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Structure of Supermanifolds and Supersymmetry Transformations

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Abstract. After giving a global, constraint-free Lagrangian formulation of the $N=1$ superspace supergravity in terms of super fibre bundles and differential forms over a supermanifold, we show that the concept of body manifold of a supermanifold provides a natural manner to reduce the theory to spacetime. This reduction, however, is not canonical, and the various ways in which it can be done give rise to transformations of the field variables which generalise the known invariances of the $N=1$ spacetime supergravity under supersymmetry transformations and spacetime diffeomorphisms.

1. Introduction

The introduction of superspace [1] allows us to regard supergravity as a geometrical theory, contrary to spacetime supergravity, which is the theory of a spin-3/2 matter field interacting with a geometrical (gravitational) field. Superspace supergravity has been first formulated by Wess and Zumino [2]; as it stands, it is a purely local theory. The analogy with general relativity suggests the introduction of a manifold M, locally modelled on superspace. This leads to the concept of supermanifold, which in recent years has been the object of an intensive research [3–7], and seems to be the key for a global geometric formulation of supergravity.

In this framework, a difficulty immediately arises: how to connect the spacetime theory with the theory formulated on the supermanifold. Actually, under weak assumptions, a supermanifold M defines an ordinary manifold M_0 together with a well-behaved projection $\Phi: M \to M_0$. It is quite natural to identify M_0 with spacetime, but, since in general an immersion $\imath: M_0 \to M$ such that $\Phi \circ \imath = \mathrm{id}_{M_0}$ fails to exist, one is not able to pull back the theory onto M_0. This can be done with ease locally: given an open set $V \subset M$ with local coordinates $(x^i, \xi^\alpha)(x^i$ even, ξ^α odd), we may look at the points of V with coordinates $(x^i \in \mathbb{R}^4, 0)$ as the image of

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$\Phi(V) \subset M_0$. Of course, this technique is local and coordinate-dependent.

In this paper we face up to this problem and show that it is related to a natural geometrical interpretation of the supersymmetry transformations. After introducing a (4, 4)-dimensional supermanifold M, carrying a super fibre bundle $\mathrm{Lor}(M)$ whose structure group is the Lorentz group, we consider on $\mathrm{Lor}(M)$ a connection form ρ with values in gp = graded Poincaré algebra. Then a rewriting of the Einstein–Rarita–Schwinger (ERS) Lagrangian 4-form on M yields field equations written in terms of differential forms, which in local coordinates are equivalent to the kinematical constraints and the field equations of Wess and Zumino's formulation. The problem of the reduction of the theory to spacetime is considered, and it is shown that two "infinitesimally different" injections $\iota, \iota' : U \subset M_0 \to M$ give rise to transformations $\delta\iota^*\rho = \iota'^*\rho - \iota^*\rho$ which leave the action integral over M_0 unchanged and generalise the usual susy transformations. Thus we may say that supersymmetries arise from the non-uniqueness of the (local) immersion of the spacetime into the supermanifold M.

Some difficulties are still present in this framework: for instance, the supermanifold Einstein equation, obtained by varying the Lagrangian with respect to the frame, is not consistent (of course, consistency is restored once the equation is pulled back onto M_0). This drawback is related to the fact that the supermanifold Lagrangian 4-form is not invariant under superdiffeomorphisms.

2. Fundamentals of Supermanifold Theory

In this section we give a brief resumé of the fundamentals of supermanifold theory [3, 4, 7] that will be needed in the sequel. The basic object for the construction of a supermanifold is a particular kind of Banach algebras, called *Banach–Grassmann algebras* [4], whose main property is graded-commutativity:

$$Q = Q_0 \oplus Q_1, \quad Q_r Q_s = (-1)^{rs} Q_s Q_r \subset Q_{r+s}, \quad r, s = 0, 1.$$

Q_0 splits into $Q_0 = \mathbb{R} \oplus Q_0'$, and the projection $\sigma : Q \to \mathbb{R}$ is called *body map*.

