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The Cauchy Problem in Extended Supergravity, $N=1$, $d=11$

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Abstract. We prove the Grassmann valued system of extended supergravity $N=1$, $d=11$ proposed by Cremmer and Julia is well proposed and causal.

Introduction

The extension of supergravity to d-dimensional models offers a synthesis between ordinary $d=4$, $N=1$ supergravity [8, 14] which unifies a bosonic and a fermionic field (graviton and gravitino) and the old ideas of Kaluza-Klein-Jordan-Thiry for unification of gravitation and electromagnetism through a fifth dimension of space time, an idea extended to Yang-Mills fields by B. DeWitt, R. Kerner etc. Among the various extended supergravity models proposed for unification of all fundamental interactions a particularly interesting one, called $N=1$, $d=11$ supergravity, is an Einstein Cartan theory in an 11 dimensional space time with source a spin 3/2 field, a spinor valued 1-form. However, it is necessary, in order to have a coherent system, to add another field called the "three index photon", a numerical valued 3-form.

We show in this paper – as we have done before for simple supergravity cf. [2], that the system of partial differential equations of the $N=1$, $d=11$ extended supergravity satisfied by the Grassmann valued fields is a well posed system for the Cauchy problem, with constraints but causal: the solution at a point depends only on the initial data which are in the past of that point, this past being determined by the isotropic cone of the numerical part of the metric.

1. Notations

$V=S\times\mathbb{R}$, 11 dimensional, C^∞ manifold, x^M, $M=0,\dots,10$ local coordinates, $\partial_M=\partial/\partial x^M$, $\mathbf{e}=(e_A{}^M)$, $\mathbf{e}_A=e_A{}^M\partial_M$: 11 dimensional moving frame, $e^A{}_M$ inverse matrix of $e_A{}^M$, $\theta^A=e^A{}_M dx^M$ moving coframe dual of $\mathbf{e}_A$.

$$\begin{aligned} g_{MN}&=e^A{}_M e^B{}_N \eta_{AB}: \quad \text{hyperbolic metric } \mathbf{g},\\ \eta_{AB}&=\operatorname{diag}(1,-1,\dots,-1) \quad \text{Minkowski metric}. \end{aligned} \tag{1.1}$$

Indices from the beginning of the alphabet are moving frame indices, from the end of the alphabet natural frame indices. Components in the natural and moving frame are interchanged through $e_A{}^M$ or $e^A{}_M$. Indices are lowered by g_{MN}, η_{AB} [respectively raised by g^{MN}, η^{AB}], for instance $\partial_A = e_A{}^M\partial_M$, $e^{AM} = g^{MN}e^A{}_N$. $e = \det(e^A{}_M)$, $\tau = e dx^0 \ldots dx^{10}$, volume element, $\tau^{M_1 \ldots M_{11}} = e^{-1}\varepsilon_{0\ldots 10}^{M_1 \ldots M_{11}}$: contravariant components of the volume 11-form, Γ^A standard Dirac matrices, $\Gamma^M = e_A{}^M\Gamma^A$, $\Gamma^A\Gamma^B + \Gamma^B\Gamma^A = 2\eta^{AB}$,

$$\Gamma^{M_1 \ldots M_p} = \frac{1}{p!}\varepsilon^{M_1 \ldots M_p}_{N_1 \ldots N_p}\Gamma^{N_1} \ldots \Gamma^{N_p} = \Gamma^{[N_1} \ldots \Gamma^{N_p]},$$

$\boldsymbol{\psi} = (\psi_N)$ 1-form with spinor values, in a spin frame associated to the Lorentz frame $\mathbf{e}$, $\bar{\boldsymbol{\psi}} = (\bar{\psi}_N) = i\tilde{\boldsymbol{\psi}}\Gamma^0$, $\tilde{}$: transposed imaginary conjugate. $\bar{\boldsymbol{\omega}} = (\bar{\omega}_M{}^A{}_B)$: Riemann connection of $\mathbf{g}$:

$$\bar{\omega}_M{}^A{}_B = e^C{}_M\bar{\omega}_C{}^A{}_B = \tfrac{1}{2}(c_M{}^A{}_B + c^A{}_{MB} + c^A{}_{BM}) \quad \text{with} \quad c_C{}^A{}_B e_A = [e_C, e_B],$$

$$\text{thus} \quad c_M{}^A{}_N = 2\partial_{[N}e^A{}_{M]} = \partial_N e^A{}_M - \partial_M e^A{}_N. \tag{1.2}$$

$\boldsymbol{\omega} = (\omega_M{}^A{}_B)$: metric connection with torsion:

$$\omega_M{}^A{}_B = \bar{\omega}_M{}^A{}_B + C_M{}^A{}_B, \qquad C = (C_M{}^A{}_B) \quad \text{contorsion tensor}, \tag{1.3}$$

$$C_M{}^A{}_B = \tfrac{1}{2}(S_M{}^A{}_B + S^A{}_{BM} + S^A{}_{MB}), \tag{1.4}$$

with $\mathbf{S} = (S_C{}^A{}_B) = -(S_B{}^A{}_C)$ the torsion tensor. $\bar{\boldsymbol{\nabla}}$, $\boldsymbol{\nabla}$ covariant derivatives in $\bar{\boldsymbol{\omega}}$ and $\boldsymbol{\omega}$, respectively. $\mathbf{D}$: Riemannian covariant derivative on tensor indices, and on spinor indices in the spin image of the connection $\boldsymbol{\omega}$

$$\tfrac{1}{4}\omega_M{}^A{}_B\Gamma_A\Gamma^B. \tag{1.5}$$

Curvature tensor:

$$R_{MN}{}^{AB} = 2(\partial_{[M}\omega_{N]}{}^{AB} + \omega_{[M}{}^A{}_C\omega_{N]}{}^{CB}). \tag{1.6}$$

3-index photon: 3-form $\mathbf{A} = \frac{1}{3!}A_{MNP}dx^M \wedge dx^N \wedge dx^P$, $\mathbf{F} = d\mathbf{A}$.

