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# Geometry of $N = 1$ Supergravity (II)

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**Abstract.** The supergravity torsion and curvature constraints are shown to be a particular case of constraints arising in a general geometrical situation. For this purpose, a theorem is proved which describes the necessary and sufficient conditions that the given geometry can be realized on a surface as one induced by the geometry of the ambient space. The proof uses the theory of nonlinear partial differential equations in superspace, Spencer cohomologies, etc. This theorem generalizes various theorems, well known in mathematics (e.g., the Gauss–Codazzi theorem), and may be of its own interest.

## 1. Introduction

In a previous paper [1] we studied the geometry of various superspace formulations of $N = 1$ supergravity. We considered a well-known family of supergravity models labelled by a parameter $\zeta$, and found that different approaches to supergravity are connected with a general geometrical problem. Suppose one has a space endowed with a fixed geometry of some type. Then, given some surface in this space, one can define the internal geometry of the surface, induced on it by the geometry of the ambient space. The relevant general definition of induced geometry uses the language of $G$-structures (see refs. [1, 2]; the present paper is not completely self-contained, but uses the notations and conventions of ref. [1]).

In this paper we prove a theorem (Sect. 2) about the necessary and sufficient conditions that the given $G'$-structure on a manifold can be realized on some surface as one induced by the trivial $G$-structure in $\mathbb{R}^N$. In general, this problem amounts to the question whether a certain system of nonlinear partial differential equations has a solution. The theorem describes the conditions of the formal integrability (Appendix B) for that system in a convenient form of constraints on the internal geometry. There is, in general, a chain of integrability conditions of increasing orders. The number of non-trivial ones, which is always finite, is controlled by certain Spencer cohomologies (Appendix A) related to the problem.

Thus, we shall see that a $G'$-structure corresponding to induced geometry is not arbitrary, but satisfies certain constraints. It turns out that the supergravity torsion and curvature constraints are just of that nature. In refs. [1, 2] it was shown that in supergravity one also encounters induced structures. In ref. [1] we have given, for each $\zeta$, formulations of supergravity in terms of induced SCR($\zeta$)-structures. The SCR($\zeta$)-structures arise on real (4|4)-dimensional surfaces in complex superspace $\mathbb{C}^{4/4}$, when it is endowed with the trivial $G(\zeta)$-structure. For the sake of these formulations to be self-contained, the constraints on induced SCR($\zeta$)-structures must be stated in terms of internal geometry. The following constraints were claimed in ref. [1], where they were shown also to be equivalent to the usual supergravity constraints in the Wess–Zumino approach. The torsion of an arbitrary connection in an induced SCR($\zeta$)-structure satisfies

$$T^{c}_{\alpha\beta}=0,\, T^{\dot\gamma}_{\alpha\beta}=0 \quad \text{if } \zeta=\infty, \text{ also } T^{b}_{\dot\alpha b}-T^{\dot\beta}_{\dot\alpha\dot\beta}=0. \tag{1.1}$$

If we choose now a connection which obeys additionally

$$\begin{aligned}
&T^{c}_{ab}=T^{\gamma}_{ab}=T^{\gamma}_{\alpha b}=T^{\gamma}_{\dot\alpha b}=T^{\gamma}_{\alpha\beta}=0,\\
&T^{c}_{\alpha b}=\tfrac14 T_{\alpha}\delta^{c}_{b} && (T_\alpha\equiv T^{d}_{\alpha d}),\\
&T^{\dot\alpha}_{\alpha\dot\beta}=\tfrac12 t_{\alpha}\delta^{\dot\alpha}_{\dot\beta} && (t_\alpha\equiv T^{\dot\delta}_{\alpha\dot\delta}),\\
&\begin{cases} T_\alpha=t_\alpha & \text{if } \zeta\neq 1,\infty\\ T_\alpha=-t_\alpha & \text{if } \zeta=\infty\\ T_\alpha=t_\alpha=0 & \text{if } \zeta=1,\end{cases}
\end{aligned} \tag{1.2}$$

then, for an induced SCR($\zeta$)-structure with $\zeta\neq\infty$, the curvature satisfies

$$R^{\alpha}_{\beta\alpha\gamma}\varepsilon^{\beta\gamma}=0. \tag{1.3}$$

In the present paper we derive (Sect. 3) these constraints from our general theorem, and prove, by investigation of relevant Spencer cohomology groups (Appendix E), that they are not only necessary, but also sufficient.

In what follows the superspaces will not be always referred to explicitly, but the corresponding generalizations will be obvious. As usual, one has only to take care of correct signs under the (anti) symmetrization of tensors in superspace. For example, if $t_{AB}$ is a second rank tensor, then (no summation)

$$\begin{aligned}
t_{(AB)}&=\tfrac12\{t_{AB}+(-)^{AB}t_{BA}\},\\
t_{[AB]}&=\tfrac12\{t_{AB}-(-)^{AB}t_{BA}\},
\end{aligned}$$

and similarly for higher rank tensors. Here the symbol $(-)^{AB}$ means minus one, when the indices are both fermionic, and plus one otherwise.

## 2. The Basic Theorem about the Induced Structure

Let us consider the manifold $\mathbb{R}^N$, endowed with the standard trivial $G$-structure, and an $M$-dimensional regular surface in $\mathbb{R}^N$ (see [1]). If $y^{\hat a}$ ($\hat a=1,\dots,N$) are the standard coordinates in $\mathbb{R}^N$, this surface is given by $y^{\hat a}=f^{\hat a}(x)$, with $x^m$ ($m=1,\dots,M$) some coordinates on it. According to ref. [1] we have a $G'$-structure induced on the surface. This structure is determined by the frame field $e^m_a(x)$, $a,m=1,\dots,M$, which satisfies the following relation

$$\frac{\partial f^{\hat{a}}(x)}{\partial x^m} e_b^m(x) = g_b^{\hat{a}}(x), \tag{2.1}$$

with $g_b^{\hat{a}}(x)$ being the $M \times N$ part of some $N \times N$ matrix function $g_{\hat{b}}^{\hat{a}}(x)$ with values in $G$. Conversely, suppose we have some $G'$-structure represented by a frame field $e_a^m(x)$ on an $M$-dimensional manifold $\mathscr{M}$. This $G'$-structure is equivalent to an induced structure if there exists a map, $y^{\hat{a}} = f^{\hat{a}}(x)$, of $\mathscr{M}$ into $\mathbb{R}^N$ satisfying the differential equation (1). The requirement of the compatibility of Eq. (1) poses some constraints on the $G'$-structure in $\mathscr{M}$. To describe the necessary and sufficient conditions of compatibility we need the following definitions.

Let $\mathfrak{g}$ be the Lie algebra of the group $G \subset GL(N, \mathbb{R})$. The algebra $\mathfrak{g}$ consists of $N \times N$ matrices corresponding to linear transformations of the vector space $\mathbb{R}^N$. It will be convenient to denote this vector space by $W$, in order to distinguish from the other cases, when $\mathbb{R}^N$ appear in our considerations. Let $k$ be an arbitrary non-negative integer. One defines $\mathfrak{g}^{(k)}$, the $k^{\text{th}}$ prolongation of $\mathfrak{g}$, as the space of tensors $t^{\hat{a}}_{\hat{b}_1 \dots \hat{b}_{k+1}}$ in $W$, symmetric in the indices $\hat{b}_1, \dots . \hat{b}_{k+1}$, and such that for any fixed values of $\hat{b}_2, \dots, \hat{b}_{k+1}$ the matrix $t^{\hat{a}}_{\hat{b}_1 \dots \hat{b}_{k+1}}$ belongs to $\mathfrak{g}$. Thus, for instance, $\mathfrak{g}^{(0)} = \mathfrak{g}$. Now we define $\mathfrak{g}_\infty$ as the formal sum of linear spaces[1]

$$\mathfrak{g}_\infty = \mathfrak{g}^{(-1)} + \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)} + \mathfrak{g}^{(2)} + \cdots,$$

where we set $\mathfrak{g}^{(-1)} = W$.

The space $\mathfrak{g}_\infty$ has the natural structure of a Lie algebra with the following property. If $X$, $Y$ are homogeneous elements of $\mathfrak{g}_\infty$, that is $X \in \mathfrak{g}^{(k)}$, $Y \in \mathfrak{g}^{(\ell)}$ for some $k$ and $\ell$, then $[X, Y] \in \mathfrak{g}^{(k+\ell)}$. To describe explicitly the relevant operation $[,]$ let us use the following correspondence. For every $X \in \mathfrak{g}^{(k-1)}$, $k \geqq 0$, one can construct a vector field on $\mathbb{R}^N$ with the following homogeneous $k^{\text{th}}$ order polynomials as the vector components

$$X^{\hat{a}}(y) = \frac{1}{k!} t^{\hat{a}}_{\hat{b}_1 \dots \hat{b}_k} \cdot y^{\hat{b}_1} \dots y^{\hat{b}_k}.$$

Here $(y^{\hat{a}}) \in \mathbb{R}^N$, while $t^{\hat{a}}_{\hat{b}_1 \dots \hat{b}_k}$ are the tensor components of $X \in \mathfrak{g}^{(k-1)}$. For an arbitrary element $X \in \mathfrak{g}_\infty$ one has in such a way a formal power series $X^{\hat{a}}(y)$ which is a formal vector field $X(y)$ on $\mathbb{R}^N$. We identify the vector fields on $\mathbb{R}^N$ with first order differential operators: $X(y) = X^{\hat{a}}(y)\partial/\partial y^{\hat{a}}$. According to the definition of $\mathfrak{g}_\infty$, a formal vector field $X$ on $\mathbb{R}^N$ corresponds to some element $X$ of $\mathfrak{g}_\infty$ if and only if it satisfies

$$\left( \frac{\partial X^{\hat{a}}}{\partial y^{\hat{b}}}(y) \right) \in \mathfrak{g}, \tag{2.2}$$

order by order in $y$. It can be easily seen that the set of vector fields satisfying Eq. (2) is closed under the usual commutator and that the resulting operation $[,]$ in $\mathfrak{g}_\infty$ obeys $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(\ell)}] \subset \mathfrak{g}^{(k+\ell)}$. Let us remark that $\mathfrak{g}_\infty$ has, in particular, the subalgebra $\mathfrak{g}^{(0)}$,

1 i.e. the linear space of formal sums $X_{-1} + X_0 + X_1 + X_2 + \cdots$, where $X_k \in \mathfrak{g}^{(k)}$ and the number of non-zero terms may be infinite

which coincides with $\mathfrak{g}$, and the subalgebra $\mathfrak{g}^{(-1)}+\mathfrak{g}^{(0)}$, which is the semidirect sum of the Abelian algebra $W$ and the algebra $\mathfrak{g}$ acting on $W$.

We denote by $G_\infty$ the group corresponding to the Lie algebra $\mathfrak{g}_\infty$. This group will be described below in terms of formal power series on $\mathbb{R}^N$.

Now, if $e_a^m$, $a, m=1,\dots,M$ is a frame field representing a $G'$-structure in $\mathscr{M}$, let $e^a=e^a_m(x)\,dx^m$ be the corresponding coframe field. Then we define on $\mathscr{M}$ the 1-form field $\Theta$ with values in $W$ as follows:

$$\Theta^{\hat{a}}=\delta^{\hat{a}}_b e^b \tag{2.3}$$

(i.e. we prolong $e^a$ to $\Theta^{\hat{a}}$, taking $\Theta^{a'}=0$ for $a'=M+1,\dots,N$). We are ready to state the following:

**Theorem.** *A $G'$-structure in $\mathscr{M}$ is equivalent to the structure induced on a surface in the space $\mathbb{R}^N$ by the trivial $G$-structure in $\mathbb{R}^N$ (i.e. Eq. (1) has a solution) if and only if there exists on $\mathscr{M}$ a 1-form field $\Omega=\Omega_m(x)dx^m (m=1,\dots,M=\dim\mathscr{M}\leqq N)$ with values in $\mathfrak{g}_\infty$, satisfying the following conditions*

$$\Omega^{(-1)}=\Theta, \tag{2.4}$$

$$d\Omega+\tfrac{1}{2}[\Omega\wedge\Omega]=0. \tag{2.5}$$

Here $\Theta$ is connected with a coframe $e$, admissible for the $G'$-structure in view, by the relation (3); $\Omega^{(k)}$ denotes the $\mathfrak{g}^{(k)}$-component of $\Omega$ in the decomposition $\mathfrak{g}_\infty=\sum_k \mathfrak{g}^{(k)}$, while $[\Omega\wedge\Omega]$ in Eq. (5) denotes the usual operation on the Lie algebra valued forms.

We remark that the criterion stated in the theorem is self-consistent, that is it does not depend on the choice of the frame field $e_a^m(x)$ (coframe field $e^a$), for the following reasons. Condition (5) is nothing but the requirement that the gauge field $\Omega$ has vanishing curvature (strength). For every $x$-dependent $G'$-rotation of $e_a^m(x)$ we can find a $G$-gauge transformation of the field $\Omega$, which preserves (4). Then condition (5) will be maintained as well. (Note that $G$ is the subgroup of $G_\infty$, corresponding to the subalgebra $\mathfrak{g}^{(0)}=\mathfrak{g}$ of $\mathfrak{g}_\infty$.)

Let us show first that the $G'$-structure induced on the surface $y^{\hat{a}}=f^{\hat{a}}(x)$ in $\mathbb{R}^N$ necessarily has the property described in the theorem. Equation (1), determining the induced structure, can be rewritten using the previous definitions

$$df^{\hat{a}}=g^{\hat{a}}_{\hat{b}}\Theta^{\hat{b}}. \tag{2.6}$$

Now there is a $\mathfrak{g}_\infty$-valued 1-form $\Theta$ defined as follows:

$$\begin{aligned}\Omega^{(-1)}&=g^{-1}df,\\ \Omega^{(0)}&=g^{-1}dg,\\ \Omega^{(k)}&=0\quad\text{for } k>0,\end{aligned}$$

where the indices have been suppressed. From (6) it is obvious that this 1-form $\Omega$ satisfies (4). One can check in a straightforward manner that $\Omega$ satisfies also Eq. (5). Instead one can observe that the so-defined $\Omega$ is merely a pure gauge $\Omega=h^{-1}dh$ with a $G_\infty$-valued gauge function $h(x)$. Then condition (5) becomes trivial. (Although in this case $h(x)$ can be chosen to lie for all $x$ in a subgroup of $G_\infty$, corresponding to the subalgebra $\mathfrak{g}^{(-1)} + \mathfrak{g}^{(0)}$, there may be some other pure gauges, which need not have such a property, but do satisfy (4).)