After introducing the Q_0-modules $Q^{m,n} = (Q_0)^m \times (Q_1)^n$, we say that a map $f : U \to Q^{m,n}$, U open in $Q^{m,n}$, is *supersmooth* [4] if it is C^∞ and its first Fréchet differential is Q_0-linear. If f is also analytic, it is said to be *superanalytic* (SA) in U; this allows us to write, for each $u, v \in U$,

$$f(u) = f(v) + \sum_{\substack{n=1..\infty \\ A_i = 1..m+n}} (u-v)^{A_n} \dots (u-v)^{A_1} C_{A_1 \dots A_n},$$

where the constants $C_{A_1 \dots A_n}$ are Q-valued.

An *(m, n)-dimensional supersmooth (superanalytic) supermanifold* is a Banach manifold [8], endowed with an atlas $A = \{(U_\alpha, \psi_\alpha); \psi_\alpha : U_\alpha \to Q^{m,n}\}$ whose transition functions are supersmooth (superanalytic).

A key concept for our interpretation of local supersymmetries is that of *body manifold* M_0 of an (m, n)-dimensional SA manifold M. Let us consider the following symmetric, reflexive relation R in $M \times M$: xRy if there exists a chart (U_α, ψ_α) such that $x, y \in U_\alpha$ and $\sigma \circ \psi_\alpha(x) = \sigma \circ \psi_\alpha(y)$. R fails to be transitive, but one can extend it to an equivalence relation $\sim$, defined as follows: $x \sim y$ if there exists a finite sequence $x_1 \ldots x_n$ such that $x = x_1, .. x_k R x_{k+1}, x_n = y$[9]. We set $M_0 = M/\sim$ and call $\Phi : M \to M_0$ the projection. M_0 is given the quotient topology and, under suitable conditions, a C^ω differentiable structure by means of the atlas $A_0 = \{(V_\alpha, \phi_\alpha)\}$, where $V_\alpha = \Phi(U_\alpha)$, $\phi_\alpha = \sigma \circ \psi_\alpha \circ \Phi^{-1}$. In this way M_0 becomes an ordinary C^ω m-dimensional manifold, and the map Φ is C^ω.

Let us remark that, if M is not superanalytic, but only supersmooth, there are hints that a similar procedure applies, but in this connection no definite result is known to us.

A *super Lie group* is an abstract group with an SA structure making it into an SA manifold, such that the group composition is SA. An example of SL-group is GL(m,n), the set of invertible $(m+n) \times (m+n)$ matrices $X_A{}^B$ with entries in Q such that $\deg(X_A{}^B) = \deg(A) + \deg(B) \bmod 2$.

Given an S-manifold M and an SL-group G, a *principal super fibre bundle* (PSFB) with base M and structure group G is a supermanifold P such that:

(i) there is a supersmooth action of G on P;

(ii) $M = P/G$, and the projection $\pi : P \to M$ is supersmooth;

(iii) P is locally trivial (in the usual sense).

Due to the similarity of these definitions to those of ordinary differential geometry, one can easily proceed to develop further the theory of PSFBs, and, in particular, the theory of connections on such bundles; thus, a connection is a supersmooth, right-invariant distribution of subspaces of $T(P)$ [7]. The covariant derivative is $D\eta = \mathrm{hor}(d\eta)\ \forall \eta \in \Lambda(P)$.

If M is an (m, n)-dimensional S-manifold, the set GL(M) of the coframes of M may be given the structure of a PSFB over M with structure group GL(m, n). Let us consider the group L formed by the matrices of GL(4, 4) of the type

$$\begin{pmatrix} \Lambda(S) & 0 \\ 0 & S \end{pmatrix},$$

where S is an element of the (real) bispinor representation of SL(2$\mathbb{C}$), and Λ is the related Lorentz matrix. If M is (4, 4)-dimensional, by reducing the structure group GL(4, 4) of the bundle GL(M) to L, we obtain a new soldered bundle, that we shall call Lor(M) (this reduction is possible whenever M_0, the body of M, admits a spinor structure) [7]. Lor(M) carries a soldering form ω^A[1].

