval J is *maximal* if every open interval properly containing J is not a d-interval. It is clear that if f is not a diffeomorphism of the circle every d-interval is contained in a unique maximal d-interval and two maximal d-intervals are either disjoint or coincide. Since for the proof of Theorem B the case of f being a diffeomorphism of the circle is trivial, we can assume that this property holds and then the set of maximal d-intervals is a countable family $J_1, J_2, \ldots$ of disjoint intervals. Moreover, since f maps d-intervals in d-intervals it follows that for all $i \geq 1$ there exists $j \geq 1$ such that $f(J_i) \subset J_j$. We shall say that J_i is *periodic* if $f^n(J_i) \subset J_i$ for some $n \geq 1$, and that it is *eventually periodic* if $f^m(J_i)$ is contained in a periodic maximal d-interval for some $m \geq 0$. The absence of fake links easily implies that for every critical point x there exist intervals (a, x) and (x, b) (only one if $N = [0, 1]$ and x is an endpoint) not intersecting the backward orbits of the critical points. Hence (a, x) and (x, b) are d-intervals. Therefore to every critical point x we can associate maximal d-intervals $J^-(x) = (a(x), x)$, $J^+(x) = (x, b(x))$ (again, only one when $N = [0, 1]$ and x is an endpoint). If we apply Lemma I.4 to $J^\sigma(x) (\sigma = + \text{ or } -)$ we get that either $J^\sigma(x)$ is eventually periodic or

$$\liminf_{n \geq 0} d(f^n(J^\sigma(x)), C(f)) = 0. \tag{8}$$

If (8) holds it means that there exists $y \in C(f)$ such that for suitable values of n, $f^n(J^\sigma(x))$ is arbitrarily near to y. Hence, for $\alpha = + \text{ or } -$, we have $f^n(J^\sigma(x)) \cap J^\alpha(y) \neq \phi$. But since $f^n(J^\sigma(x))$ is a d-interval it follows that

$$f^n(J^\sigma(x)) \subset J^\alpha(y). \tag{9}$$

Hence every $J^\sigma(x)$ is either eventually periodic or satisfies (9) for some $n \geq 0$, $y \in C(f)$ and $\alpha = + \text{ or } -$. If $J^\alpha(y)$ is eventually periodic then so is $J^\sigma(x)$. If it is not, we have $f^k(J^\alpha(y)) \subset J^\beta(z)$ for some $k \geq 1$, $y \in C(f)$, $\beta = + \text{ or } -$. Since the number of critical points is finite, an interval must appear twice in this process, thus proving that the initial $J^\sigma(x)$ is eventually periodic. Therefore every $J^\sigma(x)$ is eventually periodic. Then to every x_i we can associate a maximal d-interval A_i and integers $n_i \geq 0$, $m_i \geq 1$

satisfying

$$f^{n_i}(J^+(x_i)) \subset A_i, \quad f^{m_i}(A_i) \subset A_i.$$

Without loss of generality we can assume (replacing m_i by $2m_i$ if necessary) that f^{m_i}/A_i is order preserving. Since every periodic point of f^{m_i} is hyperbolic it follows that every $x \in \bar{A}_i$ is either a source or belongs to the basin of a sink (recall that f^{m_i}/A_i has no critical points). Then $f^{n_i}(x_i) \in \bar{A}_i$ is either a source or belongs to the basin of a sink. The second possibility cannot hold because $x_i \in \Sigma(f)$. Then $f^{n_i}(x_i)$ is a source for all $1 \leq i \leq m$.

Using that f^{m_i}/A_i has no critical points, it is easy to find a closed interval $B_i \subset \bar{A}_i$ such that one of its endpoints is $f^{n_i}(x_i)$ and the other endpoint is a sink whose basin contains Int B_i. Now we take $g \in \text{End}^r(N)$ near to f, having all its periodic points hyperbolic and satisfying

a) f and g coincide in a neighborhood of $\bigcup\limits_{i=1}^{m} B_i$,

b) $C(f) = C(g)$,

c) $g^{n_i}(x_i) \in \text{Int } B_i$,

d) g and f coincide in $\bigcup\limits_{i=1}^{m} S_i$.

From (c) and (d) follows that all the critical points in $\hat{C}(f)$ eventually under g will fall in $\bigcup\limits_{i=1}^{m} \text{Int}(B_i)$. By (a), every Int B_i is in the basin of a sink of g. Hence

$$\hat{C}(f) \subset \Sigma(g)^c. \tag{10}$$

On the other hand, points in $C(f) - \hat{C}(f)$ are in the basins of sinks of f, and if g is sufficiently near to f, are in the basins of sinks of g. Therefore:

$$C(f) - \hat{C}(f) \subset \Sigma(g)^c. \tag{11}$$

From (10), (11) and (b) we get

$$C(g) = C(f) \subset \Sigma(g)^c. \tag{12}$$

Since every periodic point of g is hyperbolic and $\Sigma(g)$ by definition doesn't contain sinks and by (12) doesn't contain critical points, we can apply Theorem A to $\Sigma(g)$ proving that it is a hyperbolic set

To prove Theorem C suppose by contradiction that there exists an infinite set S of sinks and non-hyperbolic periodic points such that any pair of points $x, y \in S$ are not homologous. First we shall prove that $\bar{S}$ contains a non-periodic point. If it doesn't there exists a periodic point $p \in \bar{S}$ and a sequence $\{p_n\} \subset S$ converging to p. Without loss of generality we can assume that $p_1 < p_2 < \cdots < p$ and take N such that $f^N(p) = p$ and $(f^N)'(p) > 0$. Then in a neighborhood of p, f^N is order preserving. Therefore we can assume that

$$p_0 \leq x \leq y \leq p \Rightarrow f^N(x) < f^N(y). \tag{13}$$

In particular $p_0 < p_n < p$ implies

$$f^N(p_n) < f^N(p) = p.$$

Then, if

$$p_n \leqq f^N(p_n),$$

we have $p_0 \leqq p_n \leqq f^N(p_n) < p$, and applying (13) we get $p_0 \leqq f^N(p_n) \leqq f^{2N}(p_n) < f^N(p) = p$. Applying again (13) we get $p_0 \leqq f^{2N}(p_n) \leqq f^{3n}(p_n) \subset f^N(p) = p$. Continuing with this method we obtain:

$$p_n \leqq f^N(p_n) \leqq f^{2N}(p_n) \leqq \cdots \leqq f^{m_n N}(p_n) = p_n$$

if $m_n N$ is a period of p_n. Hence $f^N(p_n) = p_n$. Then the integer N and the interval $[p_n, p]$ prove that p_n is homologous to p. Since there can be at most one point in S homologous to p, it follows that we can assume that no p_n is homologous to p and then

$$f^N(p_n) < p_n \qquad (14)$$

for all $n \geqq 0$. Using the fact that f^N is injective in a neighborhood of p, we can take $\delta > 0$ such that if $x \in (p_0, p)$ and $y \neq x$ satisfy $f^N(x) = f^N(y)$, then

$$d(x, y) > \delta. \qquad (15)$$

Now take $n \geqq 0$. If $f^{jm_n}(p_n) \in (p_{n-1}, p)$ for all $0 \leqq j \leqq N$, then p_n is a periodic point of the injective order preserving map $f^N/(p_{n-1}, p)$. Hence p_n is actually a fixed point, and arguing as before, p_n is homologous to p. But we have assumed that no p_n is homologous to p. Hence for all $n \geqq 0$ there exists $0 < j_n < m_n$ such that

$$f^{j_n N}(p_n) \notin (p_{n-1}, p), \qquad (16)$$

$$f^{(j_n + 1)N}(p_n) \in (p_{n-1}, p). \qquad (17)$$

Since by (14) $f^N((p_{n-1}, p)) \supset (p_{n-1}, p)$, there exists $q_n \in (p_{n-1}, p)$ such that

$$f^N(q_n) = f^{(j_n + 1)N}(p_n). \qquad (18)$$

By (16) $f^{j_n N}(p_n) \neq q_n$ and by (18) $f^N(f^{j_n N}(p_n)) = f^N(q_n)$. Hence (15) implies:

$$d(q_n, f^{j_n N}(p_n)) > \delta. \qquad (19)$$

Moreover

$$p - q_n < p - p_{n-1} \qquad (20)$$

because $q_n \in (p_{n-1}, p_n)$. Taking limit when $n \to +\infty$ (on a subsequence if necessary) we have that $q_n \to p$ and the sequence $f^{j_n N}(p_n)$ converges to a point $\bar{p}$ that is in $\bar{S}$ but that by (19) is different from p, and by (17) $f^N(\bar{p}) = p = f^N(p)$. Since p is periodic and $\bar{p} \neq p$, the relation $f^N(\bar{p}) = p$ shows that $\bar{p} \in \bar{S}$ is not periodic. Now that we know that $\bar{S}$ contains non-periodic points we can apply Lemma I.3 to $\Lambda = \bar{S}$. Observe that $f(\bar{S}) = \bar{S}$ and then $\bigcap_{n \geqq 0} f^n(\bar{S}) = \bar{S}$. Then by Lemma I.3 there exists an interval J adapted to Λ satisfying (2) for every coherent sequence $(J, \{\varphi_n\})$ associated to $\bar{S}$. Take $x \in J \cap S$ and choose the periodic $\theta \in \mathcal{S}(\Lambda, x)$ (that exists because

x is periodic). By the definition of adapted interval there exists a coherent sequence $(J, \{\varphi_n\})$ such that $\varphi_n(x) = \theta(n)$ for all $n \geq 1$. From (2) it follows that $|(f^n)'(\theta(n))|^{-1} = |\varphi'_n(x)| \to 0$ when $n \to +\infty$ and if N is the period of x we have $\theta(nN) = \theta(0)$ for all $n \geq 1$. Then:

$$|(f^{nN})'(x)| = |(f^{nN})'(\theta(0))| = |(f^{nN})'(\theta(nN))|.$$

Then $|(f^{nN})'(x)| \to \infty$ when $n \to +\infty$ thus implying that x is a source and contradicting the definition of S.