The fields $\mathbf{e}$ and $\boldsymbol{\psi}$ take their values in a Grassmann algebra $\mathscr{G}$, whose generators are denoted by ζ^I, $I = 1, \ldots, N$ (possibly $N = \infty$), and obey the anticommutative law:

$$\zeta^I\zeta^J = -\zeta^J\zeta^I.$$

Each field f admits a decomposition (formal series if $N = \infty$, cf. [11])

$$f = \sum_{p=0}^{N} f(p), \qquad f(p) = \frac{1}{p!}f_{I_1 \ldots I_p}\zeta^{I_1} \ldots \zeta^{I_p},$$

where $f_{I_1 \ldots I_p}$ is a usual (numerical) field. Usual laws of differential tensor calculus are applicable to f through their application to each $f_{I_1 \ldots I_p}$. If the series contains only even [respectively odd] powers of the generators, f is called even [respectively odd]. Two odd elements anticommute. Even elements commute with all elements. The field $\mathbf{e}$ is supposed to be even and $\boldsymbol{\psi}$ to be odd. A $\mathscr{G}$ valued matrix is invertible if and only if its body (term of zero order in $\mathscr{G}$) is invertible. We suppose $\mathbf{e}(0)$ invertible, $\mathbf{g}(0)$ is then a usual hyperbolic metric.

We suppose that ψ is a Majorana spinor, i.e. a real spinor (the Γ matrices are pure imaginary).

2. Equations

The *Lagrangian* of $d=11$, $N=1$ extended supergravity is (cf. [6, 7])

$$\mathscr{L}=\int\Big\{e_B{}^N e_A{}^M R_{MN}{}^{AB}+2i\bar{\psi}_M\Gamma^{MNP}\Big[D_N\psi_P-\frac{i}{8}(\bar{\psi}_Q\Gamma_{NAB}{}^{QR}\psi_R)\Gamma^{AB}\psi_P\Big]$$
$$+\frac{1}{12}F_{MNPQ}F^{MNPQ}-\frac{1}{3^4\cdot 2^5}\tau^{M_1M_2\dots M_{11}}F_{M_1\dots M_4}F_{M_5\dots M_8}A_{M_9\dots M_{11}}$$
$$-\frac{1}{3\cdot 4^2}(\bar{\psi}_M\Gamma^{MNWXYZ}\psi_N+12\bar{\psi}^W\Gamma^{XY}\psi^Z)(2F_{WXYZ}-3\bar{\psi}_{[W}\Gamma_{XY}\psi_{Z]})\Big\}\tau\,. \tag{2.1}$$

The equations, $\mathscr{G}$-valued, are obtained by equating to zero the coefficients of the "variations" $\delta\mathbf{e}$, $\delta\boldsymbol{\omega}$, $\delta\boldsymbol{\psi}$, and $\delta\mathbf{A}$ in the "variation" of $\mathscr{L}$ (a dot denotes the $\mathbf{g}$ scalar product):

$$\delta\mathscr{L}\equiv\int\{2\Sigma_M{}^A\delta e_A{}^M+D^{MB}{}_A\delta\omega_M{}^A{}_B+\delta A\cdot\mathscr{F}+4i\delta\bar{\psi}_M\mathscr{R}^M\}\tau$$

obtained by the classical procedures of the linearization of the integrand and integration by part, with zero boundary terms.

1) We first vary $\boldsymbol{\omega}$. We will then, as in usual Einstein-Cartan models with first derivative couplings, obtain the torsion as a function of the other fields. Indeed, one deduces easily from (1.6), setting $\delta\omega_M{}^{AB}=f_M{}^{AB}$, antisymmetric in A and B

$$\delta R_{MN}{}^{AB}=2\nabla_{[M}f_{N]}{}^{AB}+S_M{}^Q{}_N f_Q{}^{AB}.$$

The only terms in $\delta\mathscr{L}$ which contain $\delta\boldsymbol{\omega}$ come from $\delta\mathbf{R}$ and $\delta D_N\psi_P$, and are

$$\int\{e_A{}^M e_B{}^N(2\nabla_{[M}f_{N]}{}^{AB}+S^Q_{MN}f_Q{}^{AB})+\tfrac{1}{2}\bar{\psi}_M\Gamma^{MNP}f_N{}^{AB}\Gamma_A\Gamma_B\psi_P\}\tau\,.$$

Since we have identically $\nabla_M e_A{}^N\equiv 0$ we have

$$2e_A{}^M e_B{}^N\nabla_{[M}f_{N]}{}^{AB}=2\nabla_M f_B{}^{MB}=2\bar{\nabla}_M f_B{}^{MB}+2C_M{}^M{}_N f_B{}^{NB}.$$