Let us inquire whether the existence of a field $\Omega$, obeying Eqs. (4) and (5), is also sufficient for a $G'$-structure to be induced. The above consideration suggests the following arguments. Let us suppose first that the Lie algebra $\mathfrak{g}$ is of finite type, that is $\mathfrak{g}^{(p)} = \mathfrak{g}^{(p+1)} = \cdots = 0$ for some finite $p$. Then the algebra $\mathfrak{g}_\infty$ and the corresponding Lie group $G_\infty$ are finite dimensional. If there exists a field $\Omega$ satisfying the conditions of the theorem, then in the case of a finite dimensional Lie group $G_\infty$ the vanishing curvature requirement of Eq. (5) leads to a well known conclusion that this field $\Omega$ is a pure gauge. In other words,

$$\Omega = h^{-1} dh, \tag{2.7}$$

where $h$ is a function on $\mathscr{M}$ with values in the group $G_\infty$. For an arbitrary group $G$ this statement will be proved in Appendix C. Thus we proceed here with a general $G$. To complete the proof of the theorem we have to show only that the existence of a field $\Omega$, obeying Eqs. (4), (7) for a $G'$-structure on $\mathscr{M}$, implies the existence of a solution to (6).

To make the meaning of (7) clear we have to consider the structure of the group $G_\infty$ in more detail. As it was already mentioned, $\mathfrak{g}_\infty$ may be thought of as a Lie algebra of formal vector fields on $\mathbb{R}^N$ satisfying condition (2). Such vector fields form infinitesimal automorphisms of the trivial $G$-structure in $\mathbb{R}^N$ (cf. [1, 4]). Therefore the group $G_\infty$ consists of formal transformations of $\mathbb{R}^N$, which can be represented as $y^{\hat{a}} \to \phi^{\hat{a}}(y)$, where $\phi^{\hat{a}}(y)$ is a formal power series obeying

$$\left(\frac{\partial \phi^{\hat{a}}}{\partial y^{\hat{b}}}(y)\right) \in G. \tag{2.8}$$

The group $G_\infty$ is thus the formal analog of the group $\Gamma(G)$ of automorphisms of the trivial $G$-structure in $\mathbb{R}^N$ (cf. [1, 4]).

Let $X_m^{\hat{a}}(x, y)$ correspond to the $\mathfrak{g}_\infty$-valued 1-form $\Omega = \Omega_m(x) dx^m$, as explained in the text before the theorem. Then condition (7) means that there exists an $x$-dependent element $h(x) \in G_\infty$, which corresponds to such an $x$-dependent transformation $y^{\hat{a}} \to \phi^{\hat{a}}(x, y)$ of $\mathbb{R}^N$, that

$$X_m^{\hat{a}}(x, y) = -\left[\frac{\partial (\phi^{-1})^{\hat{a}}}{\partial x^m}(x, z)\right]_{z = \phi(x, y)}, \tag{2.9}$$

where $\phi^{-1}$ is the inverse of $\phi$ at each $x$. Since $(\phi^{-1}(x, \phi(x, y)))^{\hat{a}} = y^{\hat{a}}$, we have

$$\begin{aligned} 0 &\equiv \frac{d}{dx^m}(\phi^{-1}(x, \phi(x, y)))^{\hat{a}} \\ &= \frac{\partial (\phi^{-1})^{\hat{a}}}{\partial x^m}(x, \phi(x, y)) + \frac{\partial (\phi^{-1})^{\hat{a}}}{\partial y^{\hat{b}}}(x, \phi(x, y)) \cdot \frac{\partial \phi^{\hat{b}}}{\partial x^m}(x, y). \end{aligned}$$

After the substitution into (9) we obtain

$$X_m^{\hat{a}}(x, y) = \frac{\partial (\phi^{-1})^{\hat{a}}}{\partial y^{\hat{b}}}(x, \phi(x, y)) \cdot \frac{\partial \phi^{\hat{b}}}{\partial x^m}(x, y). \tag{2.10}$$

Remember that $X_m^{\hat{a}}(x,0)=\Theta_m^{\hat{a}}(x)$, due to (4). Taking into account (8) we see that Eq. (10) at $y=0$ gives just the desired relation (6) with

$$g_{\hat{b}}^{\hat{a}}(x)=\frac{\partial\phi^{\hat{a}}}{\partial y^{\hat{b}}}(x,0), \quad f^{\hat{a}}(x)=\phi^{\hat{a}}(x,0).$$

We find that Eq. (7) does imply the existence of a solution to Eq. (6). This completes the arguments.

To conclude this section, a few remarks concerning the general situation may be useful.

Let us denote the left-hand side of Eq. (5) by $R_\infty = d\Omega + \frac{1}{2}[\Omega \wedge \Omega]$. The equation $R_\infty = 0$ turns out to be an infinite set of the conditions $R^{(k)}=0$, $k=-1,0,1,2,\ldots$. Here $R_\infty = \sum_k R^{(k)}$ is the decomposition of the 2-form $R_\infty$ with values in $\mathfrak{g}_\infty = \sum_k \mathfrak{g}^{(k)}$. Using the matrix notations for $\Omega = \sum_k \Omega^{(k)}$, $\Omega^{(-1)}=\Theta$, we have explicitly

$$\begin{aligned} R^{(-1)} &\equiv d\Theta + \Omega^{(0)} \wedge \Theta = 0, \\ R^{(0)} &\equiv d\Omega^{(0)} + \Omega^{(0)} \wedge \Omega^{(0)} + \Omega^{(1)} \wedge \Theta = 0, \text{ etc.}. \end{aligned} \tag{2.11}$$

To find out whether the field $\Omega$ satisfying the conditions of the theorem for a given $G'$-structure exists, one may try to solve the equations $R^{(-1)}=0$, $R^{(0)}=0,\ldots$, successively with respect to $\Omega^{(0)}, \Omega^{(1)}$, etc. Then one has to find at first such a 1-form $\Omega^{(0)}$ with values in $\mathfrak{g}^{(0)}=\mathfrak{g}$, that $d\Theta + \Omega^{(0)} \wedge \Theta = 0$ holds. It is clear that $d\Theta$ must obey certain algebraic conditions at each point, in order to ensure the existence of a solution $\Omega^{(0)}$. Consequently, the $G'$-structure under consideration must satisfy certain conditions on the first derivatives of the frame field $e_a^m(x)$. (Remember that $\Theta$ is connected with $e_a^m$, as described before the theorem.) Suppose such a field $\Omega^{(0)}$ exists. Now one has to find $\Omega^{(1)}=(\Omega_{\hat{b}\hat{c}}^{\hat{a}})$, the 1-form with values in $\mathfrak{g}^{(1)}$, such that $R^{(0)}=0$ holds true ($\Omega^{(1)} \wedge \Theta$ in (11) must be understood as $\Omega_{\hat{b}\hat{c}}^{\hat{a}} \wedge \Theta^{\hat{c}}$). The requirement of the existence of such a field $\Omega^{(1)}$ clearly amounts to certain conditions on the second derivatives of $e_a^m(x)$ at each point, and so on. At first sight we obtain in this way an infinite chain of the integrability conditions for Eq. (1) of increasing order. However, only a finite number of these conditions is non-trivial. As is explained in Appendix C, when we solve the equations $R^{(-1)}=0$, $R^{(0)}=0$, etc., successively, we have to deal at each step with a sort of cohomology equation. The obstructions to the compatibility of these equations lie in certain Spencer cohomology groups $H^{k,2}$ $(\mathfrak{g}_V)$, which are generally vanishing for $k$ sufficiently large.

The resulting finite number of the integrability conditions on a $G'$-structure can be formulated as constraints on the so-called structure functions. The conditions of the first and second order amount to certain constraints on the torsion and the curvature in the $G'$-structure. It may occur that the higher order conditions are trivial. This is just the case, for example, in the classical problem of the induced Riemannian metric on a surface in Euclidean space. This situation is described by the Gauss–Codazzi theorem [4, 5], which turns out to be a particular case of the theorem stated in the present section (see Appendix D for details). Fortunately, in applications to $N=1$ supergravity it will be also sufficient to consider the conditions of first and second order only.

## 3. Deriving the Torsion and Curvature Constraints

Here we shall show that the supergravity torsion and curvature constraints (1.1), (1.3) follow from the theorem of the preceding section[2].

The following notations will be used. We consider a real $(4|4)$-dimensional manifold $\mathscr{M}$ and an SCR($\zeta$)-structure in it, assuming the relevant definitions of ref. [1]. Let $e^A = dx^M E_M^A(x, \theta, \bar{\theta})$ be an admissible coframe field, with $x^M = (x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})$ being coordinates in $\mathscr{M}$. The coframe may be thought of as a vector-valued 1-form, $e = (e^A)$, which takes its values in a real $(4|4)$-dimensional vector space $V$. It will be instructive to identify $V$ with a real subspace of the complex vector space $W = \mathbb{C}^{4|4}$. If $z^A = (z^a, \theta^\alpha, \bar{\varphi}^{\dot{\pi}})$ are the complex coordinates in $W$, we set $V$ to be a subspace defined by the following equations:

$$z^a = (z^a)^*, \; \theta^\alpha = (\bar{\varphi}^{\dot{\alpha}})^*, \; \bar{\varphi}^{\dot{\pi}} = (\theta^\pi)^*. \tag{3.1}$$

After such an identification one may consider the coframe 1-form as taking values in the space $W$, or, rather, denote the resulting $W$-valued 1-form field by $\Theta$. The components of $\Theta$ that correspond to coordinates in $W$ are complex valued 1-forms, which we denote, along with their conjugates, as follows,

$$(\Theta^{\hat{A}}) = (\Theta^A; \Theta^{\bar{A}}) = (Z^a, \Theta^\alpha, \Phi^{\dot{\pi}}; Z^{\bar{a}}, \Theta^{\dot{\alpha}}, \Phi^\pi). \tag{3.2}$$

Here the index $\hat{A} = (A; \bar{A}) = (a, \alpha, \dot{\pi}; \bar{a}, \dot{\alpha}, \pi)$ refers to coordinates $z^{\hat{A}} = (z^A, z^{\bar{A}})$, where $z^{\bar{A}} = (z^{\bar{a}}, \bar{\theta}^{\dot{\alpha}}, \varphi^\pi)$ are the complex conjugates of $z^A = (z^a, \theta^\alpha, \bar{\varphi}^{\dot{\pi}})$. We have just defined a $W$-valued 1-form (2) in terms of a coframe field $(e^A) = (e^a, e^\alpha, e^{\dot{\alpha}})$. This can be expressed by the following somewhat tautological relations:

$$\begin{aligned} Z^a &= Z^{\bar{a}} = e^a, \\ \Theta^\alpha &= \Phi^\alpha = e^\alpha, \\ \Phi^{\dot{\pi}} &= \Theta^{\dot{\pi}} = e^{\dot{\pi}}. \end{aligned} \tag{3.3}$$

Note, however, that $e^a$ are always real valued 1-forms, while $e^\alpha$ and $e^{\dot{\alpha}}$ are always conjugated, for $e^A$ is a coframe in the real $(4|4)$-dimensional manifold $\mathscr{M}$. Thus the definitions (3) of a $W$-valued 1-form $\Theta$ makes its values satisfy Eqs (1) of the real subspace $V$ in $W$, as it must be.

The above notations, though cumbersome, have been introduced to be close to the general context of Sect. 2 and to make its use here most straightforward. In Sect. 2 we dealt with a $W$-valued 1-form field $\Theta$ too, the vector space $W$ being arbitrary. In the theorem about induced structures we used $\Theta$ tied to some coframe field of a $G'$-structure under consideration. This forced the values of $\Theta$ to lie in a fixed subspace (also denoted by $V$) of the space $W$. The dimension of $W$ equaled the dimension of the ambient space (or superspace) while that of $V$ equaled the dimension of the manifold (a would-be surface in $\mathbb{R}^N$, $N = \dim W$), where the given $G'$-structure was defined. Although $V$ can be an arbitrary subspace of $W$, it must always coincide with the subspace, that appears in the definition of regular surfaces induced $G'$-structures and of the group $G'$ itself. When we considered in preceding

2 Of course, the statement of the Theorem must be generalized to superspace in a straightforward manner as explained in Sect. 1

sections the general case of induced structures, we choose for definiteness the standard $M$-dimensional subspace of $W=\mathbb{R}^N$. So we did, in particular, in the formulation of the theorem (cf. Eq. (2.3)). In the present case, when $G'=\mathrm{SCR}(\zeta)$, it is convenient to use $W=\mathbb{C}^{4|4}$ and its real $(4|4)$-dimensional subspace $V$ defined by Eqs. (1). Thus, for instance, Eq. (3) of the present section is parallel to Eq. (2.3). These conventions for $W$ and $V$ are coherent with those of ref. [1], which concern the definitions of $\mathrm{SCR}(\zeta)$-structures and induced $\mathrm{SCR}(\zeta)$-structures. Note, finally, that in order to be in agreement with generalities of the preceding sections, we must treat $W=\mathbb{C}^{4|4}$ as a real $(8|8)$-dimensional space. This can be achieved by the use of complex coordinates along with their conjugates. Now the application of the theorem must cause no trouble.

