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Oscillator Representations of the 2D-Conformal Algebra

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Abstract. We display irreducible representations of the Virasoro algebra (group of diffeomorphisms of the circle) for any value of the central charge c (central extension defined by a cocycle) and of the highest weight ε, where the Kač determinants do not vanish. The construction is done in terms of a simple bosonic free field. The unitarity of the representation is discussed, and it is realized with non-trivial hermiticity properties of the free field if $\varepsilon<(c-1)/24$. In the particular case of the central charge ($c=\frac{1}{2}$) corresponding to the Ising model, the three unitary irreducible representations ($\varepsilon=0, \frac{1}{16}, \frac{1}{2}$) are realized in terms of the anticommuting oscillators of the free fields of the Neveu-Schwarz-Ramond model.

A decade after they were introduced in string theories [1], the representations of the conformal group in two dimensions are the subject of a growing interest in physics as well as in mathematics. In this communication, we discuss a general form for the irreducible representations of the associated Lie algebra with central charge, the so-called Virasoro algebra:

$$[L_n, L_m]=(n-m)L_{n+m}+\frac{c}{12}(n^3-n)\delta_{n,-m}, \tag{1}$$

with the hermiticity condition

$$L_n=L^+_{-n}, \tag{2}$$

where n, m are integers, c is a real number. For a given c, the irreducible representations are characterized by the ground state (highest weight vector) $|\varepsilon\rangle$ such that

$$L_n|\varepsilon\rangle=0, \quad n>0, \qquad L_0|\varepsilon\rangle=\varepsilon|\varepsilon\rangle. \tag{3}$$

All states can, in principle, be obtained by repeated applications of $L_{-n}(n>0)$ to $|\varepsilon\rangle$. Past experience has shown that this way of building representations is often cumbersome, especially in dealing with the product of operators, because the matrix of inner products of such states is complicated. It proved much more convenient to express the L_n's as quadratic forms of creation-annihilation operators. So far, only particular cases of such quadratic realizations are widely known. In this communication, we obtain quadratic expressions for L_n for general values of ε and c following our earlier discussion of the Liouville field theory in two dimensions [2, 3].

There is a well-known irreducible representation of (1), (2), and (3) with $c=1$ in terms of a set of harmonic oscillators [1]. Introduce creation-annihilation operators a_n, a_n^+ with positive integer indices:

$$[a_n, a_m]=[a_n^+, a_m^+]=0, \qquad [a_n, a_m^+]=\delta_{n,m}.$$

Define further a zero mode a_0 such that

$$[a_0, a_n]=[a_0, a_n^+]=0, \qquad a_0=a_0^+. \tag{4}$$

Since a_0 commutes with a_n and a_n^+, we take it to be a number. These commutation relations can be summarized as follows. If one defines a_n for negative n to be equal to a_{-n}^+, one can write Eq. (3) as

$$\begin{gathered}[a_n, a_m]=\theta(n)\delta_{n,-m}, \qquad a_n^+=a_{-n}, \\ \theta(n)=\pm 1 \quad \text{for} \quad n \gtrless 0.\end{gathered} \tag{5}$$

In dual models [1], one introduced the operators

$$L_n'=\tfrac{1}{2}\sum_{r=-\infty}^{+\infty} N(a_r a_{n-r})\sqrt{|r(n-r)|}, \tag{6}$$

where N means normal ordering (all a_n with $n\geqq 0$ to the right). These operators satisfy the Virasoro algebra [Eq. (1)] with $c=1$, $\varepsilon=a_0^2/2$. Checking this is straightforward. One must only be careful with the normal ordering; otherwise one finds the classical value $c=0$.

Next, we move to other values of c. Adding a linear term to (6) is a simple way to change c. Let us try

$$L_n''=L_n'-i\lambda n\sqrt{|n|}\,a_n.$$

One obtains immediately

$$[L_n'', L_m'']=(n-m)L_{n+m}''+[(\tfrac{1}{12}+\lambda^2)n^3-n]\delta_{n,-m}.$$

This is not quite of the form (1), but the last term can be changed by an additional shift of L_0. If one lets

$$L_n^c=L_n'-i\lambda n\sqrt{|n|}\,a_n+\tfrac{1}{2}\delta_{n,0}\lambda^2, \tag{7}$$

one verifies Eqs. (1) and (2) with

$$c=1+12\lambda^2, \tag{8}$$

$$\varepsilon=\tfrac{1}{2}(a_0^2+\lambda^2). \tag{9}$$

All is well so far. However, the reality condition $L_n = L_{-n}^+$ together with $a_n^+ = a_{-n}$ implies that λ is real, and hence $c > 1$. Moreover

$$\varepsilon > \frac{\lambda^2}{2} = \frac{c-1}{24}, \tag{10}$$

so that the simple modification (7) does not work for all values of ε and c.

