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Spins and Fermions on Arbitrary Lattices

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Abstract. It is proved that the system of free fermions on arbitrary lattices is equivalent to the set of locally interacting constrained spins. The fermionic counterpart of the unconstrained spin system is also derived. The generalization to the interacting theories is possible.

1. Introduction

In spite of many impressive results [1, 2], the Monte Carlo calculation with fermions on the lattice is still limited by the non-local nature of the problem. Mapping the fermionic into bosonic degrees of freedom may considerably change this situation. There are many attempts to do that, see for example [3], where you can also find a short historical outline and more references concerning this topic.

In this paper we will discuss the generalization of the Jordan-Wigner transformation to arbitrary dimensions [4]. We will prove the equivalence between the system of fermions and the set of locally interacting constrained spins. Such an equivalence was only conjectured in [4].

To begin with, we recall the construction proposed in [4]. Sections 2 and 3 contain necessary definitions and the proof of the equivalence. In Sect. 4 the role of the constraints is clarified. The fermionic counterpart of the unconstrained spin system is also derived there.

Consider the following Hamiltonian in two dimensions:

$$H = ia \sum_{n,e} \Phi^\dagger(n)\Phi(n+e) - \Phi^\dagger(n+e)\Phi(n),$$

where $n = (n_X, n_Y)$ labels the lattice sites and the unit vector $e = e_X, e_Y$. The fermion field Φ satisfies

$$\{\Phi^\dagger(m), \Phi(n)\} = 2\delta_{mn},$$

and other anticommutators vanish. Our aim is to express H in terms of the operators which rather commute at large distances, and hence, resemble the spin or the bosonic degrees of freedom. To this end define the *link operators*

$$S^Z(n,e)=iZ(n)Z(n+e)\,,\qquad Z=X,Y,$$

with

$$X(n)=\Phi^\dagger(n)+\Phi(n)\,,$$
$$Y(n)=i(\Phi^\dagger(n)-\Phi(n))\,,$$

obeying the Clifford algebra. The link operators satisfy the following mixed relations:

$$\begin{aligned}&[S^X(l),S^Y(l')]=0\,,&&\\ &\{S^Z(l),S^Z(l')\}=0 &&\text{if } l \text{ and } l' \text{ have one common vertex}\,,\\ &[S^Z(l),S^Z(l')]=0 &&\text{otherwise}\,.\end{aligned}\tag{1.1}$$

The l and l' denote the oriented links of the form (n,e) on the cubic lattice. The Hamiltonian, when expressed in terms of $\{S\}$, reads:

$$H=\tfrac{1}{2}\sum_{n,e}S^X(n,e)+S^Y(n,e)\,.$$

It was observed in [4] that the algebra (1.1) is also fulfilled by the operators:

$$\begin{aligned}\tilde{S}^X(n,e_X)&=\Gamma^1(n)\Gamma^3(n+e_X)\,,\\ \tilde{S}^X(n,e_Y)&=\Gamma^2(n)\Gamma^4(n+e_Y)\,,\\ \tilde{S}^Y(n,e_X)&=\tilde{\Gamma}^1(n)\tilde{\Gamma}^3(n+e_X)\,,\\ \tilde{S}^Y(n,e_Y)&=\tilde{\Gamma}^2(n)\tilde{\Gamma}^4(n+e_Y)\,,\end{aligned}\tag{1.2}$$

with $\tilde{\Gamma}^k(n)=i\prod_{j\neq k}\Gamma^j(n)$, where $\Gamma^j(n), j=1,\dots,4$, are the four dimensional *Euclidean-Dirac matrices* describing the degree of freedom at the site n. This suggests that there exists a correspondence between the set of free fermions and the system of spin-like objects described by the Hamiltonian

$$\tilde{H}=\tfrac{1}{2}\sum_n\Gamma^1(n)\Gamma^3(n+e_X)+\Gamma^2(n)\Gamma^4(n+e_Y)+(\Gamma\to\tilde{\Gamma})\,.\tag{1.3}$$

In d dimensions the algebra (1.1) is satisfied by the choice of the 2^d-dimensional representation of the Clifford algebra for $\{\Gamma\}$. For $d=1$ the relation between $\{\Gamma\}$ (now $\{\sigma\}$) and $\{\Phi\}$ reduces to the well known Jordan-Wigner transformation [5, 6]. For $d>1$, however, the equivalence of H and $\tilde{H}$ was not obvious. In fact, the set of constraints which reduces the Hilbert space of Eq. (1.3), was introduced in [4] (cf. Sect. IV). Even then the relation between $\{S\}$ and $\{\tilde{S}\}$ was not clear.