The first term, a Riemann divergence, disappears by integration and we are left with

$$\int\Big\{2C_M{}^M{}_N f_B{}^{NB}+S_A{}^Q{}_B f_Q{}^{AB}+\frac{i}{2}(\bar{\psi}_M\Gamma^{MNP}\Gamma_A\Gamma_B)\psi_P f_N{}^{AB}\Big\}\tau\,,$$

that is an integral of the form

$$\int D^Q{}_{AB}f_Q{}^{AB}\tau=0\,,$$

where $D^Q{}_{AB}$ is the tensor, antisymmetric in A and B, which we shall equate to zero:

$$D^Q{}_{AB}\equiv S_A{}^Q{}_B+C_M{}^M{}_A\delta_B{}^Q-C_M{}^M{}_B\delta_A{}^Q+\frac{i}{2}\bar{\psi}_M\Gamma^{MQP}\Gamma_{AB}\psi_P=0\,, \tag{2.2}$$

which implies

$$S_A{}^A{}_B+C_M{}^M{}_B-11C_M{}^M{}_B+\frac{i}{2}\bar{\psi}_M\Gamma^{MAP}\Gamma_{AB}\psi_P=0\,.$$

Thus, since [cf. (1.4)]

$$S_A{}^A{}_B = S_M{}^M{}_B = C_M{}^M{}_B,$$

$$C_M{}^M{}_B = \frac{i}{2\cdot 9}\bar\psi_M \Gamma^{MAP}\Gamma_{AB}\psi_P,$$

$$S_A{}^Q{}_B = -\frac{i}{2}\bar\psi_M \Gamma^{MPQ}\Gamma_{AB}\psi_P + \frac{i}{9}\bar\psi_M \Gamma^{MCP}\Gamma_{C[B}\psi_P \delta_{A]}{}^Q. \tag{2.3}$$

2) Equating to zero the coefficient of $\delta e_A{}^M$ gives:

$$\Sigma_M{}^A \equiv (G_M{}^A - T_M{}^A) = 0, \tag{2.4}$$

where $G_M{}^A \equiv R_M{}^A - \frac{1}{2}e^A{}_M R$ is the (non-symmetric) Einstein-Cartan tensor which comes from the variation of $R \equiv e_A{}^M e_B{}^N R_{MN}{}^{AB}$ and $\mathbf{T} = (T_M{}^A)$ is the stress energy tensor, of the type (recall $\Gamma^M = e_A{}^M \Gamma^A$)

$$\mathbf{T}(\mathbf{e}, \partial\mathbf{e}, \mathbf{F}, \boldsymbol{\psi}, \partial\boldsymbol{\psi}),$$

even polynomial in $\boldsymbol{\psi}$, $\partial\boldsymbol{\psi}$ of order 4 (linear in $\partial\boldsymbol{\psi}$), polynomial in $\mathbf{F}$ of order 2, in $\mathbf{e}$, $\partial\mathbf{e}$ of order 6 (linear in $\partial\mathbf{e}$).

3) By equating to zero the coefficient of $\delta\mathbf{A} = \mathbf{a}$, and since $\delta\mathbf{F} = \mathbf{f} = \delta d\mathbf{A} = d\mathbf{a}$ [$\nabla\cdot$ is the coderivative operator: $(\nabla\cdot F)^{NPQ} = \nabla_M F^{MNPQ}$, $*$ denotes the metric adjunction of forms, k_1, k_2 numbers] we get an equation of the form[1]

$$\mathscr{F} \equiv k_1 \nabla\cdot\mathbf{F} + k_2^*(\mathbf{F}\wedge\mathbf{F}) + \nabla\cdot\boldsymbol{\Phi} = 0, \tag{2.5}$$

with $\boldsymbol{\Phi} \equiv \Phi(\boldsymbol{\psi}, \mathbf{e})$ a polynomial in $\boldsymbol{\psi}$, homogeneous of degree 2, polynomial in $\mathbf{e}$ of degree 6.

4) Equating to zero the coefficient of $\delta\bar\psi_M = \bar f_M$ gives the Rarita-Schwinger equation[2]

$$\mathscr{R}^M \equiv \Gamma^{MNP} D_N \psi_P + r^M = 0, \tag{2.6}$$

where $r^M = r^M(\boldsymbol{\psi}, \mathbf{e}, \mathbf{F})$ is an odd polynomial in $\boldsymbol{\psi}$ of degree 3, depending linearly on $\mathbf{F}$, and on $\mathbf{e}$ by polynomials of degree 6.

3. Identities

When the torsion is given by (2.3) the variation of $\mathscr{L}$ reduces to

$$\delta\mathscr{L} = \int (2\Sigma_M{}^A \delta e_A{}^M + 4i\delta\bar\psi_M \mathscr{R}^M + \delta A\cdot\mathscr{F})\tau. \tag{3.1}$$

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1 One used the identity (k_i some various numbers)

$$\begin{aligned}\int {}^*(F\wedge f)\cdot A\tau &= k_1\int A^*\cdot(F\wedge f)\tau = k_1\int A^*\cdot d(F\wedge a)\tau \\ &= k_2\int(\nabla\cdot A^*)\cdot(F\wedge a)\tau = -k_2\int {}^*F\cdot(F\wedge a)\tau = k_3\int {}^*(F\wedge F)\cdot a\tau\end{aligned}$$

2 One uses the identity

$$\begin{aligned}\bar\psi_M \Gamma^{MNP} D_N f_P &= D_N(\bar\psi_M \Gamma^{MNP} f_P) - D_N(\bar\psi_M \Gamma^{MNP}) f_P \\ &= \bar\psi_N{}^{(}\bar\psi_M \Gamma^{MNP} f_P) + \bar f_M(\Gamma^{MNP} D_N \psi_P + (D_N \Gamma^{MNP})\psi_M)\end{aligned}$$