To prove Theorem D we can assume that f is not topologically equivalent to an irrational rotation because in that case the property follows from a stronger property proved by Herman [2]. Suppose also that Λ doesn't satisfy the second option of Theorem D. We have to show that $\lambda(\Lambda) = 0$, where $\lambda(\cdot)$ denotes the Lebesgue measure. Let $\mathcal{F}$ be the family of compact invariant sets $\Gamma \subset \Lambda$ such that

$$\lambda(\Lambda - \{x \in \Lambda \,|\, \omega(x) \subset \Gamma\}) = 0.$$

Order $\mathcal{F}$ by inclusion. Let $\mathcal{F}_0 \subset \mathcal{F}$ be a totally ordered subfamily. We claim that $\cap \{\Gamma \,|\, \Gamma \in \mathcal{F}_0\} \in \mathcal{F}_0$. Denote $\Lambda_0 = \cap \{\Gamma \,|\, \Gamma \in \mathcal{F}_0\}$. Take a neighborhood V of Λ_0. Then $\Gamma \subset V$ for some $\Gamma \in \mathcal{F}_0$ and

$$\lambda(\Lambda - \{x \in \Lambda \,|\, \omega(x) \subset V\}) \leq \lambda(\Lambda - \{x \in \Lambda \,|\, \omega(x) \subset \Gamma\}) = 0.$$

Since this is true for every neighborhood V of Λ_0 it follows that $\lambda(\Lambda - \{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\}) = 0$. Hence $\Lambda_0 \in \mathcal{F}_0$ proving the claim. We can now apply Zorn's Lemma to $\mathcal{F}$ and obtain a *minimal* $\Lambda_0 \in \mathcal{F}$. We claim that Λ_0 has the following property: for every open interval J such that $j \cap \Lambda_0 \neq \phi$, the set

$$\tilde{J} = \{x \in J \cap \Lambda \,|\, f^n(x) \in J \text{ for infinitely many } n\text{'s}\}$$

has positive measure. Suppose by contradiction that $\lambda(\tilde{J}) = 0$ and define

$$\Lambda_1 = \{x \in \Lambda \,|\, f^n(x) \notin J \quad \text{for all } n \geq 0\}.$$

Clearly Λ_1 is compact and invariant. Moreover:

$$\{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\} = \left(\bigcup_{n \geq 0} f^n(\tilde{J}) \right) \cup \{x \in \Lambda \,|\, \omega(x) \subset \Lambda_1\}.$$

Since $\lambda\left(\bigcup_{n \geq 0} f^{-n}(\tilde{J}) \right) = 0$ (because $\lambda(\tilde{J}) = 0$) it follows that:

$$\lambda(\Lambda - \{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\}) = \lambda(\Lambda - \{x \in \Lambda \,|\, \omega(x) \subset \Lambda_1\}).$$

Hence $\Lambda_1 \in \mathcal{F}$. But then the minimality of Λ_0 implies $\Lambda_1 = \Lambda_0$. This means $J \cap \Lambda_0 = \phi$ contradicting our hypothesis on J. Now we claim that if $\lambda(\{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\}) \neq 0$ there exists an open interval J with $J \cap \Lambda_0 \neq \phi$ and $\lambda(\tilde{J}) = 0$. Since such interval cannot exist, it will follow that $\lambda(\{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\}) = 0$ and then:

$$\lambda(\Lambda) = \lambda(\Lambda - \{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\}).$$

But the measure at right is zero because $\Lambda_0 \in \mathcal{F}$, and then $\lambda(\Lambda) = 0$ as we wished to prove.

To prove the claim observe that $f(\Lambda) \in \mathcal{F}$, and then the minimality of Λ_0 implies

$\Lambda_0 = f(\Lambda_0)$. Hence $\Lambda_0 = \bigcap\limits_{n \geq 0} f^n(\Lambda_0)$. If Λ_0 is finite, every point in Λ_0 is periodic. Then $\lambda(\{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\}) = 0$ because if it is $\neq 0$ there exists a periodic orbit $\gamma \subset \Lambda_0$ such that $\lambda(x \in \Lambda \,|\, \omega(x) = \gamma\}) \neq 0$. This inequality easily implies the existence of an interval J satisfying the second option in Theorem D. Since we are assuming that this option doesn't hold it follows that $\lambda(\{x \in \Lambda \,|\, \omega(x) \subset \Lambda_0\}) = 0$. If Λ_0 is an infinite union of periodic orbits, it is easy to prove, using Theorem C, that there exists an open interval J such that f^N/J is a diffeomorphism of J for some $N \geq 1$ and containing infinitely many periodic orbit of Λ_0. Since we are assuming that the second option of Theorem D doesn't hold, it follows that $\lambda(J \cap \Lambda_0) = 0$. But on the other hand $J \cap \Lambda_0 = \tilde{J}$ because every point in Λ_0 is periodic. It remains the case when $\Lambda_0 = \bigcap\limits_{n \geq 0} f^n(\Lambda_0)$ contains non-periodic points. We can apply Lemma I.3 and obtain an interval J adapted to Λ_0 and $K_1 > 0$ satisfying (4). We shall prove that $\lambda(\tilde{J}) = 0$. Denote $F_n(J, \Lambda)$ the set of maps $\psi: \supset$ that can be written as a composition of n elements of $F(J, \Lambda)$. By Lemma I.3 there exists $\sigma < \lambda < 1$ such that:

$$|\psi'(x)| \leq \lambda^n \tag{21}$$

for all $x \in F_n(J, \Lambda)$. In particular, (21) implies:

$$\operatorname{diam} \psi(J) \leq \lambda^n \operatorname{diam}(J) \tag{22}$$

for all $\psi \in F_n(J, \Lambda)$. Denote $\tilde{F}_n(J, \Lambda)$ the set of maps ψ in $F_n(J, \Lambda)$ such that there exists a coherent sequence $(J, \{\varphi_n\})$ satisfying $\psi = \varphi_n$. By I.3 there exists $K_1 > 0$ such that

$$|\psi'(x)| \leq K_1 |\psi'(y)|$$

for all $x \in J$, $y \in J$ and $\psi \in \bigcup\limits_{n \geq 0} \tilde{F}_n(J, \Lambda)$. It follows that

$$\frac{\lambda(\psi(A))}{\lambda(\psi(J))} \leq K_2 \frac{\lambda(S)}{\lambda(J)}$$

for all $\psi \in \bigcup\limits_{n \geq 0} \tilde{F}_n(J, \Lambda)$ and every Borel set $A \subset J$. Then:

$$\frac{\lambda(\tilde{J})}{\lambda(J)} = 1 - \frac{\lambda(J - \tilde{J})}{\lambda(J)} \geq 1 - K_2 \frac{\lambda(\psi(J - \tilde{J}))}{\lambda(\psi(J))}$$

$$= 1 - K_2 \left(1 - \frac{\lambda(\psi(\tilde{J}))}{\lambda(\psi(J))}\right),$$

and since $\psi(\tilde{J}) \supset \psi(J) \cap \tilde{J}$,

$$\frac{\lambda(\tilde{J})}{\lambda(J)} \geq 1 - K_2 \left(1 - \frac{\lambda(\psi(J) \cap \tilde{J})}{\lambda(\psi(J))}\right). \tag{23}$$

If $\lambda(\tilde{J}) \neq 0$ we can take a density point $x \in \tilde{J}$. There exists a sequence $\psi_{n_j} \in \tilde{F}_{n_j}(\Lambda, J)$, with $n_j \to +\infty$ such that $x \in \psi_{n_j}(J)$ for all n_j. By (22), $\operatorname{diam} \psi_{n_j}(J) \to 0$ when $j \to +\infty$. The definition of density point grants:

$$\lim_{j \to +\infty} \frac{\lambda(\psi_{n_j}(J) \cap \tilde{J})}{\lambda(\psi_{n_j}(J))} = 1.$$