In Sect. 3 we will prove that the commutation relations (1.1) determine the link operators up to the unitary transformation. The proof works for arbitrary lattices and with the n-depending coordinate number. It will also be carried out for a general Hamiltonian bilinear in the fermion field. Hence, it will apply to the important case of the Kogut-Susskind Hamiltonian [7–9].

2. The Algebra of the Link Operators

Let us establish the notations which will be used throughout this paper. The word *lattice* will denote what is usually called a *graph*. The words: *link*, *path*, *loop* will be used in the place of words: *edge*, *chain*, and *circuit*, respectively [12–15].

Let L be a directed lattice which is symmetric and connected, N its vertex-set containing $\mathcal{N}$ points and $\{\Phi_n : n \in N\}$ a fermionic field defined on the $2^{\mathcal{N}}$-dimensional Hilbert space V. Obviously,

$$\begin{aligned}\{\Phi_n^\dagger, \Phi_m\} &= \delta_{nm}, \\ \{\Phi_n, \Phi_m\} &= 0, \qquad n, m \in N.\end{aligned} \tag{2.1}$$

It is well known that the rules (2.1) determine these operators up to the unitary isomorphism.

Consider the following Hamiltonian:

$$H = \sum_{n \neq m} (b_{(n,m)} \operatorname{Im} \Phi_n^\dagger \Phi_m + c_{(n,m)} \operatorname{Re} \Phi_n^\dagger \Phi_m) + \sum_n d_n \Phi_n^\dagger \Phi_n. \tag{2.2}$$

The lattice L will be called the *interaction lattice associated with the Hamiltonian* (2.2) iff for a pair (n, m) which is not a link of L one has:

$$b_{(n,m)} = b_{(m,n)} = c_{(n,m)} = c_{(m,n)} = 0.$$

In the following only the interaction lattices will be considered.

The operators:

$$\begin{aligned}X_n &= \Phi_n^\dagger + \Phi_n, \\ Y_n &= i(\Phi_n^\dagger - \Phi_n), \qquad n \in N,\end{aligned} \tag{2.3}$$

fulfil the algebra:

$$\begin{aligned}Z_n^\dagger &= Z_n, \\ \{W_m, Z_n\} &= 2\delta_{mn}\delta_{WZ}, \qquad W, Z = X, Y.\end{aligned}$$

As the consequence of (2.1) and (2.3) the Clifford commutation rules determine the operators Z_n up to unitary equivalence. From (2.3) one obtains:

$$\begin{aligned}Re\, \Phi_n^\dagger \Phi_m &= \tfrac{1}{4}(iX_n Y_m + iX_m Y_n), \\ \operatorname{Im} \Phi_n^\dagger \Phi_m &= -\tfrac{1}{4}(iX_n X_m + iY_n Y_m).\end{aligned}$$

Therefore, the Hamiltonian (2.2) is the sum of the terms of the form $iW_m Z_n$, where $W, Z = X, Y$ and m, n are the vertices of L. To have a compact description of these terms and their algebra we introduce the directed double lattice $\tilde{L}$. The vertex-set of $\tilde{L}$ is equal to $N \times \{X, Y\}$ and the pair (m_W, n_Z) [where m_W stands for (m, W)] is a link of L in the following cases:

i) (m, n) is a directed link of L,

ii) $m = n$ and $W \neq Z$.

Now, assigning to each link $l = (m_W, n_Z)$ of $\tilde{L}$ the operator

$$S(l) = iW_m Z_n, \tag{2.4}$$

one has:

$$H = \sum_{l \in \tilde{L}} a_l S(l),$$

where a_l are reals.

Define the operator

$$S(\gamma) = (-i)^{k-1} S(l_1) \dots S(l_k),$$

where $l_1, \dots, l_k$ form the path γ, and $S(\gamma) = i$ if γ is a degenerated path, i.e. a single vertex-point. From the previous definitions one obtains immediately

$$S(\gamma_1 \gamma_2) = -iS(\gamma_1)S(\gamma_2), \qquad (2.5)$$

where the end of the γ_1 is the beginning of γ_2 and $\gamma_1\gamma_2$ stands for the sum of both paths.

The following proposition describes the algebra of the link operators $\{S(l): l \in \tilde{L}\}$.

Proposition 1. A) $(S(l))^\dagger = S(l)$, $(S(l))^2 = 1$ *for* $l \in \tilde{L}$,
B) $\{S(l), S(l')\} = 0$ *if links* l, l' *have exactly one common point (they overlap),* $[S(l), S(l')] = 0$ *in other cases,*
C) $\mathrm{Tr} \prod_{n \in N} S(n_X, n_Y) = 0$,
D) $S(\gamma) = i$, *if* γ *is a closed path on* $\tilde{L}$.

