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Existence of Maximal Surfaces in Asymptotically Flat Spacetimes

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Abstract. We prove the existence of maximal surfaces in asymptotically flat spacetime satisfying an interior condition. This uses a priori estimates which can also be applied to prescribed mean curvature surfaces in cosmological spacetimes and the Dirichlet problem.

1. Introduction

Maximal surfaces are spacelike submanifolds of a Lorentzian manifold which locally maximize the induced area functional. This leads to a nonlinear elliptic equation which is interpreted geometrically as the vanishing of the mean extrinsic curvature. More generally, one considers surfaces of prescribed mean curvature. The main interest in such surfaces presently comes from general relativity, where there have already been many applications. For example, they have been used to prove positivity of mass [SY1], analyse the space of solutions of Einsteins equations [FMM] and in numerical integration schemes for Einsteins equations [P, ES]. Further references can be found in review papers such as [MT, ChY].

It is clear that a good understanding of the existence and regularity properties of such surfaces is needed. In this paper we prove the existence of asymptotically flat maximal surfaces in asymptotically flat spacetimes satisfying a uniformity condition in the interior (Theorem 5.4). Along the way we show that the Dirichlet problem in nonflat spacetimes is solvable (Theorem 4.2) and prove the existence of constant mean curvature surfaces in cosmological spacetimes (Theorem 4.1). The result for cosmological spacetimes was first proved by Claus Gerhardt, but our proof appears to be simpler.

These results hold under very general conditions. For example, the usual energy inequalities on the curvature tensor [HE] are not needed, the mean curvature can be nonconstant and singularities protected by barrier surfaces (i.e. crushing singularities [ES]) are permitted. The restrictions are that the spacetime admits a smooth time function and that some compactness condition is satisfied. This latter condition is needed to ensure that spacelike surfaces with given boundary do not reach arbitrarily far into the future (past). An instructive example of Brill [Br] of a spacetime having a compact surface (with boundary) with a noncompact domain of dependence, indicates that such a condition is necessary. We note that this difficulty arises even when considering solutions of the variational problem [Av, Go].

The first results in this area were due to Avez [Av] who showed that the associated variational problem is solvable, assuming some compactness conditions. However, the resulting surface is a priori only Lipschitz-continuous and may be null. This illustrates the main difficulty of the problem and arises directly from the form of the nonlinearity of the elliptic equation satisfied by the defining function of the surface. In flat space $\mathbb{R}^{n,1}$ this equation becomes

$$\frac{1}{\sqrt{1-|Du|^2}}\left(\delta_{ij}+\frac{D_iuD_ju}{1-|Du|^2}\right)D_{ij}u=0\,,$$

where $u\in C^2(\mathbb{R}^n)$ and the mean curvature is zero.

In 1968 Calabi [C] showed that for $n\leqq 4$ this equation has the Bernstein property that the only entire solutions are linear. This was later extended to all dimensions by Cheng-Yau [CY] and their estimates were used by Treibergs [T] to construct and classify constant mean curvature surfaces in $\mathbb{R}^{n,1}$. For non-flat spacetimes several authors used implicit function techniques to find solutions near known exact solutions. We mention Choquet-Bruhat [Ch] and Stumbles [St] and refer to [MT] for a review of these results. The main uniqueness theorem is due to Frankel and Brill-Flaherty [BF] and applies only to constant mean curvature surfaces in spacetimes satisfying the timelike convergence condition. In view of the very general existence theorems in this paper, it would be useful to find stronger uniqueness results.

The solvability of the Dirichlet problem and the regularity of variational extrema in $\mathbb{R}^{n,1}$ were shown in [BS]. Previously weak results for the Dirichlet problem in $\mathbb{R}^{n,1}$ had been given by Bancel [Ba] and Flaherty [F]. Independently Gerhardt obtained a gradient estimate similar to [BS] but valid for nonflat spacetimes, and applied it to solve the prescribed mean curvature problem in cosmological spacetimes. Generalizing a barrier construction of [BS], he was also able to solve the Dirichlet problem in spacetimes conformal to a product.

This paper is organized as follows: In Sect. 2 we describe notation and give a number of calculations of mean curvature and the first variation of mean curvature. These calculations all rely heavily on the slicing provided by the time function. Section 3 contains the basic gradient estimates, for surfaces with smooth or empty boundary. These estimates are much stronger than those of [BS], [Ge] since they depend only on pointwise bounds. Existence theorems are then derived in Sect. 4 for situations in which an a priori height estimate follows, either from a compactness assumption (Dirichlet problem) or from existence of barrier surfaces (cosmological problem). Even here it is not necessary that the domain be bounded, and an application giving constant mean curvature surfaces near a singularity of Kasner type appeared in [B].

The main difficulty in applying the methods of Sect. 4 to the maximal surface problem is in obtaining an a priori height bound. This is done in Sect. 5 by a test-function argument exploiting the existence of barrier surfaces at spacelike infinity.

The main result Theorem 5.4 constructs a maximal surface asymptotic to a given level set of the time function. The asymptotic conditions required are standard [ChY], and only a mild condition is needed in the interior in order to exclude behaviour such as in Brills example.

Most of the results here are contained in [B].

2. Notation and Calculations

A spacetime $\mathscr{V}$ is a smooth $(n+1)$ dimensional manifold (Hausdorf and paracompact) with smooth Lorentzian metric $ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu$ of signature $(+ + \dots + -)$. We denote the metric pairing by $\langle\cdot,\cdot\rangle$, the canonical connection by ∇ and use the summation convention with index ranges $1 \leqq i, j, \dots \leqq n$, $1 \leqq \lambda, \mu, \dots \leqq n+1$. For simplicity we assume smoothness throughout and will note weaker regularity conditions where appropriate. Since most of the calculations are purely formal unless otherwise stated, we make no assumptions about the causal topology of $\mathscr{V}$ apart from requiring that $\mathscr{V}$ has a time function.

Definition 2.1. $t \in C^\infty(\mathscr{V})$ is a *time function* if ∇t is a nonzero, everywhere timelike vector field. Thus $\mathscr{V}$ is time-oriented, with ∇t lying in the past-directed timelike cone. Associated to t we have the *reference slices* $\mathscr{S}_t = \{p \in \mathscr{V} : t(p) = t\}$, which are Riemannian submanifolds with respect to the induced metric and topology. The *lapse function* $\alpha \in C^\infty(\mathscr{V})$ of t is defined by

$$\alpha^{-2} = -\langle \nabla t, \nabla t\rangle\,, \tag{2.1}$$

and thus the future-directed unit normal vector to the reference slices is

$$T = -\alpha \nabla t\,. \tag{2.2}$$

Choosing an orthonormal frame $\{v_i\}_1^n$ on $\mathscr{S}_t$ yields an *adapted orthonormal frame* $\{v_\lambda\}_1^{n+1}$, where $v_{n+1} = T$, which defines a positive-definite norm on tensors on $\mathscr{V}$. For example, if $B(\cdot,\cdot) \in \Gamma(T^*\mathscr{V} \otimes T^*\mathscr{V})$, then

$$\|B\| = \sup\left(\sum_{\lambda,\mu=1}^{n+1} |B(v_\lambda, v_\mu)|^2\right)^{1/2},$$

$$\|B\|_k = \sum_{j=0}^{k} \|\nabla^j B\|\,.$$

A *spacelike surface* $M \subset \mathscr{V}$ is a codimension-one submanifold of $\mathscr{V}$ with Riemannian induced metric with respect to the induced topology. Denoting the induced connection by ∇^M, the operators div_M, $\operatorname{grad}_M = \nabla^M$ and Δ_M are given by

$$\operatorname{div}_M X = \langle e_i, \nabla_{e_i} X\rangle\,, \qquad X \in \Gamma(T\mathscr{V})\,,$$

$$\operatorname{grad}_M \phi = \nabla^M \phi = e_i(\phi) e_i\,, \qquad \phi \in C^\infty(M)\,,$$

$$\Delta_M \phi = \operatorname{div}_M \operatorname{grad}_M \phi = (e_i e_i - \nabla^M_{e_i} e_i)(\phi)\,,$$

where $\{e_i\}_i^n$ is any orthonormal frame on M. Let N be the future-directed unit normal to M, so the second fundamental form $A(\cdot,\cdot)$ and mean curvature $H = H_M$

are given by

$$A(e_i, e_j) = \langle e_i, \nabla_{e_j} N \rangle, \tag{2.3}$$

$$H = A(e_i, e_i) = \operatorname{div}_M N. \tag{2.4}$$

The *height function* $u \in C^\infty(M)$ of M is the restriction of the time function to M, $u = t|_M$. Denoting the component tangential to M by $(\cdot)^{\|}$ we have

$$\nabla^M u = (\nabla t)^{\|} = \nabla t + \alpha^{-1} v N = \alpha^{-1}(vN - T), \tag{2.5}$$

where $v = -\langle T, N \rangle$ measures the "angle" between M and the reference slicing. Observe that

$$|\nabla^M u|^2 = \alpha^{-2}(v^2 - 1), \tag{2.6}$$

and hence $v \geqq 1$, a result that also follows from Lemma 3.3. From (2.4) and (2.5) we have

$$Hv = \operatorname{div}_M(\alpha \nabla^M u) + \operatorname{div}_M T, \tag{2.7}$$

$$\Delta_M u = \alpha^{-1} v H + \operatorname{div}_M \nabla t. \tag{2.8}$$

Proposition 2.1. *With the above notations, we have*

$$\Delta_M v = v(|A|^2 + \operatorname{Ric}(N, N)) - \langle T, \nabla^M H \rangle + T(H_T), \tag{2.9}$$

where $\operatorname{Ric}(\cdot, \cdot)$ *is the Ricci tensor of* $\mathscr{V}$, *and* $T(H_T)$ *is the variation of mean curvature of* M *under the deformation vector field* T. *This can also be expressed in terms of the Killing tensor* $\mathscr{L}_T g$ *of* T, $\mathscr{L}_T g(X, Y) = \langle \nabla_X T, Y \rangle + \langle \nabla_Y T, X \rangle$ *by the expression*

$$T(H_T) = \tfrac{1}{2}(\nabla_N \mathscr{L}_T g)(e_i, e_i) - (\nabla_{e_i} \mathscr{L}_T g)(N, e_i) - \tfrac{1}{2} H \mathscr{L}_T g(N, N) - \mathscr{L}_T g(e_i, e_j) \cdot A(e_i, e_j). \tag{2.10}$$