Another bundle of physical interest is the *graded Poincaré bundle*, which is obtained by giving to the product GP(M) = Lor$(M) \times H$ the structure of a PSFB having the graded Poincaré group as structure group (here H is the supersymmetry group [3]). Lor(M) is a reduced bundle of GP(M); let j : Lor$(M) \to$ GP(M) be the injection. In the next section we describe a formulation of supergravity in which the dynamical variable is a connection form $(\rho_i{}^k, \rho^A)$ on GP(M)[1] such that $\omega^A = j^* \rho^A$ (so to say, ρ is an "affine" non-generalised connection). The form $\omega_i{}^k = j^* \rho_i{}^k$ is a connection form on Lor(M) [7]. After introducing the curvature and torsion forms of $\omega_i{}^k$, $\Omega_i^k = D\omega_i{}^k$ and $\Theta^A = D\omega^A$ (D being the covariant derivative in

1 Capital indices run over $Q^{4,4}$, single small latin indices over $Q^{4,0}$, greek indices over $Q^{0,4}$, pairs of small latin indices over the Lorentz algebra

Lor(M)), we have the following structure equations and Bianchi identities (we shall always omit the wedge product symbol):

$$\Omega_i{}^k = d\omega_i{}^k - \omega_i{}^h\omega_h{}^k, \tag{2.1a}$$

$$\Theta^i = d\omega^i - \omega^k\omega_k{}^i, \quad \Theta = d\omega + \tfrac{1}{2}\omega\sigma^T_{ik}\omega^{ik}, \tag{2.1b, c}$$

$$D\Omega_i{}^k \equiv d\Omega_i{}^k + \Omega_i{}^h\omega_h{}^k - \omega_i{}^h\Omega_h{}^k = 0, \tag{2.2a}$$

$$D\Theta^i \equiv d\Theta^i + \Theta^h\omega_h{}^i = \omega^h\Omega_h{}^i, \quad D\Theta \equiv d\Theta - \tfrac{1}{2}\Theta\sigma^T_{ik}\omega^{ik} = -\tfrac{1}{2}\omega\sigma^T_{ik}\Omega^{ik} \tag{2.2b, c}$$

(for conventions and notation see the Appendix).

3. A Geometrical Framework for Supergravity

Now we wish to exploit the concept of super fibre bundle over a supermanifold in order to give a global formulation of superspace supergravity in terms of differential forms which parallels the fibre-bundle formulation of the Einstein–Cartan theory [10]. A formal variational principle, based on a 4-form which is the rewriting on the supermanifold of the usual ERS Lagrangian, will yield both the superspace field equations and Wess–Zumino's constraints.

Let M be a (4, 4)-dimensional S-manifold, carrying a Lorentz bundle Lor(M) with a connection $\omega_i{}^k$ and a soldering form ω^A. We consider on M the Lagrangian 4-form:

$$\Lambda = \varepsilon_{hijk}\omega^h\omega^i\Omega^{jk} + 4\Theta C\gamma_i\gamma_5\omega^T\omega^i. \tag{3.1}$$

(Here and in the following, in order to simplify the notation, we confuse the horizontal forms on Lor(M) with the corresponding forms on M obtained through local sections.) Variation of the Lagrangian with respect to ω^{ik}, ω^i and ω yields

$$\delta\Lambda = 4\delta\omega^i(\hat{E}_i + \hat{t}_i) - 2\delta\omega^{ik}(\hat{c}_{ik} + \hat{s}_{ik}) + \delta\omega\hat{G}^T, \tag{3.2}$$

with

$$\hat{E}_i = \tfrac{1}{2}\varepsilon_{ijkh}\omega^j\Omega^{kh}, \quad \hat{t}_i = -\Theta C\gamma_i\gamma_5\omega^T, \tag{3.3a, b}$$