These rules are consequences of (2.3) and the definitions of the link and path operators.

The problem which was formulated in the Introduction can now be stated more precisely. Does the Proposition 1 determine the link operators uniquely? This question will be studied in the next section.

We will also need the following result.

Proposition 2. *If the set of operators* $\{S(l): l \in \tilde{L}\}$ *has the properties* A), B) *and* $S(l) = -S(l^{-1})$, *where* l^{-1} *is a link directed inversely to* l, *then:*

a) $\{S(\gamma_1), S(\gamma_2)\} = 0$ *if* γ_1, γ_2 *are not closed paths having exactly one common edge-point (i.e. the end or the beginning),*

b) $[S(\gamma), S(l)] = 0$ *when* $l \in \tilde{L}$ *and* γ *is a closed path,*

c) $(S(\gamma))^\dagger = S(\gamma)$, $(S(\gamma))^2 = 1$ *if* γ *is not a closed path,*

d) $(S(\gamma))^\dagger = -S(\gamma)$, $(S(\gamma))^2 = -1$ *if* γ *is a closed path, and*

e) $S(\gamma^{-1}) = -S(\gamma)^{-1}$.

3. The Generalization of the Jordan-Wigner Transformation to Arbitrary Lattices

The equivalence between the Clifford and the link variables will be shown in two steps. First, consider a tree subset of a general lattice. In this case *there exists a one-to-one correspondence between the vertices of the graph and the paths.* One simply chooses, for every vertex, the unique path connecting it to the root of the tree. Therefore, the generalization of the Jordan-Wigner transformation to arbitrary trees is possible. Secondly, a simple algebraic observation will allow for the extension of our construction to the full lattices.

We begin with the following:

Theorem. *Let L' denote the directed tree with its vertex-set $\tilde{N} = N \times \{X, Y\}$. Choose the root of the tree to be $\bar{w} = \bar{n}_X$. Further, let $\{S'(l): l \in L'\}$ denote the family of the operators, on a finite-dimensional Hilbert space V', fulfilling the rules* A) *and* B). *Finally, call γ_w the only path connecting the root point $\bar{w}$ with $w \in \tilde{N}$. If*

$$\mathrm{Tr}(S'(\gamma_{\bar{n}_Y}) \prod_{n \neq \bar{n}} S'(\gamma_{n_X}) S'(\gamma_{n_Y})) = 0, \tag{3.1}$$

then there exists a family of hermitian operators $\{X'_n, Y'_n: n \in N\}$ such that:

$$\{W'_m, Z'_n\} = 2\delta_{mn}\delta_{WZ},$$

and $S'(l) = iW'_m Z'_n$, where $l = (m_W, n_Z)$ is a link of L'.

Equation (3.1) replaces Condition C). Since there are no closed paths on the tree L', Condition D) is trivially satisfied.

Proof. Operators $\{S'(\gamma_w): w \neq \tilde{w}\}$ fulfill the following algebra:

$$\begin{aligned} &(S'(\gamma_w))^\dagger = S'(\gamma_w), \\ &\{S'(\gamma_w), S'(\gamma_u)\} = 2\delta_{wu}, \qquad w, u \neq \bar{w}. \end{aligned} \tag{3.2}$$

Denoting $S' = i^{\mathcal{N}-1} S'(\gamma_{\bar{n}_Y}) \prod_{n \neq \bar{n}} S'(\gamma_{n_X}) S'(\gamma_{n_Y})$, one has [from (3.1) and (3.2)] the identities:

$$\begin{aligned} &(S')^\dagger = S', \quad (S')^2 = 1, \quad \mathrm{Tr}\, S' = 0, \\ &[S', S'(\gamma_w)] = 0 \quad \text{for} \quad w \neq \bar{w}. \end{aligned}$$

Consider the decomposition of V' into two mutually orthogonal subspaces corresponding to the two eigenvalues (± 1) of S'. Using the block notation, we can write:

$$S' = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad S'(\gamma_w) = \begin{bmatrix} T_w^1 & 0 \\ 0 & T_w^2 \end{bmatrix}$$

for $w \neq \bar{w}$. It follows from (3.2) that the T's satisfy:

$$(T_w^i)^\dagger = T_w^i, \qquad \{T_w^i, T_u^i\} = 2\delta_{wu} \quad \text{for} \quad w, u \neq \bar{w}, \qquad i = 1, 2.$$