The whole problem is due to the reality condition (5). For $n \neq 0$, it holds by construction. For $n = 0$, it is necessary in order to ensure that $L_m = L_{-m}^+$ for all m.

We now discuss the extension to more general values of ε and c. First introduce a compact notation by letting

$$p_n = \left(\frac{3|n|}{c-1}\right)^{1/2} a_n, \quad n \neq 0, \qquad p_0 = \left(\frac{3}{c-1}\right)^{1/2} a_0. \tag{11}$$

Equations (5), (7), and (9) can be rewritten as

$$[p_n, p_m] = n\frac{3}{c-1}\delta_{n,-m}, \tag{12}$$

$$L_n = \frac{c-1}{6}\left[\sum_r N(p_r p_{n-r}) - inp_n\right] + \frac{c-1}{24}\delta_{n,0}, \tag{13}$$

$$\varepsilon = \frac{c-1}{6}\left(p_0^2 + \frac{1}{4}\right). \tag{14}$$

This structure already appeared in our quantum solution of the Liouville field theory [2], and our general study gives us hints on how to extend it to arbitrary values of ε and c. The following discussion is already implicitly contained in our earlier papers [2, 3]. Consider the p_n's as the Fourier modes of a field $P(\sigma)$, by writing

$$P(\sigma) = \sum_n p_n e^{-in\sigma}, \qquad P(\sigma + 2\pi) = P(\sigma). \tag{15}$$

If we identify L_0 with the Hamiltonian, we have at time $\tau \neq 0$:

$$P(\sigma, \tau) = \sum_n p_n e^{-in(\sigma+\tau)}, \tag{16}$$

and P is thus a simple free field. Equation (13) can be rewritten as

$$\frac{3}{\pi(c-1)}\sum_n L_n e^{-in\sigma} - \frac{1}{8\pi} = N\left[P^2(\sigma) + \frac{\partial P}{\partial \sigma}\right] \equiv U(\sigma). \tag{17}$$

In [2], we have derived a canonical transformation from $P(\sigma)$ to another free field $\tilde{P}(\sigma)$ which is defined by the implicit equation

$$N\left(P^2 + \frac{\partial P}{\partial \sigma}\right) = \tilde{N}\left(\tilde{P}^2 + \frac{\partial \tilde{P}}{\partial \sigma}\right), \tag{18}$$

where $\tilde{P}(\sigma)$ has a Fourier decomposition

$$\tilde{P}(\sigma) = \sum_n \tilde{p}_n e^{-in\sigma}, \tag{19}$$

with

$$\tilde{p}_0 = -p_0, \tag{20}$$

and where $\tilde{N}$ means normal ordering with respect to the modes of $\tilde{P}$ (all $\tilde{p}_n$ with positive n to the right). If $2ip_0$ is not an integer, we showed that Eqs. (18) and (19) perturbatively relate P and $\tilde{P}$ in a well-defined way, such that if (12) holds, P and $\tilde{P}$ satisfy the same commutation relations:

$$[p_n, p_m] = [\tilde{p}_n, \tilde{p}_m] = \frac{3n}{c-1}\delta_{n,-m}. \tag{21}$$

From Eq. (18) we can thus obtain another equivalent expression for the L_n's:

$$L_n = \frac{c-1}{26}\left[\sum_r \tilde{N}(\tilde{p}_r\tilde{p}_{n-r}) - in\tilde{p}_n\right] + \frac{c-1}{24}\delta_{n,0}. \tag{22}$$

Therefore, we now have two ways to satisfy the hermiticity condition $L_n = L^+_{-n}$. We can let

$$p_n^+ = p_{-n}, \qquad p_0^+ = p_0, \tag{23}$$

or

$$p_n^+ = \tilde{p}_{-n}, \qquad p_0^+ = \tilde{p}_0 = -p_0. \tag{24}$$

In the creation-annihilation language which we used above, the first choice is the obvious one. It leads to real values of p_0. The second choice is new and defines an unusual hermitian structure. It forces p_0 to be pure imaginary, and allows us to cover the remaining regions of the (ε, c) plane.