Let us consider an $\mathrm{SCR}(\zeta)$-structure in a real $(4|4)$-dimensional manifold $\mathscr{M}$. Let $\Theta$ be a $W$-valued 1-form field connected with some admissible coframe, $e^A=dx^M E_M^A(x,\theta,\bar\theta)$, in $\mathscr{M}$ by means of Eqs. (3). Our aim is to specify the conditions ensuring that this structure in $\mathscr{M}$ is equivalent to an $\mathrm{SCR}(\zeta)$-structure induced on some surface by the trivial $G(\zeta)$-structure in the ambient space $\mathbb{C}^{4|4}$. Let $\mathfrak{g}(\zeta)$ denote the Lie algebra of the group $G(\zeta)$. According to the theorem of Sect. 2 we must inquire into existence of a 1-form field $\Omega$ on $\mathscr{M}$ that takes values in $\mathfrak{g}(\zeta)_\infty=\sum_k \mathfrak{g}(\zeta)^{(k)}$ and satisfies (2.4) and (2.5). Let us consider the components $\Omega^{(k)}$ of $\Omega$ corresponding to the decomposition of $\mathfrak{g}(\zeta)_\infty$ into the sum of subspaces $\mathfrak{g}(\zeta)^{(k)}$; let us decompose also the infinite dimensional equation (2.5) into its $\mathfrak{g}(\zeta)^{(k)}$ components. Thus, instead of $d\Omega+\frac{1}{2}[\Omega\wedge\Omega]=0$, we consider the equations $R^{(k)}=0$ for $k=-1,0,1,2,\ldots$, where $R^{(k)}$ form the decomposition of $R_\infty=d\Omega+\frac{1}{2}[\Omega\wedge\Omega]$. An explicit expression for $R^{(k-1)}$ involves $\Omega^{(j)}$ only with $-1\leqq j\leqq k$, moreover, $\Omega^{(k)}$ enters linearly and without derivatives. (This can be seen, using the property $[\mathfrak{g}(\zeta)^{(i)},\mathfrak{g}(\zeta)^{(j)}]\subset\mathfrak{g}(\zeta)^{(i+j)}$, of the Lie algebra $\mathfrak{g}(\zeta)_\infty$, as stated in Sect. 2.) Hence one may consider each equation $R^{(k-1)}=0$ as a linear equation, to be solved for $\Omega^{(k)}$ in terms of $\Omega^{(j)}$ with $-1\leqq j<k$. Starting with $\Omega^{(-1)}=\Theta$ (cf. Eq. (2.4)) one may try to find successively $\Omega^{(k)}$ for $k=0,1,2,\ldots$, from $R^{(k-1)}=0$. If a solution exists at each step, a field $\Omega$, satisfying the requirements of the theorem, will exist too. It follows that the compatibility of arising linear equations is necessary and sufficient for the given $\mathrm{SCR}(\zeta)$-structure to be induced. That is to say, the integrability conditions for the differential equation (2.1) are rewritten as compatibility conditions of certain linear algebraic equations. In the case of the algebras $\mathfrak{g}(\zeta)$, it turns out that these equations for $\Omega^{(k)}$ with $k\geqq 2$ are always compatible, provided the first two equations, $R^{(-1)}=0$, $R^{(0)}=0$, are compatible. In other words, the existence of a solution $\Omega^{(0)}$ to $R^{(-1)}=0$ and $\Omega^{(1)}$ to $R^{(0)}=0$ implies that $\Omega^{(k)}$ for $k\geqq 2$ exist too, satisfying $R^{(k-1)}=0$. When $\zeta=\infty$, moreover, the existence of $\Omega^{(0)}$ is already sufficient for $\Omega^{(k)}$ with $k\geqq 1$ to exist. The proof can be found in Appendix E; it follows from the vanishing of certain Spencer cohomology groups related to $\mathfrak{g}(\zeta)$.

Thus in the present case it suffices to investigate the compatibility of the system $R^{(-1)}=0$, $R^{(0)}=0$. The former equation takes the following explicit form:

$$R^{(-1)\hat{A}}\equiv d\Theta^{\hat{A}}-\Theta^{\hat{B}}\wedge\Omega_{\hat{B}}^{\hat{A}},\tag{3.4}$$

where the matrix 1-form $(\Omega_{\hat{B}}^{\hat{A}})$ stands for the $\mathfrak{g}(\zeta)$-valued 1-form $\Omega^{(0)}$. That is to say, we are looking for a solution $\Omega^{\hat{A}}_{\hat{B}}$ to (4), which is a 1-form that satisfies the conditions corresponding to the algebra $\mathfrak{g}(\zeta)$. The group $G(\zeta)$ was defined as a group of complex linear transformations of $\mathbb{C}^{4|4}$ (see [1]). We are using now the space $W$, which is, strictly speaking, $\mathbb{C}^{4|4}$ considered as a real space. Hence we have to use the matrices of the form $(X^{\hat{A}}_{\hat{B}})$, which consist of blocks $(X^{A}_{B})$, $(X^{\bar{A}}_{B})$, $(X^{A}_{\bar{B}})$ and $(X^{\bar{A}}_{\bar{B}})$, where $X^{\bar{A}}_{\bar{B}}=(X^{A}_{B})^*$, $X^{A}_{\bar{B}}=(X^{\bar{A}}_{B})^*$. (Remember that an index $\hat{A}=(A;\bar{A})$ runs over the values corresponding to the complex coordinates $z^A$ and their complex conjugates $z^{\bar{A}}$.) Then according to the definition of the group $G(\zeta)$, its Lie algebra, $\mathfrak{g}(\zeta)$, consists of the matrices $(X^{\hat{A}}_{\hat{B}})$ obeying

$$X^{a}_{\dot{\pi}}=X^{\bar{a}}_{\pi}=0,\; X^{\alpha}_{\dot{\pi}}=X^{\dot{\alpha}}_{\pi}=0,\; X^{A}_{\bar{B}}=X^{\bar{A}}_{B}=0, \tag{3.5}$$

and also a $\zeta$-dependent condition

$$\mathrm{tr}_{\zeta} X=0, \tag{3.6}$$

where the following notation is used. If $\zeta\neq\infty$,

$$\mathrm{tr}_{\zeta} X=\zeta(X^{a}_{a}-X^{\alpha}_{\alpha})-X^{\dot{\pi}}_{\dot{\pi}}, \tag{3.7}$$

whereas for $\zeta=\infty$, we set

$$\mathrm{tr}_{\infty} X=X^{a}_{a}-X^{\alpha}_{\alpha}. \tag{3.8}$$

The existence of a 1-form $\Omega^{\hat{A}}_{\hat{B}}$ satisfying Eqs. (4), (5) and (6) is equivalent to certain constraints on $d\Theta$, that is on the first derivatives of a frame field representing the given SCR($\zeta$)-structure. It is convenient to express these conditions in a more covariant form.

Let us consider a connection in the SCR($\zeta$)-structure, that is an SCR($\zeta$)-valued 1-form $\omega$. The Lie algebra SCR($\zeta$) consists of linear transformations of a real (4|4)-dimensional space. We consider this real space as a subspace $V$ in $W$, given by (1). According to the general definition, the algebra $\mathfrak{g}'$ consists of transformations of $V$ that can be extended to linear transformations of the space $W$, belonging to the algebra $\mathfrak{g}$ and leaving the subspace $V$ invariant. In the present case, $\mathfrak{g}=\mathfrak{g}(\zeta)$, $\mathfrak{g}'=\mathrm{SCR}(\zeta)$, and each extension is unique. In particular, for a matrix $(Y^{A}_{B})$ in SCR($\zeta$), we get a matrix $(\tilde{Y}^{\hat{A}}_{\hat{B}})$ in $\mathfrak{g}(\zeta)$ taking $\tilde{Y}^{A}_{B}=Y^{A}_{B}$, $\tilde{Y}^{A}_{\bar{B}}=0$. (Remember that $(Y^{A}_{B})\in\mathrm{SCR}(\zeta)$ satisfies the same equation as $X^{A}_{B}$ in (5), (6) and still obeys $Y^{a}_{b}=(Y^{a}_{b})^*$ and $(Y^{\alpha}_{\beta})^*=\delta^{\dot{\alpha}}_{\dot{\pi}}\delta^{\dot{\rho}}_{\beta}Y^{\dot{\pi}}_{\dot{\rho}}$.) Let $\tilde{\omega}^{\hat{A}}_{\hat{B}}$ be the $\mathfrak{g}(\zeta)$ extension of the SCR($\zeta$)-valued 1-form $\omega^{A}_{B}$. (Were this extension not unique, we could choose an arbitrary one. This may happen for a different couple $\mathfrak{g}$, $\mathfrak{g}'$.)

For the coframe 1-form field $e^A$ and connection $\omega^{A}_{B}$ the torsion 2-form $T^A$ is defined as usual,

$$T^A=de^A-e^B\wedge\omega^{A}_{B}. \tag{3.9}$$

Replacing $e^A$ and $\omega^{A}_{B}$ by $\Theta^{\hat{A}}$ and $\tilde{\omega}^{\hat{A}}_{\hat{B}}$, it is straightforward to check that the 2-form

$$\tilde{T}^{\hat{A}}=d\Theta^{\hat{A}}-\Theta^{\hat{B}}\wedge\tilde{\omega}^{\hat{A}}_{\hat{B}} \tag{3.10}$$

is connected with $T^A$ in the same way as $\Theta^{\hat{A}}$ is connected with $e^A$ in Eq. (3). Combining Eq. (4) with the definition (10) of $\tilde{T}^{\hat{A}}$, we get $\tilde{T}^{\hat{A}}=\Theta^{\hat{B}}\wedge\Gamma^{\hat{A}}_{\hat{B}}$, where $\Gamma^{\hat{A}}_{\hat{B}}=e^C\Gamma^{\hat{A}}_{\hat{B}C}$ is a $\mathfrak{g}(\zeta)$-valued 1-form given by $\Gamma=\Omega-\tilde{\omega}$. Expanding in the basis of $e^A$ and turning back from $\Theta$, $\tilde{T}$ to $e, T$, we obtain the following expression for $T^A = \frac{1}{2}e^C \wedge e^B T_B{}^A{}_C$, the torsion of the connection

$$T_B{}^A{}_C = \Gamma^A_{CB} - (-)^{BC}\Gamma^A_{BC}. \tag{3.11}$$

Finally, the first order integrability condition for an SCR($\zeta$)-structure to be induced is satisfied (i.e. Eq. (4) is compatible) if and only if the torsion, $T_B{}^A{}_C$, of some arbitrary connection in this SCR($\zeta$)-structure can be represented in the form (11), with some quantities $\Gamma^{\hat{A}}_{\hat{B}C}$ satisfying for each fixed value of the index $C$ the conditions (5), (6) of the algebra $\mathfrak{g}(\zeta)$ [3]. Now it is easy to read off the resulting torsion constraints. The algebraic conditions on the right-hand side of Eq. (11) imply that the torsion must obey

$$T_\alpha{}^c{}_\beta = 0, \quad T_\alpha{}^{\dot\gamma}{}_\beta = 0, \tag{3.12}$$

and one more constraint, for $\zeta = \infty$ only,

$$T_\alpha{}^b{}_b - T_\alpha{}^\beta{}_\beta = 0. \tag{3.13}$$

(For example, $\Gamma^c_{\dot\pi B} = 0$ due to (5). Consequently, $T_{\dot\pi}{}^c{}_{\dot\rho}$ must vanish, since $T_{\dot\pi}{}^c{}_{\dot\rho} = \Gamma^c_{\dot\rho\dot\pi} + \Gamma^c_{\dot\pi\dot\rho}$, and we have also $T_\alpha{}^c{}_\beta = 0$, for $T_\alpha{}^c{}_\beta$ is related to $T_{\dot\pi}{}^c{}_{\dot\rho}$ by complex conjugation.) The resulting torsion constraints are just those claimed at the end of Sect. 1.

One may stop at this point if $\zeta = \infty$, as we have already mentioned. When $\zeta \neq \infty$, one has to proceed further and derive the curvature constraints from the requirement of compatibility of the equation $R^{(0)} = 0$. Explicitly one has

$$R^{(0)\hat{A}}_{\hat{B}} = d\Omega^{\hat{A}}_{\hat{B}} - \Omega^{\hat{C}}_{\hat{B}} \wedge \Omega^{\hat{A}}_{\hat{C}} - \Theta^{\hat{C}} \wedge \Omega^{(1)\hat{A}}_{\hat{B}\hat{C}} = 0, \tag{3.14}$$

where $\Omega^{\hat{A}}_{\hat{B}}$ is the same as in (4). Equation (14) is to be solved with respect to the 1-forms $\Omega^{(1)\hat{A}}_{\hat{B}\hat{C}}$, which represent a $\mathfrak{g}(\zeta)^{(1)}$-valued 1-form $\Omega^{(1)}$. Thus according to the definitions of Sect. 2, for each value of $\hat{C}$ the 1-forms $\Omega^{(1)\hat{A}}_{\hat{B}\hat{C}}$ satisfy Eqs. (5) and (6) of the matrix algebra $\mathfrak{g}(\zeta)$, and also

$$\Omega^{(1)\hat{A}}_{\hat{B}\hat{C}} = (-)^{\hat{B}\hat{C}}\Omega^{(1)\hat{A}}_{\hat{C}\hat{B}}. \tag{3.15}$$

Let us introduce the following notation:

$$\tilde{R}^{\hat{A}}_{\hat{B}} = d\Omega^{\hat{A}}_{\hat{B}} - \Omega^{\hat{C}}_{\hat{B}} \wedge \Omega^{\hat{A}}_{\hat{C}}. \tag{3.16}$$

Equation (14) may be rewritten, expanding in the basis of 1-forms $e^A$: $\tilde{R}^{\hat{A}}_{\hat{B}} = \frac{1}{2}e^C \wedge e^D \tilde{R}^{\hat{A}}_{\hat{B}CD}$, $\Omega^{(1)\hat{A}}_{\hat{B}\hat{C}} = e^D \Omega^{(1)\hat{A}}_{\hat{B}\hat{C}D}$, while $\Theta^{\hat{A}}$ is related to $e^A$ via (3). Then Eq. (14) assumes the following form:

$$\Omega^{(1)A}{}_{BCD} - (-)^{CD}\Omega^{(1)A}{}_{BDC} = \tilde{R}^A_{BCD}, \tag{3.17}$$

and similarly for complex conjugates. This equation is to be regarded as an inhomogeneous linear equation with respect to $\Omega^{(1)}$. The right-hand side, $\tilde{R}^A_{BCD}$, satisfies Eqs. (5), (6) for any fixed $C$ and $D$, and still obeys

$$\tilde{R}^A_{[BCD]} = 0. \tag{3.18}$$

3 It is easy to verify that if this condition holds for some connection, then it does for any. In the general case the first order integrability condition as a torsion constraint was derived in ref. [2], see also [3]

The last condition is a form of a Bianchi identity, which can be also written as $\Theta^{\hat{B}} \wedge \tilde{R}_{\hat{B}}^{\hat{A}}=0$. This identity can be obtained from (4) by external differentiation. Note that the condition (18) is necessary for compatibility of the linear system (17), as the left-hand side of (17) vanishes obviously due to (15) under the super-anti-symmetrization in $B$, $C$, $D$. The question[4] we are interested in is what conditions beyond (18), if any, have to be imposed on $\tilde{R}^{A}_{BCD}$ to ensure the compatibility of Eq. (17).