Since the anti-commutation relations determine the operators $\{T_w^i\}$ up to the unitary equivalence, there exists the unitary isomorphism $U: V_1 \to V_2$ such that $U T_w^1 U^{-1} = T_w^2$ for $w \neq \bar{n}_X, \bar{n}_Y$. From the relation:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = S = i^{\mathcal{N}-1} \begin{bmatrix} T_{\bar{n}_Y}^1 \prod\limits_{n \neq \bar{n}} T_{n_X}^1 T_{n_Y}^1 & 0 \\ 0 & T_{\bar{n}_Y}^2 U \prod\limits_{n \neq \bar{n}} T_{n_X}^1 T_{n_Y}^1 U^{-1} \end{bmatrix}$$

one obtains

$$T_{\bar{n}_Y}^2 = -U T_{\bar{n}_Y}^1 U^{-1}.$$

Define

$$Y'_{\bar{n}} = \left[\begin{array}{c|c} 0 & U^{-1} \\ \hline U & 0 \end{array}\right]. \tag{3.3}$$

After the simple algebra one obtains

$$\begin{aligned} (Y'_{\bar{n}})^{\dagger} &= Y_{\bar{n}}, \\ (Y'_{\bar{n}})^2 &= 1, \\ \{Y'_{\bar{n}}, S'(\gamma_{\bar{n}_Y})\} &= 0, \\ [Y'_{\bar{n}}, S'(\gamma_w)] &= 0 \end{aligned} \tag{3.4}$$

for $w \neq \bar{n}_X, \bar{n}_Y$. Now we can define the required operators:

$$\begin{aligned} X'_{\bar{n}} &= -iS'(\gamma_{\bar{n}_Y})Y'_{\bar{n}} = iY'_{\bar{n}}S(\gamma_{\bar{n}_Y}), \\ X'_n &= -iX'_{\bar{n}}S'(\gamma_{n_X}), \\ Y'_n &= -iX'_{\bar{n}}S'(\gamma_{n_Y}) \end{aligned} \tag{3.5}$$

for $n \neq \bar{n}$. From the equality $S'(\gamma_{n_Z}) = iX'_nZ'_n$ ($Z = X, Y$), formulas (3.2)–(3.5) one has the relations:

$$\begin{aligned} (Z'_n)^{\dagger} &= Z'_n, \\ \{W'_m, Z'_n\} &= 2\delta_{mn}\delta_{WZ}. \end{aligned}$$

Consider now a directed link (m_W, n_Z) on L'. If $m_W = \bar{n}_X$, then $S'(\bar{n}_X, n_Z) = S'(\gamma_{n_Z})$, and this gives the equality $S'(\bar{n}_X, n_Z) = iX'_{\bar{n}}Z'_n$. If $m_W \neq \bar{n}_X$, one has $n_Z \neq \bar{n}_X$ and $\gamma_{m_W}(m_W, n_Z) = \gamma_{n_Z}$. Using (2.5) we obtain $S'(\gamma_{n_Z}) = -iS'(\gamma_{m_W})S'(m_W, n_Z)$, and from this $S'(m_W, n_Z) = iS'(\gamma_{m_W})S'(\gamma_{n_Z})$, which gives the identity $S'(m_W, n_Z) = iW'_mZ'_n$. This ends the proof.

Now, we will generalize this result to arbitrary lattices with closed paths.

Corollary. *Let L be a directed lattice and $\tilde{L}$ its double lattice. If $\{S'(l): l \in \tilde{L}\}$ denotes a set of operators on a finite-dimensional Hilbert space V', fulfilling points* A), B), C), *and* D), *then there exists the family of operators $\{X'_n, Y'_n: n \in L\}$ for which:*

$$\begin{aligned} (Z'_n)^{\dagger} &= Z'_n, \\ \{W'_m, Z'_n\} &= 2\delta_{mn}\delta_{WZ}, \end{aligned}$$

for $W, Z = X, Y$, $m, n \in N$, and such that

$$S'(l) = iW'_mZ'_n,$$

where $l = (m_W, n_Z) \in \tilde{L}$. Moreover, if $\{S(l): l \in \tilde{L}\}$ is the set of operators defined by the equality (2.4), *then there exists a unitary isomorphism $U: V \to V'$ such that*

$$US(l)U^{-1} = S'(l) \quad \text{for} \quad l \in \tilde{L}.$$

Proof. For an arbitrary choice of the root point $\bar{n}_X$ there exists a maximal directed tree $L' \subset \tilde{L}$. Theorem applies for the family of operators $\{S'(l): l \in L'\}$ because the same consideration as in the end of the proof of Theorem reduces Eq. (3.1) to Assumption C). Now let us define the operators $\{\bar{S}(l): l \in \tilde{L}\}$ by the equality

$$\bar{S}(l) = iW'_m Z'_n$$

for any directed link $(m_W, n_Z) \in \tilde{L}$. Of course, the $\bar{S}(l)$ and $S'(l)$ coincide for $l \in L'$. In addition,

$$\begin{aligned} S'(m_W, n_Z) &= iS'(\gamma_{m_W})S'(\gamma_{n_Z}), \\ \bar{S}(m_W, n_Z) &= i\bar{S}(\gamma_{m_W})\bar{S}(\gamma_{n_Z}). \end{aligned} \tag{3.6}$$

