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Indecomposable Representations with Invariant Inner Product

A Theory of the Gupta-Bleuler Triplet

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Abstract. Consequences of the existence of an invariant (necessarily indefinite) non-degenerate inner product for an indecomposable representation π of a group G on a space $\mathfrak{H}$ are studied. If π has an irreducible subrepresentation π_1 on a subspace $\mathfrak{H}_1$, it is shown that there exists an invariant subspace $\mathfrak{H}_2$ of $\mathfrak{H}$ containing $\mathfrak{H}_1$ and satisfying the following conditions: (1) the representation $\pi_1^{\#}=\pi \bmod \mathfrak{H}_2$ on $\mathfrak{H} \bmod \mathfrak{H}_2$ is conjugate to the representation $(\pi_1, \mathfrak{H}_1)$, (2) $\mathfrak{H}_1$ is a null space for the inner product, and (3) the induced inner product on $\mathfrak{H}_2 \bmod \mathfrak{H}_1$ is non-degenerate and invariant for the representation

$$\pi_2=(\pi|_{\mathfrak{H}_2}) \bmod \mathfrak{H}_1\,,$$

a special example being the Gupta-Bleuler triplet for the one-particle space of the free classical electromagnetic field with $\mathfrak{H}_1=$ space of longitudinal photons and $\mathfrak{H}_2=$ the space defined by the subsidiary condition.

1. Introduction

In the study of massless particles, one meets [1–6, 9–11, 15, 17–19] indecomposable representations π of a Lie group G on a space $\mathfrak{H}$ (with an invariant indefinite inner product), which take, for example, the following form:

$$\left.\begin{array}{c}\pi_n \to \pi_{n-1} \to \dots \to \pi_1 \\ \mathfrak{H}=\mathfrak{H}_n \supset \mathfrak{H}_{n-1} \supset \dots \supset \mathfrak{H}_1\end{array}\right\}. \tag{1.1}$$

Here $\mathfrak{H}_j$ is a $\pi(G)$-invariant subspace of $\mathfrak{H}_{j+1}$ without any $\pi(G)$-invariant complement, namely, there are no subspaces $\mathfrak{H}_j'$ such that $\mathfrak{H}_j+\mathfrak{H}_j'=\mathfrak{H}_{j+1}$, $\mathfrak{H}_j \cap \mathfrak{H}_j'=\mathbf{0}$, $\pi(G)\mathfrak{H}_j' \subset \mathfrak{H}_j'$. The representation π_j of G on $\mathfrak{H}_j/\mathfrak{H}_{j-1}$ is obtained by first restricting π to the subspace $\mathfrak{H}_j$ of $\mathfrak{H}$ and then considering it modulo $\mathfrak{H}_{j-1}: \pi_j(g)[\xi]=[\pi(g)\xi]$ for $\xi \in \mathfrak{H}_j$, where $[\xi]=\xi+\mathfrak{H}_{j-1}$ is a vector in $\mathfrak{H}_j/\mathfrak{H}_{j-1}$. (We note that the construction of representations on a quotient space, especially with respect to a null space, is now a popular game [14].)

It has been noticed in examples [1–3, 5, 6, 15] that $\pi_1 = \pi_n$, and it has been suggested that this is a general phenomenon [7]. The purpose of this note is to formulate this into a mathematical theorem.

A representation $\pi_1^{\#}$ of G on $\mathfrak{H}_1^{\#}$ will be called conjugate to a representation π_1 of G on $\mathfrak{H}_1$ if there is a sesquilinear form (ξ, n), $\xi \in \mathfrak{H}_1^{\#}$, $\eta \in \mathfrak{H}_1$ such that $\mathfrak{H}_1$ and $\mathfrak{H}_1^{\#}$ separate each other [i.e. $(\xi, \eta) = 0$ for all $\eta \in \mathfrak{H}_1$ implies $\xi = \mathbf{0}$ and $(\xi, \eta) = 0$ for all $\xi \in \mathfrak{H}_1^{\#}$ implies $\eta = \mathbf{0}$] and the following equality holds for all $\xi \in \mathfrak{H}_1^{\#}$, $\eta \in \mathfrak{H}_1$, and $g \in G$:

$$(\pi_1(g^{-1})\xi, \eta) = (\xi, \pi_1(g)\eta). \tag{1.2}$$

Our conclusion: if π_1 is an irreducible subrepresentation of an indecomposable representation π of G on a space $\mathfrak{H}$ with a G-invariant non-degenerate (indefinite) inner product, then π is of the form (1.1) $(n \geqq 2)$ with π_n on $\mathfrak{H}_n/\mathfrak{H}_{n-1}$ dual to π_1 on $\mathfrak{H}_1$ (Theorem 1). (In examples quoted above, π_1 is selfdual.)

If $\mathfrak{H}_j$ has a complement $\mathfrak{K}_{j+1}$ in $\mathfrak{H}_{j+1}$ (i.e. $\mathfrak{H}_j + \mathfrak{K}_{j+1} = \mathfrak{H}_{j+1}$, $\mathfrak{H}_j \cap \mathfrak{K}_j = \mathbf{0}$) for $j = 1, \ldots, n-1$, then $\mathfrak{H} = \mathfrak{K}_1 + \mathfrak{K}_2 + \ldots + \mathfrak{K}_n$ with $\mathfrak{K}_1 \equiv \mathfrak{H}_1$ and $\pi(g)$ can be written in the following matrix form:

$$\pi(g) = \begin{pmatrix} \pi_1(g) & c_{12}(g) & \ldots & c_{1n}(g) \\ 0 & \pi_2(g) & \ldots & c_{2n}(g) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & \pi_n(g) \end{pmatrix}. \tag{1.3}$$

The off-diagonal elements $c_{ij}(g)$ $(i<j)$ can be studied in terms of cohomologies [8, 12, 13, 16]. For $n=2$ and 3, we give a necessary and sufficient condition on representations π_j for (1.1) to have G-invariant non-degenerate inner product in terms of cohomologies related to π_j (Theorem 2).

All mathematical arguments are algebraic, topological complications being avoided in the following way: (1) We call a subspace $\mathfrak{E}$ of $\mathfrak{H}$ closed if $\mathfrak{E} = (\mathfrak{E}^{\perp})^{\perp}$, where $\perp$ indicates the polar of $\mathfrak{E}$ in $\mathfrak{H}$ with respect to the (G-invariant) non-degenerate inner product in $\mathfrak{H}$:

$$\mathfrak{E}^{\perp} = \{\xi \in \mathfrak{H}: (\xi, \eta) = 0 \text{ for all } \eta \in \mathfrak{E}\}. \tag{1.4}$$

Irreducibility and indecomposability are defined in terms of G-invariant closed subspaces. Then Theorem 1 holds without any further assumption. (2) For discussing $c_{ij}(g)$, we introduce the assumption that subspaces $\mathfrak{H}_j$ have closed complements in $\mathfrak{H}_{j+1}$ (used in their definition). With this assumption, Theorem 2 holds.