One can verify that it is always possible to match a solution to (17) satisfying the relaxed conditions: Eqs. (5) and (15), but not (6). Let $\psi^{\hat{A}}_{\hat{B}\hat{C}D}$ be such a solution, that is

$$\psi^{A}_{BCD}=(-)^{BC}\psi^{A}_{CBD}, \tag{3.19}$$

$$\psi^{A}_{BCD}-(-)^{CD}\psi^{A}_{BDC}=\tilde{R}^{A}_{BCD}, \tag{3.20}$$

with Eqs. (5) satisfied by $\psi^{\hat{A}}_{\hat{B}\hat{C}D}$ for any $\hat{C}, D$. The desired solution, $\Omega^{(1)\hat{A}}_{\hat{B}\hat{C}D}$, however, must obey one more condition, corresponding to Eq. (6), $\zeta(\Omega^{(1)a}{}_{aCD}-\Omega^{(1)\alpha}{}_{\alpha CD})-\Omega^{(1)\dot{\pi}}{}_{\dot{\pi}CD}=0$, or, symbolically, $\mathrm{tr}_\zeta\Omega^{(1)}=0$. Let us try to find $\Omega^{(1)}$ in the form $\Omega^{(1)}=\psi-\phi$. Combining (15) and (19), (17) and (20), we see that $\phi^{\hat{A}}_{\hat{B}\hat{C}D}$ must obey $\phi^{\hat{A}}_{[BC]D}=0$ and $\phi^{A}_{B[CD]}=0$, which yields

$$\phi^{A}_{BCD}=\phi^{A}_{(BCD)}. \tag{3.21}$$

The condition $\mathrm{tr}_\zeta\Omega^{(1)}=0$ requires that $\mathrm{tr}_\zeta\phi=\mathrm{tr}_\zeta\psi$. Thus the compatibility of Eq. (17) for satisfying (5), (6), (15) amounts to the compatibility of the equation

$$\zeta(\phi^{a}_{aCD}-\phi^{\alpha}_{\alpha CD})-\phi^{\dot{\pi}}_{\dot{\pi}CD}=(\mathrm{tr}_\zeta\psi)_{CD} \tag{3.22}$$

for the unknown $\phi$ satisfying (5) and (21). To work out these conditions, let us observe that the left-hand side of Eq. (22) vanishes identically for $C=\dot{\rho}$, $D=\dot{\sigma}$. Indeed, applying successively (21) and (5) we find $\phi^{a}_{a\dot{\rho}\dot{\sigma}}=\phi^{a}_{\dot{\rho}a\dot{\sigma}}=0$, and similarly $\phi^{\alpha}_{\alpha\dot{\rho}\dot{\sigma}}=0$. Finally, $\phi^{\dot{\pi}}_{\dot{\tau}\dot{\rho}\dot{\sigma}}=0$ vanishes, for it is antisymmetric in three indices, as required by (21), whereas each of them takes two values only. By the same arguments $(\mathrm{tr}_\zeta\psi)_{\dot{\rho}\dot{\sigma}}$ reduces to a single term, namely, $(\mathrm{tr}_\zeta\psi)_{\dot{\rho}\dot{\sigma}}=-\psi^{\dot{\pi}}_{\dot{\pi}\dot{\rho}\dot{\sigma}}$. This term need not be zero *a priori*, since $\psi$, unlike $\phi$, need not satisfy (21). However, we observe immediately that it should vanish, $(\mathrm{tr}_\zeta\psi)_{\dot{\rho}\dot{\sigma}}=0$, to ensure the compatibility of Eq. (22) where $(\mathrm{tr}_\zeta\phi)_{\dot{\rho}\dot{\sigma}}=0$ identically. Thus we have to require

$$\psi^{\dot{\pi}}_{\dot{\pi}\dot{\rho}\dot{\sigma}}=0. \tag{3.23}$$

A direct inspection reveals that if (23) holds, there are no further obstructions for a solution $\phi$ of Eq. (22) to exist. Then Eq. (17) will be compatible too, with a solution given by $\Omega^{(1)}=\psi-\phi$.

Returning to Eq. (17), the condition of its compatibility given by (23) is now to be expressed in terms of $\tilde{R}^{A}_{BCD}$. By definition, $\psi$ satisfies (19) and (20). It follows from Eq. (19) that $\psi^{\dot{\pi}}_{\dot{\tau}\dot{\rho}\dot{\sigma}}$ can always be represented as

$$\psi^{\dot{\pi}}_{\dot{\tau}\dot{\rho}\dot{\sigma}}=\psi^{\dot{\pi}}_{\dot{\sigma}}\varepsilon_{\dot{\tau}\dot{\rho}}, \tag{3.24}$$

4 This is, in fact, a typical problem of the Spencer cohomology calculus, see Appendices A, C, E

where $\varepsilon_{\dot{\tau}\dot{\rho}}$ is the antisymmetric symbol. Then from the relation (20) we have

$$\tilde{R}^{\dot{\pi}}_{\dot{\tau}\dot{\rho}\dot{\sigma}} = \psi^{\dot{\pi}}_{\dot{\sigma}}\varepsilon_{\dot{\tau}\dot{\rho}} + \psi^{\dot{\pi}}_{\dot{\rho}}\varepsilon_{\dot{\tau}\dot{\sigma}}. \tag{3.25}$$

Using a notation $\psi_{\dot{\rho}\dot{\sigma}} \equiv \psi^{\dot{\pi}}_{\dot{\rho}}\varepsilon_{\dot{\pi}\dot{\sigma}}$, we obtain

$$\tilde{R}^{\dot{\pi}}_{\dot{\pi}\dot{\rho}\dot{\sigma}} = \psi_{\dot{\rho}\dot{\sigma}} + \psi_{\dot{\sigma}\dot{\rho}}. \tag{3.26}$$

Note that the 2-form $\tilde{R}^{\hat{A}}_{\hat{B}}$ obeys $(\mathrm{tr}_\zeta \tilde{R}) = 0$, because it is a $\mathfrak{g}(\zeta)$-valued 2-form, as the definition (16) shows. By arguments already stated and using, in particular, the identity (18) the equality $(\mathrm{tr}_\zeta \tilde{R})_{\dot{\rho}\dot{\sigma}} = 0$ can be reduced to

$$\tilde{R}^{\dot{\pi}}_{\dot{\pi}\dot{\rho}\dot{\sigma}} = 0. \tag{3.27}$$

Then Eqs. (26) and (27) give $\psi_{\dot{\rho}\dot{\sigma}} = \lambda\varepsilon_{\dot{\rho}\dot{\sigma}}$ for some function $\lambda$, and Eq. (15) can be rewritten,

$$\tilde{R}_{\dot{\pi}\dot{\tau}\dot{\rho}\dot{\sigma}} \equiv \varepsilon_{\dot{\rho}\dot{\varphi}}\tilde{R}^{\dot{\varphi}}_{\dot{\tau}\dot{\rho}\dot{\sigma}} = \lambda(\varepsilon_{\dot{\pi}\dot{\sigma}}\varepsilon_{\dot{\tau}\dot{\rho}} + \varepsilon_{\dot{\pi}\dot{\rho}}\varepsilon_{\dot{\tau}\dot{\sigma}}). \tag{3.28}$$

Note, however, that this expression is, in fact, a mere consequence of the Bianchi identities. Now condition (23) can be expressed in terms of $\tilde{R}$. Using (24) and $\psi^{\dot{\pi}}_{\dot{\sigma}} = \lambda\delta^{\dot{\pi}}_{\dot{\sigma}}$, we obtain first $\lambda = 0$. Then Eq. (28) gives a condition equivalent to (23),

$$\tilde{R}^{\dot{\pi}}_{\dot{\tau}\dot{\pi}\dot{\sigma}}\varepsilon^{\dot{\tau}\dot{\sigma}} = 0. \tag{3.29}$$

Let us notice that this is equivalent to $\tilde{R}^{\dot{\pi}}_{\dot{\tau}\dot{\rho}\dot{\sigma}} = 0$. The last condition could be read off already from Eqs. (23)–(25). However, Eq. (29) is the weakest form of such a condition, which is simpler to use, while $\tilde{R}^{\dot{\pi}}_{\dot{\tau}\dot{\rho}\dot{\sigma}} = 0$ can be restored by means of Bianchi identities.

We have derived Eq. (29) as a condition on the right-hand side of Eq. (17), required for its compatibility, or, equivalently, for the compatibility of the equation $R^{(0)} = 0$ (see Eq. (14)). Remember that the compatibility of this equation amounts to the second order integrability conditions in the problem of induced SCR($\zeta$)-structures. Let us notice that condition (29) is indeed a constraint on the SCR($\zeta$)-structure. The 2-form $\tilde{R}^{\hat{A}}_{\hat{B}}$ was defined in Eq. (16) in terms of a 1-form $\Omega^{\hat{A}}_{\hat{B}}$, which in turn was introduced as a solution of Eq. (4). Thus (29) implies certain restrictions of the first order in derivatives on the field $\Omega^{\hat{A}}_{\hat{B}}$, and hence, by Eq. (4), certain second order constraints on a coframe field representing the SCR($\zeta$)-structure in view. (Note that a coframe enters Eq. (4) via the relation (3), and that the resulting constraints on the SCR($\zeta$)-structure does not depend on the particular admissible coframe field.) Of course, the condition (29), as it stands, is still of little use. What we wanted is to express the second order integrability condition in terms of curvature of connections in the SCR($\zeta$)-structure, rather than of the $\mathfrak{g}(\zeta)$-valued 1-form $\Omega^{\hat{A}}_{\hat{B}}$. It is straightforward to extract this from Eq. (29).

It would be inconvenient, however, to deal with arbitrary connections. Let us instead reduce the set of connections to be considered. For this purpose one may impose certain conditions on the torsion. A possible choice of such conditions is given by Eq. (1.2) of the next section. One can see that given a coframe field $e^A$, one is always able to find such an SCR($\zeta$)-connection in Eq. (9), which satisfies (1.2). Of course, a connection obeying (1.2) is by no means unique, and we consider any such connection. Let $\omega^A_B = e^C\omega^A_{BC}$ be an SCR($\zeta$)-valued connection 1-form, which satisfies

(1.2). Let also $\tilde{\omega}^{\hat{A}}_{\hat{B}}$ be, as above, a $\mathfrak{g}(\zeta)$-valued 1-form, the extension of $\omega^{A}_{B}$. Assuming that the first order integrability conditions are satisfied, the torsion of the connection $\omega^{A}_{B}$ must obey the constraints (12) in addition to the "conventional constraints" of Eq. (1.2). Remember that the torsion tensors satisfying (12) are just those, which can be represented as in Eq. (11). That is to say, there exists a $\mathfrak{g}(\zeta)$-valued 1-form $\Gamma^{\hat{A}}_{\hat{B}} = e^{C}\Gamma^{\hat{A}}_{\hat{B}C}$ satisfying (11). A possible choice may be as follows:

$$\begin{aligned}
&\Gamma^{c}_{\alpha\dot{\beta}} = \Gamma^{\bar{c}}_{\dot{\beta}\alpha} = T_{\alpha}{}^{c}{}_{\dot{\beta}},\\
&\Gamma^{c}_{b\dot{\alpha}} = \Gamma^{\bar{c}}_{\bar{b}\dot{\alpha}} = \tfrac{1}{4}T_{\dot{\alpha}}{}^{d}{}_{d}\,\delta^{c}_{b},\\
&\Gamma^{\gamma}_{\beta\dot{\alpha}} = \tfrac{1}{2}T_{\dot{\alpha}}{}^{\delta}{}_{\delta}\,\delta^{\gamma}_{\beta},\\
&\Gamma^{\pi}_{\rho\dot{\alpha}} = \tfrac{1}{2}T_{\dot{\alpha}}{}^{\delta}{}_{\delta}\,\delta^{\pi}_{\rho},\\
&\Gamma^{\dot{\pi}}_{\dot{\rho}\dot{\alpha}} = 0,
\end{aligned}\tag{3.30}$$

while the other independent components of $\Gamma^{\hat{A}}_{\hat{B}C}$ vanish. (Note that the last line in Eq. (30) serves to ensure $\mathrm{tr}_{\zeta}\Gamma = 0$ if $\zeta \neq \infty$, as required for a $\mathfrak{g}(\zeta)$-valued 1-form). Then, if we set $\Omega^{\hat{A}}_{\hat{B}} = \tilde{\omega}^{\hat{A}}_{\hat{B}} + \Gamma^{\hat{A}}_{\hat{B}}$, it will satisfy Eq. (4). Thus it is the 1-form $\Omega^{\hat{A}}_{\hat{B}}$, that is to be substituted into the constraint (29). Taking into account the definition (16), we have

$$\tilde{R}^{A}_{B} = R^{A}_{B} + d\Gamma^{A}_{B} - \omega^{C}_{B} \wedge \Gamma^{A}_{C} - \Gamma^{C}_{B} \wedge \omega^{A}_{C} - \Gamma^{C}_{B} \wedge \Gamma^{A}_{C},$$

where

$$R^{A}_{B} = d\omega^{A}_{B} - \omega^{C}_{B} \wedge \omega^{A}_{C}$$

is the curvature of the connection $\omega^{A}_{B}$. Then using Eq. (30) we obtain finally that condition (29) is equivalent to the following curvature constraint

$$R^{\alpha}_{\beta\alpha\gamma}\varepsilon^{\beta\gamma} = 0.\tag{3.31}$$

Of course, this final form of the second order constraint depends on the choice of conventional constraints in Eq. (1.2), which could be different. (The only purpose of Eq. (1.2) was to reduce the set of connections.)