Since paths $\gamma_{m_W}, \gamma_{n_Z}$ are built from the links belonging to L', one immediately obtains

$$S'(\gamma_{m_W}) = \bar{S}(\gamma_{m_W}).$$

Taking into account Eqs. (3.6) finally one has:

$$S'(l) = US(l)U^{-1}$$

for an arbitrary link in $\tilde{L}$.

Resuming, one sees that the conditions from Proposition 1, which are fulfilled for operators $\{S(l): l \in \tilde{L}\}$ determine them uniquely up to the unitary equivalence. This is the first step towards solving the problem formulated in [4].

In the next section we will investigate the spin representation of the link operators.

4. The System of Constrained Spins

Assume that each vertex of the interaction lattice L has an even number of neighbours equal to $2k_n$ ($k_n \geqq 1$). The set of the nearest neighbours of a given vertex will be denoted L_n. Let V_n be 2^{k_n}-dimensional Hilbert space and $\{\Gamma_n^k: k \in L_n\}$ a family of $2k_n$ operators on V_n having the algebra

$$\begin{aligned} (\Gamma_n^k)^\dagger &= \Gamma_n^k, \\ \{\Gamma_n^k, \Gamma_n^l\} &= 2\delta_{kl} \end{aligned} \tag{4.1}$$

for $k, l \in L_n$. Define

$$\Gamma_n^0 = i^{k_n} \Gamma_n^{m_1} \dots \Gamma_n^{m_{2k_n}},$$

where $m_1, \dots, m_{2k_n}$ is an arbitrary order on L_n and

$$\tilde{\Gamma}_n^k = -i\Gamma_n^k \Gamma_n^0.$$

Let Γ_0 be an operator on the two dimensional Hilbert space V_0, such that $\Gamma_0^\dagger = \Gamma_0$, $(\Gamma_0)^2 = 1$, $\mathrm{Tr}\,\Gamma_0 = 0$. The Hilbert space of the whole system is

$$W = \Big(\bigotimes_{n \in N} V_n\Big) \otimes V_0.$$

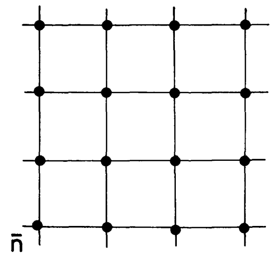

Fig. 1

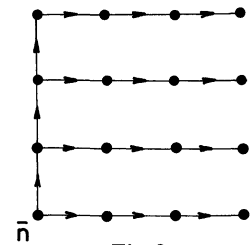

Fig. 2

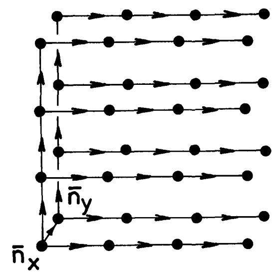

Fig. 3

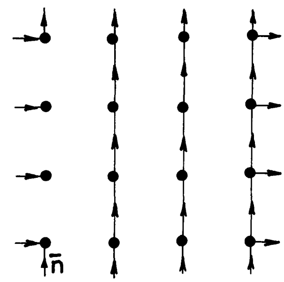

Fig. 4

For an operator B_n on the space V_n, $B_{(n)}$ will denote $\left(\bigotimes_{m \neq n} 1_m\right) \otimes B_n$. Define, for the oriented link (m, n) in L,

$$\begin{aligned}
\tilde{S}(m_X, n_X) &= -\tilde{S}(n_X, m_X) = \Gamma^n_{(m)} \Gamma^m_{(n)} A_{(0)}, \\
\tilde{S}(m_Y, n_Y) &= -\tilde{S}(n_Y, m_Y) = \tilde{\Gamma}^n_{(m)} \tilde{\Gamma}^m_{(n)} A_{(0)}, \\
\tilde{S}(m_X, n_Y) &= -\tilde{S}(n_Y, m_X) = \Gamma^n_{(m)} \tilde{\Gamma}^m_{(n)} A_{(0)}, \qquad (4.2) \\
\tilde{S}(m_Y, n_X) &= -\tilde{S}(n_X, m_Y) = \tilde{\Gamma}^n_{(m)} \Gamma^m_{(n)} A_{(0)}, \\
\tilde{S}(n_X, n_Y) &= -\tilde{S}(n_Y, n_X) = \Gamma^0_{(n)} A_{(0)},
\end{aligned}$$

where $A_0 = \Gamma_0$ iff the pair (m_W, n_Z) contains the vertex $\bar{n}_X$ and $A_0 = 1$ in other cases. Equations (4.2) are similar to (1.2). Note, however, the additional contribution from the $\bar{n}_X$ vertex[1].