Proof. Fix $p \in M$ and choose an orthonormal frame $\{e_i\}$ on M such that $(\nabla^M_{e_i} e_j)(p) = 0$. For generality let X be a vector field in a neighbourhood of M with associated flow $\phi_s : \mathscr{V} \to \mathscr{V}$, $|s| \leqq \varepsilon$. In our case we have that X is transverse to M but this is not necessary, provided we interpret $X(H_X)$ properly [SL]. Extending the frame $\{e_i\}$ by $e_i(s) = \phi_{s*}(e_i)$ gives a frame tangent to $M_s = \phi_s(M)$ with metric $g_{ij}(s) = \langle e_i(s), e_j(s) \rangle$, so $g_{ij}(0) = \delta_{ij}$ and $\mathscr{L}_X e_i = [X, e_i] = 0$. Let $N(s)$ be the future-directed unit normal to M_s, so the second fundamental form $A_{ij}(s)$ is given by $A_{ij}(s) = \langle e_i, \nabla_{e_j} N \rangle(s)$, and hence (with the s-dependence implicit)

$$\nabla_{e_i} N = A_{ij} g^{jk} e_k, \qquad \nabla_{e_i} e_j = A_{ij} N + (\nabla_{e_i} e_j)^{\|}.$$

Direct computation at p now gives

$$\begin{aligned}
\Delta_M \langle X, N \rangle (p) &= e_i e_i \langle X, N \rangle \\
&= e_i \langle \nabla_X e_i, N \rangle + e_i(\langle X, e_j \rangle A_{ij}) \\
&= \langle R(e_i, X) e_i, N \rangle + \langle \nabla_X \nabla_{ei} e_i, N \rangle + 2 \langle \nabla_X e_i, e_j \rangle A_{ij} \\
&\quad + \langle X, e_j \rangle (e_j(H) + \langle R(e_i, e_j) N, e_i \rangle) + \langle X, \nabla_{e_i} e_j \rangle A_{ij} \\
&= -\operatorname{Ric}(X, N) + X \langle \nabla_{e_i} e_i, N \rangle - X(g^{ij}) A_{ij} \\
&\quad + X^{\|}(H) + \operatorname{Ric}(X^{\|}, N) + \langle X, N \rangle |A|^2 \\
&= \langle X, N \rangle (|A|^2 + \operatorname{Ric}(N, N)) + \langle X, \nabla^M H \rangle - X(H_X),
\end{aligned}$$

where precisely,

$$X(H_X) = \frac{\partial}{\partial s}(H(s))|_{s=0}\,. \tag{2.11}$$

Setting $X = T$ gives (2.9). To motivate (2.10) observe that if X is a Killing vector field, $\mathscr{L}_X g = 0$, then ϕ_s is an isometry and hence $X(H_X) \equiv 0$. The calculation is as follows:

$$\begin{aligned}
X(H_X)(p) &= -A_{ij}X\langle e_i, e_j\rangle + X\langle e_i, \nabla_{e_i}N\rangle \\
&= -\mathscr{L}_X g_{ij}A_{ij} + e_i\langle \nabla_X e_i, N\rangle - \langle \nabla_{e_i}\nabla_X e_i, N\rangle + \langle e_i, \nabla_X \nabla_{e_i} N\rangle \\
&= -\mathscr{L}_X g_{ij}A_{ij} + e_i(\langle \nabla_X e_i, N\rangle + \langle e_i, \nabla_X N\rangle) - \langle \nabla_{e_i} e_i, \nabla_X N\rangle \\
&\quad + \langle R(X, e_i)N, e_i\rangle - e_i\langle \nabla_{e_i} X, N\rangle + \langle \nabla_{e_i} X, \nabla_{e_i} N\rangle \\
&= -\mathscr{L}_X g_{ij}A_{ij} + \langle \nabla_N \nabla_{e_i} X, e_i\rangle - \langle \nabla_{e_i}\nabla_N X, e_i\rangle - \langle \nabla_{\nabla_N e_i} X, e_i\rangle \\
&\quad + \langle \nabla_{\nabla_{e_i} N} X, e_i\rangle - e_i\langle \nabla_{e_i} X, N\rangle + \langle \nabla_{e_i} X, \nabla_{e_i} N\rangle \\
&= \tfrac{1}{2} N(\mathscr{L}_X g(e_i, e_i)) - \mathscr{L}_X g(\nabla_N e_i, e_i) - e_i(\mathscr{L}_X g(N, e_i)) + \langle \nabla_N X, \nabla_{e_i} e_i\rangle \\
&= \tfrac{1}{2} \nabla_N \mathscr{L}_X g(e_i, e_i) - e_i(\mathscr{L}_X g(N, e_i)) + \tfrac{1}{2} H \mathscr{L}_X g(N, N) \\
&= \tfrac{1}{2} \nabla_N \mathscr{L}_X g(e_i, e_i) - \nabla_{e_i}\mathscr{L}_X g(N, e_i) - \mathscr{L}_X g_{ij}A_{ij} - \tfrac{1}{2} H \mathscr{L}_X g(N, N) \qquad \square
\end{aligned}$$

Observe that the calculations thus far have been *intrinsic*, in that they depend only on the choice of time function and not on some local coordinate system. However we will later need to do locaul calculations, so let (x^i, t) be local coordinates (t is still the time function) in which the metric has the form

$$ds^2 = -(\alpha^2 - \beta^2)dt^2 + 2\beta_i dx^i dt + g_{ij}dx^i dx^j\,, \tag{2.12}$$

where α is the lapse function (2.1) and $\beta = \beta_i g^{ij}\partial_j$ is the shift vector. We write ∂_i, ∂_t for coordinate tangent vectors and denote partial derivatives by subscripts, so the tangential gradient operator on the slices $\mathscr{S}_t$ is $D\phi = g^{ij}\phi_i\partial_j = \phi^i\partial_i$, $\phi \in C^\infty(\mathscr{V})$. The future-directed unit normal vector T is given by

$$T = -\alpha\nabla t = \alpha^{-1}(\partial_t - \beta)\,, \tag{2.13}$$

and then the second fundamental form A^o_{ij} and mean curvature H^o of the slices $\mathscr{S}_t$ are

$$A^o_{ij} = \langle \partial_i, \nabla_{\partial_j} T\rangle = \tfrac{1}{2}\alpha^{-1}\partial_t g_{ij} - \tfrac{1}{2}\alpha^{-1}\mathscr{L}_\beta g_{ij}\,, \tag{2.14}$$

$$H^o = g^{ij}A^o_{ij} = \tfrac{1}{2}\alpha^{-1}g^{ij}\partial_t g_{ij} - \alpha^{-1}\operatorname{div}^o(\beta)\,, \tag{2.15}$$

where div^o is the divergence on the slices $\mathscr{S}_t$.

The height function $u \in C^\infty(M)$ can be extended to $\mathscr{V}$ by requiring $\partial_t u = 0$. Since M is then a level set of $(u - t)$, we have

$$N = v(U + T)\,, \tag{2.16}$$

where $U = (1 + \beta \cdot Du)^{-1}\alpha Du$, and $v = (1 - |U|^2)^{-1/2}$. Choosing an orthonormal frame on M with $e_1 = |\nabla^M u|^{-1}\nabla^M u$, we calculate H,

$$\nabla^M u = \alpha^{-1}v^2(U + |U|^2 T)\,, \qquad e_1 = v(\tilde{e}_1 + |U|T)\,,$$

where $\tilde{e}_1 = |Du|^{-1}Du$ if $Du \neq 0$. Then

$$\begin{aligned}
H &= v\,\mathrm{div}_M(U+T)\\
&= v\,\mathrm{div}^o(U+T) + v(v^2-1)\left(\langle \tilde{e}_1, \nabla_{\tilde{e}_1} U\rangle + A^o_{11}\right)\\
&\quad + v^3|U|\left(\langle \tilde{e}_1, \nabla_T U\rangle + \langle \tilde{e}_1, \nabla_T T\rangle + \langle T, \nabla_{\tilde{e}_1+|U|T} U\rangle\right)\\
&= v\,\mathrm{div}^o U + vH^o + v^3|U|^2\tilde{e}_1(|U|) + v|U|\langle \tilde{e}_1, \nabla_T T\rangle + \tfrac{1}{2}v^3 T(|U|^2),\\
H &= \mathrm{div}^o\left(\frac{U}{\sqrt{1-|U|^2}}\right) + vH^o + v\langle U, \nabla_T T\rangle + \tfrac{1}{2}v^3 T(|U|^2). \qquad (2.17)
\end{aligned}$$

For later reference, we also calculate in the same manner

$$\begin{aligned}
\mathrm{div}_M T &= H^o + v^2 U^i U^j A^o_{ij} + v^2\langle U, \nabla_T T\rangle\\
&= H^o + (v^2-1)\left(A^o_{11} - \alpha^{-1}T(\alpha)\right) + \langle \nabla^M u, \nabla^M \alpha\rangle, \qquad (2.18)
\end{aligned}$$

since $\langle U, \nabla_T T\rangle = \alpha^{-1}U(\alpha) = \alpha^{-1}U \cdot D\alpha$.