$$\hat{c}_{ik} = \varepsilon_{ikjh}\Theta^h\omega^j, \quad \hat{s}_{ik} = -(1/2)\varepsilon_{ikjh}(\omega C\gamma^j\omega^T)\omega^h, \tag{3.3c, d}$$

$$\hat{G} = 2(2\omega^i\Theta - \omega\Theta^i)C\gamma_i\gamma_5, \tag{3.3e}$$

(here use of Fierz identities is necessary). Obviously, we obtain the following field equations, written in terms of 3-forms on M:

$$\hat{E}_i = -\hat{t}_i, \quad \hat{c}_{ik} = -\hat{s}_{ik}, \quad \hat{G} = 0 \tag{3.4a, b, c}$$

In order to show that Eqns. (3.4) are actually equivalent to Wess and Zumino's equations, it is necessary to introduce the components of the geometrical quantities (2.1) over the basis of $T^*(M)$ given by the ω^A's:

$$\Theta^i = \tfrac{1}{2}\omega^j\omega^h\hat{T}_{hj}{}^i + \omega^j\omega^\mu\hat{T}_{\mu j}{}^i + \tfrac{1}{2}\omega^\mu\omega^\nu\hat{T}_{\nu\mu}{}^i,$$

$$\Theta^\lambda = \tfrac{1}{2}\omega^j\omega^h\hat{T}_{hj}{}^\lambda + \omega^j\omega^\mu\hat{T}_{\mu j}{}^\lambda + \tfrac{1}{2}\omega^\mu\omega^\nu\hat{T}_{\nu\mu}{}^\lambda,$$

$$\Omega^{ik} = \tfrac{1}{2}\omega^j\omega^h\hat{R}_{hj}{}^{ik} + \omega^j\omega^\mu\hat{R}_{\mu j}{}^{ik} + \tfrac{1}{2}\omega^\mu\omega^\nu\hat{R}_{\nu\mu}{}^{ik}.$$

Equations (3.4) then yield

$$\hat{T}_{\mu\nu}{}^{i} = -(C\gamma^i)_{\mu\nu}, \quad \hat{T}_{jh}{}^{i} = \hat{T}_{j\mu}{}^{i} = \hat{T}_{\mu\nu}{}^{\lambda} = \hat{T}_{j\mu}{}^{\lambda} = 0 \tag{3.5}$$

$$\varepsilon^{ijhk}\hat{T}_{jh}{}^{\mu}(C\gamma_k)_{\mu\nu} = 0 \Leftrightarrow (\gamma^j)_{\nu\mu}\hat{T}_{jh}{}^{\mu} = 0, \tag{3.6a}$$

$$\hat{R}_{hj}{}^{ij} - \tfrac{1}{2}\delta^i_h\hat{R}_{kj}{}^{kj} = 0, \quad \hat{R}_{\mu\nu}{}^{i} = 0, \tag{3.6b, c}$$

$$\hat{R}_{j\mu}{}^{ik} = -\tfrac{1}{2}\varepsilon^{iknh}\hat{T}_{nh}{}^{\nu}(C\gamma_j\gamma_5)_{\nu\mu} \tag{3.6d}$$

In particular Eqns. (3.5), here obtained from the formal variational principle based on Λ, are the kinematical constraints of Wess and Zumino's formulation.

We conclude this section giving some differential identities we shall need later on. They are:

$$D\hat{c}_{ik} = \omega_i\hat{E}_k - \omega_k\hat{E}_i, \tag{3.7a}$$

$$D\hat{s}_{ik} = \omega_i\hat{t}_k - \omega_k\hat{t}_i - \tfrac{1}{4}\omega\sigma^T_{ik}\hat{G}^T, \tag{3.7b}$$

$$D\hat{G} = -2\Theta C\gamma_i\gamma_5(\Theta^i + \tfrac{1}{2}\omega C\gamma^i\omega^T) + 2\omega C\gamma^i(\hat{E}_i + \hat{t}_i). \tag{3.7c}$$

Equation (3.7a) is a rewriting of the Bianchi identity (2.2b). Eqns. (3.7b, c) are the analogs on M of the identities that on ordinary spacetime yield respectively the conservation of spin–angular momentum [10] and the consistency of the Rarita–Schwinger equation ([11], Eqn. (11)). Finally, let us note that the identities (3.7) allow us to prove the consistency of Eqs. (3.4b, c) while the supermanifold Einstein equation (3.4a) is not consistent. (Here by consistency we mean only the formal requirement that the equations obtained by exterior differentiation of the field equations are fulfilled on shell.)