In the following we will explain the relation between the S's and the standard fermionic operators.

Points A) and B) are valid for the family $\{\tilde{S}(l): l \in \tilde{L}\}$ and moreover $\tilde{S}(l^{-1}) = -\tilde{S}(l)$, hence Proposition 2 is satisfied. However, Condition D) is not fulfilled for $k_n > 1$. The dimension of W is greater than 2^N in this case. Therefore, we must restrict ourselves to the relevant subspaces. We will construct them in a few steps. The construction is illustrated in the figures for the two dimensional toroidal lattice (Fig. 1).

First, let us choose, from the lattice L, the directed tree R, with the root at the vertex $\bar{n}$ (Fig. 2). R has exactly $N-1$ links. Then we build the directed tree L' in the following way: the vertex-set of L' is the set $N \times \{X, Y\}$, and the set of links is equal to $\{(n_Z, m_Z): (n, m) \text{ is a link of } R, Z = X, Y\} \cup \{(\bar{n}_X, \bar{n}_Y)\}$ (Fig. 3). The point $\bar{n}_X$ is

1 Γ^0 is related to the total number of fermions in the system. The precise connection depends on the topology of the lattice

chosen as the root of L'. Define also the oriented complement of R to L (cf. Fig. 4): $K=\{l: l\in\tilde{L} \text{ and } l, l^{-1}\notin R\}$. It is easy to see that K has $\mathcal{M}=\sum_{n\in N} k_n-(\mathcal{N}-1)$ elements. For every link $\lambda=(m,n)\in K$ one can construct the loop $\gamma_\lambda=\gamma_{m_X}(m_X,n_X)(\gamma_{n_X})^{-1}$, and associated with it the operator $P_\lambda=-i\tilde{S}(\gamma_\lambda)$.

The following equalities are consequences of (4.1), (4.2), and Proposition 2:

$$\begin{aligned}
&\text{a)} && (P_\lambda)^\dagger=P_\lambda, \quad (P_\lambda)^2=1,\\
&\text{b)} && [P_\lambda,\tilde{S}(l)]=0 \quad \text{for} \quad l\in\tilde{L}, \quad \lambda\in K, \quad \text{which implies}\\
&\text{c)} && [P_\lambda,P_{\lambda'}]=0, \quad \text{where} \quad \lambda,\lambda'\in K,\\
&\text{d)} && \operatorname{Tr}(P_{\lambda_1}\dots P_{\lambda_k})=0 \quad \text{if} \quad \lambda_i\neq\lambda_j \quad \text{when} \quad i\neq j.
\end{aligned} \tag{4.3}$$

The family of operators $\{P_\lambda: \lambda\in K\}$ is the maximal family of the loop operators satisfying (4.3). The main reason for introducing the complement K was to extract explicitly such a family.

Let $F=\{\varepsilon: K\to\{-1,1\}\}$ stand for the set of all sign functions on K. There are $2^{\mathcal{M}}$ elements of F. We will identify a function ε with its extension to L by the conditions: $\varepsilon(\lambda)=\varepsilon(\lambda^{-1})$ if $\lambda\in L$, $\varepsilon(\lambda)=1$ when $\lambda\in R$, and with the function on $\tilde{L}$ defined in the following way: $\varepsilon(n_X,n_Y)=\varepsilon(n_Y,n_X)=1$, $\varepsilon(m_W,n_Z)=\varepsilon(m,n)$ where $(m,n)\in L$.

Let ε be an arbitrary element of F. Associate with ε the hermitian projector

$$\Xi_\varepsilon=\prod_{\lambda\in K}\frac{1+\varepsilon(\lambda)P_\lambda}{2}$$

onto the subspace W_ε. It follows from (4.3) that the subspaces $\{W_\varepsilon: \varepsilon\in F\}$ are mutually orthogonal and $W=\bigoplus_{\varepsilon\in F}W_\varepsilon$. Moreover, Eq. (4.3b) implies that $[\Xi_\varepsilon,S(l)]=0$, i.e. that W_ε is an invariant subspace of $\tilde{S}(l)$. In what follows the superscript ε will denote the restriction of a given operator to the subspace W_ε.