There is an alternative expression for H which illustrates the essential difficulty of our problem. First observe that it is always possible to construct coordinates (x^i, t) in which the shift vector β vanishes. Lifting the coordinates (x^i) to M gives coordinate tangent vectors $X_i = \partial_i + u_i\partial_t$ with induced metric

$$\bar{g}_{ij} = g_{ij} - \alpha^2 u_i u_j, \qquad \bar{g}^{ij} = g^{ij} + \alpha^2 v^2 u^i u^j, \qquad (2.19)$$

which shows that the ellipticity of Δ_M is controlled by v. In particular, a calculation based on (2.7) shows that

$$H = \alpha v \bar{g}^{ij} u_{|ij} + v(H^o + Du \cdot D\alpha) + v^3(Du \cdot D\alpha + |Du|^2\alpha_t - \alpha^2 u^i u^j A^o_{ij}).$$

It is clear from this expression that the prescribed mean curvature equation is nonuniformly elliptic, since

$$g^{ij}\zeta_i\zeta_j \leqq \bar{g}^{ij}\zeta_i\zeta_j \leqq v^2 g^{ij}\zeta_i\zeta_j, \qquad \zeta \in \mathbb{R}^n,$$

and hence an a priori estimate is needed for v.

3. Gradient Estimates

In this section we give some estimates for v. The techniques employed are simpler than those in [BS, Ge] but the final estimates are significantly stronger. It is interesting to note that the key inequality involving $|A|^2$ occurs in some form in all of the preceding works: [BS, CY, Ge].

The previous notation will apply. We use c to denote constants depending only on n, and C for constants depending on geometric data. Important constants are denoted $C_1, C_2, \ldots, m, \delta, \ldots$.

Definition. Let $\mathcal{T}_+\mathcal{V}$ denote the bundle over $\mathcal{V}$ with fibre at $p \in \mathcal{V}$ consisting of future-directed unit (timelike) vectors at p. Then $F \in C^\infty(\mathcal{T}_+\mathcal{V})$ satisfies the *mean curvature structure conditions* ("the structure conditions") with constant Λ if, for any spacelike surface $M \subset \mathcal{V}$, the function $\tilde{F} \in C^\infty(M)$, $\tilde{F}(p) = F(p, N(p))$, $p \in M$

satisfies

$$|\tilde{F}(p)| \leqq \Lambda v, \tag{3.1}$$

$$|\langle T, \nabla^M \tilde{F}\rangle| \leqq \Lambda(v^3 + v^2|A|) \quad \text{for all} \quad p \in M. \tag{3.2}$$

If the mean curvature H_M of a fixed surface M satisfies the structure conditions we say that H_M satisfies the structure conditions along M (with constant Λ).

Some examples of functions satisfying (3.1), (3.2) are:

$$F(p, v(p)) = \phi(p), \qquad \phi \in C^\infty(\mathscr{V}), \tag{3.3}$$

with $\|\phi\|_1 < \infty$, since by (2.19)

$$|\langle T, \nabla^M \phi\rangle| = |\bar{g}^{ij} X_i(\phi) \langle X_j, T\rangle| \leqq v^2 \|\nabla \phi\|,$$

so the structure conditions hold with $\Lambda = \|\phi\|_1$;

$$F(p, v(p)) = \langle X, v\rangle, \tag{3.4}$$

where $X \in \Gamma(T\mathscr{V})$ is a smooth vector field on $\mathscr{V}$ with $\|X\|_1 < \infty$. Using (2.16) gives

$$|\tilde{F}(p)| \leqq v(|\langle U, X\rangle| + \langle T, X\rangle|) \leqq v\|X\|, \tag{3.5}$$

and (2.3) gives

$$|\langle \nabla^M \tilde{F}, T\rangle| = |\langle \nabla_{T^{\|}} X, N\rangle + A(T^{\|}, X^{\|})| \leqq cv^3 \|\nabla X\| + cv^2 \|X\| \, |A|, \tag{3.6}$$

and hence $\Lambda = c\|X\|_1$.

The main gradient estimates are given by the following:

Theorem 3.1. *Let $(\mathscr{V}, g)$ be a spacetime with time function t such that*

$$\|\mathrm{Ric}\|, \ \|\alpha\|, \ \|\alpha^{-1}\nabla\alpha\|, \ \|A^o\|, \ \|\mathscr{L}_T g\|_1 \leqq C_1, \tag{3.7}$$

and suppose that M is a compact spacelike surface with height function u and mean curvature H satisfying the structure conditions along M with constant Λ.

(i) *If $\partial M = \emptyset$ then*

$$v(p) \leqq 2\exp\{K \min(m_+ - u(p), u(p) - m_-)\} \quad \textit{for all} \quad p \in M, \tag{3.8}$$

where $K = K(C_1, \Lambda)$ and $m_+ = \sup_M u$, $m_- = \inf_M u$.

(ii) *If $\partial M \neq \emptyset$, then*

$$v(p) \leqq 2\exp\{K \min(m_+ - u(p), u(p) - m_-)\} \sup_{\partial M} v \quad \textit{for all} \quad p \in M, \tag{3.9}$$

where $K = K(C_1, \Lambda)$.

(iii) *If $\partial M \neq \emptyset$ and ∂M satisfies the conditions*

$$\|H_{\partial M}\| = \sup_{\partial M} \left(\sum_{\lambda=1}^{n+1} |\langle H_{\partial M}, v_\lambda\rangle|^2 \right)^{1/2} \leqq C_1, \tag{3.10}$$

$$u|_{\partial M} = \text{const}, \tag{3.11}$$

where $H_{\partial M}$ is the mean curvature vector of ∂M, then

$$v(p) \leqq 2\exp(Km) \quad \textit{for all} \quad p \in M, \tag{3.12}$$

where $K = K(C_1, \Lambda)$ and $m = \sup_M |u|$.

(iv) *If* $\partial M \neq \emptyset$ *and* $p \in M$ *is such that*

$$u(p) - \sup_{\partial M} u \geqq \varepsilon > 0, \tag{3.13}$$

then

$$v(p) \leqq 2\exp\{K(m_+ - u(p))\}, \tag{3.14}$$

where $K = K(C_1, \Lambda, \varepsilon^{-1})$. *The analogous statement when* $u < \inf_{\partial M} u$ *is also true.*

Remarks. (1) The condition (3.7) could be replaced by

$$\|\mathrm{Ric}\|, \quad \|\log\alpha\|_1, \quad \|\nabla T\|_1 \leqq C_1.$$

(2) The main estimate (3.12) is extended to nonconstant boundary data in Corollary 3.4. Note the very weak dependence on the geometry of $\mathscr{V}$ – this is essential for the applications to unbounded domains.

(3) It will be clear that the proof requires only that $ds^2 \in C^2$, $t \in C^3$, and $M \in C^3$. In addition, the Einstein field equations are not assumed in any form.

Proof. Let K be a (large) constant to be fixed later, and consider the maximum point $q \in M$ of $e^{Ku}v$. First suppose $q \in \mathrm{Int}(M)$, so

$$\begin{aligned} 0 &= Kv\nabla^M u + \nabla^M v, \\ 0 &\geqq Kv\Delta_M u - K^2 v|\nabla^M u|^2 + \Delta_M v \quad \text{at} \quad q. \end{aligned} \tag{3.15}$$

Using (2.8), (2.9), (2.10) and the structure conditions, we estimate

$$\begin{aligned} \Delta_M u &\geqq -\alpha^{-1} C v^2, \\ \Delta_M v &\geqq v|A|^2 - C(v^3 + v^2|A|) \\ &\geqq (1-\varepsilon)v|A|^2 - C(\varepsilon^{-1})v^3 \quad \text{for any} \quad \varepsilon > 0. \end{aligned} \tag{3.16}$$

Let $\{\lambda_i\}_1^n$ be the eigenvalues of $A(\cdot,\cdot)$ with $|\lambda_1| = \max|\lambda_i|$, then

$$|A|^2 = \Sigma_1^n \lambda_i^2 \geqq \lambda_1^2 + \frac{1}{n-1}(\Sigma_n^2 \lambda_i)^2 \geqq \left(1 + \frac{1}{n}\right)\lambda_1^2 - H^2 \tag{3.17}$$

by the Schwarz and arithmetic-geometric mean inequalities, since $H = \Sigma_1^n \lambda_i$.