2. Invariant Hermitian Form

Let π be a representation of a group G on a complex vector space $\mathfrak{H}$. Let (ξ, η), $\xi \in \mathfrak{H}$, $\eta \in \mathfrak{H}$ be a hermitian form on $\mathfrak{H}$, linear in ξ and conjugate linear in η $[\overline{(\xi, \eta)} = (\eta, \xi)]$, $\pi(G)$-invariant [i.e. $(\pi(g)\xi, \pi(g)\eta) = (\xi, \eta)$ for all $g \in G$ and all $\xi \in \mathfrak{H}$, $\eta \in \mathfrak{H}$] and non-degenerate [i.e. $(\xi, \eta) = 0$ for all $\eta \in \mathfrak{H}$ implies $\xi = \mathbf{0}$].

We recall (cf. Sect. 1) that a subspace $\mathfrak{E}$ is defined to be closed if $\mathfrak{E} = (\mathfrak{E}^{\perp})^{\perp}$.

Lemma 2.1. *$\mathfrak{E}$ is a closed subspace if and only if there is a subset Δ of $\mathfrak{H}$ such that $\mathfrak{E}=\Delta^{\perp}$.*

Proof. If $\mathfrak{E}$ is a closed subspace, we may choose $\Delta=\mathfrak{E}^{\perp}$. To prove the converse, we note (i) $\mathfrak{K}_1\subset\mathfrak{K}_2$, then $\mathfrak{K}_1^{\perp}\supset\mathfrak{K}_2^{\perp}$; (ii) from definition, $\mathfrak{K}\subset(\mathfrak{K}^{\perp})^{\perp}$. By taking $\mathfrak{K}=\mathfrak{E}$, we have $\mathfrak{E}\subset(\mathfrak{E}^{\perp})^{\perp}$ by (ii). By taking $\mathfrak{K}=\Delta$, we obtain $\Delta\subset(\Delta^{\perp})^{\perp}$ by (ii) and hence $\Delta^{\perp}\supset((\Delta^{\perp})^{\perp})^{\perp}$ by (i). Therefore, $\mathfrak{E}=(\mathfrak{E}^{\perp})^{\perp}$ *for* $\mathfrak{E}=\Delta^{\perp}$. Q.E.D.

We note that any finite dimensional subspace is closed by the non-degeneracy of the inner product.

Remark 2.2. The collection of seminorms $p_{\xi}(\eta)=|(\xi,\eta)|$ on $\eta\in\mathfrak{H}$ with ξ varying over $\mathfrak{H}$ defines the weak topology $\sigma(\mathfrak{H},\mathfrak{H})$ on $\mathfrak{H}$. A subspace $\mathfrak{E}$ of $\mathfrak{H}$ is closed in the above sense [i.e. $\mathfrak{E}=(\mathfrak{E}^{\perp})^{\perp}$] if and only if $\mathfrak{E}$ is $\sigma(\mathfrak{H},\mathfrak{H})$-closed due to the bipolar theorem.

We introduce the following definitions:

Definition 2.3. A representation π of G on $\mathfrak{H}$ is irreducible if there is no closed subspace $\mathfrak{E}$ of $\mathfrak{H}$ such that $\mathfrak{E}\neq\mathfrak{H}$, $\mathfrak{E}\neq\mathbf{0}$ and $\pi(g)\mathfrak{E}\subset\mathfrak{E}$ (i.e. $\mathfrak{E}$ is G-invariant) for all $g\in G$. A representation π of G on $\mathfrak{H}$ is topologically indecomposable if there are no non-zero closed G-invariant subspaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ of $\mathfrak{H}$ such that $((\mathfrak{H}_1+\mathfrak{H}_2)^{\perp})^{\perp}=\mathfrak{H}$ and $\mathfrak{H}_1\cap\mathfrak{H}_2=\mathbf{0}$ (i.e. $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are topologically complementary).

In terms of the notion of conjugate representations introduced in Sect. 1, we can now present one of our main results:

Theorem 1. *Let π be a representation of a group G on a complex vector space $\mathfrak{H}$ with a G-invariant non-degenerate hermitian form. Let $\mathfrak{H}_1$ be a G-invariant closed subspace of $\mathfrak{H}$ such that the restriction π_1 of π to $\mathfrak{H}_1$ is irreducible and there are no G-invariant, closed topological complements of $\mathfrak{H}_1$ in $\mathfrak{H}$. Then* (1) *$\mathfrak{H}_1$ is a null space,* (2) *π is of the form* (1.1) *with $n=2$ or 3 such that π_n on $\mathfrak{H}_n/\mathfrak{H}_{n-1}$ is conjugate to π_1 on $\mathfrak{H}_1$, and* (3) *$\mathfrak{H}_2/\mathfrak{H}_1$ has a $\pi_2(G)$-invariant non-degenerate hermitian form.*

Proof. Consider $\mathfrak{K}=\mathfrak{H}_1\cap\mathfrak{H}_1^{\perp}$. Due to G-invariance of the hermitian form, the G-invariance of $\mathfrak{H}_1$ implies that of $\mathfrak{H}_1^{\perp}$ and hence that of $\mathfrak{K}$. As an intersection of closed subspaces, $\mathfrak{K}$ is closed. More explicitly, $(\mathfrak{H}_1+\mathfrak{H}_1^{\perp})^{\perp}=\mathfrak{H}_1^{\perp}\cap(\mathfrak{H}_1^{\perp})^{\perp}=\mathfrak{H}_1^{\perp}\cap\mathfrak{H}_1=\mathfrak{K}$. By the irreducibility assumption for π_1, the G-invariant closed subspace $\mathfrak{K}$ of $\mathfrak{H}_1$ must be either $\mathfrak{H}_1$ or $\mathbf{0}$.

If $\mathfrak{K}=\mathbf{0}$, then $(\mathfrak{H}_1+\mathfrak{H}_1^{\perp})^{\perp}=\mathfrak{K}=\mathbf{0}$ while $\mathfrak{H}_1\cap\mathfrak{H}_1^{\perp}=\mathbf{0}$. Hence $\mathfrak{H}_1^{\perp}$ is a G-invariant, closed, topological complement of $\mathfrak{H}_1$. This contradicts the assumption and $\mathfrak{K}=\mathbf{0}$ is excluded. [The assumption excludes the case of the zero complement (i.e. $\mathfrak{H}_1^{\perp}=\mathbf{0}$, $\mathfrak{H}_1=\mathfrak{H}$, π irreducible) as well.]