In all cases, the ground state $|\varepsilon\rangle$ is such that

$$p_n|\varepsilon\rangle = 0, \qquad n > 0, \tag{25}$$

or equivalently

$$\tilde{p}_n|\varepsilon\rangle = 0, \qquad n > 0. \tag{26}$$

Excited states can be obtained by applying monomials of L_{-n}'s, or of p_{-n}'s, or of $\tilde{p}_{-n}$'s with $n > 0$ to $|\varepsilon\rangle$. In the generic case, each method provides an equivalent basis for the Hilbert space and the representation is irreducible because no subspace is left invariant. We shall elaborate on this point later.

According to Eq. (14), there are four cases:

$$c > 1,\ \varepsilon > \frac{c-1}{24},\ p_0 = p_0^+, \quad p_n = p^+_{-n}, \tag{27a}$$

$$c < 1,\ \varepsilon < \frac{c-1}{24},\ p_0 = p_0^+, \quad p_n = p^+_{-n}, \tag{27b}$$

$$c > 1,\ \varepsilon < \frac{c-1}{24},\ p_0 = -p_0^+,\ p_n = \tilde{p}^+_{-n}, \tag{28a}$$

$$c < 1,\ \varepsilon > \frac{c-1}{24},\ p_0 = -p_0^+,\ p_n = \tilde{p}^+_{-n}. \tag{28b}$$

Case (27a) was already discovered by the simple modification of the operator with $c=1$. The corresponding Hilbert space is automatically positive definite.

In case (27b), on the contrary, one has the usual hermiticity property, but the commutation relation (21) has the wrong sign and there are ghosts. This was obvious from the beginning since the state $L_{-1}|\varepsilon\rangle$ has the norm.

$$\langle\varepsilon|L_1 L_{-1}|\varepsilon\rangle = 2\varepsilon\,, \tag{29}$$

which is negative if (27b) holds.

If (28a) or (28b) holds, one has to introduce the new hermitian structure. It is convenient to write

$$p_0 = -i\frac{k}{2}, \qquad k \text{ real}, \tag{30}$$

$$\varepsilon = \frac{c-1}{24}(1-k^2)\,. \tag{31}$$

According to (29), ε must be positive, and hence there are obviously ghosts unless

$$k^2<1 \quad \text{if} \quad c>1,\ \varepsilon<\frac{c-1}{24}, \tag{32a}$$

$$k^2>1 \quad \text{if} \quad c<1,\ \varepsilon>\frac{c-1}{24}. \tag{32b}$$

In order to proceed, we recall Kač's result [4] on the matrix of inner products of the states generated by the L_{-n}'s applied to $|\varepsilon\rangle$. This matrix naturally diagonalizes into submatrices, one for each eigenvalue of L_0. Kač gave the determinant of each such submatrix. If $(c-1)(c-25)>0$, these determinants vanish at some level for

$$k = \pm k_{pq} = \pm\left[\frac{p+q}{2} + \frac{p-q}{2}\left(\frac{c-25}{c-1}\right)^{1/2}\right], \tag{33}$$

where p and q are two positive integers. If $(c-1)(c-25)<0$, the vanishing occurs only for

$$k = \pm p,\ p \text{ positive integer}. \tag{34}$$

In both cases, k is real, and one has to introduce our non-trivial hermitian conjugation $p_n = \tilde{p}^{+}_{-n}$.

For $c>25$, Eq. (33) indicates that all k_{pq} are outside the interval [0, 1], except k_{11} which is equal to 1. For $25>c>1$, one sees from Eq. (34) that this remains trivially true. Thus, there is no zero in case (32a). One can therefore obtain all the matrix of inner products by continuation [5] from the case $c>1$, $\varepsilon>(c-1)/24$, where we know that they are positive definite, without any change of sign. Therefore, the matric is positive definite for $c>1$, $0<\varepsilon<(c-1)/24$. The case $c>25$, $\varepsilon<(c-1)/24$ appears in the weak coupling regime of the Liouville field theory [3].