Thus

$$\begin{aligned} |\nabla^M v|^2 &= -A(T^{\|}, \nabla^M v) - \langle N, \nabla_{\nabla^M v} T\rangle \\ &\leqq |\nabla^M v|\,(v|\lambda_1| + Cv^2) \\ &\leqq (1+\varepsilon)v^2\lambda_1^2 + C(\varepsilon^{-1})v^4 \quad \text{for any} \quad \varepsilon > 0, \end{aligned}$$

and combining this with (3.17) gives

$$v^2|A|^2 \geqq \left(1 + \frac{1}{2n}\right)|\nabla^M v|^2 - Cv^4 \geqq \left(1 + \frac{1}{2n}\right)K^2 v^2 |\nabla^M u|^2 - Cv^4 \quad \text{at} \quad q.$$

Substituting this and (3.16), in (3.15) gives

$$0 \geqq \frac{1}{2n}K^2 v|\nabla^M u|^2 - C(K\alpha^{-1} + 1)v^3 \quad \text{at} \quad q,$$

and since $|\nabla^M u|^2 = \alpha^{-2}(v^2-1)$, we can choose $K = K(C_1, \Lambda)$, so that this gives $v(q) \leqq 2$, and hence

$$v(p) \leqq 2\exp\{K(\sup_M u - u(p))\} \quad \text{for all} \quad p \in M. \tag{3.18}$$

Applying the same argument to $e^{-Ku}v$ gives (3.8) and (3.9). To show the main estimate (3.12), let q_+, q_- denote the maximum points of $e^{Ku}v$, $e^{-Ku}v$, respectively. If either of $q_+, q_- \in \mathrm{Int}(M)$, then by the above argument we are done, so we consider the remaining case $q_+, q_- \in \partial M$. Since $u|_{\partial M} = 0$, we can take $q_+ = q_- = q$. Letting e_1 be the inner normal to ∂M in M, we have

$$e_1(e^{\pm Ku}v) \leqq 0 \quad \text{at} \quad q. \tag{3.19}$$

But ∂M is a level set of u, so

$$e_1 = \pm |\nabla^M u|^{-1} \nabla^M u \tag{3.20}$$

for some choice of sign [if $|\nabla^M u|\,(q) = 0$, we are done], so choosing the appropriate sign in (3.19) gives

$$Kv|\nabla^M u|^2 \leqq |\langle \nabla^M u, \nabla^M v\rangle| \quad \text{at} \quad q. \tag{3.21}$$

The right-hand side is estimated using (2.8) and $\Delta_M u = (e_i e_i - \nabla_{e_i} e_i)(u)$ as follows: the term with e_1 is calculated using (3.20), (2.6)

$$\begin{aligned}(e_1 e_1 - \nabla^M_{e_1} e_1)(u) &= |\nabla^M u|^{-1} \langle \nabla^M u, \nabla^M(|\nabla^M u|)\rangle \\ &= -\alpha^{-1}\langle \nabla^M u, \nabla^M \alpha\rangle + v(v^2-1)^{-1}\langle \nabla^M u, \nabla^M v\rangle,\end{aligned}$$

and the terms in $e_2, \ldots, e_n$ are estimated by

$$\Sigma_2^n (e_i e_i - \nabla^M_{e_i} e_i)(u) = \Delta_{\partial M} u - \langle H_{\partial M}, e_1\rangle e_1(u) = -\langle H_{\partial M}, \nabla^M u\rangle,$$

since $u|_{\partial M} = \text{const}$. Here $H_{\partial M}$ is the mean curvature vector of ∂M,

$$H_{\partial M} = -\Sigma_2^n (\nabla_{e_i} e_i)^\perp,$$

where $(\cdot)^\perp$ denotes the component orthogonal to ∂M. Collecting terms gives

$$v\langle \nabla^M u, \nabla^M v\rangle = (v^2-1)\,(\alpha^{-1}\langle \nabla^M \alpha, \nabla^M u\rangle + \langle H_{\partial M}, \nabla^M u\rangle + \Delta_M u),$$

and from (2.6), (2.8) and the structure conditions we have

$$|\langle \nabla^M u, \nabla^M v\rangle| \leqq \alpha C v |\nabla^M u|^2,$$

where $C = C(C_1, \Lambda, \|H_{\partial m}\|)$, and hence at q,

$$Kv|\nabla^M u|^2 \leqq \alpha C v |\nabla^M u|^2.$$

For $K = K(C_1, \Lambda, \|H_{\partial M}\|)$ chosen large enough this is a contradiction, and hence the interior estimates (3.18) must hold. This proves (3.12). Finally the estimate (3.14) follows from the interior maximum argument applied to the function $\phi(u)v$, where $\phi(u) = (e^u - e^{\sup_{\partial M} u})_+^K$. □

To handle nonconstant boundary data we show that the time function can be modified to incorporate a given spacelike surface as a level set. This result is also useful in barrier arguments.

Proposition 3.2. *Let $\partial_t = -\alpha^2 \nabla t$ be the coordinate vector in the zero-shift coordinate system and let the associated flow be $\phi_s : \mathscr{V} \to \mathscr{V}$. Suppose $\mathscr{S}$ is a given spacelike surface with height function w, and define $\mathscr{S}_s = \{p \in \mathscr{V} : \phi_{-s}(p) \in \mathscr{S}\}$.*

(i) *Suppose $\delta > 0$, $m, B < \infty$ are such that $\phi_{-s} : \mathscr{S}_s \to \mathscr{S}$ is a diffeomorphism for $|s| < \delta$, and*

$$\sup_{\mathscr{S}} w - \inf_{\mathscr{S}} w = 2m , \qquad \sup_{|s| \leqq \delta} \sup_{\mathscr{S}_s} v \leqq B , \tag{3.22}$$

where $v = v(s) = -\langle N(s), T \rangle$ and $N(s)$ is the future-directed unit normal to $\mathscr{S}_s$ (hence $\mathscr{S}_s$ is spacelike).

(ii) *Suppose $\mathscr{V}$ can be written as the disjoint union*

$$\mathscr{V} = \mathscr{U}_\delta \cup I^+(\mathscr{S}_\delta) \cup I^-(\mathscr{S}_{-\delta}) , \tag{3.23}$$

where I^+, I^- denote the chronological future, past with respect to $\mathscr{V}$ [HE], p. 182, *and*

$$\mathscr{U}_\delta = \bigcup_{|s| \leqq \delta} \mathscr{S}_s .$$

Then there is a time function $\tilde{t} \in C^\infty(\mathscr{V})$ such that

$$\mathscr{S} = \{p \in \mathscr{V} : \tilde{t}(p) = 0\} , \tag{3.24}$$

$$\tilde{t}(p) = t(p) \quad \textit{for} \quad \begin{cases} t(p) > \sup_{\mathscr{S}} w + 3\delta \\ t(p) < \inf_{\mathscr{S}} w - 3\delta . \end{cases} \tag{3.25}$$

Further, letting $\tilde{\alpha}$, $\tilde{T}$ denote the lapse function and future unit normal with respect to $\tilde{t}$, we have $|\langle T, \tilde{T} \rangle| \leqq B$,

$$(m + \delta + \tfrac{1}{2})^{-1} \leqq \alpha^{-1} \tilde{\alpha} \leqq 2B ,$$

$$\| \tilde{T} \|_k \leqq C_2(k, m, \delta, B, \| T \|_k, \| Dw \|_{k; \mathscr{U}_\delta}, \| \alpha \|_k) , \tag{3.26}$$

where w is extended to $\mathscr{U}_\delta$ by $\partial_t w = 0$, and

$$\| Dw \|_{k; \mathscr{U}_\delta} = \sum_{j=0}^{k} \sup_{\mathscr{U}_\delta} |D^{j+1} w| .$$

Remark. Condition (i) is trivially satisfied if $\mathscr{S}$ is compact. Condition (ii) says that $\mathscr{S}$ is "large enough" in $\mathscr{V}$.

Proof. Normalize t by $\sup_{\mathscr{S}} w = -\inf_{\mathscr{S}} w = m$, and coordinatize $\mathscr{U}_\delta$ by

$$\mathscr{U}_\delta = \{p = (y, t) : p = \phi_{(t - w(y))}(y), y \in \mathscr{S}, |t - w(y)| \leqq \delta\} .$$

Let $\chi \in C_c^\infty(\mathbb{R})$, $h \in C^\infty(\mathbb{R})$ be functions satisfying

$$\chi \geqq 0 , \qquad spt \chi \subset (0; \tfrac{1}{2}\delta) , \qquad \int_0^\delta \chi = 1$$

$$0 \leqq h \leqq 1, h(s) = \begin{cases} 0 & \text{if} \quad s \leqq -\delta , \quad \int_{-\delta}^{\delta} h = \delta , \\ 1 & \text{if} \quad s \geqq \delta \end{cases}$$

and define functions $\chi_+, \chi_- \in C^\infty(\mathbb{R})$, $F \in C^\infty(\mathbb{R}^2)$ by

$$\chi_\pm(s) = \chi(s) \pm \chi(-s),$$
$$F(z,t) = \tfrac{1}{2}(t-m-2\delta) + \tfrac{1}{2}\int_{-\infty}^{t-z} [(m+2\delta)\chi_+(s) + z\chi_-(s)]\,ds.$$