Now we will show that the operators $\{S^\varepsilon(l): l\in L'\}$ satisfy the assumptions of Theorem. They obviously fulfill points A) and B). To prove (3.1) it is sufficient to note that

$$\tilde{S}(n_X,n_Y)=i\tilde{S}(\gamma_{n_X})\tilde{S}(\gamma_{n_Y}) \tag{4.4}$$

for $n\neq\bar{n}$. One can show this inductively using the formula

$$\tilde{S}(m_W,n_Z)=-\tilde{S}(m_W,m_X)\tilde{S}(m_X,n_X)\tilde{S}(n_X,n_Z), \tag{4.5}$$

where (m,n) is a directed link of L. From (4.4) one obtains the identity:

$$\tilde{S}^\varepsilon(\gamma_{\bar{n}_Y})\prod_{n\neq\bar{n}}(\tilde{S}^\varepsilon(\gamma_{n_X})\tilde{S}^\varepsilon(\gamma_{n_Y}))=\frac{1}{i^{\mathcal{N}-1}}\Xi_\varepsilon\prod_{n\in N}\tilde{S}(n_X,n_Y)\circ j_\varepsilon, \tag{4.6}$$

where $j_\varepsilon: W_\varepsilon\to W$ is a natural inclusion. Thus the trace of the left-hand side of (4.6) is equal to the trace of

$$\frac{1}{i^{\mathcal{N}-1}}\Xi_\varepsilon\prod_{n\in N}\tilde{S}(n_X,n_Y).$$

This trace is equal to zero due to the $A_{(0)}$ factor introduced in (4.2). One can now see why we have introduced the additional space V_0. There exists a unitary isomorphism $U_\varepsilon: V \to W_\varepsilon$ such that $\tilde{S}^\varepsilon(l) = U_\varepsilon S(l) U_\varepsilon^{-1}$ for $l \in L'$. Operators $\{\bar{S}^\varepsilon(l): l \in \tilde{L}\}$, defined by the formula $\bar{S}^\varepsilon(l) = U_\varepsilon S(l) U_\varepsilon^{-1}$, obey Conditions A) to D), and $\bar{S}^\varepsilon(l) = \tilde{S}^\varepsilon(l)$ for $l \in L'$. Further $P^\varepsilon_\lambda = \varepsilon(m,n)$ for $\lambda = (m,n) \in K$, and from the definition of P^ε_λ one has:

$$\tilde{S}^\varepsilon(\gamma_{m_X}(m_X, n_X)\gamma_{n_X}^{-1}) = i\varepsilon(m,n).$$

Proposition 2e) gives:

$$\tilde{S}^\varepsilon(m_X, n_X) = i\varepsilon(\lambda)(\tilde{S}^\varepsilon(\gamma_{m_X}))^{-1}\tilde{S}^\varepsilon(\gamma_{n_X}).$$

On the other hand, we have for $\bar{S}^\varepsilon(l)$:

$$\bar{S}^\varepsilon(m_X, n_X) = i(\bar{S}^\varepsilon(\gamma_{m_X}))^{-1}\bar{S}^\varepsilon(\gamma_{n_X}).$$

Hence, since $\bar{S}^\varepsilon(l) = \tilde{S}^\varepsilon(l)$ for $l \in L'$, we have:

$$\tilde{S}^\varepsilon(m_X, n_X) = \varepsilon(m,n)\bar{S}^\varepsilon(m_X, n_X)$$

for $(m,n) \in K$. From (4.4) and the analogous equality for $\{\bar{S}^\varepsilon(l): l \in \tilde{L}\}$, one obtains

$$\tilde{S}^\varepsilon(n_X, n_Y) = \bar{S}^\varepsilon(n_X, n_Y) = \varepsilon(n_X, n_Y)\bar{S}^\varepsilon(n_X, n_Y).$$

The identities $\tilde{S}^\varepsilon(l) = -\tilde{S}^\varepsilon(l^{-1})$, $\bar{S}^\varepsilon(l) = -\bar{S}^\varepsilon(l^{-1})$ give

$$\tilde{S}^\varepsilon(l) = \varepsilon(l)\bar{S}^\varepsilon(l)$$

for a link $l = (m_X, n_X)$ or $l = (m_W, m_Z)$, where $W \neq Z$. Using (4.5) and a similar equation valid for $\{\bar{S}^\varepsilon(l): l \in \tilde{L}\}$, we get finally:

$$\tilde{S}^\varepsilon(l) = \varepsilon(l)\bar{S}^\varepsilon(l) = \varepsilon(l)U_\varepsilon S(l)U_\varepsilon^{-1} \tag{4.7}$$

for an arbitrary link l of $\tilde{L}$. Equation (4.7) is the main result of this section.