We now have $\mathfrak{H}_1\cap\mathfrak{H}_1^{\perp}=\mathfrak{K}=\mathfrak{H}_1$ and hence $\mathfrak{H}_1\subset\mathfrak{H}_1^{\perp}$. Therefore, $\mathfrak{H}_1$ is a null space [i.e. $\xi\in\mathfrak{H}$ and $\eta\in\mathfrak{H}$ imply $(\xi,\eta)=0$].

Let $\mathfrak{H}_2=\mathfrak{H}_1^{\perp}$ and $\mathfrak{H}_1^{\#}\equiv\mathfrak{H}/\mathfrak{H}_2$. For $\xi\in\mathfrak{H}$, denote the class $\xi+\mathfrak{H}_2$ of ξ in $\mathfrak{H}/\mathfrak{H}_2$ by $\xi^{\#}$. Denote $\pi^{\#}(g)\xi^{\#}\equiv(\pi(g)\xi)^{\#}$ which does not depend on the choice of ξ in the equivalence class $\xi^{\#}$ due to the G-invariance of $\mathfrak{H}_2$. Define $\langle\xi^{\#},\eta\rangle=(\xi,\eta)$ for $\xi\in\mathfrak{H}$ and $\eta\in\mathfrak{H}_1$. Again this does not depend on the choice of $\xi\in\xi^{\#}$ due to $\mathfrak{H}_2=\mathfrak{H}_1^{\perp}$. Due to the non-degeneracy of the hermitian form on $\mathfrak{H}$, $\mathfrak{H}_1^{\#}$ separates $\mathfrak{H}_1$, i.e. $\langle\xi^{\#},\eta\rangle=0$ for all $\xi^{\#}\in\mathfrak{H}_1^{\#}$ implies $\eta=0$. On the other hand, $\langle\xi^{\#},\eta\rangle=0$ for all $\eta\in\mathfrak{H}_1$ implies $\xi\in\mathfrak{H}_1^{\perp}=\mathfrak{H}_2$, and hence $\xi^{\#}=0$. Namely, $\mathfrak{H}_1$ separates $\mathfrak{H}_1^{\#}$. By G-invariance of the hermitian form, we have

$$\langle\pi_1^\#(g^{-1})\xi^\#,\eta\rangle=(\pi_1(g^{-1})\xi,\pi_1(g^{-1})\pi_1(g)\eta)=(\xi,\pi_1(g)\eta)=\langle\xi^\#,\pi_1(g)\eta\rangle.$$

Therefore, $\pi_1^\#$ on $\mathfrak{H}_1^\#$ is conjugate to π_1 on $\mathfrak{H}_1$.

If $\mathfrak{H}_2=\mathfrak{H}_1^\perp$ coincides with $\mathfrak{H}_1$, we have the form (1.1) with $n=2$. If $\mathfrak{H}_2\neq\mathfrak{H}_1$, restrict $\pi(g)$ to $\mathfrak{H}_2$, and then consider it modulo $\mathfrak{H}_1$. Denote the representation so obtained on $\mathfrak{H}_2/\mathfrak{H}_1$ by π_2. We then have the form with $n=3$. In either case $\pi_n=\pi_1^\#$ is conjugate to π_1 and (2) is proved.

For $\xi\in\mathfrak{H}_2$, denote $[\xi]=\xi+\mathfrak{H}_1\in\mathfrak{H}_2/\mathfrak{H}_1$, and define $([\xi],[\eta])=(\xi,\eta)$. Due to $\mathfrak{H}_2=\mathfrak{H}_1^\perp$, this does not depend on representative vectors $\xi\in[\xi]$ and $\eta\in[\eta]$. Furthermore, $([\xi],[\eta])=0$ for all $[\eta]\in\mathfrak{H}_2/\mathfrak{H}_1$ implies $\xi\in\mathfrak{H}_2^\perp=(\mathfrak{H}_1^\perp)^\perp=\mathfrak{H}_1$, and hence $[\xi]=0$. Since the hermitian form is $\pi_2(G)$-invariant, we have (3).

Remark 2.4. If the representation π is topologically indecomposable and has a proper irreducible subrepresentation π_1, then the assumption of Theorem 1 is satisfied.

3. Cohomology

In order to discuss $c_{ij}(g)$, we introduce a standard notation for a cohomology.

Definition 3.1. The set $C^n(\pi_1,\pi_2)$ of n-cochains for representations $\pi_i^\#$ on $\mathfrak{H}_i^\#(i=1,2)$ is the set of all functions $c(g_1,\dots,g_n)$ of $g_k\in G$ $(k=1,\dots,n)$, taking values in the set $L(\mathfrak{H}_1,\mathfrak{H}_2)$ of all everywhere-defined linear maps from $\mathfrak{H}_1$ into $\mathfrak{H}_2$.

In our application, there are representations π_j on $\mathfrak{H}_j$ $(j=1,2)$ conjugate to π_j on $\mathfrak{H}_j$, and we need the following restriction:

Definition 3.2. The set $C_b^n(\pi_1,\pi_2)$ of bounded n-cochains is the set of $c\in C^n(\pi_1,\pi_2)$ such that there exists $c^*\in C^n(\pi_2^\#,\pi_1^\#)$ satisfying

$$\langle\xi,c(g_1,\dots,g_n)\eta\rangle=\langle c^*(g_n^{-1},\dots,g_n^{-1})\xi,\eta\rangle \tag{3.1}$$

for all $\eta\in\mathfrak{H}_1$ and $\xi\in\mathfrak{H}_2^\#$. The set $C_s^n(\pi_1\pi_1^\#)$ of self-adjoint cochains is the set of $c\in C^n(\pi_1,\pi_1^\#)$ such that (3.1) is satisfied for $c^*=c$ and $\xi,\eta\in\mathfrak{H}_1$.

The coboundary operation δ is defined by the following and satisfies $\delta^2=0$.

Definition 3.3.

$$\begin{aligned}(\delta c_n)(g_1,\dots,g_{n+1})&=\pi_2(g_1)c_n(g_2,\dots,g_n)\\&\quad+\sum_{i=1}^{n}(-1)^i c(g_1,\dots,g_ig_{i+1},\dots,g_{n+1})\\&\quad+(-1)^{n+1}c_n(g_1,\dots,g_n)\pi_1(g_{n+1}).\end{aligned} \tag{3.2}$$

Cocycles, coboundaries, and cohomologies are defined as usual.