The time function $\tilde{t}$ is defined explicitly by

$$\tilde{t}(p) = \begin{cases} F(w(p), t(p)); & p \in \mathscr{U}_\delta, \\ t(p) \pm \tfrac{1}{2} \int_{-\infty}^{\mp t(p)+m+2\delta} h(s)\,ds; & p \in I^\pm(\mathscr{S}_{\pm\delta}). \end{cases}$$

Then $F(z,z)=0$ gives (3.24), while (3.25) follows from $h(s)=0$ for $s \leqq -\delta$. If $p \in \mathscr{U}_\delta$ and $t(p) - w(p) \geqq \tfrac{1}{2}\delta$ then since $\int_{-\infty}^{t} h(s)\,ds = t$ for $t \geqq \delta$,

$$F(w(p), t(p)) = \tfrac{1}{2}(t+m+2\delta) = t + \tfrac{1}{2}\int_{-\infty}^{-t+m+2\delta} h(s)\,ds,$$

and similarly for $t - w \leqq -\tfrac{1}{2}\delta$, so $\tilde{t}$ is C^∞. If $p \notin \mathscr{U}_\delta$

$$\nabla\tilde{t} = (1 - \tfrac{1}{2}h(\mp t + m + 2\delta))\nabla t, \quad p \in I^\pm(\mathscr{S}_{\pm\delta}),$$

which gives (3.26), and if $p \in \mathscr{U}_\delta$

$$\nabla\tilde{t} = \tfrac{1}{2}(1+\psi)\nabla t + \tfrac{1}{2}\left(\int_{-\infty}^{t-w} \chi_-(s)\,ds - \psi\right) Dw, \tag{3.27}$$

where $\psi = (m+2\delta)\chi_+(t-w) + w\chi_-(t-w)$. Then

$$\tilde{\alpha} = 2\alpha(1+\psi)^{-1}|\langle T, \tilde{T}\rangle|,$$
$$|\langle T, \tilde{T}\rangle| = \left(1 - \alpha^2\left(\frac{\psi - \int\chi_-}{\psi+1}\right)^2 |Dw|^2\right)^{-1/2} \leqq B,$$

since $\int_{-\infty}^{t} \chi_- \geqq -1$, and the first two estimates of (3.26) follow now from $0 \leqq \psi \leqq 2(m+\delta)$. From (3.27) and (2.2) we have

$$\tilde{T} = \left(1 - \alpha^2\left(\frac{\psi-\int\chi_-}{\psi+1}\right)^2 |Dw|^2\right)^{-1/2} \left(T + \alpha\left(\frac{\psi - \int\chi_-}{\psi+1}\right) Dw\right),$$

which yields the estimates for $\nabla^k T$. □

The following linear algebra exercise shows that the norms $\|\cdot\|_k$, $\|\cdot\|_{\tilde{k}}$ defined by the reference slicings of t, $\tilde{t}$ respectively are equivalent.

Lemma 3.3. *Let T_1, T_2, T_3 be future-directed unit timelike vectors. Then*

$$1 \leqq -\langle T_1, T_2\rangle \leqq 2\langle T_1, T_3\rangle\langle T_2, T_3\rangle - 1.$$

Thus in the situation of Proposition 3.2, the tilt functions $v = -\langle T, N\rangle$, $\tilde{v} = -\langle \tilde{T}, N\rangle$ are related by

$$(2B)^{-1} v \leqq \tilde{v} \leqq 2Bv. \tag{3.28}$$

Corollary 3.4. *Suppose $\mathscr{V}$, M satisfy the conditions of Theorem 3.1 with ∂M nonempty and smooth, and suppose $\mathscr{S} \subset \mathscr{V}$ is a spacelike surface satisfying the conditions of Proposition 3.2 with respect to some subset $\mathscr{V}' \subset \mathscr{V}$ such that $M \subset \mathscr{V}'$, and $\partial\mathscr{S} = \partial M$. Then there is a constant $C = C(C_1, \Lambda, C_2(\mathscr{S}), \|H_{\partial M}\|, \sup_M |u|)$ such that $v(p) \leqq C$ for all $p \in M$.*

Proof. The assumptions on $\mathscr{S}$ give a time function $\tilde{t}$ in $\mathscr{V}'$ such that $\tilde{t}|_{\partial M} = 0$ and Theorem 3.1 (iii) estimates $\tilde{v}$. The conclusion (3.26) of Proposition 3.2 and (3.28) give the estimate for v. □

4. Existence Theorems

Existence and regularity results for surfaces of prescribed mean curvature follow from standard elliptic theory once a uniform gradient bound is established. The estimates of the previous section provide this once an a priori estimate for the height function u is given. In this section such an estimate follows from compactness or barrier assumptions.

Theorem 4.1. *Let $(\mathscr{V}, ds^2)$ be a C^∞ cosmological spacetime with time function $t \in C^\infty(\mathscr{V})$ satisfying the estimate* (3.7) *and coordinatized by*

$$\mathscr{V} = \{(x, t), x \in \mathscr{S}, t_1 \leqq t \leqq t_2\},$$

so $\mathscr{S}$ is a compact n-manifold without boundary.

Suppose that $F \in C^\infty(\mathscr{T}_+\mathscr{V})$ satisfies the structure conditions (3.1), (3.2) *with constant Λ, and there are C^∞ spacelike Cauchy surfaces $\mathscr{S}_+$, $\mathscr{S}_-$ in $\mathscr{V}$ such that*

$$\mathscr{S}_+ \subset I^+(\mathscr{S}_-), \tag{4.1}$$

$$\begin{aligned} F(p, N_+) &> H_{\mathscr{S}_+}(p) \quad \text{for all} \quad p \in \mathscr{S}_+, \\ F(p, N_-) &< H_{\mathscr{S}_-}(p) \quad \text{for all} \quad p \in \mathscr{S}_-, \end{aligned} \tag{4.2}$$

where N_+, N_- are the future unit normals and $H_{\mathscr{S}_+}$, $H_{\mathscr{S}_-}$ are the mean curvatures of $\mathscr{S}_+$, $\mathscr{S}_-$ respectively. Then there is a C^∞ spacelike Cauchy surface $M \subset \mathscr{V}$ such that $H_M(p) = F(p, N(p))$ for all $p \in M$.

Proof.[1] We will use the Leray-Schauder fixed point theory in the form of [LL, Theorem 4.4.3]. The method is essentially that of [Ge] with some modifications. We work in zero-shift coordinates, considering surfaces as defined by graphs over $\mathscr{S}$, and by applying Proposition 3.2 twice we can assume that

$$\mathscr{S}_\pm = \{p \in \mathscr{V} : t(p) = \pm t_0\}.$$

Define the Banach space $\mathfrak{B} = C^{1,\lambda}(\mathscr{S})$ and the subset

$$\mathfrak{K} = \{w \in \mathfrak{B} : |w|_{1,\lambda} \leqq K_1, \max_{\mathscr{S}} v(w) \leqq K_2, \max_{\mathscr{S}} |w| \leqq t_0\},$$

where λ, K_1, K_2 are constants to be fixed later and $|\cdot|_{1,\lambda}$ is the Hölder norm. We are using the notation

$$v(w) = v(x, w) = (1 - \alpha^2 |Dw|^2)^{-1/2}|_{(x, w(x))}.$$

1 I would like to thank G. Galloway for pointing out an error in the previous version of this proof

For $0<\varepsilon\leqq 1$ define the operator $\mathfrak{T}_\varepsilon:\mathfrak{B}\cap\mathfrak{K}\to\mathfrak{B}$ by letting $u=\mathfrak{T}_\varepsilon w$ be the solution of

$$\operatorname{div}_w(\alpha Du)+\langle G(w),Du\rangle-\varepsilon v(w)^{-1}u=v(w)^{-1}F_w-J(w)\,, \tag{4.3}$$

where div_w, F_w are evaluated with respect to graph(w) and

$$G(w)_i=v(w)^2(\alpha^2 w^j A^o_{ij}+\alpha_i)|_{(x,w(x))}\,,$$
$$J(w)=H^o(x,w(x))\,.$$

Since $w\in\mathfrak{K}$ this equation is uniformly elliptic with $C^{0,\lambda}$ coefficients, so the Schauder estimates [GT, Chap. 6] show that $\mathfrak{T}_\delta$ is well defined and compact, provided that the operator

$$\phi\mapsto\operatorname{div}_w(\alpha D\phi)+\langle G(w),D\phi\rangle-\varepsilon v(w)^{-1}\phi$$

has trivial kernel. This follows from the Hopf maximum principle after noting that the operator is of the form $a^{ij}\phi_{ij}+b^i\phi_i+c\phi$, with $c<0$.