1) $\mathscr{L}$ is invariant under Lorentz transformations of the moving frame $\mathbf{e}$ and associated transformations of the spin frame. That is, $\delta\mathscr{L}=0$ whenever

$$\delta e^A{}_M = U^{AB} e_{BM}\,, \qquad \delta\bar{\psi}_M = \tfrac{1}{4}\bar{\psi}_M U^{AB}\Gamma_A\Gamma_B\,, \qquad \delta A = 0\,,$$

with $U^{AB}_{(x)}$ a generator of Lorentz transformation in the tangent space T_xV, i.e. an antisymmetric 2-tensor. We therefore have, identically

$$2\Sigma_{[AB]} + i\bar{\psi}_M \Gamma_{AB}\mathscr{R}^M \equiv 0\,. \tag{3.2}$$

2) From the invariance of $\mathscr{L}$ by the gauge transformation[3] $\delta\mathbf{A}=d\boldsymbol{\varphi}$, $\boldsymbol{\varphi}$ arbitrary 2-form, we deduce the identity

$$\nabla\cdot\mathscr{F}\equiv 0\,. \tag{3.3}$$

3) By diffeomorphism (or change of local coordinates) the Lagrangian is also invariant. That is, $\delta\mathscr{L}=0$ whenever[4] (ξ generator of local diffeomorphisms, i.e. arbitrary vector field)

$$\delta e_A{}^M = \xi^L \bar{\nabla}_L e_{(A)}{}^M - e_A{}^L \bar{\nabla}_L \xi^M\,, \qquad \delta\bar{\psi}_M = \xi^L\bar{\nabla}_L\bar{\psi}_M + \bar{\psi}_M\bar{\nabla}_L\xi^M\,,$$

where $\bar{\nabla}_L$ acts only on the natural coordinate index

$$\delta A = \xi\cdot dA + d(\xi\cdot A)\,.$$

If we take into account the previous identities[5], we obtain

$$2\bar{\nabla}_L\Sigma_M{}^L - 4i[(\bar{\nabla}_M\bar{\psi}_L - \bar{\nabla}_L\bar{\psi}_M)\mathscr{R}^L - \bar{\psi}_M\bar{\nabla}_L\mathscr{R}^L] + F\cdot\mathscr{F}\equiv 0\,. \tag{3.4}$$

4) The infinitesimal invariance of $\mathscr{L}$ by the supersymmetry

$$\delta e_A{}^M = i\bar{\varepsilon}\Gamma^M\psi_A\,, \qquad \delta\psi_M = D_M\varepsilon + \varphi_M\varepsilon\,, \qquad \delta A_{MNP} = \tfrac{3}{2}\bar{\varepsilon}\Gamma_{[MN}\psi_{P]}$$

(where φ_M is a given polynomial[6] in $\boldsymbol{\psi}$, even of degree 2, polynomial in e of degree 6 and linear in $\mathbf{F}$), is equivalent to the identity

$$-4iD_M\mathscr{R}^M + 4i\varphi_M\mathscr{R}^M + 2i\Gamma^M\psi_A\Sigma_M{}^A + \tfrac{3}{2}\Gamma_{[MN}\psi_{P]}\mathscr{F}^{MNP} = 0\,. \tag{3.5}$$

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3 F is unchanged by the transformation $A\mapsto A+d\varphi$ and we have

$$\int {}^*(F\wedge F)\cdot(d\varphi)\tau = k_1\int(\nabla\cdot{}^*(F\wedge F))\cdot\varphi\tau = k_2\int({}^*d(F\wedge F))\cdot\varphi\tau = 0$$

4 The parenthesis on A means that this index here is just a label, not to be covariantly derived

5 We use the fact that

$$\nabla_M e_{(A)}{}^L = \omega_M{}^B{}_A e_B{}^L\,.$$

Thus

$$\Sigma_L{}^A\nabla_M e_{(A)}{}^L = \Sigma_{BA}\omega_M{}^{BA} = \Sigma_{[BA]}\omega_M{}^{BA}\,.$$

Also

$$\int d(\xi\cdot A)\cdot\mathscr{F}\tau = k\int(\xi\cdot A)\cdot(\nabla\cdot\mathscr{F})\tau\,,$$

6 $$\varphi_M = \frac{i}{32}(\bar{\psi}_N\Gamma_{MAB}{}^{NP}\psi_P)\Gamma^{AB} + \frac{i}{(12)^2}(\Gamma^{NPQ}{}_{RM} - \Gamma^{PQR}\delta^N_M)(F_{NPQR} - 3\bar{\psi}_N\Gamma_{PQ}\psi_R)$$

4. Gauges

1) The Rarita-Schwinger gauge, introduced by these authors in 4-dimensional Minkowski space time to separate pure spin states is the condition

$$\chi \equiv \Gamma^M \psi_M = 0 . \tag{4.1}$$

In such a gauge the Rarita-Schwinger equation can be shown to be equivalent to an equation with the hyperbolic principal part $\not{D}\psi$. Indeed, we have the identity:

$$\mathscr{A}_M \equiv \not{D}\psi_M - \tfrac{1}{2}\Gamma_M[\Gamma^N \not{D}\psi_N + (\not{D}\Gamma^P)\psi_P - \not{D}\chi] + (D_M\Gamma^P)\psi_P - {}_M\chi , \tag{4.2}$$

where

$$\not{D} \equiv \Gamma^M D_M , \qquad \mathscr{A}_M \equiv \Gamma^{MNP} D_N \psi_P ,$$

from which we deduce [7]