It turns out that the link operators in the spin representation are equivalent to the standard one in the Fermi representation. The only effect of the constraints are the appropriate Z_2 factors residing on the links of the original lattice. For the particular choice of the constraints $\varepsilon = 1$ one has $\tilde{S}^\varepsilon(l) = U_\varepsilon S(l) U_\varepsilon^{-1}$.

Let us now derive the fermionic equivalent of the unconstrained spin system. To this end we number the elements of $F: \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2^{\mathcal{M}}}$ such that $\varepsilon_1 = 1$. If $\{e_i: i = 1, \ldots, 2^{\mathcal{N}}\}$ is an (orthonormal) basis of V, then $\{U_{\varepsilon_j}(e_i): i = 1, \ldots, 2^{\mathcal{N}}, j = 1, \ldots, 2^{\mathcal{M}}\}$ is obviously the (orthonormal) basis of W. Let $[S(l)]$ denote the matrix of the operator $S(l)$ with respect to the basis $\{e_i\}$. The matrix of $\tilde{S}(l)$ in the basis $\{U_{\varepsilon_j}(e_i)\}$ is the following:

$$[\tilde{S}(l)] = \begin{bmatrix} [S(l)] & 0 & & \\ 0 & \varepsilon_2(l)[S(l)] & & \\ & & \ddots & 0 \\ & & & \varepsilon_{2^{\mathcal{M}}}(l)[S(l)] \end{bmatrix}. \tag{4.8}$$

If we put $\tilde{H}=\sum_{l\in\tilde{L}} a_l\tilde{S}(l)$, then (in the basis $\{U_{\varepsilon_j}(e_i)\}$)

$$[\tilde{H}]=\begin{bmatrix} [H_1] & 0 & & \\ 0 & [H_2] & & \\ & 0 & \ddots & 0 \\ & & & [H_{2^{\mathscr{M}}}] \end{bmatrix}, \tag{4.9}$$

where $[H_i]$ stands for the matrix of the operator

$$H_i=\sum_{l\in\tilde{L}} \varepsilon_i(l)a_lS(l)$$

in the basis $\{e_i: i=1,\ldots,2^{\mathscr{N}}\}$. Moreover, one can see that

$$H_1=U_{\varepsilon_1}HU_{\varepsilon_1}^{-1}.$$

Equations (4.7) and (4.9) answer the question about the relations between the unconstrained spins and fermions on an arbitrary lattice.

It follows from the commutation relations (4.1) that Γ's can be considered as Green components of a parafermi field [16, 17]. However, our case differs in two points:

1) The roles of internal degrees of freedom and the space-time variables are exchanged.

2) Lattices discussed here can have n-dependent coordination number.

For a constant coordination number one can consider the relation between our approach and the Klein transformations which in this case would directly relate Γ's with anticommuting objects. The transformation (4.7) preserves locality of the interactions between Γ's while, as it is well known, the Klein transformations would lead to the non-local interactions in higher dimensions [16]. Therefore, there are no Klein transformations on Γ's which would give the link operators similar to those proposed in Eqs. (4.2). This can also be seen from Eq. (4.7). Only for $\varepsilon=1$ one may expect the existence of such transformations. This is the well known case of one-dimensional lattices.

5. Summary and Conclusions

The replacement of the fermionic operators $\{S(l)\}$ has one main advantage. The later anticommute only locally contrary to the Fermi operators. Thus it might be useful, from the point of view of future numerical calculations, to deal only with the link operators. However, the matrix representation of $\{S(l)\}$ is quite complicated if one uses the traditional Jordan-Wigner transformation in more than one dimension.

In [4] a quite simple representation for the link operators was proposed. However, the precise relation between the two was not clear. The existence of too many paths on a lattice in more than one dimension caused the difficulty. Reducing the lattice to the tree, on the first stage of the proof, was the important simplification. As a consequence, the more precise understanding of the relation between spins and fermions in higher dimensions becomes possible.

In this paper we have proved that, with a small modification, the system of constrained spins considered in [4] is equivalent to the set of fermions on rather general lattices. The role of the constraints was also clarified. The fermionic equivalent of the system of unconstrained spins was also derived [Eqs. (4.7) and (4.9)]. It was shown that the effect of the constraints is to introduce some Z_2 factors on the links of the original lattice with fermionic degrees of freedom. Our considerations include the Kogut-Susskind Hamiltonian in the three dimensional space as a special case. They can be also generalized to the interacting theories.

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