From the calculations of Sect. 2 we verify that

$$H(w)=v(w)\,(\operatorname{div}_w(\alpha Dw)+\operatorname{div}_w(T))\,,$$
$$\operatorname{div}_w(T)=\langle G(w),Dw\rangle+J(w)\,,$$

so if u is a fixed point of $\sigma\mathfrak{T}_\varepsilon$, $0\leqq\sigma\leqq 1$, then

$$H(u)=\sigma F_u+(1-\sigma)v(u)H^o+\varepsilon u\,. \tag{4.4}$$

This satisfies the mean curvature structure conditions, so from Theorem 3.1(i) and standard Hölder estimates [GT, Theorem 12.6], we have the estimates

$$v(u)\leqq C(C_1,\Lambda,t_0)\,,$$
$$|u|_{1,\lambda'}\leqq C'(C_1,\Lambda,t_0)\,,$$

for some $0<\lambda'<1$. Setting $K_1=C'+1$, $K_2=C+1$ and $\lambda=\lambda'$ shows that $u\in\operatorname{Int}(\mathfrak{K})$ unless u touches $\mathscr{S}_+$ or $\mathscr{S}_-$. But from (4.2), if $u=\sigma\mathfrak{T}_\varepsilon u$ and $\max_{\mathscr{S}}u=t_0$, then

$$H(u)-H^o=\sigma(F_u-H^o)+\varepsilon t_0>0$$

at the maximum point. This contradicts the maximum principle so $u\in\operatorname{Int}(\mathfrak{K})$, and the Leray–Schauder theory shows that $\mathfrak{T}_\varepsilon$ has a fixed point u_ε with mean curvature $H(u_\varepsilon)=F_{u_\varepsilon}+\varepsilon u_\varepsilon$. By the Schauder estimates again, the fixed point $u_\varepsilon\in\mathfrak{K}$ is in fact smooth. Since all the above estimates are independent of ε, we can take a subsequence $\{u_{\varepsilon_i};\ \varepsilon_i\downarrow 0\}$ converging to $u\in C^\infty(\mathscr{S})$, which defines the required surface. □

To state a theorem about the Dirichlet problem some terminology is needed:

Definition. The *domain of dependence* of $\mathscr{S}\subset\mathscr{V}$ is

$$D(\mathscr{S})=\overline{D^+(\mathscr{S})}\cup\overline{D^-(\mathscr{S})}\,, \tag{4.5}$$

where D^+, D^- are the future, past domains of dependence with respect to $\mathscr{V}$ [HE, p. 201].

Theorem 4.2. *Let* $(\mathscr{V}, ds^2)$ *be a* C^∞ *spacetime with time function* $t \in C^\infty(\mathscr{V})$ *satisfying* (3.7), *and suppose that* $\mathscr{S}$ *is a* C^∞ *connected spacelike surface with* $\partial\mathscr{S}$ *smooth and nonempty, satisfying*

$$D^+(\mathscr{S}) \cap D^-(\mathscr{S}) = \mathscr{S}, \tag{4.6}$$

$$\partial D(\mathscr{S}) \text{ is generated by null geodesic segments with endpoints on } \partial\mathscr{S} \text{ (cf. [HE], Proposition 6.5.3)}, \tag{4.7}$$

$$D(\mathscr{S}) \text{ is compact.} \tag{4.8}$$

Suppose that $F \in C^\infty(\mathscr{T}_+\mathscr{V})$ *satisfies the structure conditions, then there is a spacelike surface,* $M \subset \mathscr{V}$ *satisfying the Dirichlet problem:*

$$\begin{aligned} H_M(p) &= F(p, N(p)) \quad \text{for all} \quad p \in M \\ \partial M &= \partial\mathscr{S}. \end{aligned} \tag{4.9}$$

Remark. Assumption (4.6) is designed to avoid possible pathological cases, (4.7) roughly says that $D(\mathscr{S})$ has no "holes" and (4.8) is needed to apply the gradient estimate. Note that (4.8) does not follow from compactness of $\mathscr{S}$, as the Brill example mentioned in the introduction illustrates.

Proof. Compactness of $D(\mathscr{S})$ shows that Proposition 3.2 can be applied with $\mathscr{V} = D(\mathscr{S})$, so we may assume $\mathscr{S} = \{p \in D : t(p) = 0\}$. By virture of (4.7) and the definition of $D(\mathscr{S})$, we can introduce zero-shift coordinates (x, t) in $D(\mathscr{S})$ by $p \leftrightarrow (x, t)$; $x = \phi_{-t}(p) \in \mathscr{S}$, where ϕ_s is the flow of $-\alpha^2 \nabla t = \partial_t$. Compactness of $D(\mathscr{S})$ gives a uniform height estimate and hence uniform gradient estimates for fixed points of the operator $\mathfrak{T}_\varepsilon$ of Theorem 4.1. The Leray–Schauder theory applies, since any spacelike surface $M \subset D$ with $\partial M = \partial\mathscr{S}$ cannot touch ∂D except at $\partial\mathscr{S}$ because of (4.7). Setting $\varepsilon = 0$ gives the result. □

Remark. The above proof can be generalized to include upper and lower barriers, exactly as in Theorem 4.1. This gives solvability of the Dirichlet problem near crushing singularities [ES]. Formally, we have

Theorem 4.3. *Suppose that the conditions of Theorem 4.2 hold, but with* (4.8) *replaced by*

$$D(\mathscr{S}) \cap I^-(\mathscr{S}_+) \text{ is compact.} \tag{4.8$'$}$$

where $\mathscr{S}_+$ *is a future barrier surface. That is,* $\mathscr{S}_+$ *is a smooth spacelike surface satisfying the topological conditions*

$$\begin{aligned} &\mathscr{S} \subset I^-(\mathscr{S}_+), \\ &\partial(D(\mathscr{S}) \cap I^-(\mathscr{S}_+)) = (\partial D(\mathscr{S}) \cap I^-(\mathscr{S}_+)) \cup (\mathscr{S}_+ \cap D(\mathscr{S})), \end{aligned} \tag{4.10}$$

and has mean curvature H_+ *satisfying*

$$F(p, N_+(p)) > H_+(p) \quad \text{for all} \quad p \in \mathscr{S}_+$$

where N_+ *is the future unit normal to* $\mathscr{S}_+$.

Under these conditions there is a smooth spacelike surface $M \subset D(\mathscr{S}) \cap I^-(\mathscr{S}_+)$ *satisfying the Dirichlet problem*

$$\begin{cases} H_M(p) = F(p, N(p)) \quad \text{for all} \quad p \in M \\ \partial M = \partial\mathscr{S}. \end{cases}$$

Proof. Condition (4.10) ensures that Proposition 3.2 can be applied to make $\mathscr{S}_+$ a level set of the time function. The remainder of the proof follows Theorems 4.1 and 4.2 and is omitted. □

5. Asymptotically Flat Maximal Surfaces

In this section $\mathscr{V}$ is a 4-dimensional spacetime with radius function $r \in C^\infty(\mathscr{V})$ and time function $t \in C^\infty(\mathscr{V})$ satisfying the estimates (3.7).

Definition. $\mathscr{V}$ is *asymptotically flat* if there is a constant $R_0 \geqq 1$ such that the *exterior region* $\mathscr{V}_E = \{p \in \mathscr{V},\ r(p) \geqq R_0\}$ has coordinates (x^i, t) such that

$$\begin{aligned} &r = (\Sigma_1^3 x^{i2})^{1/2}, \\ &ds^2 = -(\alpha^2 - \beta^2)dt^2 + 2\beta_i dx^i dt + g_{ij}dx^i dx^j = g_{\lambda\mu}dx^\lambda dx^\mu, \end{aligned} \tag{5.1}$$

and there are constants C_3, C_4 such that

$$r \sum_{\lambda,\mu} |g_{\lambda\mu} - \eta_{\lambda\mu}| + r^2 \sum_{\kappa,\lambda,\mu} |\partial_\kappa g_{\lambda\mu}| \leqq C_3, \tag{5.2}$$

$$r^3 |H^o| \leqq C_4, \tag{5.3}$$

with $C_3/R_0 \leqq 10^{-2}$, where $\eta_{\lambda\mu}$ is the Minkowski metric and H^o is the mean curvature of the slices $\mathscr{S}_t$.

The *interior region* of $\mathscr{V}$ is $\mathscr{V}_I = \{p \in \mathscr{V},\ r(p) \leqq R_0\}$. $\mathscr{V}$ satisfies the *uniform interior condition* if there is constant C_5 such that for all $q \in \mathscr{V}$ with $r(q) = R_0$,

$$\begin{aligned} &\sup_{p \in \mathscr{V}_I - I^+(q)} (t(p) - t(q)) \leqq C_5 \quad \text{if} \quad t(q) \geqq 0, \\ &\sup_{p \in \mathscr{V}_I - I^-(q)} (t(q) - t(p)) \leqq C_5 \quad \text{if} \quad t(q) \leqq 0. \end{aligned} \tag{5.4}$$

Remarks. (1) The uniform interior condition is slightly stronger than saying the Brill phenomenon doesn't occur. In particular it implies that $D(\mathscr{S})$ is compact for compact $\mathscr{S} \subset \mathscr{S}_0$.

(2) For simplicity we have assumed that $\mathscr{V}$ has only one end but this is not necessary – the generalization to a finite number of ends is immediate. Combining this with the barriers provided by crushing singularities shows that the results apply, for example, to spacetimes close to the maximally-extended Schwarzschild solution.

We start by constructing barrier surfaces at spacelike infinity (Proposition 5.1), then prove the fundamental height estimate (Theorem 5.3) from which the existence of maximal slices follows easily (Theorem 5.4).