4. Geometric Derivation of Supersymmetry Transformations

Let us assume that the supermanifold M of the previous section admits a body manifold M_0; it is quite natural to identify M_0 with the spacetime. Now, the geometric structure of M (frames, connection) induces (locally) geometric structure on M_0 too. However, this procedure is not canonical, and the various ways in which it can be done give rise to certain transformations of the field variables, that we now proceed to deduce.

As we have already noticed in the Introduction, given an open set $U \subset M_0$, we may always define an immersion $\imath: U \to M$. Thus we may pull back the forms ω^i, ω_i^k and ω on U, obtaining the coframe e^i, the connection form Γ_i^k and the gravitino 1-form ψ:

$$e^i = \imath^*\omega^i, \quad \Gamma_i{}^k = \imath^*\omega_i{}^k, \quad \psi = \imath^*\omega. \tag{4.1}$$

Moreover, setting $D\imath^*\eta = \imath^*D\eta$ for each $\eta \in \Lambda(M)$, we have

$$\imath^*\Omega_i{}^k = D\Gamma_i{}^k \equiv R_i{}^k, \tag{4.2a}$$

$$\imath^*\Theta^i = De^i \equiv T^i, \tag{4.2b}$$

$$\imath^*\Theta = D\psi \equiv T. \tag{4.2c}$$

Pulling back the field equations (3.4) we have on U the equations

$$E_i = -t_i, \quad c_{ik} = -s_{ik}, \quad G = 0. \tag{4.3a, b, c}$$

Equations (4.3) are the usual field equations of the spacetime supergravity, and are the Euler equations of the ERS Lagrangian 4-form,

$$L = \varepsilon_{ijhk} e^i e^j R^{hk} + 4TC\gamma_i\gamma_5\psi^T e^i \equiv \imath^* \Lambda \tag{4.4}$$

Now, let $\imath' : U \to M$ be a different immersion. We expect that the physics in U is the same as in the previous case. Indeed, we can show that the changes induced on $e^i, \psi, \Gamma_i{}^k$ by changing the immersion are symmetries of the Lagrangian (4.4). Since we want to consider only "infinitesimal changes," we take a 1-parameter family of immersion $\{\imath_t : U \to M\}$. It is easy to see that, at least locally, there exists a 1-parameter group of S-diffeomorphisms of M, $\{\chi_t\}$, such that $\imath_t = \chi_t \circ \imath$. Defining, for each $\eta \in \Lambda(M)$.

$$\delta\imath^*\eta = \lim_{t\to 0} \frac{1}{t}(\imath_t^*\eta - \imath^*\eta),$$

one has

$$\delta\imath^*\eta = \imath^* \lim_{t\to 0} \frac{1}{t}(\chi_t^*\eta - \eta) = \imath^* \pounds_X \eta, \tag{4.5}$$

where £ is the Lie derivative. The field $X \in T(M)$ generates $\{\chi_t\}$ and has vanishing body ($\Phi_* X = 0$). Applying Eq. (4.5) to the field variables (4.1) we obtain

$$\delta e^i = \imath^*[D(X \lrcorner \omega^i) + X \lrcorner \Theta^i - \omega^k (X \lrcorner \omega_k{}^i)], \tag{4.6a}$$

$$\delta\psi = \imath^*[D(X \lrcorner \omega) + X \lrcorner \Theta + \tfrac{1}{2}\omega\sigma_{ik}^T (X \lrcorner \omega^{ik})], \tag{4.6b}$$

$$\delta\Gamma_i{}^k = \imath^*[X \lrcorner \Omega_i{}^k + D(X \lrcorner \omega_i{}^k)]. \tag{4.6c}$$