Finally, in case (32b), the situation is complicated because the determinants vanish in the interval. Friedan et al. have shown [5] that there are ghosts except if

$$c = 1 - \frac{6}{r(r+1)}, \qquad r=2,3,4,\ldots, \tag{35}$$

and

$$k = \pm k_{pq}, \quad q \leqq p \leqq r-1. \tag{36}$$

This is possible because the vanishing of the determinant is not due to the existence of a zero norm state, but rather reflects the fact that some of the states generated by the L_{-n}'s are linearly dependent. Hence, the space of the representation becomes smaller than the bosonic Fock spaces we have been using. The only simple case where we can exhibit irreducible unitary representations is for $c = \frac{1}{2}$ $(r=3)$, for the three allowed values of $\varepsilon: 0, \frac{1}{16}, \frac{1}{2}$. These representations use free fermions, and we introduce the following sets of anticommuting operators:

$$\{b_m, b_n\} = \delta_{m,-n}, \quad m, n = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \ldots, \tag{37}$$

$$\{d_m, d_n\} = \delta_{m,-n}, \quad m, n = 0, \pm 1, \pm 2, \ldots, \tag{38}$$

$$b_{-m} = b_m^+, \; d_{-m} = d_m^+.$$

Define by normal ordering:

$$L_n^b = \tfrac{1}{2} \sum_m m N(b_{n-m} b_m), \tag{39}$$

$$L_n^d = \tfrac{1}{2} \sum_m m N(d_{n-m} d_m) + \tfrac{1}{16}. \tag{40}$$

Both L_n^b and L_n^d satisfy Eqs. (1) and (2) with $c = \frac{1}{2}$ (see [1]). The ground states $|0_b\rangle$ and $|0_d\rangle$ of the fermionic harmonic oscillators are defined by

$$b_m |0_b\rangle = 0, \quad m > 0, \tag{41}$$

$$d_m |0_d\rangle = 0, \quad m > 0. \tag{42}$$

The d representation immediately leads to $\varepsilon = 1/16$, the highest weight vector being $|0_d\rangle$. The b representation is trivially reducible, because L_n commutes with $G = (-1) \sum_{m=-\frac{1}{2}}^{\infty} b_{-m} b_m$. It splits into two irreducible representations, depending on the sign of G. The two highest weight vectors are $|0_b\rangle$ and $b_{-1/2}|0_b\rangle$, and their respective values of ε are 0 and $\frac{1}{2}$. Thus, we find naturally the only three allowed values of ε at this value of c [5], which corresponds to the Ising model. The energy density and spinor operators of the Ising model are built with the pion emission vertex operator [1] of the Neveu-Schwarz-Ramond model, while the order operator is built with the fermion emission vertex [6].

References

1. Jacob, M. (ed.): Dual theory. Amsterdam: North-Holland 1974
   Scherk, J.: An introduction to the theory of dual models and strings. Rev. Modern Phys. **47**, 123 (1975)
2. Gervais, J.-L., Neveu, A.: New quantum treatment of Liouville field theory. Nucl. Phys. B **224**, 329 (1983)
3. Gervais, J.-L., Neveu, A.: Novel triangle relation and absence of tachyons in Liouville strong field theory. Nucl. Phys. B **238**, 125 (1984)

4. Kač, V.G.: Proceedings of the International congress of mathematicians, Helsinki 1978, and references therein; Lecture Notes in Physics, Vol. 94, p. 441. Berlin, Heidelberg, New York: Springer 1979
5. Friedan, D., Qiu, Z., Shenker, S.: Conformal invariance, unitarity and two-dimensional critical exponents. Enrico Fermi Institute preprint 83–66 (1983)
6. Thorn, C.B.: Embryonic dual model for pions and fermions. Phys. Rev. D **4**, 1112 (1971)
Schwarz, J.H.: Dual quark-gluon model of hadrons. Phys. Lett. **37** B, 315 (1971)
Brink, L., Olive, D., Rebbi, C., Scherk, J.: The missing gauge conditions for the dual fermion emission vertex and their consequences. Phys. Lett. **45** B, 379 (1973)
Olive, D., Scherk, J.: Towards satisfactory scattering amplitudes for dual fermions. Nucl. Phys. B **64**, 334 (1973)

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