Applying (23) to $\psi = \psi_{n_j}$ and taking limits we obtain $\lambda(\tilde{J}) \geq \lambda(J)$. Hence $\lambda(\tilde{J}) \neq 0$ implies that $\tilde{J}$ is dense in J. In particular Λ contains J. But Λ doesn't contain critical points. Therefore f^n/J has no critical points for all $n \geq 0$. Moreover $f^n(J) \subset \Lambda \neq N$ for all $n \geq 1$. This excludes the possibility $f^n(J) = N = S^1$ and shows that every power of f is injective in J and has no critical points. In other words J is a d-interval. But by Lemma I.4 this means that there exists an open interval J_1 such that $f^n(J_1) \subset J_1$, f^n has no critical points in J_1 and $f^N(J) \subset J_1$ for some $N \geq 1$. Therefore the interval $f^n(J) \subset \Lambda \cap J_1$ proves that f satisfies the second option of Theorem D. Since we are assuming that this option doesn't hold, it follows that $\tilde{J}$ is not dense in J and $\lambda(\tilde{J}) = 0$.

To prove Theorem E we take the interval J obtained applying Lemma I.3 to $\Lambda = S^1$. We shall first prove that $\lambda(\Gamma \cap J) \neq 0$ implies $\lambda(\Gamma \cap J) = \lambda(J)$. Denote Γ_0 the set of points $x \in S^1$ whose forward orbit doesn't intersect J. Then Γ_0 is compact and invariant. By Theorem D, $\lambda(\Gamma_0) = 0$. Denote J_0 the set of points $x \in J$ such that there exists $N > 0$ satisfying $f^n(x) \notin J$ for all $n \geq N$. Then $J_0 \subset \bigcup_{n \geq 0} f^{-n}(\Gamma_0)$ and $\lambda(J_0)$

Define $J_1 = J - J_0$. Since $\lambda(J_0) = 0$, to prove that $\lambda(\Gamma \cap J) = J$ it suffices to show that $\lambda(\Gamma \cap J_1) = 0$. Denote $F_n(J, S^1)$ the set of maps $\psi: J \supset$ that can be written as a composition of n elements of $F(J, S^1)$. Then for every $x \in J_1$ there exists a sequence $\psi_{n_i} \in F_n(J, S^1)$ such that

$$x \in \psi_{n_i}(J) \tag{24}$$

for all $i \geq 1$. Moreover, I.3 implies that:

$$|\psi'(x)| \leq \lambda^n$$

for all $\psi \in F_n(J, S^1)$ and $n \geq 1$. In particular:

$$\operatorname{diam} \psi(J) \leq \lambda^n \operatorname{diam} J \tag{25}$$

for all $n \geq 1$ and $\psi \in F_n(J, S^1)$. Moreover, by Lemma I.3

$$|\psi'(x)| \leq K_2 |\psi'(y)|$$

for all $n \geq 1$, $\psi \in F_n(J, S^1)$, $x \in J$, $y \in J$. Then

$$\frac{\lambda(\psi(A))}{\lambda(\psi(J))} \geq K_1^{-1} \frac{\lambda(A)}{\lambda(J)}$$

for all $\psi \in F_n(J, S^1)$, $n \geq 1$ and every Borel set $A \subset J$. Therefore, if $x \in \Gamma \cap J$ and the sequence $\{\psi_{n_i}\}$ is chosen satisfying (24) for all n_i, we obtain:

$$\frac{\lambda(\psi_{n_i}(\Gamma^c \cap J_1))}{\lambda(\psi_{n_i}(J))} \geq K^{-1} \frac{\lambda(\Gamma^c \cap J_1)}{\lambda(J)}.$$

But $\psi_{n_i}(\Gamma^c) \subset f^{-n_i}(\Gamma^c) \subset \Gamma^c$ and $\psi_{n_i}(J_1) \subset J_1$. Then

$$\Gamma^c \cap J_1 \cap \psi_{n_i}(J) \supset \psi_{n_i}(\Gamma^c \cap J_1).$$

Hence:

$$\frac{\lambda(\Gamma^c \cap J_1 \cap \psi_{n_i}(J))}{\lambda(\psi_{n_i}(J))} \geqq K_1^{-1} \frac{\lambda(\Gamma^c \cap J_1)}{\lambda(J)}. \tag{26}$$

But if x is a density point of $\Gamma \cap J_1$, it follows from (25) and (24) that:

$$\lim_{i \to +\infty} \frac{\lambda(\Gamma^c \cap J_1 \cap t_{n_i}(J))}{\lambda(\psi_{n_i}(J))} = 0.$$

Then, by (26), $\lambda(\Gamma^c \cap J_1) = 0$ and $\lambda(\Gamma \cap J_1) = \lambda(J_1)$. Now we know that $\lambda(\Gamma \cap J) \neq 0$ implies $\lambda(\Gamma \cap J) = \lambda(J)$. Suppose that $\lambda(\Gamma) = 0$. Observe that $\bigcup_{n \geqq 0} f^{-n}(J)$ has full measure because its complement is a proper compact invariant set. Then:

$$0 < \lambda(\Gamma) = \lambda\left(\Gamma \cap \left(\bigcup_{n \geqq 0} f^{-n}(J)\right)\right) \leqq \lambda\left(\bigcup_{n \geqq 0} f^{-n}(\Gamma \cap J)\right).$$

This inequality shows that $\lambda(\Gamma \cap J) > 0$. But we proved that $\lambda(\Gamma \cap J) > 0$ implies $\lambda(\Gamma \cap J) = \lambda(J)$. Hence $\lambda(\Gamma^c \cap J) = 0$ and then $\lambda(f^n(\Gamma^c \cap J)) = 0$ for all $n \geqq 0$. Taking n such that $f^n(J) = S^1$ we obtain:

$$\lambda(\Gamma^c) = \lambda(\Gamma^c \cap f^n(J)) \leqq \lambda(f^n(\Gamma^c \cap J)) = 0.$$

II. Proof of the Lemmas

We shall begin by proving Lemma I.4 because it is in fact previous to the other Lemmas. It is essentially an easy reformulation of Denjoy's theorem and therefore we shall only outline those parts of the proof that are only straightforward modifications of the proof of Denjoy's theorem.

We shall use the concept of maximal interval introduced in Sect. I to prove Theorem B. Given a d-interval J of $f \in \mathrm{End}^{1+\varepsilon}(N)$ we can take (because f is not a differmorphism of the circle), unique maximal d-intervals $J_i \supset f^i(J)$, $i = 0, 1, \ldots$. Suppose that there exist $m > n \geqq 0$ such that $J_m \cap J_n \neq \phi$. Then $J_m = J_n$ and

$$f^{m-n}(J_m) \cap J_m \supset f^{m-n}(f^n(J_0)) \cap J_m = f^m(J_0) \cap J_m = f^m(J_0) \neq \phi.$$

But $f^{m-n}(J_m)$ is a d-interval. Hence

$$f^{m-n}(J_m) \subset J_m,$$

and the theorem is proved. Now suppose that

$$J_m \cap J_n = \phi \tag{1}$$

for all $m > n \geqq 0$. Then

$$\sum_{n=0}^{\infty} \operatorname{diam} J_n < +\infty. \tag{2}$$

Suppose that

$$\liminf_{n \to +\infty} d(J_n, C(f)) = 0. \tag{3}$$

We also have:

$$d(f^n(J_0), C(f)) \leqq d(J_n, C(f)) + \text{diam } J_n. \tag{4}$$

But (2) implies diam $J_n \to 0$. This together with (3) and (4) implies

$$\liminf_{n \to +\infty} d(f^n(J_0), C(f)) = 0,$$

and the Lemma is proved. It remains to consider the case

$$\liminf_{n \to +\infty} d(J_n, C(f)) > 0.$$

This means that there exists $n_0 \geqq 0$ such that

$$\inf_{n \geqq n_0} d(J_n, C(f)) > 0.$$

Since $f^n(J_{n_0}) \subset J_{n+n_0}$, we obtain:

$$\inf_{n \geqq n_0} d(f^n(J_{n_0}), C(f)) > 0, \tag{5}$$

$$\sum_{n=0}^{\infty} \text{diam } f^n(J_{n_0}) < +\infty. \tag{6}$$

We shall prove that (5) and (6) together contradict the maximality of J_{n_0}. From (5) and (6), as in the proof of Denjoy's theorem, we get

$$\sum_{n=0}^{\infty} |(f^n)'(x)| < +\infty \tag{7}$$

for all $x \in \bar{J}_{n_0}$.