$$\mathscr{R}_M \equiv \mathscr{A}_M + r_M \equiv (\not{D}\psi_M + \varrho_M) - \tfrac{1}{2}\,\Gamma_M\,\Gamma^N(\not{D}\psi_N + \varrho_N) - D_M\chi + \tfrac{1}{2}\Gamma_M \not{D}\chi , \tag{4.3}$$

if

$$\varrho_M = r_M + \frac{\Gamma^N}{2-d}\Gamma_M r_N + (D_M\Gamma^P)\psi_P .$$

2) The harmonic gauge (γ^P_{MN} Riemannian connection in the natural frame)

$$\Phi^P \equiv \gamma^P_{MN} g^{MN} = 0 \tag{4.4}$$

is well known to turn Einstein equations into a hyperbolic system due to the identity, for the Ricci tensor of a Riemannian connection

$$\begin{aligned} \bar{R}_{MN} &\equiv \bar{R}^{(h)}_{MN} + \tfrac{1}{2}(g_{MP}\partial_N \Phi^P + g_{NP}\partial_M \Phi^P) , \\ \bar{R}^{(h)}_{MN} &= -\tfrac{1}{2}\,\Box\, g_{MN} + r_{MN}(g, \partial g) , \qquad \Box = g^{LP}\partial^2_{LP} . \end{aligned} \tag{4.5}$$

3) The $O(1,10)$ gauge introduced in simple supergravity by Bao et al. [1] limits the moving frame by the condition

$$e_{A[M}\,\Box\, e^A{}_{N]} = 0 . \tag{4.6}$$

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7 The following identity holds irrespective of the dimension ($\Gamma_M\Gamma_N + \Gamma_N\Gamma_M = 2g_{MN}$)

$$\Gamma_{MNP} = \Gamma_M\Gamma_{NP} - g_{MN}\Gamma_P + g_{MP}\Gamma_N .$$

From it we deduce

$$\Gamma_M \mathscr{A}^M \equiv \Gamma_M \Gamma^{MNP} D_N \psi_P \equiv (d-2)\Gamma^{NP} D_N \psi_P$$

and

$$\Gamma^M \Gamma_L \mathscr{A}^L \equiv (d-2)(\Gamma^{MNP} D_N \psi_P + \Gamma^P D^M \psi_P - \not{D}\psi^M) .$$

Thus

$$\Gamma_L \mathscr{A}^L \equiv \frac{(d-2)}{2}(\not{D}\chi - (\not{D}\Gamma^P)\psi_P - \Gamma^P \not{D}\psi_P)$$

and

$$\mathscr{A}^M = \frac{\Gamma^M}{2}(\not{D}\chi - (\not{D}\Gamma^P)\psi_P - \Gamma^P \not{D}\psi_P) - D^M\chi + (D^M\Gamma^P)\psi_P + \not{D}\psi^M$$

In such a frame we have

$$\Box\, g_{MN} \equiv \Box\,(e_{AM}e^A{}_N) = 2e_{AM}\,\Box\, e^A{}_N + 2g^{PQ}\partial_P e_{AM}\partial_Q e^A{}_N\,. \tag{4.7}$$

4) On the "3 index photon" we shall impose the gauge condition, analogous to the Lorentz one[8]

$$\partial\cdot A \equiv (\partial_M A^{MNP}) = 0\,. \tag{4.8}$$

In this "Lorentz gauge" the principal part of $\mathscr{F}$, $\nabla\cdot\mathbf{F}$ reduces to $\Box\,\mathbf{A}$. Under all these gauge conditions the system takes the form (where t stands for "truncated")

$$^t\Sigma_N{}^A - \tfrac{1}{2}e^A{}_N\,{}^t\Sigma \equiv -\Box\, e^A{}_N + f^A{}_N(\mathbf{e},\partial\mathbf{e},\psi,\partial\psi,\mathbf{F}) = 0\,, \tag{4.9}$$

$$^t\mathscr{R}_M - \frac{\Gamma_M}{d-2}\Gamma_N\,{}^t\mathscr{R} \equiv \not{D}\psi_M + \varrho_M \equiv \Gamma^N\partial_N\psi_M + f_M(\mathbf{e},\partial\mathbf{e},\psi,\mathbf{F}) = 0\,, \tag{4.10}$$

$$^t\mathscr{F} \equiv \Box\,\mathbf{A} + f(\mathbf{e},\partial\mathbf{e},\mathbf{A},\mathbf{F},\psi,\partial\psi) = 0\,. \tag{4.11}$$

If $\mathbf{e}$, ψ, $\mathbf{A}$ were numerical valued this system would be a non-strict hyperbolic[9] system in the sense of Leray-Ohya (cf. criterion in [4]) with causal propagation governed by the light cone of the metric $\mathbf{g}$.

If the truncated equations (4.9)–(4.11) hold, the identities of the previous paragraph give the following equations for the left-hand side of the gauge conditions:

1) We have

$$^t\Sigma_{MN} \equiv \Sigma_{MN} - \tfrac{1}{2}(g_{MP}\partial_N\Phi^P + g_{NP}\partial_M\Phi^P - g_{MN}\partial_P\Phi^P) - e_{A[M}\,\Box\, e^A{}_{N]}\,. \tag{4.12}$$

In particular:

$$\Sigma_{[MN]} \equiv R_{[MN]} - T_{[MN]} = {}^t\Sigma_{[MN]} - \tfrac{1}{2}e_{A[M}\,\Box\, e^A{}_{N]}\,.$$

Therefore, when (4.9) and (4.10) are satisfied, we deduce from (3.2) the equation

$$-\tfrac{1}{2}e_{A[M}\,\Box\, e^A{}_{N]} - i\bar\psi_P\Gamma_{MN}(-\tfrac{1}{2}D^P\chi + \tfrac{1}{2}\Gamma^P\not{D}\chi) = 0\,. \tag{4.13}$$

2) When (4.11) is satisfied we have

$$k_1^{-1}\mathscr{F} = d(\partial\cdot\mathbf{A})\,.$$

The identity $\nabla\cdot\mathscr{F} = 0$ gives therefore an equation linear and homogeneous in $\partial\cdot\mathbf{A}$, of the type

$$\Box\,(\partial\cdot\mathbf{A}) + \mathbf{h}\cdot\partial(\partial\cdot\mathbf{A}) = 0\,, \tag{4.14}$$

where $\mathbf{h}$ is a given function of $\mathbf{e}$ and $\partial\mathbf{e}$.