Definition 3.4.

$$Z^n(\pi_1,\pi_2)=\{c_n\in C^n;\ \delta c_n=0\},$$

$$B^n(\pi_1,\pi_2)=\delta C^{n-1}(\pi_1,\pi_2)\quad(=0\text{ for }n=0),$$

$$H^n(\pi_1,\pi_2)=Z^n(\pi_1,\pi_2)/B^n(\pi_1,\pi_2).$$

Similarly, Z^n, B^n, H^n with suffices b and s are defined. Note that $(\delta c_n)^* = (-1)^{n+1}\delta(c_n^*)$, as is seen by a straightforward calculation. Thus $B_s^n \equiv i^n \delta C_s^{n-1} = B_b^n \cap C_s^n$.

Definition 3.5. For $c_{32} \in C^m(\pi_2, \pi_3)$ and $c_{21} \in C^n(\pi_1, \pi_2)$, define $c_{32} \times c_{21} \in C^{m+n}(\pi_1, \pi_2)$ by

$$(c_{32} \times c_{21})(g_1, \ldots, g_{m+n}) = c_{32}(g_1, \ldots, g_m)c_{21}(g_{m+1}, \ldots, g_{m+n}). \tag{3.3}$$

Lemma 3.6.

$$Z^m(\pi_2, \pi_3) \times Z^n(\pi_1, \pi_2) \subset Z^{m+n}(\pi_1, \pi_3),$$

$$B^m(\pi_2, \pi_3) \times Z^n(\pi_1, \pi_2) \subset B^{m+n}(\pi_1, \pi_3),$$

$$Z^m(\pi_2, \pi_3) \times B^n(\pi_1, \pi_2) \subset B^{m+n}(\pi_1, \pi_3),$$

$$H^m(\pi_2, \pi_3) \times H^n(\pi_1, \pi_2) \subset H^{m+n}(\pi_1, \pi_3).$$

Proof. Straightforward consequences of the following formula:

$$\delta(c_{32} \times c_{21}) = (\delta c_{32}) \times c_{21} + (-1)^m c_{32} \times (\delta c_{21}). \tag{3.4}$$

4. Cohomological Conditions on $c_{ij}(g)$

In the rest of the paper, we assume that each $\mathfrak{H}_j$ has a complement so that $\pi(g)$ can be written in the form of (1.3) and discuss the off-diagonal elements $c_{ij}(g)$.

Lemma 4.1. *In order that* (1.2) *be a representation of a group G, it is necessary and sufficient that* (1) *each* π_i *is a representation and* (2) c_{ij}, $1 \leqq i < j \leqq n$, *satisfy*

$$\delta c_{kl} = -\sum_{j=k+1}^{l-1} c_{kj} \times c_{jl} \quad (1 \leqq k < l \leqq n), \tag{4.1}$$

where the right-hand side is understood to be 0 *if* $l = k+1$.

Proof. Follows from a straightforward computation of $(\pi(g_1)\pi(g_2))_{kl} = \pi(g_1 g_2)_{kl}$.

Remark 4.2. The right-hand side of (4.1) belongs to $Z^2(\pi_l, \pi_k)$ already due to other constraints involving only c_{ij} with $i+j < k+l$. [By (3.4), we have

$$\delta(\sum c_{kj} \times c_{jl}) = \sum (\delta c_{kj}) \times c_{jl} - \sum c_{kj} \times \delta c_{jl}.$$

By substituting (4.1) into δc_{kj} and δc_{jl}, we see that this vanishes by cancellation.] Equation (4.1) requires it to belong to $B^2(\pi_l, \pi_k)$.

For given π and an increasing sequence of G-invariant subspaces $\mathfrak{H}_1, \mathfrak{H}_2, \ldots, \mathfrak{H}_n$, $c_{ij}(g)$ depends on the choice of a complement $\mathfrak{K}_{k+1}$ of $\mathfrak{H}_k$ in $\mathfrak{H}_{k+1}$. Change of $\mathfrak{K}$'s can be described in terms of a triangular matrix S ($S_{kl} = 0$ for $k > l$) which leaves every $\mathfrak{H}_j$ invariant. A general triangular matrix S is generated by the following specific ones

$$S_i = \mathbb{1} + u_{ii}(R_i - \mathbb{1}_{\mathfrak{H}_i}), \qquad S_{ij} = \mathbb{1} + u_{ij}R_{ij}, \tag{4.2}$$

where u_{ij} is a matrix unit [i.e. $(u_{ij})_{kl} = \delta_{ik}\delta_{jl}$], R_i is an operator on $\mathfrak{H}_i$ and R_{ij} is a linear map from $\mathfrak{H}_j$ into $\mathfrak{H}_i$. Changes by transformation $\pi(s) \to S_*^{-1}\pi(g)S_*$ are as follows, where π's and c's not mentioned are not changed at all:

$$S_i: \pi_i(g) \to R_i^{-1}\pi_i(g)R_i, \quad c_{ij}(g) \to R_i^{-1}c_{ij}(g), \quad c_{ki}(g) \to c_{ki}(g)R_i,$$

$$S_{ij}: c_{ij}(g) \to c_{ij}(g) + (\delta R_{ij})(g), \quad c_{ik}(g) \to c_{ik}(g) - R_{ij}c_{jk}(g),$$

$$c_{kj}(g) \to c_{kj}(g) + c_{ki}(g)R_{ij}.$$

We now consider a specific case of $n=2$ and assume that the G-invariant subspace $\mathfrak{H}_1$ $(=\mathfrak{K}_1)$ has a complement $\mathfrak{K}_2$ so that π is of the form (1.3) with $n=2$. By Lemma 4.1, c_{12} belongs to $Z^1(\pi_2, \pi_1)$.