To conclude, we have proved that the given SCR($\zeta$)-structure is induced on a surface by the trivial $G(\zeta)$-structure in $\mathbb{C}^{4|4}$ if and only if (i) the torsion of some (and hence any) connection in this SCR($\zeta$)-structure satisfies (12) for $\zeta \neq \infty$, or (12) and (13) for $\zeta = \infty$; (ii) when $\zeta \neq \infty$, the curvature of some (and hence any) connection obeying (1.2) satisfies (31). This is just what we claimed at the end of Sect. 1.

## Appendix A. Spencer Cohomologies[5]

Let us consider some linear space $\mathfrak{g}$ that consists of $M \times N$ matrices, or linear maps from $V$ to $W$, where $V$ and $W$ are vector spaces: $V \simeq \mathbb{R}^{M}$ with coordinates $(x^{b})$, while $W \simeq \mathbb{R}^{N}$ with coordinates $(y^{\hat{a}})$. (The generalization of the considerations that follow to the case of superspaces is straightforward.) That is to say, consider some subspace $\mathfrak{g} \subset \mathrm{Hom}(V, W)$. The $k^{\text{th}}$ prolongation, $\mathfrak{g}^{(k)}$, of the subspace $\mathfrak{g}$ is defined as the space of

5 See, e.g. [6, 7]

tensors, $t^{\hat{a}}_{b_1 \dots b_{k+1}}$, symmetric in the lower indices and such that the matrix $t^{\hat{a}}_{b_1 \dots b_{k+1}}$ corresponds to an element of $\mathfrak{g}$ for any fixed values of the indices $b_2, \dots, b_{k+1}$. Furthermore, we set $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(-1)} = W$. For $i, k = 0, 1, 2, \dots$, consider also the space $C^{k,i}(\mathfrak{g})$, which consists of tensors, $t^{\hat{a}}_{b_1 \dots b_k; c_1 \dots c_i}$, that are, roughly speaking, arbitrary tensors from $\mathfrak{g}^{(k-1)}$ supplied with additional antisymmetric indices $c_1, \dots, c_i$. The elements of $C^{k,i}(\mathfrak{g})$ will be called cochains. The Spencer differential $\partial$ is an operator acting from $C^{k,i}(\mathfrak{g})$ to $C^{k-1,i+1}(\mathfrak{g})$ as follows:

$$\begin{aligned} t^{\hat{a}}_{b_1 \dots b_k; c_1 \dots c_i} &\to (\partial t)^{\hat{a}}_{b_1 \dots b_{k-1}; c_1 \dots c_{i+1}} \\ &= t^{\hat{a}}_{b_1 \dots b_{k-1}[c_1; c_2 \dots c_{i+1}]}, \end{aligned} \tag{A.1}$$

where the antisymmetrization in the $i+1$ last indices (with a factor of $1/(i+1)!$ included) is assumed. The operator (1) satisfies clearly $\partial\partial = 0$. One says that a cochain $\alpha \in C^{k,i}(\mathfrak{g})$ is closed if $\partial\alpha = 0$, and exact if $\alpha = \partial\beta$ for some $\beta \in C^{k+1,i-1}(\mathfrak{g})$. If we denote by $Z^{k,i}(\mathfrak{g})$ the subspace of closed cochains and by $B^{k,i}(\mathfrak{g})$ the subspace of exact cochains in $C^{k,i}(\mathfrak{g})$, then from $\partial\partial = 0$ it follows that $B^{k,i}(\mathfrak{g}) \subset Z^{k,i}(\mathfrak{g})$. Therefore we may define the cohomology group of degree $(k, i)$ as a coset space $H^{k,i}(\mathfrak{g}) = Z^{k,i}(\mathfrak{g})/B^{k,i}(\mathfrak{g})$. The cosets $H^{k,i}(\mathfrak{g})$, $k, i = 0, 1, 2, \dots$, are called the Spencer cohomology groups of the space $\mathfrak{g} \subset \mathrm{Hom}(V, W)$.

The space $\mathfrak{g}$ is called $p$-acyclic, if $H^{k,i}(\mathfrak{g}) = 0$ for $k > 0$ and $0 \leqq i \leqq p$. Furthermore, $\mathfrak{g}$ is called involutive, if $H^{k,i}(\mathfrak{g}) = 0$ for $k > 0$ and $i \geqq 0$. (Note that every space $\mathfrak{g}$ is 1-acyclic).

Let us consider, as an example, the space of all $M \times N$ matrices, i.e. $\mathfrak{g} = \mathrm{Hom}(V, W)$. It can be demonstrated in a direct way, using, for instance, the Young tableau calculus, that the space $\mathfrak{g} = \mathrm{Hom}(V, W)$ is always involutive. We remark in passing that this statement is essentially the Poincaré lemma for $W$-valued differential forms on $\mathbb{R}^M$ with arbitrary polynomial coefficients. Another example is $\mathfrak{g} = \mathrm{gl}(n, \mathbb{C})$, that is the space of all complex linear maps of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ into itself, considered as a subspace in $\mathrm{gl}(2n, \mathbb{R}) = \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. To see that $\mathrm{gl}(n, \mathbb{C})$ is involutive too, it is convenient to arrange the coordinates of $\mathbb{R}^{2n}$ into two sets of complex numbers, $(z^a)$, $a = 1, \dots, n$ and $(z^{\bar{a}})$, which are the complex coordinates in $\mathbb{C}^n$ and their conjugates. Then $\mathrm{gl}(n, \mathbb{C})$ becomes the space of complex $2n \times 2n$ matrices $X^{\hat{a}}_{\hat{b}}$ with $\hat{a}$, $\hat{b} = 1, \dots, n; \bar{1}, \dots, \bar{n}$, such that $X^{\bar{a}}_{\bar{b}} = (X^a_b)^*$ and $X^{\bar{a}}_b = X^a_{\bar{b}} = 0$. In this representation the problem is factorized in an obvious way and the computation of cohomologies of $\mathrm{gl}(n, \mathbb{C})$ is reduced essentially to the previous example.

The same factorization occurs in a more general case of a subspace $\mathfrak{g} \subset \mathrm{Hom}(V, W)$, which consists of matrices satisfying the only condition that some of the matrix elements on certain places must vanish for all matrices in $\mathfrak{g}$. Following the authors of ref. [8], we call such spaces multifoliate[6]. Multifoliate subspaces in $\mathrm{Hom}(V, W)$ are always involutive. Let us consider an example of this sort, which proves to be useful in our study of supergravity (Sect. 3 and Appendix E). Let us take the superspaces $V = W = \mathbb{C}^{4|4}$ with coordinates arranged into two groups: $(z^{\hat{A}}) = (z^A; z^{\bar{A}})$, where $(z^A) = (z^a, \theta^\alpha, \bar{\varphi}^{\dot{\pi}})$, $a = 1, 2, 3, 4$; $\alpha = 1, 2$; $\dot{\pi} = \dot{1}, \dot{2}$, are the complex coordinates in $\mathbb{C}^{4|4}$, while $(z^{\bar{A}}) = (z^{\bar{a}}, \bar{\theta}^{\dot{\alpha}}, \varphi^\pi)$ are their complex conjugates. Consider the

6 In [8] the matrix Lie algebras of this type were considered

space $\mathfrak{g}$, which consists of matrices $X^{\hat{A}}_{\hat{B}}$ obeying the following conditions

$$X^{a}_{\bar{\pi}} = X^{\bar{\alpha}}_{\pi} = 0, \quad X^{A}_{\bar{B}} = X^{\bar{A}}_{B} = 0. \tag{A.2}$$

(Of course, $X^{\bar{A}}_{\bar{B}} = (X^{A}_{B})^{*}$, while the last line of Eq. (2) means that $\mathfrak{g}$ consists of complex linear maps.) The so defined space $\mathfrak{g}$, being, in fact, a Lie algebra, is multifoliate and, hence, involutive. This example is intimately connected with the algebras $\mathfrak{g}(\zeta)$ appearing in the context of supergravity (see, in particular, Appendix E).

## Appendix B. Criterion of Formal Integrability

Our aim in this appendix is to make it manifest that some mathematical methods, used in the theory of nonlinear partial differential equations, survive the advance of superspaces. For this purpose we review at an elementary level the relevant mathematical constructions.

The modern status of the subject and further references can be found in refs. [6] and [7], that we shall follow in the main. However, we will use throughout local coordinates and make no attempts to account for the conditions like non-degeneracy or constancy of ranks of some maps, etc. The advantage of such a cavalier treatment will be that the superspace generalization becomes self-evident.

Let us consider a system of partial differential equations, with $y^a$ ($a = 1, \ldots, N$) being the unknown functions of the arguments $x^m$ ($m = 1, \ldots, M$). Without loss of generality we may study a first order system of the following form:

$$F\left(\frac{\partial y^a}{\partial x^m}, y^a, x^m\right) = 0, \tag{B.1}$$

where $F = (F^1, \ldots, F^\tau)$, with $\tau$ being the number of distinct equations in the system. The function $F$ will be understood throughout to satisfy the condition that the equation

$$F(p^a_m, p^a, x^m) = 0 \tag{B.2}$$

is compatible with respect to $p^a_m$ for any given $p^a$ and $x^m$. We can proceed from the first order equations to an equivalent system of the second order by adding the equations that correspond to

$$F_n(p^a_{\ell m}, p^a_m, p^a, x^m) = 0 \tag{B.3}$$

for $\ell, n = 1, \ldots, M$, where we set, as a definition,

$$F_n = \frac{d}{dx^n} F \equiv p^a_{mn} \frac{\partial F}{\partial p^a_m} + p^a_n \frac{\partial F}{\partial p^a} + \frac{\partial F}{\partial x^n}. \tag{B.4}$$

The resulting second order system consists of Eq. (2) together with Eqs. (3), where the second derivatives of the unknown function $p^a = y^a(x)$ are to be inserted in place of $p^a_{mn}$, and so on. Proceeding further in an obvious way, at the $i^{\text{th}}$ step we obtain a system of the order $(i + 1)$, called the $i^{\text{th}}$ prolongation of the system (1). The top order equation is of the form

$$F_{n_1 \ldots n_i}(p^a_{m_1 \ldots m_{i+1}}, \ldots, p^a_{m_1}, p^a, x^m) = 0, \tag{B.5}$$

with $F_{n_1 \dots n_i}$ defined by applying $i$ times $d/dx$ to the function $F$ (cf. Eq. (4)).

Let us try to find a solution to Eq. (1) as a formal power series,

$$y^a(x) = p^a + \sum_k \frac{1}{k!} p^a_{m_1 \dots m_k}(x^{m_1} - x_0^{m_1}) \dots (x^{m_k} - x_0^{m_k}). \tag{B.6}$$

Then the coefficients of this expansion must obviously satisfy an infinite chain of equations, namely, Eq. (2) at $x = x_0$ and Eq. (5) at $x = x_0$, for $i = 1, 2, 3, \dots$. Note that $p^a_{m_1 \dots m_{i+1}}$ enters Eq. (5) linearly. Therefore, if some $p^a$, $p^a_m$ satisfying (2) at $x = x_0$ have been found and we are looking for higher coefficients successively, then at each step we have to solve a certain system of linear algebraic equations. The following definition is in order.

The system of partial differential equations (1) is called formally integrable if, given arbitrary $p^a_m$, $p^a$, $x_0^m$ satisfying (2), the arising linear algebraic equations are compatible at each step in an order by order search of higher coefficients for every choice of solutions at the preceding steps.