3) When (4.9) is satisfied we have

$$\mathscr{R}_M \equiv -D_M\chi + \tfrac{1}{2}\Gamma_M\not{D}\chi\,.$$

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8 We choose $\partial\cdot A$ instead of $\nabla\cdot A$ in order not to introduce unnecessary non-diagonal terms in the coupled system

9 The characteristic matrix is non-diagonal due to the appearance of terms in $\partial\mathbf{e}$ in the Rarita-Schwinger equation. For such a system the existence theorem is valid only in a Gevrey class (C^∞ functions with restrictions on growth of derivatives), but the domain of dependence properties, which give the causality, hold

Then when (4.9) and (4.11) are satisfied the identity (3.5) gives a linear homogeneous equation of the type

$$-\tfrac{1}{2}D^M D_M \chi + \mathrm{lin}(\chi, \partial\chi, \Phi, \partial\Phi, d(\partial\cdot\mathbf{A})) = 0, \tag{4.15}$$

where lin(...) denotes an expression linear in its arguments, with coefficients which are functions of $\mathbf{e}$, ∂e, $\partial^2\mathbf{e}$, $\boldsymbol{\psi}$, $\partial\boldsymbol{\psi}$.

4) Using (4.12), (4.13), and also (4.14) we deduce from the Bianchi identity that the solutions of Eqs. (4.8)–(4.10) satisfy also a linear homogeneous equation of the form

$$\Box\,\mathbf{\Phi} + \mathrm{lin}(\mathbf{\Phi}, \partial\mathbf{\Phi}, \partial\cdot\mathbf{A}, \partial(\partial\cdot\mathbf{A})) = 0. \tag{4.16}$$

If the unknown were scalar valued the system (4.14)–(4.16) would be a linear homogeneous hyperbolic system for the gauge conditions.

5. Cauchy Problem. Constraints

The Cauchy data, on the submanifold $S_0 = S \times \{0\}$ are:

The moving frame $\mathbf{e}$ and its time derivative $\partial_0\mathbf{e}$.

The spin 3/2 field $\boldsymbol{\psi}$.

The 3-form $\mathbf{A}$ and its time derivative $\partial_0\mathbf{A}$.

These quantities must satisfy on S_0 the following equation which depends only on them

$$\mathscr{R}^O \equiv \Gamma^{ONP}\partial_N \psi_P + f^O(\mathbf{e}, \partial\mathbf{e}, \boldsymbol{\psi}, \partial\mathbf{A}) = 0. \tag{5.1}$$

We suppose also that $\boldsymbol{\psi}$ satisfies the Rarita-Schwinger initial gauge condition

$$\chi \equiv \Gamma^M \psi_M = 0 \quad \text{on } S_0. \tag{5.2}$$

We determine $\partial_0\boldsymbol{\psi}$ on S_0 by the equation

$$\not{D}\boldsymbol{\psi} + \mathbf{f} = 0 \quad \text{on } S_0, \tag{5.3}$$

and we suppose that we have then

$$\Sigma^{MO} = 0 \quad \text{on } S_0, \tag{5.4}$$

$$\mathscr{F}^{ONP} = 0 \quad \text{on } S_0 \tag{5.5}$$

(these quantities depend only on $\mathbf{e}$, $\boldsymbol{\psi}$, $\mathbf{A}$ and their first derivatives, now known on S_0).

We deduce from (5.1), (5.4), and (5.3) [cf. identity (4.3)]

$$\partial_0\chi = 0 \quad \text{on } S_0. \tag{5.6}$$

We suppose that the initial data for $\mathbf{e}$ and $\partial_0\mathbf{e}$ are such that the corresponding metric satisfies the harmonicity conditions

$$\mathbf{\Phi} = 0 \quad \text{on } S_0, \tag{5.7}$$

and that the initial data for $\mathbf{A}$, $\partial_0\mathbf{A}$ satisfy

$$\partial\cdot\mathbf{A} = 0 \quad \text{on } S_0. \tag{5.8}$$

For a solution of the truncated equations (4.9) and (4.11) with initial data satisfying the constraints (5.4) and (5.5) we then have also

$$\partial_0 \mathbf{\Phi} = 0, \qquad \partial_0(\partial \cdot \mathbf{A}) = 0 \quad \text{on } S_0. \tag{5.9}$$

The homogeneous system (4.14)–(4.16) and the vanishing on S_0 of $\chi, \partial \cdot \mathbf{A}, \mathbf{\Phi}$ and their first derivatives would insure by known theorems the vanishing of these quantities in all the domain of dependence of the solution, if the unknown were numerical values.