As in the proof of Denjoy's theorem, we can find, using (5) and (7), an open interval $A \supset \{a\}$, where a is an endpoint of $\bar{J}_{n_0}$, such that

$$\lim_{n \to +\infty} \text{diam } f^n(A) = 0. \tag{8}$$

Observe that the case $\bar{J}_{n_0} = N = [0,1]$ doesn't arise because of (1). The case $\bar{J}_{n_0} = N = S^1$ is also impossible for the same reason. Then we can take an open interval $A \supset A' \supset \{a\}$ so small that

$$A' \cup J_{n_0} \neq N. \tag{9}$$

Moreover, since we cannot have $J_{n_0} = N = [0,1]$, we can take the endpoint a different from 0 and 1. Hence:

$$A' \cup J_{n_0} \neq J_{n_0}. \tag{10}$$

Finally (8) implies that we can take A' so small that

$$\inf_{n \geqq 0} \text{diam } f^n(A') < \tfrac{1}{2} \inf_{n \geqq 0} d(f^n(J_{n_0}), C(f)),$$

and then

$$\inf_{n \geqq 0} d(f^n(A' \cup J_{n_0}), C(f)) > 0. \tag{11}$$

By (9), $A' \cup J_{n_0}$ is an interval. By (11) it is a d-interval and by (16) it properly contains J_{n_0}. Hence J_{n_0} is not a maximal d-interval and this contradiction completes the proof of I.4.

The proof of Lemma I.2 requires a preliminary result:

Lemma II.1. *If* $f \in \mathrm{End}^2(N)$ *and* $\Lambda \subset N - \partial N$ *is a compact invariant set not containing critical points, then, for every non-periodic point* $a \in \bigcap_{n \geq 0} f^n(\Lambda)$ *and* $\delta > 0$, *there exists* $\varepsilon > 0$ *such that if* $J \supset \{a\}$ *is an open interval with* $\mathrm{diam}\,(J) \leq \varepsilon$ *then, for all* $x \in J \cap \Lambda$ *and* $\theta \in \mathscr{S}(\Lambda, x)$ *there exists a coherent sequence* $(J, \{\varphi_n\})$ *satisfying*

$$\varphi_n(x) = \theta(n), \tag{12}$$

$$\mathrm{diam}\,\varphi_n(J) \leq \delta \tag{13}$$

for all $n \geq 0$.

Proof. Take a non-periodic point $a \in \bigcap_{n \geq 0} f^n(\Lambda)$ and $\delta > 0$. Let $\varepsilon_n > 0$ be the maximum positive number such that if $J_n = (a - \varepsilon_n, a + \varepsilon_n)$ then, for every $x \in \Lambda \cap J_n$ and every $\theta \in \mathscr{S}(\Lambda, x)$ there exists branches $\varphi_j : J_n \to N$ of $f^{-j}/J, j = 1, \ldots, n$ such that

$$\varphi_j(x) = \theta(j),$$
$$\mathrm{diam}\,\varphi_j(J_n) \leq \delta$$

for all $1 \leq j \leq n$ and

$$f\,\varphi_{j+1} = \varphi_j$$

for all $1 \leq j \leq n$. Without loss of generality we can suppose

$$0 < \delta < \tfrac{1}{2} d(\Lambda, C(f) \cup f(N)).$$

This implies that there exists $0 \leq j_n \leq n$ such that

$$\mathrm{diam}\,\varphi_{j_n}(J_n) = \delta.$$

If we prove that $\liminf_{n \to +\infty} \varepsilon_n = \varepsilon > 0$, the Lemma is proved just taking $J = (a - \varepsilon, a + \varepsilon)$. Suppose that $\liminf_{n \to +\infty} \varepsilon_n = 0$. Then $j_n \to +\infty$. Now define $U_n = \varphi_{j_n}(J_n)$. A subsequence of the sequence of intervals $\{U_n\}$ converges to an open interval U that satisfies diam $U = \delta$. We shall assume to simplify the notation that $U_n \to U$. Then U is a d-interval. By I.4 there exists a d-interval V and integers $n \geq 1$, $N \geq 1$ such that

$$f^m(U) \subset V, \tag{14}$$
$$f^N(V) \subset V, \tag{15}$$

$$f^N/V \text{ has no critical points.} \tag{16}$$

Take a point $q \in U$ such that, $q \in U_n$ for every large value of n, say for all $n \geq n_0$. By (14), (15), (16) the ω-limit set of q is a periodic orbit γ. Then

$$d(a, \gamma) \leq \varepsilon_n + d((a - \varepsilon_n, a + \varepsilon_n), \gamma) = \varepsilon_n + d(f^{j_n}(U_n), \gamma) \leq \varepsilon_n + d(f^{j_n}(q), \gamma). \tag{17}$$

But $j_n \to + \infty$ when $n \to + \infty$. Then

$$\lim_{n \to + \infty} d(f^{j_n}(q), \gamma) = 0.$$

Moreover $\liminf_{n \to + \infty} \varepsilon_n = 0$. Hence (17) proves that $a \in \gamma$, contradicting that a is non-periodic.

To prove I.2 we shall consider first the case when f is an immersion of S^1. If its degree is 1 or -1, it is a diffeormorphism. Moreover it has periodic points because it is not topologically equivalent to an irrational rotation. Then, using that x is not a periodic point it is trivial to construct the adapted interval J (in fact any open interval $J \supset \{x\}$ such that $f^{-n}(J) \cap J = \phi$ for all $n > 0$ works). If the degree d of f is not 1 or -1 we define $g: S^1 \supseteq$ by $g(z) = z^d$ and, as we explained in the Introduction, there exists a monotone map $h: S^1 \supseteq$ satisfying $gh = hf$. Now observe that if J is an interval whose endpoints are fixed points of some power g^n of g (i.e. roots of the equation $z^{d^n-1} - 1 = 0$) and doesn't contain fixed points of g^n, then J satisfies (c) (with respect to g). Condition (b) is obvious since $C(g) = \phi$ and the existence condition (condition (a)) is easy to check (even for $A = S^1$). Therefore J is an interval adapted to S^1, in particular adapted to Λ. Now, given $x \in \Lambda$, we take $h(x)$ and an interval $J \supset \{h(x)\}$ as described above. Using h it is easy to show that $h^{-1}(J)$ is an interval adapted to S^1 (in particular to Λ).

Now suppose that f is not an immersion of S^1. Given the non-periodic point $x \in \bigcap_{n \geq 0} f^n(\Lambda)$, let $\varepsilon > 0$ be given by Lemma II.1, taking as $\delta > 0$ a number satisfying

$$0 < \delta < \tfrac{1}{2} d(\Lambda, C(f)). \tag{18}$$

If $J \supset \{x\}$ is an open interval with diam $J < \varepsilon$ then, by Lemma II.1 the first condition of the definition of interval adapted to Λ is satisfied. Condition (b) is also satisfied because if $(J, \{\varphi_n\})$ is a coherent sequence associated to Λ, every $\varphi_n(J)$ contains a point in Λ and, by Lemma II.1, diam $\varphi_n(J) < \delta$. Then

$$d(\varphi_n(J), C(f)) \leq d(\Lambda, C(f)) - \text{diam } \varphi_n(J) \leq d(\Lambda, C(f)) - \delta \leq \delta.$$

Therefore conditions (a) and (b) are granted just by taking J having diameter $\leq \varepsilon$. The problem is condition (c). If J is an open interval denote $\mathscr{C}(J)$ the set of all the coherent sequences $(J, \{\varphi_n\})$ associated to J and denote $\hat{J}$ the connected component containing x of the open set:

$$J \cup (U(\{\varphi_n(J) | n \geq 1, \quad (J, \{\varphi_n\}) \in \mathscr{C}(J)\}).$$

Take a sequence $J_1 \supset J_2 \supset \cdots \supset \{x\}$ of open intervals with $\text{diam}(J_n) \to 0$. Then $\hat{J}_1 \supset \hat{J}_2 \supset \cdots \{x\}$. If diam $(\hat{J}_m) \to 0$ we are done because it is clear from the definition of $\hat{J}$ that if $(\hat{J}, \{\varphi_n\}) \in \mathscr{C}(\hat{J})$, then $\varphi_n(\hat{J}) \cap \hat{J} \neq \phi$ implies $\varphi_n(\hat{J}) \subset \hat{J}$. Hence any $\hat{J}_m$ with diameter less than ε will satisfy condition (c) (and, as we observed before (a) and (b) are implied by $\text{diam}(\hat{J}_m) < \varepsilon$). Then suppose that $\bigcap_{m \geq 0} \hat{J}_m$ is not $\{x\}$. Then its interior is an open interval U. We claim that $U \subset \Lambda$. Given $y \in U$ and $\varepsilon_0 > 0$, take m so large that $\text{diam}(J_m) < \varepsilon_0$ and

$$\sup \{\text{diam } \varphi_n(J_m) | n \geq 1, (J_m \{\varphi_n\}) \in \mathscr{C}(J_m)\} < \varepsilon_0.$$