Proposition 4.3. *$\mathfrak{H}_1$ does not have a G-invariant complement if and only if $c_{12} \notin B^1(\pi_2, \pi_1)$.*

Proof. First assume that there exists a G-invariant complement $\mathfrak{K}_2'$. Then $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{K}_2 = \mathfrak{H}_1 + \mathfrak{K}_2'$. Hence any $\xi \in \mathfrak{H}$ has two decompositions

$$\xi = \xi_1 + \xi_2 = \xi_1' + \xi_2', \quad (\xi_1, \xi_1' \in \mathfrak{H}_1, \ \xi_2 \in \mathfrak{K}_2, \ \xi_2' \in \mathfrak{K}_2'). \tag{4.3}$$

Define $W = (W_{ij})_{i,j=1,2}$, $W_{ij}\xi_j = \xi_j'$, $(W^{-1})_{ij}\xi_j' = \xi_j$. Then $W_{21} = (W^{-1})_{21} = 0$, $W_{11} = (W^{-1})_{11} = 1$, $(W_{22})^{-1} = (W^{-1})_{22}$, and $(W^{-1})_{12} = -W_{12}W_{22}^{-1}$. Since $\pi(g)$ has a diagonal form $\pi_1(g) \oplus \pi_2'(g)$ for the decomposition $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{K}_2'$, we have

$$\pi(g) = W^{-1}\begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2'(g) \end{pmatrix} W \tag{4.4}$$

for the decomposition $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{K}_2$. Hence $\pi_2(g) = W_{22}^{-1}\pi_2'(g)W_{22}$ and

$$c_{12} = \delta W_{12}. \tag{4.5}$$

Conversely, assume that $c_{12} = \delta W_{12}$ for some $W_{12} \in C^0(\pi_2, \pi_1)$. Then define $\mathfrak{K}_2' = (1 - W_{12})\mathfrak{K}_2$. If $(1 - W_{12})\xi_2 \in \mathfrak{H}_1$ for $\xi_2 \in \mathfrak{K}_2$, then $\xi_2 = (1 - W_{12})\xi_2 + W_{12}\xi_2 \in \mathfrak{H}_1$ (due to $W_{12}\xi_2 \in \mathfrak{H}_1$) and $\xi_2 \in \mathfrak{H}_1 \cap \mathfrak{K}_2 = \mathbf{0}$. Hence $\mathfrak{H}_1 \cap \mathfrak{K}_2' = \mathbf{0}$. Furthermore,

$$W = \begin{pmatrix} 1 & W_{12} \\ 0 & 1 - W_{12} \end{pmatrix} \tag{4.6}$$

obviously intertwines $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{K}_2$ into $\mathfrak{H}_1 + \mathfrak{K}_2'$, so that $\mathfrak{K}_2'$ is a complement of $\mathfrak{H}$. Furthermore, since $W_{12}\xi_2 \in \mathfrak{H}_1$ for $\xi_2 \in \mathfrak{K}_2$,

$$\begin{aligned} \pi(g)(1 - W_{12})\xi_2 &= \pi_2(g)\xi_2 - \pi_1(g)W_{12}\xi_2 + c_{12}(g)\xi_2 \\ &= (1 - W_{12})\pi_2(g)\xi_2 \in \mathfrak{K}_2', \end{aligned}$$

due to $c_{12} = \delta W_{12}$. Therefore, $\mathfrak{K}_2'$ is a $\pi(G)$-invariant complement of $\mathfrak{H}_1$. Q.E.D.

For our application, we would be interested in closed subspaces. Let $\mathfrak{H}^\#$ be a subspace of the algebraic dual of $\mathfrak{H}$ and assume that $\mathfrak{H}^\#$ separates $\mathfrak{H}$. We consider the weak topology $\sigma(\mathfrak{H}, \mathfrak{H}^\#)$ on $\mathfrak{H}$ given by seminorms $\xi \in \mathfrak{H} \to |\zeta(\xi)| \equiv p_\zeta(\xi)$, $\zeta \in \mathfrak{H}^\#$.

Assumption. The $\pi(G)$-invariant subspace $\mathfrak{H}_1$ of $\mathfrak{H}$ has a closed complement $\mathfrak{K}_2 \equiv \mathfrak{H}_2$ such that $\mathfrak{H}_2^\perp + \mathfrak{H}_1^\perp = \mathfrak{H}^\#$, where $\mathfrak{H}_1^\perp$ and $\mathfrak{H}_2^\perp$ are the polar of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ in $\mathfrak{H}^\#$.

Since $\mathfrak{H}$ separates its dual and hence $\mathfrak{H}^{\#}$ by definition, $\mathfrak{H}_1^{\perp} \cap \mathfrak{H}_2^{\perp} = 0$. Then $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology on $\mathfrak{H}_i$ is the $\sigma(\mathfrak{H}_i, \mathfrak{H}_i^{\#})$ topology, where $\mathfrak{H}_1^{\#} = \mathfrak{H}_2^{\perp}$, $\mathfrak{H}_2^{\#} \equiv \mathfrak{H}_1^{\perp}$. Let $\pi(g)$ be of the form (1.3) with $n=2$ for the decomposition $\mathfrak{H} = \mathfrak{H}_1 + \mathfrak{H}_2$.

Proposition 4.3′. *Under the assumption above, $\mathfrak{H}_1$ does not have a G-invariant closed complement $\mathfrak{K}_2'$ satisfying $\mathfrak{H}_1^{\perp} + (\mathfrak{K}_2')^{\perp} = \mathfrak{H}^{\#}$ if and only if $c_{12} \notin B_b^1(\pi_2, \pi_1)$.*