6. Grassmann-valued Cauchy Problem

We have supposed, and it is necessary for the identity (3.5) to hold, that the fields take their values in the Grassmann algebra $\mathscr{G}$, with $\mathbf{e}$ and $\mathbf{A}$ even valued, $\boldsymbol{\psi}$ odd valued. The numerical equations satisfied are obtained by equating to zero each component in $\mathscr{G}$ of the $\mathscr{G}$ valued equations. The identities obtained in Sect. 3 hold in $\mathscr{G}$, therefore, they give a set of numerical valued identities. The Cauchy data on S_0 are even valued for $\mathbf{e}, \partial_0\mathbf{e}, \mathbf{A}, \partial_0\mathbf{A}$, odd valued for $\boldsymbol{\psi}$, and supposed to satisfy the $\mathscr{G}$ valued constraints (5.1)–(5.7). Equations (5.8) and (5.9), with values in $\mathscr{G}$ are then satisfied.

Theorem. *The Cauchy problem for the equations of $d=11$ extended supergravity, with values in a Grassmann algebra $\mathscr{G}$ is well posed and causal. If $\mathscr{G}$ has N generators and the Cauchy data (satisfying the constraints and the initial gauge conditions) are such that*[10], *on S_0 (with s a non-negative integer and $0 \leqq p \leqq [(N+1)/2]$),*

$$\mathbf{A}(2p), \mathbf{e}(2p), \boldsymbol{\psi}(2p-1) \in H^{\mathrm{loc}}_{7+[N/2]-p+s},$$

$$\partial_0\mathbf{A}(2p), \partial_0\mathbf{e}(2p) \in H^{\mathrm{loc}}_{6+[N/2]-p+s},$$

the solution exists in a neighborhood Ω of S_0, globally hyperbolic for $\mathbf{g}(0)$, *and belongs to H^{loc}_{6+s}.*

If $\mathscr{G}$ has an infinite number of generators the Cauchy data must belong to C^∞, and the solution exists as a formal series of C^∞ functions.

Proof.[11] 1) The "body", terms of order zero in $\mathscr{G}$, of the original equations (2.4)–(2.6) reduces to the Einstein equation for the body $\mathbf{g}(0)$ of the metric $\mathbf{g}$ with source the Maxwell field $\mathbf{F}(0) = d\mathbf{A}(0)$ of the body $\mathbf{A}(0)$ of the 3-index photon, and the Maxwell equation for $\mathbf{F}(0)$, namely[12]

$$\Sigma_{MN}(0) \equiv G_{MN}(0) - \tfrac{1}{3}(F_{MPQR}(0)F^{NPQR}(0) - \tfrac{1}{8}g_{MN}(0)F_{PQRS}(0)F^{PQRS}(0)), \tag{6.1a}$$

$$\mathscr{F}^{NPQ}(0) \equiv \nabla_M(0)F^{MNPQ}(0) + 18c\tau^{M_1 \dots M_8 NPQ}_{(0)} F^{(0)}_{M_1 \dots M_4} F^{(0)}_{M_5 \dots M_8}. \tag{6.1b}$$

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10 A function S_0 [respectively Ω] belongs to H^{loc}_s if its restriction to any compact set of S_0[respectively Ω] belongs to H_s.

$\mathbf{A}(2p)$, $\mathbf{e}(2p)$, $\boldsymbol{\psi}(2p-1)$ belong on each S_t to the same Sobolev space as on S_0, and the dependence on t is as given in [16], or [5]. Thus if $N=2n$ we have $\mathbf{A}(2p), \mathbf{e}(2p), \boldsymbol{\psi}(2p-1) \in H^{\mathrm{loc}}_{7+s}$, $p \leqq n$, and if $N=2n+1$ we have again $\mathbf{A}(2p), \mathbf{e}(2p) \in H^{\mathrm{loc}}_{7+s}$, $p \leqq n$ while $\boldsymbol{\psi}(N) \in H^{\mathrm{loc}}_{6+s}$

11 A field $f(p)$ is a set of $N!/p!$ numerical fields. In a numerical equation of finite order only a finite number of the numerical fields of order $\leqq p$ appears, only one of order p if $p=0$. We say that $f(p) \in H_s$ if each $f_{I_1 \dots I_p} \in H_s$

12 Various exact solutions of these equations, candidates for a "ground state" of the theory have been obtained (cf. [12, 13, 15] and references in [18])

The Cauchy problem for these equations, with data $\mathbf{e}(0)$, $\partial_0\mathbf{e}(0)$, $\mathbf{A}(0)$, $\partial_0\mathbf{A}(0)$ satisfying the constraints (verified as a consequence of the $\mathscr{G}$-valued constraints)

$$\Sigma^{MO}(0)=0\,,\qquad \mathscr{F}^{OPQ}(0)=0\quad \text{on } S_0 \tag{6.2}$$

is well posed and causal, with propagation determined by $\mathbf{g}(0)$, because[13] when $\Phi(0)=0$ and $\partial\cdot\mathbf{A}(0)=0$, they reduce to a strictly hyperbolic system (of second order on an 11 dimensional manifold), and because the gauge conditions are preserved by evolution for initial data satisfying the constraints, due to the identities, deduced from (3.3), (3.4)

$$2\bar{V}_L(0)\Sigma_M{}^L(0)+\mathbf{F}(0)\cdot\mathscr{F}(0)=0\,,\qquad \bar{V}(0)\cdot\mathscr{F}(0)=0\,,$$

which give, when Eqs. (6.1) are satisfied in the gauges, an ordinary linear homogeneous hyperbolic system for $\mathbf{\Phi}(0)$ and $(\partial\cdot\mathbf{A})(0)$, body of the system (4.15), (4.16), while the body of Eqs. (5.9) shows that the Cauchy data on S_0 for these quantities vanish. These results hold (cf. [16, 5]), for Cauchy data $\mathbf{e}(0)$, $\mathbf{A}(0)$ in H_{7+a} and $\partial\mathbf{e}(0)$, $\partial\mathbf{A}(0)$ in H_{6+a} with a non-negative integer. The solution is in H_{7+a}, where Ω is some neighborhood of S_0 in $S\times\mathbb{R}$, globally hyperbolic for $\mathbf{g}(0)$. The Sobolev spaces can be taken to be local Sobolev spaces.