We can take such m by Lemma II.1 By the definition of $\hat{J}_m$ and since $y \in \hat{J}_m$, either $y \in J_m$ or $y \in \varphi_n(J_m)$ for some $n \geq 1$ and $(J_m, \{\varphi_n\}) \in \mathscr{C}(J_m)$. In the first case $d(y, x) \leq \operatorname{diam} J_m \leq \varepsilon_0$. In the second case, observing that $\varphi_n(J_m)$ contains points of Λ because $(J_m\{\varphi_n\})$ is associated to Λ, we obtain again

$$d(y, \Lambda) \leq d(y, \varphi_n(J_m)) \leq \operatorname{diam} \varphi_n(J_m) < \varepsilon_0.$$

Hence $d(y, \Lambda) \leq \varepsilon_0$ for all ε_0 and then $y \in \Lambda$. This completes the proof of the claim. But $U \subset \Lambda$ implies $f^n(U) \subset \Lambda$ for all $n > 0$ and then f^n/U has no critical points for all $n > 0$. Since f is not an immersion of S^1 it follows that U is a d-interval. Let V be the maximal d-interval containing U. By Lemma I.4 it is eventually periodic. Moreover the construction of U shows that $x \in \bar{U}$. We have now two cases to consider

I) $x \in V$. From the fact that V is eventually periodic follows the existence of an open interval $\{x\} \subset J \subset V$ satisfying

$$f^n(J) \cap J = \phi$$

for all $n \geq 1$. This implies that

$$\varphi_n(J) \cap J = \phi$$

for all $n \geq 1$, $(J, \{\varphi_n\}) \in \mathscr{C}(J)$. Hence condition (c) is satisfied and we are done.

II) $x \in \partial V$. An endpoint of an eventually periodic interval is either eventually periodic or its forward orbit contains a critical point (maybe both). The second possibility cannot hold for x because x is contained in Λ that doesn't contain critical points. Then x is eventually periodic. But it cannot be periodic. Therefore V itself is eventually periodic but not periodic and:

$$\liminf_{n \to +\infty} d(f^n(V), V) > 0. \tag{19}$$

On the other hand for every m there exists $n_m \geq 1$ and $(J_m, \{\varphi_n\}) \in \mathscr{C}(J_m)$ such that

$$U \cap \varphi_{n_m}(J_m) \cap J_m \neq \phi. \tag{20}$$

In particular

$$d(x, \varphi_{n_m}(x)) \leq \operatorname{diam} J_m + \operatorname{diam} \varphi_{n_m}(J_n).$$

But $\operatorname{diam} J_m \to 0$ when $m \to +\infty$ and, by Lemma II.1.

$$\operatorname{Sup}\{\operatorname{diam} \varphi_n(J_m) | n \geq 1, (J_m, \{\varphi_n\}) \in \mathscr{C}(J_m)\}$$

converges to zero when $m \to +\infty$. Hence $d(x, \varphi_{n_m}(x)) \to 0$ when $m \to +\infty$. If the sequence $\{n_m\}$ is bounded, this implies that x is periodic, contradicting our hypothesis. If it is unbounded we use (20) to obtain

$$\phi \neq f^{n_m}(U \cap \varphi_{n_m}(J_m) \cap J_n) \subset f^{n_m}(U) \cap J_m \cap f^{n_m}(J_m),$$

which implies:

$$f^{n_m}(V) \cap J_m \neq \phi.$$

Then,

$$d(f^{n_m}(V), V) \leqq d(f^{n_m}(V), x) \leqq \operatorname{diam} J_m,$$

contradicting (19).

The proof of Lemma I.3 will require different methods according to whether $\Lambda = N, \Lambda \neq N$. To prove it in the case when $\Lambda \neq N$ we need the three following lemmas. In its statements Λ and f will be as in the statement of I.3 and the notation $\mathscr{C}(J_0, \Lambda)$ will have the same meaning as that of the proof of II.1.

Lemma II.2. *If $x \in \bigcap\limits_{n \geqq 0} f^n(\Lambda)$ is non-periodic there exists an interval $J_0 \supset \{x\}$ adapted to Λ such that for all $r \geqq 0$ there exists an arbitrarily small interval $\{x\} \subset J \subset J_0$ adapted to Λ satisfying*

$$\operatorname{diam} \varphi_n(J_0) \leqq r$$

for all $n \geqq N$ and $(J_0, \{\varphi_n\}) \in \mathscr{C}(J_0, \Lambda)$ such that

$$\varphi_n(J) \cap J \neq \phi.$$

Proof. By Lemma I.2 there exist arbitrarily small intervals $J_0 \supset \{x\}$ adapted to Λ. If the ω-limit set of x is a periodic orbit, γ, take J_0 satisfying

$$\bar{J}_0 \cap \gamma = \phi. \tag{21}$$

This can be done because since x is not periodic $x \notin \gamma$. If the ω-limit set of x is not a periodic orbit, choose any $J_0 \supset \{x\}$ adapted to Λ. Suppose that J_0 doesn't satisfy the required property. Then there exists $r > 0$, a sequence of coherent sequences $(J_0, \{\varphi_n^{(i)}\}) \in \mathscr{C}(J_0, \Lambda)$ $i = 1, 2, 3, \ldots$, a sequence of integers $n_1 < n_2 < \ldots$ and a sequence of intervals $J_1 \supset J_2 \supset \cdots \supset \{x\}$, with $\bigcap\limits_{n \geqq 1} J_n = \{x\}$, such that

$$\varphi_{n_i}^{(i)}(J_i) \cap J_i \neq \phi \tag{22}$$

$$\operatorname{diam} \varphi_{n_i}^{(i)}(J_0) > r \tag{23}$$

for all $i \geqq 1$. Also from the fact that J_0 is adapted to Λ it follows that for any pair of coherent sequences $(J_0, \{\varphi_n\}) \in \mathscr{C}(J_0, \Lambda)$ and $(J_0, \{\psi_n\}) \in \mathscr{C}(J_0, \Lambda)$, and every $n \geqq l \geqq 1$, one of the following relations holds:

$$\varphi_n(J_0) \cap \psi_l(J_0) = \phi,$$

$$\varphi_n(J_0) \subset \psi_l(J_0) \quad \text{or} \quad \psi_l(J_0) \subset \varphi_n(J_0).$$

Using this property it is easy to see that the sequence above can be chosen satisfying

$$\varphi_{n_i}^{(i)}(J_0) \subset \varphi_{n_j}^{(j)}(J_0) \subset J_0 \tag{24}$$

for all $i \geqq j \geqq 1$. Therefore (23) and (24) imply that the set

$$U = \operatorname{Int} \bigcap\limits_{i \geqq 1} \varphi_{n_i}^{(i)}(J_0) \tag{24}$$

is a non-empty open interval contained in J_0. It is easy to check that it is a d-interval. Since f is not topologically equivalent to an irrational rotation, U is eventually

periodic by Lemma I.4. On the other hand

$$\lim_{i \to +\infty} \operatorname{diam} \varphi_{n_i}^{(i)}(J_i) = 0 \qquad (25)$$

by Lemma II.1. Moreover, by (22) and (24):

$$d(x, U) = \lim_i d(x, \varphi_{n_i}^{(i)}(J_0)) \leqq \lim_i d(x, \varphi_{n_i}^{(i)}(J_i))$$

$$\leqq \lim_i (\operatorname{diam} J_i + \operatorname{diam} \varphi_{n_i}^{(i)}(J_i)).$$

Since $\lim_i \operatorname{diam} J_i = 0$, this inequality together with (25) implies that $x \in \bar{U}$. Moreover from the definition of U it follows that $f^{n_i}(U) \subset J_0$. Hence $\omega(x) \cap \bar{J}_0 \neq \phi$. But since $x \in \bar{U}$ and U is eventually periodic it follows that $\omega(x)$ is a periodic orbit. Then the relation $\omega(x) \cap \bar{J}_0 \neq \phi$ contradicts the way we choose J_0.

Lemma II.3. *If* $f \in \operatorname{End}^2(N)$ *for all* $\delta > 0$ *there exists* $K_0 = K_0(\delta, f) > 0$ *such that* $(J, \{\varphi_n\})$ *is a coherent sequence satisfying:*

$$\inf_{n \geqq 0} d(\varphi_n(J), C(f)) > 0,$$

then

$$\frac{|\varphi_n'(x)|}{|\varphi_n'(y)|} \leqq \exp K_0 \sum_{j=0}^{n-1} \operatorname{diam} \varphi_j(J),$$

$$|\varphi_n'(x)| \leqq \frac{\operatorname{diam} \varphi_n(J)}{\operatorname{diam} J} \exp K_0 \sum_{j=0}^{n-1} \operatorname{diam} \varphi_j(J)$$

for all $n \geqq 0$, $x \in J$.