The radial mean curvature equation in $\mathbb{R}^{3,1}$ for $w = w(r)$,

$$r^{-2}\left(\frac{r^2 w'}{\sqrt{1 - w'^2}}\right)' = H_{\text{flat}},$$

can be solved with $H = -\Lambda r^{-3}$ in $r \geqq R$, giving

$$w'(r) = -\left(1 + \delta^{-2}\left(\frac{r}{R}\right)^4 \left(1 + \delta^{-1} R^{-2} \Lambda \log\left(\frac{r}{R}\right)\right)^{-2}\right)^{-1/2}, \tag{5.5}$$

with $\delta = -w'(1-w'^2)^{-1/2}(R)$. A calculation shows that $w'' \geqq 0$ for $r \geqq R$ when $\Lambda \leqq 2\delta R^2$ so we set $\delta = 4/3$, $\Lambda = 8R^2/3$, and then $(1-w'^2)^{-1/2} \leqq 5/3$ for all $r \geqq R$. Now setting $w(\infty)=0$ gives

$$w(r) = R \int_{r/R}^{\infty} \left(1 + \frac{9}{16} s^4 (1+2\log s)^{-2}\right)^{-1/2} ds, \tag{5.6}$$

and then $w(R) = C_6 R$, where $C_6 \cong 2$ is an absolute constant.

Then in the metric (5.1) by (2.17) we have

$$H(w) = \operatorname{div}^o\left(\frac{W}{\sqrt{1-|W|^2}}\right) + v(w)(H^o + \langle W, \nabla_T T\rangle) + \tfrac{1}{2} v^3 T(|W|^2). \tag{5.7}$$

By asymptotic flatness there are constants R_1, C such that

$$v(w) \leqq (1-w'^2)^{-1/2} + Cr^{-1}, \qquad r \geqq R \geqq R_1,$$

and the last two terms of (5.7) can be estimated by

$$v|\langle W, \nabla_T T\rangle| + \tfrac{1}{2} v^3 |T(|W|^2)| \leqq C(|w'| r^{-2} + |w'w''| r^{-1}).$$

Since $g^{ij} = \delta_{ij} + O(r^{-1})$, $\nabla = \nabla^{\text{flat}} + O(r^{-2})$ by asymptotic flatness (5.2), we can estimate

$$|\operatorname{div}^o(W(1-|W|^2)^{-1/2}) - H_{\text{flat}}(w)| \leqq C(|w''| r^{-1} + |w'| r^{-2}),$$

and thus

$$H(w) \leqq -\Lambda r^{-3} + 2C_4 r^{-3} + C\Lambda r^{-4}\left(1 + \log\left(\frac{r}{R}\right)\right)$$

since by (5.5),

$$w'(r) = -\Lambda r^{-2}(1-w'^2)^{1/2}\left(\frac{1}{2} + \log\left(\frac{r}{R}\right)\right),$$

$$w''(r) = 2\Lambda r^{-3}(1-w'^2)^{3/2} \log\left(\frac{r}{R}\right).$$

Since $\Lambda = \frac{8}{3} R^2$, we can find $R_1(R_0, C_3, C_4)$ such that $H(w) \leqq -R^2 r^{-3}$ for $r \geqq R \geqq R_1$. We have shown

Proposition 5.1. *There are constants $R_1 = R_1(R_0, C_3, C_4)$, C_6, C_7 such that for any $R \geqq R_1$ the surface w_R defined by (5.6) is spacelike for $r \geqq R$ and satisfies*

$$H(w_R) \leqq -R^2 r^{-3}, \qquad r \geqq R, \tag{5.8}$$

$$v(w_R) \leqq 2, \qquad r \geqq R, \tag{5.9}$$

$$w_R(\infty) = 0, \qquad w_R(R) = C_6 R, \qquad C_6 \cong 2, \tag{5.10}$$

$$C_7^{-1} \leqq r(R\log r)^{-1} w_R(r) \leqq C_7; \qquad r \geqq 2R. \tag{5.11}$$

The above construction generalizes to handle $H^o = O(r^{-2-\varepsilon})$, $\varepsilon > 0$. For simplicity we now redefine R_0 to the R_1 given by the proposition – it is not difficult to check that this only changes C_5 by no more than $2(R_1 - R_0)$. Write $w = w_{R_0}$.

Corollary 5.2. *Suppose that M is a compact maximal surface with height function u such that $\partial M \subset \mathscr{V}_E \cap \mathscr{S}_0$. Then*

$$\text{either} \quad \sup_M |u| \leqq C_6 R_0, \quad \text{or} \quad \sup_M |u| \leqq \sup_{M \cap \mathscr{V}_I} |u|. \tag{5.12}$$

Proof. It is clear that the translated surfaces defined by $w+\tau$, $\tau \in \mathbb{R}$ also satisfy (5.8)–(5.11). Since similar considerations hold for the functions $-w+\tau$, $\tau \in \mathbb{R}$, it will suffice to prove the estimates for $\sup u$.

Consider $M \cap \mathscr{V}_E$ as a graph over $\mathscr{S}' \subset \mathscr{S}_0 \cap \mathscr{V}_E$, and define

$$\tau_0 = \inf\{\tau \in \mathbb{R} : w(x) + \tau \geqq u(x) \text{ for all } x \in \mathscr{S}' \cap \mathscr{V}_E\}.$$

Since $H(w+\tau_0) < H_M$ the maximum principle shows that the contact point $x_0 \in \mathscr{S} \cap \mathscr{V}_E$ must satisfy either $r(x_0) = R_0$ or $x_0 \in \partial M$, and then $\tau_0 \leqq 0$. The conclusion (5.12) follows from (5.10). □

Notice that this says we may assume $\sup |u|$ is attained in $\mathscr{V}_I \cap M$.

Theorem 5.3. *Let $\mathscr{V}$ be an asymptotically flat spacetime with uniform interior satisfying (3.7) and (5.3–5), and suppose M is a compact maximal ($H=0$) surface with $\partial M \subset \mathscr{V}_E \cap \mathscr{S}_0$ and $\|H_{\partial M}\| \leqq C_1$. Then there is a constant $C_8 = C_8(R_0, C_1, C_3, C_4, C_5)$ such that $\sup_M |u| \leqq C_8$, where u is the height function of M.*

Proof. It suffices to estimate $\sup_M u$ and for simplicity we scale $\mathscr{V}$ so that $R_0 = 1$. Constants depending on R_0, C_1, C_3, C_4, C_5 but not on $\sup_M u$ will be denoted by C.

Define the modified time function on $\mathscr{V}_E$, $\tilde{t}(p) = t(p) - w(r(p))$, $p \in \mathscr{V}_E$, and use $\tilde{}$ to indicate quantities defined by the $\tilde{t}$-slicing.

Let $\psi \in C^1(\mathbb{R}_+)$ satisfy $\psi(0) = 0$ and $\psi'(s), \psi(s) > 0$ for $s > 0$, and let $\tilde{M} = M \cap \mathscr{V}_E \cap \{p : \tilde{t}(p) \geqq 0\}$. Then M has boundary consisting of components where $\tilde{u} = 0$ and a set $\tilde{B} \subset M \cap \{p : r(p) = R_0 = 1\}$. Multiplying (2.7) by $\tilde{\alpha}\psi(\tilde{u})$ and integrating by parts over $\tilde{M}$ gives

$$\int_{\tilde{B}} \tilde{\alpha}^2 \psi(\tilde{u}) \langle \nabla^M \tilde{u}, \sigma \rangle d\mathscr{H}^{n-1} = \int_{\tilde{M}} \tilde{\alpha}\psi(\tilde{u}) [H\tilde{v} - \operatorname{div}_M \tilde{T} + \langle \nabla^M \tilde{u}, \nabla^M \tilde{\alpha} \rangle] dv_M$$
$$+ \int_{\tilde{M}} \tilde{\alpha}^2 \psi'(\tilde{u}) |\nabla^M \tilde{u}|^2 dv_M,$$

where σ is the outer normal of $\partial \tilde{M}$, dv_M is the volume form on M and $\mathscr{H}^{n-1}$ is Hausdorf measure on $\partial \tilde{M}$. From (2.6), (2.18) and $H=0$ this can be written as

$$\int_{\tilde{M}} \tilde{\alpha}\psi(\tilde{u}) H^o \tilde{v} \, dv_M = \int_{\tilde{M}} \psi(\tilde{u})(\tilde{v}^2 - 1)\left(\frac{\psi'}{\psi}(\tilde{u}) + \tilde{T}(\tilde{\alpha}) - \tilde{\alpha}\tilde{A}^o_{11} + \frac{\tilde{\alpha}}{\tilde{v}+1}\tilde{H}^o\right) dv_M$$
$$- \int_{\tilde{B}} \tilde{\alpha}^2 \psi(\tilde{u}) \langle \nabla^M \tilde{u}, \sigma \rangle d\mathscr{H}^{n-1}, \tag{5.13}$$

where $\tilde{A}^o_{11} = |\tilde{U}|^{-2} \tilde{A}^o(\tilde{U}, \tilde{U})$ in the notation of Sect. 2. By asymptotic flatness (5.2) and Proposition 5.1 there is a constant $C_9 = C_9(R_0, C_3, C_4)$ such that

$$|\tilde{\alpha}\tilde{A}^o_{11} - \tilde{\alpha}(\tilde{v}+1)^{-1}\tilde{H}^o - \tilde{T}(\tilde{\alpha})| \leqq C_9 r^{-2} \quad \text{for} \quad r \geqq 1. \tag{5.14}$$