2) The terms of order 1 in the system are only $\mathscr{R}(1)$, and $\mathscr{R}(1)=0$ is a linear Rarita-Schwinger type system [14] for $\boldsymbol{\psi}(1)$ when $\mathbf{e}(0)$ and $\mathbf{A}(0)$ are known in $H^{\text{loc}}_{7+a}(\Omega)$,

$$\mathscr{R}^M(1)\equiv\Gamma^{MNP}(0)V_N(0)\psi_P(1)+r^M(1)=0\,.$$

The Cauchy problem for this system is well posed and causal, with propagation determined by $\mathbf{g}(0)$ because the system

$$\not{V}(0)\psi_M(1)+\varrho_M(1)=0$$

[where $\varrho_M(1)$ is of the form $a_M{}^L(0)\psi_L(1)$, $a_M{}^L(0)$ function of $e(0)$, $\partial e(0)$, $F(0)$] is strictly hyperbolic, and the gauge condition $\chi(1)\equiv\Gamma^L(0)\psi_L(1)=0$ is preserved through evolution[15] due to the identity (4.14) and Eqs. (5.4), (5.18) taken at the order 1 in $\mathscr{G}$. These results hold for Cauchy data $\boldsymbol{\psi}(1)$ on S_0 in[16] $H_{6+a}(S_0)$, the solution exists in all the globally hyperbolic domain Ω, and is in $H_{6+a}(\Omega)$.

3) Suppose we have solved up to order $2n$ Eq. (2.4)–(2.6), as well as the gauge conditions (4.1), (4.4), (4.6), (4.8).

The equations of order $2n$ in $\mathscr{G}$ are

$$\Sigma_{MN}(2n)\equiv G_{MN}(2n)-T_{MN}(2n)=0\,, \tag{6.3a}$$

$$\mathscr{F}(2n)\equiv(V\cdot F)(2n)-k*(F\wedge F)(2n)+(V\cdot\Psi)(2n)=0\,. \tag{6.3b}$$

If the gauge conditions

$$\Phi(2n)=0\,,\qquad (e_{A[N}\,\square\, e^A{}_{M]})(2n)=0\,,\qquad (\partial\cdot A)(2n)=0\,, \tag{6.4}$$

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13 These results are well known for ordinary Einstein Maxwell equations

14 $\psi(1)$ is a set of N numerical fields ψ_I

15 The integrability condition $\bar{V}_M(0)\mathscr{R}^M(1)=0$ is satisfied mod $\Sigma_{MN}(0)=0$, $\mathscr{F}(0)=0$

16 We lose one derivative here because $a_M{}^L(0)$ is only in $H_{6+a}(\Omega)$ when $\mathbf{e}(0)$, $\mathbf{A}(0)$ are in $H_{7+a}(\Omega)$

are satisfied, these equations reduce to a system of the type:

$$\square(0)e^A{}_N(2n)+\varphi^A{}_N=0, \tag{6.5a}$$

$$\square(0)A(2n)+\varphi=0, \tag{6.5b}$$

where $\varphi^A{}_N$ and φ are affine functions in $e(2n)$, $\partial e(2n)$, $A(2n)$, $\partial A(2n)$ whose coefficients are known functions of $e(2p)$, $A(2p)$, $0\leqq p\leqq n$, and their derivatives of order $\leqq 2$, and of $\psi(2p+1)$, $0\leqq p\leqq n$, and its first derivatives.

The linear hyperbolic system (6.5a), (6.5b) has a global solution in Ω, $A(2n)$, $e(2n)\in H_{7+a_n}$, for Cauchy data in $H_{7+a_n}\times H_{6+a_n}$, if $e(2p)$, $A(2p)$, H_{8+a_n}, $\psi(2p+1)\in H_{7+a_n}$, $0\leqq p\leqq n$, with a_n a non-negative integer. This solution satisfies the gauge conditions (6.4) by the identity of Sect. 3 and the constraints and initial gauge conditions of Sects. 4 and 5 written at that order and the properties supposed satisfied at the lower orders.

The equations of order $2n+1$ reduce to the linear system in $\psi(2n+1)$

$$\mathscr{R}^M(2n+1)=0, \tag{6.6}$$

while Eq. (4.10) gives, at this order the linear hyperbolic system

$$\nabla\!\!\!/(0)\psi(2n+1)+f=0, \tag{6.7}$$

where f is an affine function of $e(2p)$, $A(2p)$, $\psi(2p-1)$, $0\leqq p\leqq n$, and their first derivatives. A solution of the Cauchy problem for (6.7) satisfies (6.6) [by the identity (3.5) written at the order $2n+1$] if the Cauchy data satisfy the constraints, and the initial gauge conditions. These results hold for Cauchy data $\psi(2n+1)$ on S_0 in H_{6+a_n}, if $e(2p)$, $A(2p)$, $\psi(2p-1)\in H_{7+a_n}$ for $0\leqq p\leqq n$; the solution is in H_{6+a_n} on Ω.

The conclusion follows by induction on n, choosing $a_n=s+[N/2]-n$.

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