This Lemma is proved using a trivial adaptation of the method of Schwarz, proof of Denjoy's theorem. For a proof see Jacobson [1] Lemma Ia, Ib.

Lemma II.4. *If* J *is an interval adapted to* Λ *such that either* $F(J, \Lambda) = \phi$ *or there exists* $0 < \lambda < 1$ *satisfying:*

$$|\psi'(x)| < \lambda$$

for all $x \in J$ *and* $\psi \in F(J, \Lambda)$, *then there exists* $K_1 > 0$ *such that*

$$|\varphi_n'(x)| \leqq K_1 |\varphi_n'(y)|,$$

$$\sum_{n=1}^{\infty} |\varphi_n'(x)| \leqq K_1$$

for all x, y *in* J *and* $(J, \{\varphi_n\}) \in \mathscr{C}(J, \Lambda)$.

Proof. We shall first prove that there exists $K > 0$ such that:

$$\sum_{n=0}^{\infty} \operatorname{diam} \varphi_n(J) \leqq K \qquad (26)$$

for all $(J, \{\varphi_n\}) \in \mathscr{C}(J, \Lambda)$. We note that here φ_0 stands for the identity and the same

notation will be used in all this proof. Take $(J, \{\varphi_n\}) \in \mathscr{C}(J, \Lambda)$. Suppose that

$$\varphi_n(J) \cap J = \phi$$

for all $n \geq 1$ (this is the only possibility if $F(J, \Lambda) = \phi$). Then:

$$\varphi_n(J) \cap \varphi_m(J) = \phi$$

for all $1 \leq n \leq m$, and then

$$\sum_{n=0}^{\infty} \text{diam } \varphi_n(J) \leq 2 \text{ diam } N. \tag{27}$$

Now suppose that $\varphi_n(J) \cap J \neq \phi$ for infinitely many values of n, and let $n_1 < n_2 < \cdots$ be those integers for which $\varphi_{n_i}(J) \cap J \neq \phi$. Since J is adapted to Λ,

$$\varphi_{n_i}(J) \subset J$$

for all $i \geq 1$. Clearly $\varphi_n \in F(\Lambda, J)$. Moreover it is easy to see (using again that J is adapted to Λ) that for all $i > 1$ there exists $\psi_i \in F(J, \Lambda)$ such that

$$\psi_i / \varphi_{n_{i-1}}(J) = \varphi_{n_i} f^{n_i - 1}$$

Set $\psi_1 = \varphi_{n_1}$. Define $m_1 = n_1$ and $m_i = n_i - n_{i-1}$ if $i > 1$. The maps ψ_i can be described as follows. There exist branches $\varphi_n^{(i)}$ of f^{-n}/J, where $1 \leq n \leq m_i$ satisfying the following properties:

a) $\varphi_n^{(i)} = f \varphi_{n+1}^{(i)}$ for all $i \geq 1$, $1 \leq n < m_i$,
b) $\varphi_{n_i}^{(i)} = \psi_i$ for all $i \geq 1$.

Now observe that the definition of the sequence $\{n_i\}$ plus the fact of J being adapted to Λ imply:

$$\varphi_n^{(i)}(J) \cap J = \phi$$

for all $1 \leq n < m_i$ and $i \geq 1$. Then

$$\varphi_n^{(i)}(J) \cap \varphi_m^{(i)}(J) = \phi$$

for all $1 \leq n < m_i$. Hence

$$\sum_{n=0}^{m_1 - 1} \varphi_n^{(i)}(J) \leq 2 \text{ diam } N$$

for all $i \geq 1$. By II.3 there exists a constant K_0 depending only on J such that

$$\frac{|(\varphi_n^{(i)})'(x)|}{|(\varphi_n^{(i)})'(y)|} \leq \exp K_0 \sum_{j=0}^{n-1} \text{diam } \varphi_j^{(i)}(J) \leq \exp 2K_0 \text{ diam } N. \tag{28}$$

Set $K_1 = \exp 2K_0 \text{ diam } N$. From (28) it follows easily that if $A \subset J$ is any subset:

$$\frac{\text{diam } \varphi_n^{(i)}(A)}{\text{diam } A} \leq K_1 \frac{\text{diam } \varphi_n^{(i)}(J)}{\text{diam } J},$$

and then

$$\sum_{n=0}^{m_1 - 1} \text{diam } \varphi_n^{(i)}(A) \leq \text{diam } A \frac{K_1}{\text{diam } J} \sum_{n=0}^{m_1 - 1} \text{diam } \varphi^{(i)}(J)$$

$$\leq \operatorname{diam} A \frac{K_1}{\operatorname{diam} J} 2 \operatorname{diam} N. \tag{29}$$

Set

$$K_2 = 2K_1 \operatorname{diam} N.$$

If $i > 1$, applying (29) to $A = \varphi_{n_{i-1}}(J)$, we obtain:

$$\sum_{n=0}^{m_i-1} \operatorname{diam} \varphi_n^{(i)}(\varphi_{n_{i-1}}(J)) < K_2 \operatorname{diam} \varphi_{n_{i-1}}(J) \operatorname{diam}(J).$$

But $\varphi_{n_{i-1}}$ is a composition of $i - 1$ maps in $F(J, \Lambda)$, namely the maps $\psi_1, \ldots, \psi_{i-1}$. Then:

$$|\varphi_{n_{i-1}}(x)| \leq \lambda^{i-1}$$

for all $x \in J$. Hence

$$\operatorname{diam} \varphi_{n_{i-1}}(J) \leq \operatorname{diam}(J)\lambda^{i-1},$$

and then

$$\sum_{n=0}^{m_i-1} \operatorname{diam} \varphi_n^{(i)}(\varphi_{n_{i-1}}(J)) \leq K_2 \lambda^{i-1}.$$

In a similar way we obtain:

$$\sum_{n=0}^{m_1-1} \operatorname{diam} \varphi^{(1)}(J) \leq K_2.$$

Then

$$\sum_{n=0}^{\infty} \operatorname{diam} \varphi_n(J) \leq \sum_{i=1}^{\infty} \sum_{n=0}^{m_i-1} \operatorname{diam} \varphi_n^{(1)}(\varphi_{n_i-1}(J))$$

$$\leq \sum_{i=1}^{\infty} K_2 \lambda^i. \tag{30}$$

With minor modifications the same methods can be applied to the case when $\varphi_n(J) \cap J \neq \phi$ holds for a finite non-empty set of values of n, and the result is

$$\sum_{n=0}^{\infty} \varphi_n(J) \leq 2 \operatorname{diam} N + \sum_{i=0}^{\infty} K_2 \lambda^i. \tag{31}$$

Then (27), (30) and (31) prove (26). The proof of II.4 is now an easy corollary of II.3. In fact, applying II.3 to $(J, \{\varphi_n\}) \in \mathscr{C}(J, \Lambda)$, we obtain

$$\frac{|\varphi_n'(x)|}{|\varphi_n'(y)|} \leq \exp K_0 \sum_{j=0}^{n-1} \operatorname{diam} \varphi_j(J) \leq \exp K_0 K \tag{32}$$

for all $x, y \in J$ and $n \geq 1$. Moreover

$$|\varphi_n'(x)| \leq \frac{\operatorname{diam} \varphi_n(J)}{\operatorname{diam} J} \exp K_0 \sum_{j=0}^{n-1} \operatorname{diam} \varphi_j(J) \leq \frac{\operatorname{diam} \varphi_n(J)}{\operatorname{diam} J} \exp K_0 K.$$

Hence:

$$\sum_{n=0}^{\infty} |\varphi_n'(x)| \leqq \frac{\exp K_0 K}{\operatorname{diam} J} \sum_{n=0}^{\infty} \operatorname{diam} \varphi_n(J) \leqq \frac{K \exp K_0 K}{\operatorname{diam}(J)}.$$

This inequality and (32) prove the Lemma.