Proof. First assume that there exists a G-invariant closed complement $\mathfrak{K}_2'$ of $\mathfrak{H}_1$ satisfying $\mathfrak{H}_1^{\perp} + (\mathfrak{K}_2')^{\perp} = \mathfrak{H}^{\#}$. By the same procedure as before we have (4.5), and we have only to show that W_{12}^* exists so that $W_{12} \in C_b^0(\pi_2, \pi_1)$. Let $\xi_\alpha \in \mathfrak{H}$ be a net tending to 0 in $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology. Let $\xi_\alpha = \xi_{\alpha 1}' + \xi_{\alpha 2}'$, $\xi_{\alpha 1}' \in \mathfrak{H}_1$, $\xi_{\alpha 2}' \in \mathfrak{K}_2'$. For any $\zeta \in \mathfrak{H}^{\#}$, we have $\zeta = \zeta_1' + \zeta_2'$ with $\zeta_1' \in (\mathfrak{K}_2')^{\perp}$, $\zeta_2' \in \mathfrak{H}_1^{\perp}$, and $(\zeta, \xi_{\alpha 1}') = (\zeta_1', \xi_{\alpha 1}') = (\zeta_1', \xi_\alpha) \to 0$. Hence W_{12} is $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ continuous. For any $\zeta \in \mathfrak{H}^{\#}$, we have $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in \mathfrak{K}_2^{\perp}$, $\zeta_2 \in \mathfrak{H}_1$, and $(\zeta, \xi) = (\zeta_2, \xi)$ for $\xi \in \mathfrak{K}_2$. Thus $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology on $\mathfrak{K}_2$ is the same as $\sigma(\mathfrak{K}_2, \mathfrak{H}_1^{\perp})$ topology. Similarly, $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology on $\mathfrak{H}_1$ is the same as $\sigma(\mathfrak{H}_1, \mathfrak{H}_1^{\#})$ topology with either $\mathfrak{H}_1^{\#} = \mathfrak{K}_2^{\perp}$ or $\mathfrak{H}_1^{\#} = (\mathfrak{K}_2')^{\perp}$. For $\zeta \in \mathfrak{H}_1^{\#}$, $(\zeta, W_{12}\xi)$ is $\sigma(\mathfrak{K}_2, \mathfrak{H}_1^{\perp})$ continuous (due to the continuity of W_{12} shown above), and hence there exists a unique $W_{12}^*\zeta \in \mathfrak{H}_1^{\perp} = \mathfrak{K}_2^{\#}$ satisfying $(W_{12}^*\zeta, \xi) = (\zeta, W_{12}\xi)$. Here the uniqueness follows from the separating property of $\mathfrak{K}_2$ on $\mathfrak{K}_2^{\#}$ ($\mathfrak{H}_1^{\perp} \cap \mathfrak{K}_2^{\perp} = 0$) and implies the linearity of W_{12}^*. Therefore, $W_{12} \in C_b^0(\pi_2, \pi_1)$.
Conversely, assume that $c_{12} = \delta W_{12}$ for some $W_{12} \in C_b^0(\pi_2, \pi_1)$. We then prove that $\mathfrak{K}_2' = (1 - W_{12})\mathfrak{K}_2$ is closed and $\mathfrak{H}^{\#} = \mathfrak{H}_1^{\perp} + (\mathfrak{K}_2')^{\perp}$. Let $(1 - W_{12})\xi_\alpha$ ($\xi_\alpha \in \mathfrak{K}_2$) be a net in $\mathfrak{K}_2'$ tending to $\bar{\xi} \in \mathfrak{H}$. Let $\bar{\xi} = \bar{\xi}_1 + \bar{\xi}_2$, $\bar{\xi}_1 \in \mathfrak{H}_1$, $\bar{\xi}_2 \in \mathfrak{K}_2$. For any $\zeta \in \mathfrak{H}_1^{\perp} = \mathfrak{K}_2^{\#}$, we have $(\zeta, \xi_\alpha - \bar{\xi}_2) = (\zeta, (1 - W_{12})\xi_\alpha - \bar{\xi}) \to 0$, and hence $\xi_\alpha \to \bar{\xi}_2$ in $\sigma(\mathfrak{K}_2, \mathfrak{K}_2^{\#})$ topology, which is the same as $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology on $\mathfrak{K}_2$. For any $\zeta \in \mathfrak{H}_1^{\#}$, we have $(\zeta, W_{12}(\xi_\alpha - \bar{\xi})) = (W_{12}^*\zeta, \xi_\alpha - \bar{\xi}_2) \to 0$. Hence $W_{12}(\xi_\alpha - \bar{\xi}_2) \to 0$ in $\sigma(\mathfrak{H}_1, \mathfrak{H}_1^{\#})$ topology which is the same as $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology on $\mathfrak{H}_1$. Hence $(1 - W_{12})\xi_\alpha \to (1 - W_{12})\bar{\xi}_2 \in \mathfrak{K}_2'$ and $\mathfrak{K}_2'$ is $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ closed.

Let $(1 - W_{12})\xi_\alpha$ ($\xi_\alpha \in \mathfrak{K}_2$) be a net in $\mathfrak{K}_2'$ tending to 0 in $\sigma(\mathfrak{K}_2', \mathfrak{H}_1^{\perp})$ topology. Then, for any $\zeta \in \mathfrak{H}_1^{\perp}$, $(\zeta, \xi_\alpha) = (\zeta, (1 - W_{12})\xi_\alpha) \to 0$. Since $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology on $\mathfrak{K}_2$ is the same as $\sigma(\mathfrak{K}_2, \mathfrak{H}_1^{\perp})$ topology due to the assumption, we have $\xi_\alpha \to 0$. Hence $(\zeta, W_{12}\xi_\alpha) = (W_{12}^*\zeta, \xi_\alpha) \to 0$ for all $\zeta \in \mathfrak{K}_2^{\perp}$, which implies $W_{12}\xi_\alpha \to 0$ in $\sigma(\mathfrak{H}_1, \mathfrak{K}_2^{\perp})$ topology or, equivalently, in $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology. Therefore, we have $(1 - W_{12})\xi_\alpha \to 0$ in $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topology. This means that $\sigma(\mathfrak{K}_2', \mathfrak{H}_1^{\perp})$ and $\sigma(\mathfrak{H}, \mathfrak{H}^{\#})$ topologies are the same on $\mathfrak{K}_2'$. Hence, for any $\zeta \in \mathfrak{H}^{\#}$, there exists $\zeta_2' \in \mathfrak{H}_1^{\perp}$ such that $(\zeta, \xi) = (\zeta_2', \xi)$ for all $\xi \in \mathfrak{K}_2'$, i.e. $\zeta_1' \equiv \zeta - \zeta_2' \in (\mathfrak{K}_2')^{\perp}$. Therefore, $\mathfrak{H}^{\#} = \mathfrak{H}_1^{\perp} + (\mathfrak{K}_2')^{\perp}$. Q.E.D.

Proposition 4.3 is generalized to the case of $\mathfrak{H} = \mathfrak{K}_1 + \ldots + \mathfrak{K}_n$ with $\mathfrak{K}_i \cap \mathfrak{K}_j = 0$ for $i \neq j$ and $\mathfrak{H}_j = \mathfrak{K}_1 + \ldots + \mathfrak{K}_j$.

Corollary 4.4. *For a representation π of the form (1.3), $\mathfrak{H}_j$ does not have any G-invariant complement in $\mathfrak{H}_{j+1}$ for $j = 1, \ldots, n-1$ if and only if*

$$c_{k,k+1} \notin B^1(\pi_{k+1}, \pi_k), \qquad k = 1, \ldots, n-1, \tag{4.7}$$

$$\delta c_{k,l} = -\sum_{j=k+1}^{l-1} c_{kj} \times c_{jl}, \qquad 1 \leqq k < l+1 \leqq n+1. \tag{4.8}$$

Proof. Equation (4.8) is necessary and sufficient for π of the form (1.3) to be a representation by Lemma 4.1. Then (4.7) is necessary and sufficient by Proposition 4.3.