We set

$$p^i = \imath^*(X \lrcorner \omega^i), \quad \imath^*(X \lrcorner \omega) = \alpha + p^i \partial_i \lrcorner \psi, \quad \varepsilon_i{}^k = \imath^*(X \lrcorner \omega_i{}^k),$$

where $\{\partial_i \in T(M_0)\}$ is the frame dual to $\{e^k\}$. Note that the quantities p^i and α are $Q^{4,0}$- and $Q^{0,4}$-valued, respectively. From Eqs. (4.6) one has

$$\delta e^i = Dp^i + p^k T_k{}^i + \alpha^\mu (\imath^* \hat{T}_{\mu\nu}{}^i)\psi^\nu - \alpha^\mu (\imath^* \hat{T}_{\mu k}{}^i) e^k - e^k \varepsilon_k{}^i, \tag{4.7a}$$

$$\delta\psi = D(\alpha + p^i \partial_i \lrcorner \psi) + p^h T_h + \alpha^\mu (\imath^* \hat{T}_{\mu\nu})\psi^\nu - \alpha^\mu (\imath^* \hat{T}_{\mu k}) e^k + \tfrac{1}{2}\psi\sigma_{ik}^T \varepsilon^{ik}, \tag{4.7b}$$

$$\delta\Gamma^{ik} = p^h R_h{}^{ik} + \alpha^\mu (\imath^* \hat{R}_{\mu\nu}{}^{ik})\psi^\nu - \alpha^\mu (\imath^* \hat{R}_{\mu h}{}^{ik}) e^h + D\varepsilon^{ik}, \tag{4.7c}$$

where we have introduced the 1-forms $R_i{}^{jh}, T_i^k, T_i$ according to $R_i{}^{jh} = \partial_i \lrcorner R^{jh}$, etc. Insertion of the field Eqs. (3.5, 6) yields

$$\delta e^i = Dp^i + p^h T_h{}^i - \alpha C\gamma^i \psi^T - e^h \varepsilon_h{}^i, \tag{4.8a}$$

$$\delta\psi = D(p^i \partial_i \lrcorner \psi) + p^h T_h + D\alpha + \tfrac{1}{2}\psi\sigma_{ik}^T \varepsilon^{ik}, \tag{4.8b}$$

$$\delta\Gamma^{ik} = e^h B_h^{ik} - \tfrac{1}{2} e^k B_h^{ih} + \tfrac{1}{2} e^i B_h^{ih} + D\varepsilon^{ik} + p^h R_h{}^{ik}, \tag{4.8c}$$

with $B_h^{ik} = -1/2\ \varepsilon^{ikjn} (\alpha C\gamma_5\gamma_h)_\mu T_{jn}{}^\mu$.

Note that $B_h^{ih} = 0$ by virtue of the field equations, and the quantities p^i are subject to the constraint $\Phi_* X = 0 \Rightarrow \sigma(p^i) = 0$. On the other hand, a different way of varying the immersion $\iota: U \to M$ is given by $\iota' = \iota \circ f$, f being a diffeomorphism of U. The calculation of the variations of the field variables in this case leads to Eqs. (4.8) again, but with $\alpha = 0$, $p^i \in \mathbb{R}^4$, $\varepsilon^{ik} = p^h \partial_h \lrcorner \Gamma^{ik}$. Thus, provided that we leave aside the constraint $\sigma(p^i) = 0$, we may consider the variations due to (infinitesimal) diffeomorphisms of spacetime as already included into Eqs. (4.8).