Now we are ready to prove Lemma I.3 in the case $\Lambda \neq N$. This property ensures the existence of a point $x \in \Lambda$ that is an endpoint of an open interval U contained in Λ^c. Let $J_0 \supset \{x\}$ be the interval adapted to Λ given by II.2. Suppose that $U = (x, b)$. Choose J_0 so small that $b \notin \bar{J}_0$. Since J_0 is adapted to Λ, there exists $\delta > 0$ such that $d(\varphi_n(J_0), C(f)) > \delta$ for all $n \geqq 1$ and $(J_0, \{\varphi_n\}) \in \mathscr{C}(J_0, \Lambda)$. Let $K_0 = K_0(\delta, f)$ be the constant given by II.3. Let J_0 be the interval (b^-, b^+) and take $r > 0$ satisfying:

$$\exp 4K_0 \operatorname{diam} N \cdot \frac{r}{b^+ - x} < \frac{1}{2}. \tag{33}$$

By Lemma II.2 there exists an interval $\{x\} \subset J \subset J_0$ adapted to Λ and such that if $(J_0, \{\varphi_n\}) \in \mathscr{C}(J_0, \Lambda)$ and

$$\varphi_n(J) \cap J \neq \phi$$

for some $n \geqq 1$, then

$$\operatorname{diam} \varphi_n(J_0) \leqq r.$$

Suppose that $J = (a^-, a^+)$. Define:

$$J_1 = (x, b^+).$$

Let ψ be an element of $F(J, \Lambda)$. By definition of $F(J, \Lambda)$ there exists $x_0 \in J \cap (\bigcap_{n \geqq 0} f^n(\Lambda))$ such that $\psi(x_0) \in \bigcap_{n \geqq 0} f^n(\Lambda)$. Take $k \geqq 1$ satisying $f^k(\psi(x_0)) = x_0$, and $\theta \in \mathscr{S}(\Lambda, x_0)$ such that $\theta(k) = \psi(x_0)$. Take $(J_0, \{\varphi_n\}) \in \mathscr{C}(J_0, \Lambda)$ satisfying $\theta(n) = \varphi_n(x_0)$ for all $n \leqq 1$. Then:

$$\varphi_k/J = \psi, \tag{34}$$

and

$$\varphi_n(J) \cap J = \phi \tag{35}$$

for all $1 \leqq n < k$. We claim that for all $n \geqq 1$

$$\varphi_n(J) \cap J = \phi \Rightarrow \varphi_n(J_1) \cap J_1 = \phi. \tag{36}$$

Suppose that $\varphi_n(J) \cap J = \phi$ and $\varphi_n(J_1) \cap J_1 \neq \phi$. The last property implies $\varphi_n(J_0) \cap J_0 \neq \phi$, and then $\varphi_n(J_0) \subset J_0$. In particular $\varphi_n(J) \subset J_0$, and since $\varphi_n(J) \cap J = \phi$, either $\varphi_n(J) \subset (b^-, a^-)$ or $\varphi_n(J) \subset (a^+, b^+)$. The last relation is not possible because $\varphi_n(J)$ contains $\varphi_n(x_0)$, and $\varphi_n(x_0) = \lambda(n)$ is contained in Λ. But $(a^+, b^+) \subset (x, b^+)$ doesn't contain points in Λ. Then $\varphi_n(J) \subset (b^-, a^-)$. One of the endpoints of $\varphi_n(J_1)$ is contained in $\overline{\varphi_n(J)}$. Since $\varphi_n(J) \subset (b^-, a^-)$ it follows that one endpoint of

$\varphi_n(J_1)$ is contained in $[b^-, a^-]$. Moreover $\varphi_n(J_1) \subset J_0$ (because $\varphi_n(J_1) \subset \varphi_n(J_0)$) and, as we observed above, $\varphi_n(J_0) \subset J_0$). Hence $\varphi_n(J_1)$ is an interval contained in (b^-, b^+) with an endpoint in $[b^-, a^-]$ and non-empty intersection with $(x, b^+) = J_1$. It follows that $\varphi_n(J_1) \supset (x, a^+)$. But then $x_0 \in \varphi_n(J_1)$. This implies $f^n(x_0) \in J_1$. Then $\Lambda \cap J_1 \neq \phi$ because $f^n(x_0)$ belongs to Λ and J_1. On the other hand $J_1 \subset (x, b)$. Therefore it doesn't intersect Λ. This contradiction completes the proof of (36). From (35) and (36) follows that $\varphi_n(J_1) \cap J_1 = \phi$ for all $1 \leq n < k$. Hence $\varphi_n(J_1) \cap \varphi_m(J_1) = \phi$ for all $1 \leq n < m < k$ and then

$$\sum_{n=0}^{k-1} \operatorname{diam} \varphi_n(J_1) \leq 2 \operatorname{diam} N.$$

By Lemma II.3:

$$|\varphi'_k(y)| \leq \frac{\operatorname{diam} \varphi_k(J_1)}{\operatorname{diam} J_1} \cdot \exp K_0 \sum_{n=0}^{k-1} \operatorname{diam} \varphi_n(J_1)$$

$$\leq \exp 2K_0 \operatorname{diam} N \cdot \frac{\operatorname{diam} \varphi_k(J_0)}{b^+ - x} \tag{37}$$

for all $y \in \bar{J}_1$. Moreover (35) implies $\varphi_n(J) \cap \varphi_m(J) = \phi$ for all $1 \leq n < m$. Then, by II.3:

$$\frac{|\varphi'_k(z)|}{|\varphi'_k(y)|} \leq \exp K_0 \sum_{n=0}^{k-1} \operatorname{diam} \varphi_n(J) \leq \exp 2K_0 \operatorname{diam} N \tag{38}$$

for all y and z in $\bar{j}$. Given $z \in J$ and using (33), (37) and (38):

$$|\psi'(z)| = |\varphi'_k(z)| = |\varphi'_k(a^+)| \cdot \frac{|\varphi'_k(z)|}{|\varphi'_k(a^+)|} \leq \exp 4K_0 \operatorname{diam} N \cdot \frac{r}{b^+ - x} < \frac{1}{2}.$$

This completes the proof in the case $\Lambda \neq N$. Now let us consider the case $\Lambda = N$. In the case f has no critical points. If it is also injective, it will be a diffeormorphism. Since f is not topologically equivalent to a rotation, it will be a diffeomorphism with periodic points. Moreover by hypothesis not every point in $N = \bigcap_{n \geq 0} f^n(N) = \bigcap_{n \geq 0} f^n(\Lambda)$ will be periodic. Under this condition it is trivial to find an open interval $J \subset N$ such that $J \cap f^{-n}(J) = \phi$ for all $n \geq 1$. This means that J is adapted and $F(J, N) = \phi$. By II.4, J satisfies the properties required in Lemma I.3:

If f in not injective, then $N = S^1$ and $f : S^1 \supset$ is an immersion with degree $d \neq 1$ or -1. If $J(f) \neq S^1$ (where $J(f)$ is defined as in the introduction) there exists a plateau J satisfying $f^{-n}(J) \cap J = \phi$ for all $n \geq 1$ (any non-periodic plateau satisfies this property). Then J is adapted and $F(J, \Lambda) = \phi$. Hence, by Lemma II.4, J satisfies the thesis of I.3. It remains to consider the case $J(f) = S^1$. This means that f is topologically equivalent to the map $z \to z^d$. We claim that there exists an interval J_0 adapted to S^1 such that for all $\varepsilon > 0$ there exists $\psi_0 \in F(J_0, S^1)$ satisfying

$$\operatorname{diam} \psi_0(J_0) < \varepsilon \tag{39}$$

and

$$\operatorname{diam} \varphi \psi_0(J_0) < \varepsilon \tag{40}$$

for all $n \geq 1$ and every branch φ of f^{-n}/J_0. First let us prove this property for $f(z) = z^d$. Take as J_0 an open interval whose endpoints are fixed points of some power f^m and not containing fixed points of f^m. It is easy to verify that if $m|d| \geq 3$ then $F(J_0, S^1)$ contains infinitely many maps. Moreover $\psi_1(J_0) \cap \psi_2(J_0) = \phi$ if ψ_1 and ψ_2 are different maps in $F(J_0, \Lambda)$. Therefore there exists $\psi_0 \in F(J_0, \Lambda)$ satisfying (39). Property (40) also holds because any branch of f^{-n}/J_0 is a contraction. This completes the proof of the claim for the map $z \rightarrow z^d$. The general case of an immersion $f : S^1 \supset$ topologically equivalent to $z \rightarrow z^d$ follows easily from this case using the conjugacy between f and $z \mapsto z^d$.

On the other hand, since f is a C^2 immersion, we can apply Lemma II.3 to obtain a constant K_0 such that the inequalities of Lemma II.3 hold for every open interval $J \subset S^1$ and every coherent sequence $(J, \{\varphi_n\}) \in \mathscr{C}(J, S^1)$.