Corollary 4.5. *Let π_1 on $\mathfrak{H}_1$ and $\pi_1^\#$ on $\mathfrak{H}_1^\#$ be irreducible representations of a group G which are mutually conjugate with respect to a mutually separating sesquilinear form $\langle\xi,\eta\rangle$, $\eta\in\mathfrak{H}_1$, $\xi\in\mathfrak{H}_1^\#$. (For irreducibility, the closedness of invariant subspaces is relative to the weak topologies on $\mathfrak{H}_1$ by $\mathfrak{H}_1^\#$ and on $\mathfrak{H}_1^\#$ by $\mathfrak{H}_1$.) In order that there exists a representation of the form $\pi=\pi_1^\#\to\pi_1$ on $\mathfrak{H}=\mathfrak{H}_1+\mathfrak{H}_1^\#$ with a $\pi(G)$-invariant hermitian form*

$$(\eta_1+\xi_1,\eta_2+\xi_2)=\langle\xi_1,\eta_2\rangle+\overline{\langle\xi_2\eta_1\rangle} \tag{4.9}$$

for $\eta_j\in\mathfrak{H}_1$, $\xi_j\in\mathfrak{H}_1^\#$ $(j=1,2)$ and with no $\pi(G)$-invariant closed complement $\mathfrak{K}_2'$ of $\mathfrak{H}_1$ satisfying $(\mathfrak{K}_2')^\perp+\mathfrak{K}_1^\perp=\mathfrak{H}$, it is necessary and sufficient that $H_s^1(\pi_1^\#,\pi_1)\neq 0$.

Proof. If such a representation exists, we have a nonzero $c_{12}\in H_b^1(\pi_1^\#,\pi_1)$, which satisfies for $\xi_j\in\mathfrak{H}_1^\#$ $(j=1,2)$

$$\begin{aligned}\langle\xi_1,c_{12}(g)\xi_2\rangle&=(\xi_1,\pi(g)\xi_2)=(\pi(g^{-1})\xi_1,\xi_2)\\&=\overline{\langle\xi_2,c_{12}(g^{-1})\xi_1\rangle}\equiv\langle c_{12}(g^{-1})\xi_1,\xi_2\rangle,\end{aligned}$$

where the second equality is due to the $\pi(G)$-invariance of the inner product and the last equality is the definition of the sesquilinear form between $\xi_2\in\mathfrak{H}_1^\#$ and $c_{12}(g^{-1})\xi_1\in(\mathfrak{H}_1^\#)^\#\equiv\mathfrak{H}_1$. Hence $c_{12}=c_{12}^*\in Z_s^1(\pi_1^\#,\pi_1)$ and $c_{12}\notin B_s^1=B_b^1\cap C_s^1$.

Conversely, if there exists a nonzero $c_{12}\in\mathfrak{H}_s^1(\pi_1^\#,\pi_1)$, then π given by (1.3) on $\mathfrak{H}=\mathfrak{H}_1+\mathfrak{H}_1^\#$ is a representation of G, under which (4.9) is invariant, as is easily checked, and $\mathfrak{H}_1$ does not have an invariant complement by Proposition 4.3. (Note that c_{12} is nonzero in H_b^1 due to $B_b^1\cap C_s^1=B_s^1$.)

5. A Standard Form of the Gupta-Bleuler Triplet

The triplet of the form (1.1) with $n=3$ given in the preceding section (i.e. $\mathfrak{H}_j$ does not have any G-invariant closed complement in $\mathfrak{H}_{j+1}$, $j=1,2$, and $\mathfrak{H}_2=\mathfrak{H}_1^\perp$) will be called the Gupta-Bleuler triplet. We assume that $\mathfrak{H}_i$ has a closed complement $\mathfrak{K}_{i+1}$ in $\mathfrak{H}_{i+1}$ so that the representation is of the form (1.3).

Definition 5.1. A Gupta-Bleuler triplet

$$\pi=\pi_1^\#\to\pi_2\to\pi_1 \quad \text{on} \quad \mathfrak{H}=\mathfrak{K}_1\oplus\mathfrak{K}_2\oplus\mathfrak{K}_3 \tag{5.1}$$

with $\mathfrak{K}_1=\mathfrak{H}_1$, $\mathfrak{K}_3=\mathfrak{H}_1^\#$,

$$\pi(g)=\begin{pmatrix}\pi_1(g) & c_{12}(g) & c_{13}(g)\\ 0 & \pi_2(g) & c_{23}(g)\\ 0 & 0 & \pi_1^\#(g)\end{pmatrix},$$

and the inner product

$$(\xi_1+\xi_2+\xi_3,\xi_1'+\xi_2'+\xi_3')=\langle\xi_3,\xi_1'\rangle+\overline{\langle\xi_3',\xi_1\rangle}+(\xi_2,\xi_2')_2 \tag{5.2}$$

for $\xi_j,\xi_j'\in\mathfrak{K}_j$ $(j=1,2,3)$ is said to be in a standard form, where $(\xi_2,\xi_2')_2$ is a $\pi_2(G)$-invariant non-degenerate hermitian form on $\mathfrak{K}_2$.

Theorem 2. *Let conjugate irreducible representations π_1 on $\mathfrak{H}_1$ and $\pi_1^\#$ on $\mathfrak{H}_1^\#$ as in Corollary 4.5 and a representation π_2 on $\mathfrak{K}_2$ with an invariant non-degenerate hermitian form be given. In order that a Gupta-Bleuler triplet $\pi=\pi_1^\#\to\pi_2\to\pi_1$ on $\mathfrak{H}=\mathfrak{H}_1+\mathfrak{K}_2+\mathfrak{H}_1^\#$ in a standard form to exist, it is necessary and sufficient that there exists a $c_{12}\in H_b^1(\pi_2,\pi_1)$ ($\pi_2^\#$ on $\mathfrak{H}_2^\#$ taken to be π_2 on $\mathfrak{H}_2$ with the duality given by the G-invariant hermitian form on $\mathfrak{H}_2$) such that $c_{12}\times c_{12}^*\in B_s^2(\mathfrak{H}_1^\#,\mathfrak{H}_1)$ and c_{12} and c_{12}^* are nonzero in $H^1(\pi_2,\pi_1)$ and $H^1(\pi_1^\#,\pi_2)$.*

Proof. Assume the existence of the triplet $\pi=\pi_1^\#\to\pi_2\to\pi_1$ on $\mathfrak{H}=\mathfrak{H}_1+\mathfrak{K}_2+\mathfrak{H}_1^\#$ in a standard form. By Corollary 4.5, $c_{12}\in H^1(\pi_2,\pi_1)$ and $c_{23}\in H^1(\pi_1^\#,\pi_2)$ are nonzero and $c_{12}\times c_{23}=\delta\psi$ with $\psi=-c_{13}\in C^1(\pi_1^\#,\pi_1)$. Furthermore, the $\pi(G)$-invariance of (5.2) implies $c_{23}=c_{12}^*$ and $c_{13}=c_{13}^*$. This proves the necessity.