Let $m = \sup_M u - C_5 - C_6$ and suppose $m > 1$, so $\sup_M u$ is attained in $\mathscr{V}_I$, and by (5.4) $\tilde{u}(p) \geqq 1$ for all $p \in M$, $r(p) = 1$. The gradient estimate (3.14) applies and gives $v(p) \leqq C$ for all $p \in M$, $r(p) = 1$, where $C = C(C_1, C_5)$ does not depend on m. The boundary term in (5.13) is estimated now using Lemma 3.3, Proposition 5.1 and (5.2),

$$\left|\int_{\tilde{B}} \tilde{\alpha}^2\psi(\tilde{u})\langle \nabla^M\tilde{u},\sigma\rangle d\mathscr{H}^{n-1}\right| \leqq C\sup_M\psi(\tilde{u}). \tag{5.15}$$

To estimate the left-hand side of (5.13) we first calculate the volume form dv_M in local coordinates (5.1). The induced metric on $M\cap\mathscr{V}_E$ is given by

$$\bar{g}_{ij}=g_{ij}+u_i\beta_j+u_j\beta_i-(\alpha^2-\beta^2)u_iu_j,$$

where the height function u has been extended by $\partial_t u=0$. A short calculation shows that

$$\det(\bar{g}_{ij})=\det(g_{ij})(1+\beta\cdot Du)^2v^{-2},$$

and the volume form on M can be written

$$dv_M=\sqrt{\det g_{ij}}\,(1+\beta\cdot Du)v^{-1}d^3x,$$

with respect to the lifted coordinates (x^i) on $M\cap\mathscr{V}_E$. Considering $\tilde{M}$ as a graph over $\Omega\subset\mathscr{S}_0\cap\mathscr{V}_E$ and using Proposition 5.1, we can estimate

$$\int_{\tilde{M}} \tilde{\alpha}\psi(\tilde{u})\tilde{H}^o\tilde{v}dv_M\leqq -C\int_\Omega \psi(\tilde{u})r^{-3}d^3x, \tag{5.16}$$

where $C=C(R_0,C_3)>0$. Inserting (5.14), (5.15), (5.16) into (5.13) now gives

$$\int_\Omega \psi(\tilde{u})r^{-3}d^3x+C\int_{\tilde{M}} \psi(\tilde{u})(\tilde{v}^2-1)\left(\frac{\psi'}{\psi}(\tilde{u})-C_9r^{-2}\right)dv_M\leqq C\sup_M\psi(\tilde{u}), \tag{5.17}$$

and we now choose ψ appropriately.

Since M is spacelike, in $\mathscr{V}_E\cap M$ we have $|Du|\leqq 2$, and (5.4) gives

$$u(p)\geqq \sup_M u-C_5-2(r(p)-1)\quad \text{for all}\quad p\in\tilde{M},$$

which in terms of m and $\tilde{u}$ can be written

$$r(p)\geqq \tfrac{1}{2}(m+2-\tilde{u}(p))\quad\text{for all}\quad p\in\tilde{M}. \tag{5.18}$$

Now define ψ by

$$\log\psi(s)=\begin{cases}4C_9(m+1)^{-2}\log s, & 0<s\leqq 1\\ 4C_9(m+1)^{-1}(m+2-s)^{-1}(s-1), & 1\leqq s\leqq m\\ C_9(s+2-m-4(m+1)^{-1}), & m\leqq s,\end{cases}$$

and $\psi(0)=0$, so $\psi\in C^1(\mathbb{R}_+)$ and

$$\frac{\psi'}{\psi}(s)=\begin{cases}4C_9(m+1)^{-2}s^{-1}, & 0<s\leqq 1\\ 4C_9(m+2-s)^{-2}, & 1\leqq s\leqq m\\ C_9, & m\leqq s.\end{cases}$$

From (5.18) we have $C_9r^{-2}\leqq 4C_9(m+2-\tilde{u})^{-2}$ in $\tilde{M}$ and then $C_9r^{-2}\leqq \frac{\psi'}{\psi}(\tilde{u})$ in $\tilde{M}$, so the second term in (5.17) is positive and may be discarded. The definition of m shows that $\sup_M\tilde{u}\leqq m+C_5$, so $\sup_M\psi(\tilde{u})\leqq\exp(C_9(2+C_5))=C$, and (5.17) becomes

$$\int_\Omega \psi(\tilde{u})r^{-3}d^3x\leqq C.$$

Now $\psi(\tilde{u}) \geqq 1$ for $\tilde{u} \geqq 1$, so (5.18) shows that

$$\int_{1 \leqq r \leqq \frac{1}{2}m} r^{-3} d^3x \leqq C.$$

Integrating shows that $\log m \leqq C$ which gives the desired estimate. □

The main existence theorem follows from this estimate and the techniques of Sect. 4.

Theorem 5.4. *Let $\mathscr{V}$ be a spacetime satisfying the asymptotic flatness conditions (5.2), (5.3) and uniform interior condition (5.4) and suppose there are coordinates (x,t) in $\mathscr{V}$ which cover the region*

$$\mathscr{V}' = \{p = (x,t) : |t(p)| \leqq C_8 \quad \text{if} \quad r(p) \leqq R_2, \quad |t(p)| \leqq w_{R_2}(r(p)) \quad \text{if} \quad r(p) \geqq R_2\}, \tag{5.19}$$

where C_8 is the height estimate of Theorem 5.3, $R_2 = C_8/C_6$ and C_6, R_0 are given by Proposition 5.1. Then there is an entire maximal surface M satisfying

$$\begin{cases} H_M = 0 \\ |u(p)| \leqq C r^{-1} \log r \quad \text{for} \quad p \in M,\ r(p) \geqq R_2. \end{cases} \tag{5.20}$$

If the additional decay

$$r^3 \Sigma |\partial_\kappa \partial_\lambda g_{\mu\nu}| \leqq C_3 \tag{5.20$'$}$$

holds, then

$$r|Du| + r^2 |D^2 u| \leqq C, \tag{5.21}$$

and the mass of M [SY2] *equals the mass of $\mathscr{S}_0$.*

Remarks. (1) The coordinate condition (5.19) can be weakened to allow for crushing singularities in $\mathscr{V}_I$.

(2) This result does not need the Einstein equations, but to assert uniqueness of the resulting surface the timelike convergence condition is needed [BF].

Proof. The uniform interior condition (5.4) and the global coordinates (5.19) ensure that the Dirichlet problem

$$\begin{cases} H(u) = 0 \quad \text{in} \quad B_R = \{p \in \mathscr{S}_0 : r(p) \leqq R\} \\ u(p) = 0 \quad \text{for} \quad r(p) = R \end{cases}$$

is solvable when $R \geqq R_0$, with solution u_R satisfying

$$\sup_{B_R} (|u_R| + v(u_R)) \leqq C$$

for C independent of R. The argument outlined in Theorem 4.2 shows that u_R is smooth, with derivatives estimated independent of R so there is a subsequence u_{R_i}, $R_i \to \infty$ converging uniformly to $u \in C^\infty(\mathscr{S}_0)$ with $H(u) = 0$. The decay estimate follows from the comparison principle applied to u_R, $R \geqq R_2$ and the surfaces w_{R_2}. The additional decay (5.21) follows from a standard scaling argument (e.g. [SY2, Proposition 3]); for completeness we describe this proof, using the Schauder interior estimates of [GT, Chap. 6].

Let $x_0 \in \mathscr{S}_0$ with $r(x_0) = 2R \gg R_0$, and set $\Omega = \{x \in \mathscr{S}_0 : |x - x_0| \leqq R\}$. The uniform estimates for $|u|$, v applied to Eq. (2.17) with [GT, Theorem 12.1] give the estimate $[Du]_{\varepsilon;\Omega} \leqq C$ for some $0 < \varepsilon < 1$. We write Eq. (2.17) as $a^{ij}u_{ij} + b^i u_i = -H^0$, where a^{ij}, b^i depend on α, β^i, g_{ij}, Du and a^{ij} is uniformly elliptic, again by virtue of the uniform estimates for $|u|$, v. Let $x \in \Omega$, and set $B_k = \{y \in \mathscr{S}_0 : |x - y| \leqq k\}$, $k = 1, 2$. Then one checks easily that

$$|a^{ij}|_{0,\varepsilon;B_2} + |b^i|_{0,\varepsilon;B_2} \leqq C(1 + [Du]_{0,\varepsilon;B_2}) \leqq C\,.$$

The Schauder estimates imply

$$|u|_{2,\varepsilon;B_1} \leqq C(|u|_{0;B_2} + |H^0|_{0,\varepsilon;B_2}) \leqq CR^{-1} \log R$$

for any $x \in \Omega$, and hence

$$|a^{ij}|^{(0)}_{0,\varepsilon;\Omega} + |b^i|^{(1)}_{0,\varepsilon;\Omega} \leqq C(1 + [Du]^{(0)}_{0,\varepsilon;\Omega}) \leqq C\,.$$

Applying [GT, Theorem 6.2] now gives

$$|u|^*_{2,\varepsilon;\Omega} \leqq C(|u|_{0;\Omega} + |H^0|^{(2)}_{0,\varepsilon;\Omega}) \leqq C(R^{-1} \log R + 1)\,,$$

where we use $|\nabla H^0| = O(r^{-3})$. This gives the estimate (5.21). Note that if $|\nabla H^0| = O(r^{-4})$, then (5.21) can be strengthened to

$$r|Du| + r^2|D^2u| \leqq Cr^{-1} \log r\,. \quad \square \tag{5.21'}$$

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