Let us consider the conditions for the formal integrability. For this purpose we assume that the formal integrability holds up to the $(k+1)^{\mathrm{st}}$ order. Then we have to ascertain the conditions for this to be valid also in the next order. Suppose some solutions $p^a, p^a_{m_1}, \dots, p^a_{m_1 \dots m_{k+1}}$, have been chosen at the steps from the first to the $k^{\mathrm{th}}$ one. With the choice made in all points $x$, these coefficients may be considered as functions of $x$. Let $x_0$ be an arbitrary point. Under the above assumption it is always possible to arrange it so that the functions $p^a(x), p^a_{m_1}(x), \dots, p^a_{m_1 \dots m_{k+1}}(x)$ obey the following conditions. They satisfy at every $x$ Eq. (2) and Eqs. (5) with $i = 1, \dots, k$, whereas $p^a(x_0), p^a_{m_1}(x_0), \dots, p^a_{m_1 \dots m_{k+1}}(x_0)$ correspond to an arbitrary given solution of these equations at $x = x_0$, and, moreover,

$$\left.\frac{\partial p^a}{\partial x^{m_1}}\right|_{x = x_0} = p^a_{m_1}(x_0), \dots, \left.\frac{\partial p^a_{m_1 \dots m_k}}{\partial x^{m_{k+1}}}\right|_{x = x_0} = p^a_{m_1 \dots m_{k+1}}(x_0). \tag{B.7}$$

At the next step we have to be interested in the compatibility of Eq. (5) for $i = k+1$ at the point $x = x_0$. This equation may be written symbolically as

$$A_a^{m_1} p^a_{m_1 m_2 \dots m_{k+2}} + B_{m_2 \dots m_{k+2}} = 0, \tag{B.8}$$

where the following notation is used,

$$A_a^m = A_a^m(p_n^b, p^b, x_0^n) \equiv \left.\frac{\partial F}{\partial p^a_m}\right|_{x = x_0}. \tag{B.9}$$

Equation (8) is a system of linear inhomogeneous equations with respect to $p^a_{m_1 \dots m_{k+2}}$, while $B_{m_2 \dots m_{k+2}}$ in Eq. (8) stands for the terms depending on $p$'s of order less than $k+2$. The solution of Eq. (8) is requested to be symmetric in the indices $m_i$ $(i = 1, \dots, k+2)$. Note, however, that the existence of solutions at the preceding steps guarantees at any rate the compatibility of the equations

$$A_a^{m_1} v^a_{m_1 m_2 \dots m_{k+1} n} + B_{m_2 \dots m_{k+1} n} = 0, \tag{B.10}$$

where $v^a_{m_1 \dots m_{k+1} n}$ is required only to be symmetric in $k+1$ indices $m_i$, but not necessarily in all $k+2$ indices. The compatibility of Eq. (10) can be shown by substituting the above functions $p(x)$ into Eq. (5) for $i = k$, giving zero on the right-hand side and by the subsequent differentiation of the left-hand side with respect to all entries of $x$. Then $v^a_{m_1 \ldots m_{k+1} n} = \partial p^a_{m_1 \ldots m_{k+1}} / \partial x^n |_{x=x_0}$ gives, on account of (7), a particular solution to Eq. (10). Furthermore, from the explicit form of the terms $B_{m_2 \ldots m_{k+1} n}$ in (10), one can conclude that

$$A^{m_1}_a v^a_{m_1 \ldots m_k [ln]} = 0. \tag{B.11}$$

(Here [ ] denotes the antisymmetrization, whereas ( ) will denote the symmetrization). Since $v^a_{m_1 \ldots m_k l n} = v^a_{(m_1 \ldots m_k l) n}$, we have also

$$v^a_{m_1 \ldots m_{k-1} [m[ln]]} = 0. \tag{B.12}$$

Any solution of the linear inhomogeneous equation (10) differs from the above particular solution, $v^a_{m_1 \ldots m_{k+1} n}$, by an arbitrary solution, $u^a_{m_1 \ldots m_{k+1} n}$, of the corresponding homogeneous equations:

$$\begin{aligned} &A^{m_1}_a u^a_{m_1 \ldots m_{k+1} n} = 0, \\ &u^a_{m_1 \ldots m_{k+1} n} = u^a_{(m_1 \ldots m_{k+1}) n}. \end{aligned} \tag{B.13}$$

One may try to match some solution of Eqs. (13) to the particular solution of (10) so as to get

$$v^a_{m_1 \ldots m_k [ln]} + u^a_{m_1 \ldots m_k [ln]} = 0. \tag{B.14}$$

If one succeeded in doing so, one would obtain another solution of (10), which is symmetric in all $k + 2$ indices due to (14). That is to say, Eq. (8) would be compatible in such a case, and, hence, the hypothesis concerning the preceding steps would be valid at the $(k + 1)^{\text{st}}$ step as well. Now we need the following definition.

The symbol of a system of differential equations is the vector space $\sigma$ of solutions of the linear algebraic equations

$$A^m_a q^a_m = 0, \tag{B.15}$$

where $A^m_a$ is defined in (9). This space $\sigma$ is to be considered as a subspace in $\mathrm{Hom}(V, W)$, where $\dim W = N$, the number of the unknown functions $y^a(x)$, and $\dim V = M$, the number of the arguments $x^m$. (Note that since $A^m_a$ may depend on $p^b_n, p^b, x^n$, so may the space $\sigma$.)

One can observe that the obstruction to the compatibility of Eq. (8) corresponds to an element of the Spencer cohomology group $H^{k,2}(\sigma)$ (see Appendix A). Indeed, from Eq. (11) we see that $h^a_{m_1 \ldots m_k, ln} \equiv v^a_{m_1 \ldots m_k [ln]}$ corresponds to a cochain, $h \in C^{k,2}(\sigma)$. Moreover, Eq. (12) means that this cochain is closed, i.e. $\partial h = 0$. Similarly, any solution of (13) corresponds to some $u \in C^{k+1,1}(\sigma)$. Since $v^a_{m_1 \ldots m_k [ln]}$, a solution of (10), is defined only up to an arbitrary solution of (13), we see that $h$ is defined up to an exact cochain (because $u^a_{m_1 \ldots m_k [ln]}$ corresponds to $\partial u$). Therefore only the cohomology class, $[h] \in H^{k,2}(\sigma)$, of the closed cochain $h$ is relevant. Now Eq. (14), if compatible, implies that $h = \partial u$ for some $u$, or, equivalently, $[h] = 0$. From the above reasoning we conclude that, under the assumptions made, Eq. (8) is compatible if and only if the corresponding cohomology class $[h]$ is trivial. Remember that Eq. (8), being identical with (5) for $i = k + 1$, is considered as an equation on the $(k + 2)^{\text{nd}}$ order coefficients of a formal solution (6), provided some coefficients have been already found in all orders up to $k + 1$.

Finally, for a system of partial differential equations (1) to be formally integrable, the obstructions arising in solving it in terms of a formal power series must vanish at each step. This gives a set of integrability conditions of increasing order imposed on the functions $F$ in Eq. (1). We can conclude, that if $H^{k,2}(\sigma) = 0$ for $k$ greater than some $K$, one must take into account the integrability conditions of order not greater than $K + 1$ in derivatives of $F$. (It can be shown that, in general, only a finite number of the cohomology groups $H^{k,2}(\sigma)$ may be non-zero.) In particular, one has the following theorem.

If the symbol of the system (1) is 2-acyclic (i.e. $H^{k,2}(\sigma) = 0$ for $k > 0$) and Eqs. (3) are compatible for any $x^m$, $p^a$, $p^a_m$ satisfying (2), then the system (1) is formally integrable.

We see, that, in the case of a 2-acyclic symbol, only the first order integrability conditions are required; these are the constraints on the first derivatives of the functions $F$ ensuring the compatibility of Eq. (3). When $H^{k,2}(\sigma)$ become zero for higher $k$ only, more integrability conditions must be involved. Till now we have nothing to do with the convergence of our formal solutions. As to the existence of genuine solutions, we remark only the following: If the system (1) is formally integrable and the functions $F$ are real-analytic, then it is known that there are solutions which correspond to convergent power series [6, 7]. If, moreover, the symbol is involutive (i.e. $H^{k,i}(\sigma) = 0$ for $i \geqq 0, k > 0$[7]), then more detailed information is available converning the number of different solutions. Such partial differential equations correspond to the involutive systems of Cartan. In applications to our problems, however, we will use only the mere existence of solutions.

## Appendix C. The Proof of the Theorem

We continue the proof of the theorem of Sect. 2, using the results recapitulated in Appendices A and B. Remember that the theorem is intended to describe the necessary and sufficient conditions for the induced structure. These coincide, as we have already seen, with the conditions ensuring that a certain system of partial differential equations has a solution. The aim of the theorem is to give an equivalent formulation of these conditions, which can be more easily applied. Before proceeding with the proof let us study directly the above-mentioned system of differential equations by means of the methods discussed in Appendix B.

When considering the induced structure, we have to fix a vector space $W$, a subgroup $G$ of the group of linear transformations of $W$ and a subspace $V$ in $W$. Let us consider the restrictions to $V$ of linear maps defined on $W$ and belonging to the group $G$. Let $G_V$ denote the manifold of the resulting maps from $V$ into $W$. It is a submanifold in the space $\mathrm{Hom}(V, W)$ of all linear maps from $V$ into $W$. The notations of Sect. 2 correspond to the following particular choice made for simplicity: $W = \mathbb{R}^N$, while $V$ must be the standard $M$-dimensional subspace generated by the first $M$ vectors of the standard basis in $\mathbb{R}^N$. In this case $G$ is to be considered as consisting of $N \times N$ matrices. Assuming these conventions, the system of differential equations, to be studied in the problem of the induced

7 Note that $H^{k,0}(\sigma) = H^{k,1}(\sigma) = 0$ for $k > 0$ and for any $\sigma$

structure, takes the form of Eq. (2.1). Then the manifold $G_V$ corresponds to the rectangular $M \times N$ matrices that appear on the right-hand side of Eq. (1.1). Then (1.1) gives a system of equations with respect to the unknown functions $y^{\hat{a}}(x)$, $\hat{a}=1,\dots,N$, defined by the requirement that the rectangular matrix $g_b^{\hat{a}}= e_b^m(x)\,\partial y^{\hat{a}}(x)/\partial x^m$ should belong to $G_V$ for each $x$. In this case the symbol of the resulting system (see Appendix B) can be identified with the subspace in $\mathrm{Hom}(V, W)$ that corresponds to the tangent space of the submanifold $G_V \subset \mathrm{Hom}(V, W)$. Let $\mathfrak{g}_V$ denote the linear subspace in $\mathrm{Hom}(V, W)$ that corresponds to the Lie algebra $\mathfrak{g}$ of the group $G$ in the same way, as $G_V$ corresponds to $G$. One can identify the tangent space to $G_V$ at each point with this subspace $\mathfrak{g}_V$. Thus $\mathfrak{g}_V \subset \mathrm{Hom}(V, W)$ can be considered as the symbol of our system of differential equations. To see all this explicitly one can write this system locally as

$$\tilde{\Phi}\left(\frac{\partial y^{\hat{a}}}{\partial x^m}\, e_b^m(x)\right)=0, \tag{C.1}$$

where $\tilde{\Phi}=(\tilde{\Phi}^1,\dots,\tilde{\Phi}^{\chi})$ are some functions singling out $G_V$ as submanifold defined (locally) by the equation $\tilde{\Phi}(g_b^{\hat{a}})=0$ in the space of all $M \times N$ matrices (i.e. in $\mathrm{Hom}(V, W)$). Then the correspondence with Appendix B (in particular, with Eq. (B.1)) can be established by setting $F(p_m^{\hat{a}}, p^{\hat{a}}, x^m)=\tilde{\Phi}(p_m^{\hat{a}} e_b^m(x))$, the rest being straightforward. According to Appendix C we are able, in principle, to specify the conditions ensuring that system (1) has a solution. In this way we would obtain a finite number of integrability conditions corresponding to non-vanishing groups in the chain of Spencer cohomology groups $H^{k,2}(\mathfrak{g}_V)$, $k \geqq 0$ of the subspace $\mathfrak{g}_V$ in $\mathrm{Hom}(V, W)$. These integrability conditons would then arise in the form of differential constrains on the frame field $e_a^m(x)$ in Eq. (1), the orders of these constrains being equal to those $k$'s for which $H^{k-1,2}(\mathfrak{g}_V) \neq 0$. Our final aim is, however, to obtain the constraints on the induced structure in the form established in the statement of the theorem. In order to avoid somewhat tedious calculations that might be encountered at this point, let us proceed differently.

In Sect. 2 we succeeded in reducing the problem to another equation, which was written symbolically as Eq. (2.7). That is to say, we have shown the following: for an induced $G'$-structure, that is if Eq. (2.1) has a solution, we constructed explicitly a 1-form field $\Omega$ that satisfies the conditions of the theorem, thus proving one part of the statement. Assuming, conversely, that such a field $\Omega$ really exists, we showed that if Eq. (2.7) has a solution, so does Eq. (2.1), i.e. the $G'$-structure under consideration does correspond to an induced one. To complete the proof of the theorem, it must be shown that the condition $d\Omega+\frac{1}{2}[\Omega \wedge \Omega]=0$ is sufficient for the existence of such a $G_\infty$-valued function $h(x)$ that satisfies $\Omega=h^{-1}\,dh$. For an arbitrary matrix group $G$ (i.e. $G \subset GL(N, \mathbb{R})$), the corresponding group $G_\infty$ may be infinite dimensional. The exact meaning of the expression $\Omega=h^{-1}\,dh$ was already explained, allowing in the general case for infinite dimensional groups $G_\infty$. This led us to a system of equations (2.8), (2.9). Now we are going to find the conditions that are sufficient for the compatibility of these equations.