For the converse, we take the given c_{12} and define $c_{23}\equiv c_{12}^*$ and $c_{13}=-\psi$. Then $\pi(g)$ given by (1.3) is a representation of G on $\mathfrak{H}$ and (5.2) is $\pi(G)$-invariant. Furthermore, the non-existence of $\pi(G)$-invariant closed complement follows from Proposition 4.3 due to $c_{12}\neq 0$ and $c_{12}^*\neq 0$.

Proposition 5.3. *A Gupta-Bleuler triplet $\pi=\pi_1^\#\to\pi_2\to\pi_1$ on $\mathfrak{H}=\mathfrak{H}_1+\mathfrak{K}_2+\mathfrak{K}_3$ can be transformed into a standard form by a proper choice of the closed subspaces $\mathfrak{K}_2$ and $\mathfrak{K}_3$ (with $\mathfrak{H}_2=\mathfrak{H}_1+\mathfrak{K}_2=\mathfrak{H}_1^\perp$ fixed) if the weak topology given by $\mathfrak{H}_1$ on $\mathfrak{H}_1^\#=\mathfrak{H}/\mathfrak{H}_2=\mathfrak{K}_3$ coincides with the weak topology given by $\mathfrak{H}$ on $\mathfrak{K}_3$.*

Remark. The assumption on the weak topology is equivalent to $\mathfrak{K}_3^\perp+\mathfrak{H}_1=\mathfrak{H}$.

Proof. Assume the coincidence of two topologies. For each $\xi\in\mathfrak{K}_2$ and $\eta\in\mathfrak{K}_3$, there exist $\sigma_{12}\xi$ and $\sigma_{13}\eta$ in $\mathfrak{H}_1$ satisfying

$$(\xi,\eta')=(\sigma_{12}\xi,\eta'),\qquad (\eta,\eta')=(\sigma_{13}\eta,\eta') \tag{5.3}$$

for all $\eta'\in\mathfrak{K}_3$. Define

$$\mathfrak{K}_2'=(1-\sigma_{12})\mathfrak{K}_2,\qquad \mathfrak{K}_3'=(1-\sigma_{13}/2)\mathfrak{K}_3. \tag{5.4}$$

We have $\mathfrak{H}_1+\mathfrak{K}_2'=\mathfrak{H}_1+\mathfrak{K}_2=\mathfrak{H}_2$, $\mathfrak{H}_2+\mathfrak{K}_3'=\mathfrak{H}_2+\mathfrak{K}_3=H$. Furthermore, $\mathfrak{K}_2'$ is orthogonal to $\mathfrak{K}_3$, hence to $\mathfrak{K}_3'$ (due to $\mathfrak{H}_1\perp\mathfrak{H}_2\supset\mathfrak{K}_2'$). Further,

$$((1-\sigma_{13}/2)\eta,(1-\sigma_{13}/2)\eta')=(\eta,\eta')-((\sigma_{13}\eta,\eta')+(\eta,\sigma_{13}\eta'))/2=0,$$

i.e. $\mathfrak{K}_3'$ is a null space. Therefore, the inner product in $\mathfrak{H}$ takes the form of (5.2) relative to the decomposition $\mathfrak{H}=\mathfrak{H}_1+\mathfrak{K}_2'+\mathfrak{K}_3'$ and the Gupta-Bleuler triplet is now in a standard form.

Remark 5.4. Let $R_{12}\in C_b^0(\pi_2,\pi_1)$, $R_{13}=-R_{13}^*\in L(\mathfrak{H}_1^\#,\mathfrak{H}_1)$, and set

$$W=\begin{pmatrix}1 & -R_{12} & -R_{12}R_{12}^*/2-R_{13}\\ 0 & 1 & R_{12}^*\\ 0 & 0 & 1\end{pmatrix}. \tag{5.5}$$

Then W preserves the inner product (5.2) of the normal form and

$$W^{-1}=\begin{pmatrix}1 & R_{12} & -R_{12}R_{12}^*/2+R_{13}\\ 0 & 1 & -R_{12}^*\\ 0 & 0 & 1\end{pmatrix}. \tag{5.6}$$

The coordinate change $\pi(g)\to W\pi(g)W^{-1}$ changes c_{12} by a coboundary:

$$c_{12}\to c_{12}+\delta R_{12}, \tag{5.7}$$

$$c_{13}\to c_{13}-\{(\delta R_{12})R_{12}^{*}-R_{12}\delta(R_{12}^{*})\}/2-c_{12}R_{12}^{*}-R_{12}c_{12}^{*}+\delta R_{13}. \tag{5.8}$$

Note that $(\delta R)^{*}=-\delta(R^{*})$, so that the new c_{13} is again selfadjoint. The relation $\delta c_{13}=-c_{12}c_{12}^{*}$ is also preserved.

6. Discussion

Our results suggest the following method of analysis for a representation π of a group G on a vector space $\mathfrak{H}$ with a $\pi(G)$-invariant, nondegenerate indefinite inner product. The first step is to decompose the representation as a sum of indecomposable representations. Obviously, there would be cases, where this is not possible. (An example is a direct integral of non-equivalent indecomposable representations.)

Suppose we succeed in this first step and suppose $(\pi, \mathfrak{H})$ is now topologically indecomposable. The second step is then to find an irreducible subrepresentation $(\pi_1, \mathfrak{H}_1)$ unless $(\pi, \mathfrak{H})$ is already irreducible.

The third step is then to apply our results. Especially, if $\mathfrak{H}_1$ has a closed complement $\mathfrak{K}_2$ in $\mathfrak{H}_1^{\perp}$ ($\mathfrak{H}_1^{\perp}\supset\mathfrak{H}_1$ by Theorem 1) and if $\mathfrak{H}_1^{\perp}$ has a closed complement $\mathfrak{K}_3$ such that $\mathfrak{H}_1+\mathfrak{K}_3^{\perp}=\mathfrak{H}$, then π is of the standard form of Definition 5.1 due to Proposition 5.3, and the analysis of Theorem 2 is applicable. (It may happen that $\mathfrak{H}_1=\mathfrak{H}_1^{\perp}$, in which case we have $\mathfrak{K}_2=0$.) Now we are left with the analysis of π_2 on $\mathfrak{K}_2$ with a non-degenerate invariant inner product, and we can follow the same procedure as before.

While there are several points in the above procedure for possible obstructions, there seem to be many examples for which the above procedure works fine.

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