5. Discussion

Finally, we want to show that the transformations (4.8) are a symmetry of the spacetime supergravity, and that they contain the usual local supersymmetries as a particular case. First we note that, varying the Lagrangian according to Eqs. (3.1, 3.2, 4.8), the terms with ε^{ik} cancel each other (indeed they "simulate" a Lorentz transformation). Moreover, since the parameters p^i and α are GL(4,4)-related to the components of the field $X \in T(M)$, they are independent, so that the case $\alpha = 0$ and $p^i = 0$ may be treated separately.

1) $p^i = 0$. We have

$$\delta e^i = -\alpha C \gamma^i \psi^T, \quad \delta\psi = D\alpha, \tag{5.1a}$$

$$\delta \Gamma^{ik} = e^h B_h^{ik} - \tfrac{1}{2} e^k B_h^{ih} + \tfrac{1}{2} e^i B_h^{kh}. \tag{5.1b}$$

Thus for $p^i = 0$ the transformations (4.8) yield the ordinary supersymmetries of supergravity.

2) $\alpha = 0$. In this case we obtain

$$\delta L = 4(Dp^i + p^h T_h{}^i)(E_i + t_i) - 2p^h R_h{}^{ik}(c_{ik} + s_{ik}) + [p^h T_h + D(p^h \partial_h \lrcorner \psi)] G^T$$
$$+ \text{an exact form} = 4p^i[-DE_i + T_i{}^h E_h - \tfrac{1}{2} R_i{}^{hk} c_{hk}] +$$
$$+ 4p^i[-Dt_i + T_i{}^h t_h - \tfrac{1}{2} R_i{}^{hk} s_{hk} + (1/4) T_i G - (1/4)(\partial_i \lrcorner \psi) D G^T] + \text{an e.f.}.$$

The identities

$$DE_i = T_i{}^k E_k - \tfrac{1}{2} R_i{}^{hk} c_{hk} \text{ (contracted Bianchi identity) [10]}, \tag{5.2a}$$

$$Dt_i = T_i{}^k t_k - \tfrac{1}{2} R_i{}^{hk} s_{hk} + (1/4) T_i G^T - (1/4)(\partial_i \lrcorner \psi) D G^T, \tag{5.2b}$$

yield δL = an exact form, thus proving that the transformations (4.8) are a symmetry of the ERS Lagrangian. Let us note that both identities do not hold on the supermanifold M.

Thus we come to the conclusion that the supersymmetry transformations of the $N = 1$ spacetime supergravity are a particular case of a more general symmetry; this enlarged symmetry is to be related to the various ways in which space-time can be locally immersed in a (4, 4) dimensional S-manifold where the supergravity field equations hold. It should be noticed that in this scheme susies are not regarded as gauge transformations (at least not in the fibre-bundle sense). This is in partial agreement with some remarks of other researchers [12, 13].

Appendix

We use a Majorana representation for the Dirac matrices, so that the γ^i are purely immaginary, a Majorana spinor is real and the bispinor representation of SL(2$\mathbb{C}$) is real too. Spinors are to be regarded as row "vectors." Other conventions are:

$$C\gamma^i C = -\gamma^{iT}, \quad C^2 = 1, \quad C = -C^T,$$
$$\sigma_{ik} = (1/4)[\gamma_i, \gamma_k], \quad \gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4, \quad \varepsilon_{1234} = 1,$$
$$\eta_{ik} = \text{Minkowski metric} = \text{diag}(-1, -1, -1, +1).$$

The following Fierz identity and Fierz rearrangement are often used:

$$2\sigma_{ij}\gamma_5\gamma_h = \varepsilon_{hijq}\gamma^q + \eta_{jh}\gamma_5\gamma_i - \eta_{ih}\gamma_5\gamma_j,$$
$$\gamma^i\omega^T\hat{t}_i = (1/2)\gamma_i\gamma_5\Theta^T(\omega C\gamma^i\omega^T).$$

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Note added in proof. Since this paper was written, the question of the existence of the body of a supermanifold has been clarified in the paper by R. Catenacci et al., "On the body of supermanifolds", Preprint Dip. Matematica Univ. Pavia.