Let us rewrite Eqs. (2.8) and (2.9) as follows:

$$\Phi\left(\frac{\partial \phi^{\hat{a}}(x, y)}{\partial y^{\hat{b}}}\right)=0, \tag{C.3}$$

$$\frac{\partial \phi^{\hat{a}}(x,y)}{\partial x^m} + X^{\hat{a}}_m(x,\phi(x,y)) = 0, \tag{C.4}$$

where $\Phi = (\Phi^1, \ldots, \Phi^\tau)$ are some functions singling out the group $G$ as a submanifold defined (locally) by $\Phi(g^{\hat{a}}_{\hat{b}}) = 0$ in the space of all $N \times N$ matrices. Remember that $X^{\hat{a}}_m(x,y)$ depends on $y^{\hat{b}}$ $(\hat{a}, \hat{b} = 1, \ldots, N)$ as a formal power series corresponding to the given $\mathfrak{g}_\infty$-valued 1-form field $\Omega(x)$. Analogously the unknown "functions" $\phi^{\hat{a}}(x,y)$ are, in fact, formal power series in $y^{\hat{b}}$, with coefficients being usual functions of $x^m$, $m = 1, \ldots, M$. Equations (3) and (4) are to be understood, of course, order by order in $y$. According to Appendix B, to obtain the desired result we have to find out whether or not the system (3)–(4) is formally integrable. It means that we must consider the compatibility of that system, regarding $\phi^{\hat{a}}(x,y)$ there as a formal power series in both variables, $x$ and $y$. It doesn't matter that $X^{\hat{a}}_m(x,y)$ itself is a formal power series in $y$, the methods of Appendix B are still applicable. A straightforward calculation reveals that there are only the following integrability conditions:

$$\frac{\partial \Phi(g)}{\partial g^{\hat{c}}_{\hat{b}}}\, g^{\hat{c}}_{\hat{a}} \frac{\partial X^{\hat{a}}_m(x,y)}{\partial y^{\hat{b}}} = 0 \quad \text{if } \Phi(g) = 0, \tag{C.5}$$

$$\frac{\partial X^{\hat{a}}_m(x,y)}{\partial x^n} - \frac{\partial X^{\hat{a}}_n(x,y)}{\partial x^m} - \frac{\partial X^{\hat{a}}_m(x,y)}{\partial y^{\hat{b}}} X^{\hat{b}}_n(x,y)$$

$$+ \frac{\partial X^{\hat{a}}_n(x,y)}{\partial y^{\hat{b}}} X^{\hat{b}}_m(x,y) = 0. \tag{C.6}$$

As it can be easily seen, the condition (5) means that for each fixed $m$ the matrix $(Y^{\hat{a}}_{\hat{b}})_m = \partial X^{\hat{a}}_m / \partial y^{\hat{b}}$ should belong to the Lie algebra $\mathfrak{g}$ of the group $G$. This requirement is, however, satisfied merely by definition of $X^{\hat{a}}_m(x,y)$ which is the counterpart of the $\mathfrak{g}_\infty$-valued 1-form field $\Omega$ (cf. Sect. 2). Recalling the description of the Lie algebra $\mathfrak{g}_\infty$ in terms of formal vector fields on $\mathbb{R}^N$, we observe also that Eq. (6) can be rewritten as $d\Omega + \frac{1}{2}[\Omega \wedge \Omega] = 0$, using the above-mentioned correspondence between $X^{\hat{a}}_m(x,y)$ and $\Omega$. We obtain finally, that $d\Omega + \frac{1}{2}[\Omega \wedge \Omega] = 0$ is indeed the integrability condition for the equation $\Omega = h^{-1}\, dh$ with $h$ being a $G_\infty$-valued function. Then, in view of the reasoning of Sect. 2, the theorem is proved.

We remark that the proof is valid if everything is real-analytic. However, it is valid also in the smooth case if the group $G$ is of finite type, i.e. when $G_\infty$ is finite dimensional. (In the latter case the formal integrability need not to be involved; one can rather use the Frobenius theorem for the equation $h^{-1}\, dh = \Omega$).

Some particular examples of how to apply the theorem are contained in Sect. 3 and Appendix D. There only the constraints on the torsion and the curvature of an induced structure appeared. Such constraints correspond to the integrability conditions of the first and second order if dealing with Eq. (1), or to the terms of order not greater than two with respect to $y$ if dealing with Eq. (6). These may not suffice, however, in the general case. As it was discussed at the end of Sect. 2, generally one has a set of constraints of increasing order. To establish them, one has to find the compatibility conditions for the equation $d\Omega + \frac{1}{2}[\Omega \wedge \Omega] = 0$. Maintaining the notations of Sect. 2, this equation, when solved successively with respect to $\Omega^{(0)}, \Omega^{(1)}, \Omega^{(2)}$, etc., amounts to an infinite chain of linear algebraic equations. Here

$\Omega^{(k)}$ is a 1-form field with values in $\mathfrak{g}^{(k)}$; it must be a solution of the equation $R^{(k-1)}=0$ that arises at the $(k+1)^{\text{st}}$ step. Specifically, one has $\Omega^{(k)\hat{a}}{}_{\hat{b}_1\ldots\hat{b}_{k-1}\hat{b}_k} \wedge \Theta^{\hat{b}_k} = P^{(k-1)\hat{a}}{}_{\hat{b}_1\ldots\hat{b}_{k-1}}$, where $P^{(k-1)}$ is a 2-form with values in $\mathfrak{g}^{(k-1)}$. which depends on $\Omega^{(\ell)}$'s with $\ell<k$. Since $\Theta^{b'}=0$ for $b'=M+1,\ldots,N$, we observe about the equations on those components of $\Omega^{(k)}$ for which at least one of $\hat{b}_j$'s assumes a value from the set $M+1,\ldots,N$, that such equations are always compatible. It remains to consider the truncated components of $\Omega^{(k)}$ and $P^{(k-1)}$. These obviously correspond to the forms taking values respectively in $\mathfrak{g}_V^{(k)}$ and $\mathfrak{g}_V^{(k-1)}$ (see the definition above). Now, a $q$-form $\omega$ with values in $\mathfrak{g}_V^{(p)}$ corresponds some Spencer cochain $\tilde{\omega}\in C^{p,q}(\mathfrak{g}_V)$. Furthermore, the $(q+1)$-form $\omega^{\hat{a}}_{b_1\ldots b_p}\wedge\Theta^{b_p}$ corresponds then to $\partial\tilde{\omega}\in C^{p-1,q+1}(\mathfrak{g}_V)$. Hence the equation on $\Omega^{(k)}$ can be rendered in terms of $C^{p,q}(\mathfrak{g}_V)$-valued functions, giving $\partial\tilde{\Omega}(x)=\tilde{P}(x)$, where $\tilde{\Omega}(x)\in C^{k,1}(\mathfrak{g}_V)$ and $\tilde{P}(x)\in C^{k-1,2}(\mathfrak{g}_V)$. Note that the consistency requires $\partial\tilde{P}=0$. This is, however, always satisfied, by virtue of Bianchi identities (namely, $R^{(k-1)}\wedge\Theta=0$, provided $R^{(\ell)}=0$ for $\ell<k-1$). If $H^{k-1,2}(\mathfrak{g}_V)=0$, the condition $\partial\tilde{P}=0$ is sufficient for the compatibility of the equation $\partial\tilde{\Omega}=\tilde{P}$. Consequently, the non-trivial constraints may arise only in orders $k$, for which $H^{k-1,2}(\mathfrak{g}_V)\neq 0$. (This agrees, of course, with what we discussed in connection with Eq. (1).) It is known that for any given $\mathfrak{g}_V$ one can find, in principle, such a finite number $k_0$ (depending on $\mathfrak{g}_V$), that $H^{k,2}(\mathfrak{g}_V)=0$ if $k\geqq k_0$. (This is valid in the superspace too.) Thus in order to verify the existence of the field $\Omega$ that satisfies the conditions of the theorem it suffices to examine the compatibility of a finite number of linear algebraic systems from the infinite chain contained in the equation $d\Omega+\frac{1}{2}[\Omega\wedge\Omega]=0$. The resulting necessary and sufficient conditions on the induced structure are of orders not greater than $k_0$, where $k_0$ depends on the particular properties of the space $\mathfrak{g}_V$.

## Appendix D. Gauss–Codazzi Theorem

Let us consider an example, well known in mathematics, from the point of view of the general results obtained in Sect. 2. In Riemannian geometry one deals with the metric, which defines a non-degenerate scalar product in the tangent space at each point. Therefore one can define the fields of orthonormal frames (vierbein fields in General Relativity). These orthonormal frames are defined up to arbitrary rotations of the group $O(n)$ (if the dimension of the manifold equals $n$) and determine the corresponding metric uniquely. Hence every Riemannian metric corresponds to an $O(n)$-structure and vice versa. The flat metric corresponds to the trivial $O(n)$-structure.

Let us consider the flat Euclidean space of dimension $n+p$, i.e. the space $\mathbb{R}^{n+p}$ with the trivial $O(n+p)$-structure. Every $n$-dimensional surface in this space receives an induced metric in a familiar way. It can be easily seen that this metric corresponds precisely to the induced $G'$-structure, where $G'=O(n)$ in this case. The necessary and sufficient conditions for a given $n$-dimensional Riemannian geometry to be isometrically (locally) embedded in the flat $(n+p)$-dimensional Euclidean space are described by the theorem of Gauss and Codazzi (see, e.g. [5]). We are now able to show that this is a particular case of our theorem stated in Sect. 2. To make contact with the notations of that section we set $W = \mathbb{R}^{n+p}$ with coordinates $y^{\hat{a}} = (y^a, y^{a'})$, $\hat{a} = 1, \ldots, n+p$; $a = 1, \ldots, n$; $a' = n+1, \ldots, n+p$. We choose $V$ to be the space $\mathbb{R}^n$ considered as the subspace of $\mathbb{R}^{n+p}$ defined by the equations $y^{a'} = 0$.

First of all, we observe that $o(n+p)^{(1)} = 0$ that is all the prolongations of the algebra $o(n+p)$ vanish (see, e.g. [4]). Consequently the only integrability conditions that might be non-trivial are $R^{(-1)} = 0$, $R^{(0)} = 0$. According to Sect. 2 we must introduce a $W$-valued 1-form, $(\Theta^{\hat{a}}) = (e^a, 0)$, where $e^a$ are the 1-forms of an orthonormal (i.e. admissible) coframe field corresponding to the given Riemannian geometry on an $n$-dimensional manifold $\mathscr{M}$. Next we must find out under what conditions there exists an $o(n+p)$-valued 1-form field $\Omega^{\hat{a}}_{\hat{b}}$, that obeys

$$R^{(-1)\hat{a}} \equiv d\Theta^{\hat{a}} + \Omega^{\hat{a}}_{\hat{b}} \wedge \Theta^{\hat{b}} = 0, \tag{D.1}$$

$$R^{(0)\hat{a}}_{\hat{b}} \equiv d\Omega^{\hat{a}}_{\hat{b}} + d\Omega^{\hat{a}}_{\hat{c}} \wedge \Omega^{\hat{c}}_{\hat{b}} = 0. \tag{D.2}$$

Then the compatibility conditions for the systems of Eqs. (1), (2) will give the Gauss–Codazzi theorem. Explicitly Eq. (1) reads

$$de^a + \Omega^a_b \wedge e^b = 0, \tag{D.3}$$

$$\Omega^{a'}_b \wedge e^b = 0. \tag{D.4}$$

These equations, however, are always compatible. Indeed, if we denote the components of $\Omega^{a'}_b$ by $\Gamma^{a'}_{bc}$, i.e. $\Omega^{a'}_b = \Gamma^{a'}_{bc} e^c$, then Eq. (4) amounts to $\Gamma^{a'}_{bc} = \Gamma^{a'}_{cb}$. Notice also that since $\Omega^{\hat{a}}_{\hat{b}}$, corresponds to $o(n+p)$, its part, $\Omega^a_b$, is a matrix 1-form with values in $o(n)$. It is well known that a connection $\omega^a_b$ compatible with the metric can always be chosen to have vanishing torsion: $T^a \equiv de^a + \omega^a_b \wedge e^b = 0$. We see that Eqs. (3), (4) and, hence, Eq. (1), always have a solution. Thus only Eq. (2) remains non-trivial, giving a constraint for the induced metric.

Finally, it is necessary and sufficient for a Riemannian geometry to be realized on a surface in Euclidean space, that there exists an $o(n+p)$-valued 1-form field

$$(\Omega^{\hat{a}}_{\hat{b}}) = \begin{pmatrix} \omega^a_b & \Gamma^{a'}_{bc} e^c \\ -\Gamma^{b'}_{ac} e^c & \Omega^{a'}_{b'} \end{pmatrix}$$

satisfying (2) and such that $\Gamma^{a'}_{bc} = \Gamma^{a'}_{cb}$, while $\omega^a_b$ is the unique torsionless connection 1-form corresponding to the given orthonormal coframe $e^a$. Equations (2) for $\omega^a_b$, $\Gamma^{a'}_{bc}$, $\Omega^{a'}_{b'}$ are precisely those of Gauss and Codazzi. To see this clearly consider the simplest case, $p = 1$. Then $\Omega^{a'}_{b'} \equiv 0$. Furthermore, no explicit indices of type $a'$ are needed. In this case Eq. (2) reads as follows,

$$R_{abcd} = \Gamma_{ac} \Gamma_{bd} - \Gamma_{ad} \Gamma_{bc}, \tag{D.5}$$

$$D\Gamma_a = 0, \tag{D.6}$$

where $\frac{1}{2} R_{abcd} e^c \wedge e^d = d\omega_{ab} + \omega_{ae} \wedge \omega_{eb}$ is the Riemannian curvature, and $D\Gamma_a \equiv D(\Gamma_{ab} e^b) \equiv d\Gamma_a - \omega^b_a \wedge \Gamma_b$. If a symmetric tensor, $\Gamma_{ab} = \Gamma_{ba}$, satisfying (5) and (6) does exist, then the given Riemannian geometry corresponds to the internal geometry of a hypersurface in Euclidean space. In that case $\Gamma_{ab}$ defines a quadratic form called the second fundamental form of the hypersurface.

## Appendix E. A Calculation of Spencer Cohomologies in Supergravity

Here we intend to prove that in the problem of the induced SCR($\zeta$)-structure it suffices to consider the integrability conditions of the first and second order only. (That is to say, no higher order conditions beyond the algebraic constraints on the torsion and the curvature arise in $N=1$ supergravity.) In order to prove this, we have to show that certain Spencer cohomology groups vanish, namely $H^{k,2}(\mathfrak{g}(\zeta)_V)=0$ for $k>1$. (The definitions of Appendices A and C will be used throughout.)