Fix $\varepsilon > 0$ satisfying:

$$\varepsilon \frac{\exp 4K_0 \operatorname{diam} N}{\operatorname{diam} J} < \frac{1}{2}. \tag{41}$$

By the claim there exist an adapted interval J_0 and $\psi_0 \in F(J_0, \Lambda)$ satisfying (39) and (40). Set $J = \psi_0(J_0)$. We claim that:

$$|\psi'(x)| < \tfrac{1}{2} \tag{42}$$

for all $\psi \in F(J, \Lambda)$ and every $x \in J$. First we shall prove that:

$$|\psi'(x)| \leq \exp 2K_0 \operatorname{diam} N \frac{\operatorname{diam} \psi(J_0)}{\operatorname{diam} J_0} \tag{43}$$

for all $\psi \in F(J_0, \Lambda)$. Take a coherent sequence $(J_0, \{\varphi_n\})$ and $k \geq 1$ that satisfies:

$$\psi = \varphi_k$$

and

$$\varphi_n(J_0) \cap J_0 = \phi$$

for all $1 \leq n < k$. Then $\varphi_n(J_0) \cap \varphi_m(J_0) = \phi$ for all $1 \leq n < m \leq k$. Hence

$$\sum_{n=1}^{k} \operatorname{diam} \varphi_n(J_0) < 2 \operatorname{diam} N,$$

and by II.3:

$$|\psi'(x)| = |\varphi_k'(x)| \leq \frac{\operatorname{diam} \psi(J_0)}{\operatorname{diam} J_0} \exp K_0 \sum_{j=0}^{n-1} \operatorname{diam} \varphi_j(J_0)$$

$$\leq \frac{\operatorname{diam} \psi(J_0)}{\operatorname{diam} J_0} \exp 2K_0 \operatorname{diam} N,$$

completing the proof of (43). Now take a coherent sequence $(J_0, \{\varphi_n\})$ and $k \geq 1$ satisfying $\psi = \varphi_k/J$ and $\varphi_n(J) \cap J = \phi$ for $1 \leq n < k$. Suppose first that $\varphi_n(J_0) \cap$

$J_0 = \phi$ for all $1 \leq n < k$. Then $\varphi_n \in F(J_0, S^1)$ and by (43), (39) and (41):

$$|\psi'(x)| = |\varphi'_k(x)| \leq \exp 4K_0 \operatorname{diam} N \frac{\operatorname{diam} \psi(J_0)}{\operatorname{diam} J_0}$$

$$\leq \exp 2K_0 \operatorname{diam} N \cdot \frac{\varepsilon}{\operatorname{diam} J_0} < \frac{1}{2} \tag{44}$$

for all $x \in J$. Now suppose that there exist integers $1 \leq n < k$ such that $\varphi_n(J_0) \cap J_0 \neq \phi$. Let n_0 be the maximum of such integers. Consider the maps:

$$\psi_n = \varphi_{n_0 + n} f^{n_0} / \varphi_{n_0}(J_0) \tag{45}$$

with $n \geq 1$. Then $(\varphi_{n_0}(J_0), \{\psi_n\})$ is a coherent sequence. We can extend it to a coherent sequence $(J_0, \{\psi_n\})$. Moreover

$$\psi_n(J_0) \cap J_0 = \phi \tag{46}$$

for $1 \leq n < k - n_0$ because $\psi_n(J_0) \cap J_0 \neq \phi$ implies $\psi_n(J_0) \subset J_0$ because J_0 is adapted. and then

$$\varphi_{n_0 + n}(J_0) \cap J_0 = \psi_n(\varphi_{n_0}(J_0)) \cap J_0 \neq \phi.$$

By the way we choose n_0, this implies $n_0 + n \geq k$, hence $n \geq k - n_0$ and (46) is proved. Property (46) implies that $\psi_{k-n_0} \in F(J_0, S^1)$. Then, by (43),

$$|\psi'_{k-n_0}(x)| \leq \exp 2K_0 \operatorname{diam} N \frac{\operatorname{diam} \psi_{k-n_0}(J_0)}{\operatorname{diam} J_0}$$

for all $x \in J_0$. But by (45)

$$\psi_{k-n_0}(J_0) \subset J.$$

Hence

$$|\psi'_{k-n_0}(x)| \leq \exp 2K_0 \operatorname{diam} N \frac{\operatorname{diam} J}{\operatorname{diam} J_0}. \tag{47}$$

Moreover:

$$\psi = \psi_{k-n_0} \varphi_{n_0},$$
$$\varphi_n(J) \cap \varphi_m(J) = \phi \tag{48}$$

for all $1 \leq n < m < n_0$. Then:

$$\sum_{n=1}^{n_0 - 1} \operatorname{diam} \varphi_n(J) \leq 2 \operatorname{diam} N.$$

Applying II.3:

$$|\varphi'_{n_0}(x)| \leq \exp 2K_0 \operatorname{diam} N \frac{\operatorname{diam} \varphi_{n_0}(J)}{\operatorname{diam} J}. \tag{49}$$

Then, using (48), (49) and (47) we obtain:

$$|\psi'(x)| \leqq |\psi'_{k-n_0}(\varphi_{n_0}(x))\| \varphi'_{n_0}(x))|$$

$$\leqq \exp 4K_0 \operatorname{diam} N \frac{\operatorname{diam} \varphi_{n_0}(J)}{\operatorname{diam} J_0}$$

$$\leqq \exp 4K_0 \operatorname{diam} N \frac{\varepsilon}{\operatorname{diam} J_0} \leqq \frac{1}{2}.$$

We have thus proved that J satisfies the hypothesis of Lemma II.4. Then J satisfies the properties required in Lemma I.3.

III. Appendix. Isolated Hyperbolic Sets

Here we shall prove property (IV) of the Introduction. Let $\Lambda \subset N - \partial N$ be a hyperbolic set of $f \in \operatorname{End}^r(N)$. Let V be a neighborhood of Λ. Define

$$U(\lambda) = \{x \in N | d(x, \Lambda) < \lambda\}, \quad \Lambda(\lambda) = \bigcap_{n \geq 0} f^n(\overline{U(\lambda)}).$$

Then $\bigcap_{\lambda > 0} \overline{U(\lambda)} = \Lambda$ and $\bigcap_{\lambda > 0} \Lambda(\lambda) = \Lambda$. It follows that there exists $\varepsilon > 0$ such that

$$\Lambda \subset \Lambda(\lambda) \subset V, \tag{1}$$

$$\Lambda(\lambda) \text{ is hyperbolic} \tag{2}$$

for every $0 < \lambda < \varepsilon$. Since $U(\lambda)$ is open it can be written as a union of intervals $(a^{2i}(\lambda), a^{2i+1}(\lambda))$, $i = 0, \ldots, m(\lambda)$. Clearly $m(\lambda'') \geqq m(\lambda')$ when $0 < \lambda'' < \lambda'$. Then it is easy to see that there exist $0 < a < b < \varepsilon$ such that $m(\lambda)$ is constant for $a < \lambda < b$. Call m this constant. It follows that the functions $a^{(i)}(\lambda)$ are continuous and monotone for $1 \leqq i \leqq m$ and $\lambda \in (a, b)$. Suppose that $\overline{U(\lambda)}$ is a neighborhood of $\Lambda(\lambda)$, is hyperbolic, contains Λ, is contained in V and is isolated (with $\overline{U(\lambda)}$ as an isolating block). Then property (IV) is proved. Now suppose that $\overline{U(\lambda)}$ is not a neighborhood of $\Lambda(\lambda)$ for all $\lambda \in (a, b)$. This means that for all $\lambda \in (a, b)$ there exists $1 \leqq i \leqq m$ such that $a^{(i)}(\lambda) \in \Lambda(\lambda)$. Therefore

$$(a, b) = \bigcup_{i=1}^{m} S_i, \tag{3}$$

where S_i is define by

$$S_i = \{\lambda \in (a, b) | a^{(i)}(\lambda) \in \Lambda(\lambda)\}.$$

Since every S_i is closed, (3) implies that there exists $1 \leqq i \leqq m$ such that S_i has non-empty interior. Let (c, d) be an interval contained in S_i. Then

$$a^{(i)}(\lambda) \in \Lambda(\lambda) \subset \Lambda(d)$$

for all $\lambda \in (c, d)$, and it follows that the interval $J = a^{(i)}((c, d))$ is contained in $\Lambda(d)$. If f^n/J is injective for all $n \geq 1$, the hyperbolicity of $\Lambda(d)$ implies that diam $f^n(J) \to + \infty$ when $n \to + \infty$. But since $\Lambda(d) \supset f^n(J)$ for all $n \geqq 0$, this is impossible. Therefore

f^n/J is not injective for some $n \geq 0$. Since for all $n \geq 1$ f^n/J has no critical points, because for all $n \geq 0$, $f^n(J) \subset \Lambda(d)$, and $\Lambda(d)$ contains no critical points because it is hyperbolic, the non-injectivity of f^n/J implies that $N = S^1$ and $f^n(J) = S^1$. Then $\Lambda(d) \supset f^n(J) = S^1$. If $\Lambda = S^1$ without loss of generality we can take the neighborhood V satisfying $V \neq S^1$. But then (1) implies $\Lambda(d) \subset V$. On the other hand we have proved that $S^1 < \Lambda(d)$. Then $V = S^1$. This contradiction shows that $\overline{U(\lambda)}$ must be a neighborhood of $\Lambda(\lambda)$ for some $\lambda \in (a, \beta)$ and completes the proof of property (IV).

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