Let $z^{\hat{A}}=(z^A;z^{\bar{A}})$ be the coordinates in $W\simeq\mathbb{C}^{4/4}$, which is considered as a real (8|8)-dimensional space. Here $z^A=(z^a,\theta^\alpha,\bar{\varphi}^{\dot{\pi}})$ and $z^{\bar{A}}=(z^{\bar{a}},\bar{\theta}^{\dot{\alpha}},\varphi^{\pi})$ are, as in Sect. 3, the complex coordinates in $\mathbb{C}^{4/4}$ and their conjugates respectively. We must consider also the real (4|4)-dimensional subspace $V$ in $W$. Let $x^A=(x^a,v^\alpha,\bar{v}^{\dot{\alpha}})$ be the coordinates in $V$, where $x^a$, $a=1,\ldots,4$, are real, while $v^\alpha=(\bar{v}^{\dot{\alpha}})^*$, $\alpha=1,2$, are complex. We shall use the same fixed subspace $V$, as in Sect. 3, see Eq. (3.1). It can be defined also by the embedding into $W$, so that $(x^A)$ is mapped to $(z^{\hat{A}})\equiv(z^A;z^{\bar{A}})$ with $z^A=x^A$, or, explicitly, $z^a=x^a$, $\theta^\alpha=v^\alpha$, $\bar{\varphi}^{\dot{\pi}}=\bar{v}^{\dot{\pi}}$. The Lie algebra $\mathfrak{g}(\zeta)$ consists of linear maps of $W$ into itself, with matrices $X^{\hat{A}}_{\hat{B}}$ satisfying Eqs. (3.5–6). Then, by definition, $\mathfrak{g}(\zeta)_V$ consists of maps from $V$ into $W$, that are the restrictions to the subspace $V\subset W$ of maps belonging to $\mathfrak{g}(\zeta)$. In other words, $\mathfrak{g}(\zeta)_V$ consists of maps, $(x^A)\to(z^{\hat{A}})=(X^{\hat{A}}_B x^B)$, where the rectangular matrix $X^{\hat{A}}_B$ obeys

$$X^{a}_{\dot{\beta}}=0,\quad X^{\alpha}_{\dot{\beta}}=0, \tag{E.1}$$

and

$$\mathrm{tr}_\zeta X=0, \tag{E.2}$$

as well as the reality conditions: $X^A_b=(X^{\bar{A}}_b)^*$ $X^A_\beta=(X^{\bar{A}}_{\dot{\beta}})^*$ and $X^A_{\dot{\beta}}=(X^{\bar{A}}_\beta)^*$. The notation $\mathrm{tr}_\zeta X$ is to be understood as before: $\mathrm{tr}_\zeta X=\zeta(X^a_a-X^\alpha_\alpha)-X^{\dot{\pi}}_{\dot{\pi}}$ if $\zeta\neq\infty$ and $\mathrm{tr}_\zeta X=X^a_a-X^\alpha_\alpha$ if $\zeta=\infty$.

In order to work out the Spencer cohomologies of the subspace[8] $\mathfrak{g}(\zeta)_V\subset\mathrm{Hom}(V,W)$, it is instructive to consider first $\bar{\mathfrak{g}}\subset\mathrm{Hom}(V,W)$, where $\bar{\mathfrak{g}}$ corresponds to matrices satisfying (1), but not (2). Thus $\bar{\mathfrak{g}}$ includes $\mathfrak{g}(\zeta)_V$ as a subspace, the kernel of the map $\mathrm{tr}_\zeta$ of $\bar{\mathfrak{g}}$ into[9] $\mathbb{C}$. The space $\bar{\mathfrak{g}}$ is multifoliate and, hence, involutive, that is $H^{k,i}(\bar{\mathfrak{g}})=0$ for $i\geqq 0$, $k>0$ (see Appendix A and, particularly, the last example therein). As a matter of course, $\mathrm{tr}_\zeta$ maps $\bar{\mathfrak{g}}$ on $\mathbb{C}$, and we have an exact sequence[10], $0\to\mathfrak{g}(\zeta)_V\to\bar{\mathfrak{g}}\to\mathbb{C}\to 0$. We can extend this to a sequence for cochains. Indeed, let us introduce the following notations for the spaces of Spencer cochains: $\mathscr{C}^{k,i}(\zeta)=C^{k,i}(\mathfrak{g}(\zeta)_V)$ and $\mathscr{D}^{k,i}=C^{k,i}(\bar{\mathfrak{g}})$. Then $\mathrm{tr}_\zeta$ can be extended, for any[11] $i,k$, to a map of $\mathscr{D}^{k,i}$ into $\mathscr{S}^{k-1,i}$ where $\mathscr{S}^{k,i}=C^{k,i}(\tilde{V})$ for $\tilde{V}=\mathrm{Hom}(V,\mathbb{C})$, the space of linear maps of the real space $V$ into $\mathbb{C}$. It follows that $\mathscr{C}^{k,i}(\zeta)$ is the kernel of the map $\mathrm{tr}_\zeta\colon\mathscr{D}^{k,i}\to\mathscr{S}^{k-1,i}$. Let $\mathscr{R}^{k-1,i}(\zeta)$ be the image of this map. Thus, for any

8 The superspace $\mathrm{Hom}(A,B)$, with $A$ and $B$ being vector superspaces, may be thought of as the space of all supermatrices of size $(p|q)\times(r|s)$, where $(p|q)=\dim A$, $(r|s)=\dim B$

9 Strictly speaking, we should have written $\mathbb{C}^{1|0}$. In what follows, we will continue to refer, somewhat incorrectly, to this (1|0)-dimensional complex superspace, as $\mathbb{C}$

10 One says that $0\to A\xrightarrow{i}B\xrightarrow{\pi}C\to 0$ is an exact sequence of vector spaces $A$, $B$, $C$, if $A\subset B$, with $i$ being the inclusion map, while $C=B/A$, with $\pi$ being the projection

11 If some of the integers $i,k$ is negative, we set $C^{k,i}(\sigma)=0$ for any space $\sigma$

$i, k = 0, 1, 2, \ldots$, we obtain exact sequences

$$0 \to \mathscr{C}^{k,i}(\zeta) \to \mathscr{D}^{k,i} \overset{\mathrm{tr}_\zeta}{\to} \mathscr{R}^{k-1,i}(\zeta) \to 0. \tag{E.3}$$

The action of the Spencer differential $\partial$ is defined on the complexes $\mathscr{C}(\zeta) = \bigoplus_{i,k} \mathscr{C}^{k,i}(\zeta)$, $\mathscr{D} = \bigoplus_{i,k} \mathscr{D}^{k,i}$ and $\mathscr{S} = \bigoplus_{i,k} \mathscr{S}^{k,i}$. It is easy to check also that $\mathscr{R}(\zeta) = \bigoplus_{i,k} \mathscr{R}^{k,i}$ is a subcomplex in $\mathscr{S}$ (i.e. the subspace $\mathscr{R}(\zeta)$ in $\mathscr{S}$ is closed under the action of $\partial$). Moreover, the sequences (3) are compatible with the operation $\partial$ (in particular, $\partial \circ \mathrm{tr}_\zeta = \mathrm{tr}_\zeta \circ \partial$). In other words, there is an exact sequence of differential complexes that corresponds to (3). In such cases it is known that if the cohomology groups of the middle complex in the sequence are trivial, those of the peripheral complexes are related by certain isomorphisms. In our case, recalling that $H^{k,i}(\mathscr{C}(\zeta)) \equiv H^{k,i}(\mathfrak{g}(\zeta)_V)$ and

$$H^{k,i}(\mathscr{D}) \equiv H^{k,i}(\tilde{\mathfrak{g}}) = 0 \quad \text{for } i \geqq 0, k > 0,$$

we have the following isomorphisms:

$$H^{k,i}(\mathfrak{g}(\zeta)_V) = H^{k,i-1}(\mathscr{R}(\zeta)), \quad i \geqq 0, k > 0, \tag{E.4}$$

The problem is reduced thus to the properties of the complex $\mathscr{R}(\zeta)$.

Let us consider the case $\zeta = \infty$. First of all $\tilde{V} = \mathrm{Hom}(V, \mathbb{C})$ consists of all complex covectors $v_A$. Consequently, every cochain $s \in \mathscr{S}^{k,i} = C^{k,i}(\tilde{V})$ corresponds to a tensor $s_{\ldots,\ldots}$, where the dots stand for the set of $k$ symmetric indices (before the comma) and of $i$ antisymmetric indices (after the comma); $\mathscr{S}^{k,i}$ corresponds to the space of all complex tensors $s_{\ldots,\ldots}$ with the above symmetry properties. Of course, the symmetries of indices are being understood properly to superspace (an obvious modification of the definitions cited in Appendix A). Now, the space $\mathscr{D}^{k,i} = C^{k,i}(\tilde{\mathfrak{g}})$ consists of tensors $d^{\hat{A}}_{B\ldots,\ldots}$ (where $B$ is one among $k$ symmetric indices), satisfying $d^{a}_{\beta\ldots,\ldots} = 0$ and $d^{\alpha}_{\dot{\beta}\ldots,\ldots} = 0$, in correspondence with Eq. (1). Then the space $\mathscr{R}^{k,i}(\infty)$, being a subspace in $\mathscr{S}^{k,i}$, consists of tensors $r_{\ldots,\ldots}$, that can be represented as $r_{\ldots,\ldots} = d^{b}_{b\ldots,\ldots} - d^{\beta}_{\beta\ldots,\ldots}$ for some cochain $d \in \mathscr{D}^{k+1,i}$ (that is $r = \mathrm{tr}_\zeta d$ for $\zeta = \infty$). We observe immediately that the components $r_{\ldots\dot{\alpha}\ldots,\ldots}$ of any $r$ in $\mathscr{R}^{k,i}(\infty)$ must vanish, since $d^{b}_{b\ldots\dot{\alpha}\ldots,\ldots} = d^{\beta}_{\beta\ldots\dot{\alpha}\ldots,\ldots} = 0$ according to the definition of $\mathscr{D}^{k,i}$. By a careful examination one can find that these tensors $r_{\ldots,\ldots}$ are otherwise arbitrary. That is to say, $\mathscr{R}^{k,i}(\infty)$ consists of all tensors satisfying $r_{\ldots\dot{\alpha}\ldots,\ldots} = 0$. Consequently, $\mathscr{R}^{k,i}(\infty)$ coincides with a space of Spencer cochains $C^{k,i}(h)$, where $h$ is a subspace in $\mathrm{Hom}(V, \mathbb{C})$, that consists of complex covectors $v_A$ satisfying $v_{\dot{\alpha}} = 0$. The space $h$ is obviously multifoliate, hence it is involutive. On account of $\mathscr{R}^{k,i}(\infty) = C^{k,i}(h)$ and in view of the isomorphism (4), we conclude finally that $H^{k,i}(\mathfrak{g}(\infty)_V) = 0$ for $i \geqq 0$, $k > 0$. Thus the space $\mathfrak{g}(\infty)_V$ is involutive.

In the case $\zeta \neq \infty$ we have to specify the tensors $r_{\ldots,\ldots}$, that constitute the space $\mathscr{R}^{k,i}(\zeta)$. These are tensors that can be represented as $r_{\ldots,\ldots} = \zeta(d^{b}_{b\ldots,\ldots} - d^{\beta}_{\beta\ldots,\ldots}) - d^{\dot{\pi}}_{\dot{\pi}\ldots,\ldots}$, for some $d^{\hat{A}}_{B\ldots,\ldots}$ belonging to $\mathscr{D}^{k+1,i}$. (Note that the upper index $\dot{\pi} = \dot{1}, \dot{2}$ refers to the coordinate $\bar{\varphi}^{\dot{\pi}}$ in the space $W$, while $\dot{\pi} = \dot{1}, \dot{2}$ at the bottom of $d^{\dot{\pi}}_{\dot{\pi}\ldots,\ldots}$ corresponds to $\bar{v}^{\dot{1}}, \bar{v}^{\dot{2}}$, the complex conjugates of the coordinates $v^1, v^2$ in $V$.) In analogy with the previous case, we have $r_{\ldots\dot{\alpha}\ldots,\ldots} = d^{\dot{\pi}}_{\dot{\pi}\ldots\dot{\alpha}\ldots,\ldots}$ for some $d \in \mathscr{D}^{k+1,i}$. Since for $d \in \mathscr{D}^{1,i}$ the components $d^{\dot{\pi}}_{\dot{\pi},\ldots}$ are quite arbitrary (except for symmetries prescribed), we obtain, that, unlike the previous case, $\mathscr{R}^{1,i}(\zeta) = \mathscr{S}^{1,i}$ if $\zeta \neq \infty$. On the other hand, we still have

$r_{\ldots\dot{\alpha}\beta\ldots,\ldots} = 0$ for any $r \in \mathscr{R}^{k,i}(\zeta)$ with $k > 1$. Indeed, $r_{\ldots\dot{\alpha}\beta\ldots,\ldots}$ must be equal to $d^{\pi}_{\dot{\pi}\ldots\dot{\alpha}\beta\ldots,\ldots}$, but these vanish for any $d \in \mathscr{D}^{k+1,i}$ due to antisymmetry in the fermionic indice $\dot{\alpha}, \beta, \dot{\pi}$. We observe that $\mathscr{R}^{k,i}(\zeta) = \mathscr{R}^{k,i}(\infty)$ for $k > 1$, while $\mathscr{R}^{1,i}(\zeta)$, $\zeta \neq \infty$, is larger than $\mathscr{R}^{1,i}(\infty)$. Consequently, the cohomology groups $H^{k,i}(\mathscr{R}(\zeta))$ for $\zeta \neq \infty$ coincide with $H^{k,i}(\mathscr{R}(\infty))$ if $k > 1$, while for $k = 1$ they may differ. On account of the results concerning $\zeta = \infty$ and of the isomorphism (4), we conclude that $H^{k,i}(\mathfrak{g}(\zeta)_V) = 0$ for $i \geqq 0$, $k > 1$. The cohomology group $H^{1,2}(\mathfrak{g}(\zeta)_V)$, $\zeta \neq \infty$, which is needed in the problem of induced structure, can be calculated and proves to be non-trivial. (This is what we have done implicitly in Sect. 3.)

Finally, $H^{k,i}(\mathfrak{g}(\zeta)_V) = 0$ for $i \geqq 0, k > 1$ and for all $\zeta$, while for $\zeta = \infty$, moreover, $H^{k,i}(\mathfrak{g}(\infty)_V) = 0$ for $i \geqq 0, k > 0$. That is why in minimal supergravity ($\zeta = \infty$) the torsion constraints alone suffice, whereas the non-minimal case ($\zeta \neq \infty$) requires also the curvature constraints.

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