Commun. Math. Phys. 92, 485–490 (1984)

Communications in Mathematical Physics

A Note on $D(k, 0)$ Killing Spinors

Gerardo F. Torres del Castillo

Departamento de Física Matemática, Instituto de Ciencias de la Universidad Autónoma de Puebla, M.A.C. No. 219, 72000 Puebla, Pue. México, and Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 México, D.F., México

Abstract. The equations for the $D(k,0)$ Killing spinor fields are integrated assuming that the left conformal curvature does not vanish and that either $k \neq 2, 4, 6, \ldots$, or the Einstein vacuum field equations are satisfied.

1. Introduction

In a remarkable paper, Walker and Penrose [1] showed that every type D solution of the Einstein vacuum field equations admits a quadratic first integral of the null geodesic equations. Their result, later generalized by Hughston et al. [2] to a class of type D solutions of the Einstein-Maxwell equations, is based on the existence of a Killing spinor, from which a conformal Killing tensor of valence two is constructed. The proof given by Walker and Penrose follows from the Bianchi identities and provides a method to find explicitly the above mentioned conformal Killing tensor.

The equations for the Killing spinors have been studied by Hacyan and Plebañski [3] in the context of complex Riemannian geometry, which contains the case of real spacetimes. A direct integration of the equations for Killing spinors of type $D(k,0)$ has been done by Finley and Plebañski [4] in the case of $\mathscr{H}$ spaces (left-flat spaces). In the present work the equations fort Killing spinors of type $D(k,0)$ are integrated under some restrictions. The results apply to complexified space times as well as to real ones. The formalism and notation used here follow those of Plebañski [5]. All the spinorial indices are manipulated according to the convention $\psi_A = \varepsilon_{AB}\psi^B$, $\psi^A = \psi_B \varepsilon^{BA}$, and similarly for dotted indices.

2. Integrability Conditions

Let $L_{AB\ldots D}$ be a $D(k,0)$ Killing spinor [1], that is, $L_{AB\ldots D}$ is a totally symmetric spinor field with $2k$ indices that satisfies the equation[1]

$$\nabla^{\dot{S}}_{(R} L_{AB\ldots D)} = 0\,. \tag{1}$$

1 Round brackets denote symmetrization of the indices enclosed

According to the Ricci identities [2] $\nabla_{(T|\dot{S}|}\nabla^{\dot{S}}_{R}L_{AB\ldots D)} = -4kC^{S}_{(TRA}L_{B\ldots D)S}$. Therefore, an integrability condition of Eq. (1) is given by

$$C^{S}_{(TRA}L_{B\ldots D)S}=0\,. \qquad (2)$$

Denoting the components of $L_{AB\ldots D}$ by $L_{(j)}$, where $j=0,1,\ldots,2k$ is the number of indices taking the value two, i.e., $L_{(0)}=L_{11\ldots 1}$, $L_{(1)}=L_{11\ldots 2}, \ldots, L_{(2k)}=L_{22\ldots 2}$, and $L_{(j)}\equiv 0$ for $j<0$ or $j>2k$, the integrability conditions (2) may then be written as the set of $2k+3$ equations

$$\begin{aligned}&(2k+2-j)(2k+1-j)(2k-j)C^{(5)}L_{(j+1)}+2(2j-k)(2k+2-j)(2k+1-j)C^{(4)}L_{(j)}\\&\quad+6j(2k+2-j)(j-k-1)C^{(3)}L_{(j-1)}+2j(j-1)(2j-3k-4)C^{(2)}L_{(j-2)}\\&\quad-j(j-1)(j-2)C^{(1)}L_{(j-3)}=0, \qquad j=0,1,\ldots,2k+2,\end{aligned} \qquad (3)$$

where

$$C^{(5)}=2C_{1111},\ C^{(4)}=2C_{1112},\ C^{(3)}=2C_{1122},\ C^{(2)}=2C_{1222},\ C^{(1)}=2C_{2222}\,.$$

In a spinor frame such that $L_{(0)}=0$, it follows from Eq. (3) that $C^{(5)}=0$. This means that each principal spinor of $L_{AB\ldots D}$ is a Debever-Penrose (DP) spinor. Substituting the values $L_{(0)}=0$ and $C^{(5)}=0$ in (3), one finds that if $k\neq 2,4,6,\ldots$, then $C^{(4)}=0$. Therefore, when k is not an even integer, each principal spinor of $L_{AB\ldots D}$ is, at least, a double DP spinor and (assuming $C_{ABCD}\neq 0$) there are at most two principal spinors of $L_{AB\ldots D}$ which are not proportional to one another. If there are two of these then C_{ABCD} must be of type D, while if there is just one then $L_{(2k)}$ is the only nonvanishing component of $L_{AB\ldots D}$, and from (3) one concludes that C_{ABCD} is of type N.

When $k=2$ the condition (2) implies that L_{ABCD} is proportional to C_{ABCD}. Hence, in this case, Eq. (2) imposes no restriction on the algebraic type of C_{ABCD}.

As a consequence of Eq. (1) it follows that each principal spinor l_A of $L_{AB\ldots D}$ satisfies the condition [2]

$$l^A l^B \nabla_{A\dot{C}} l_B = 0\,. \qquad (4)$$

This means that, in a complexified spacetime, the vector fields $l^A\partial_{A\dot{B}}$ are tangent to a congruence of null strings [6] (two-dimensional totally null surfaces), while in a real spacetime (i.e., with Lorentzian signature) the vector field $l^A l^{\dot{B}}\partial_{A\dot{B}}$ (where $l^{\dot{B}}=\overline{l^B}$) is tangent to a congruence of shearfree null geodesics. If the Einstein vacuum field equations are satisfied, then Eq. (4) implies that l_A is a multiple DP spinor [6] (and conversely). Therefore, if $C_{ABCD}\neq 0$, C_{ABCD} must be of type D or N.

Thus, the existence of a $D(k,0)$ Killing spinor with $k\neq 2,4,\ldots$, implies the existence of a spinor l_A which is a solution of (4) and at the same time a multiple DP spinor. When the Einstein vacuum field equations are satisfied the conclusion applies for any value of k. In the forthcoming the discussion will be restricted to these cases with the further assumption that C_{ABCD} does not vanish.

The existence of a solution of Eq. (4) which is a repeated DP spinor implies the existence of coordinates $q^{\dot{A}}$, $p^{\dot{A}}$ such that [7]

$$\begin{aligned}g^{1\dot{A}}&=-\sqrt{2}(dp^{\dot{A}}-Q^{\dot{A}\dot{B}}\,dq_{\dot{B}}),\\ g^{2\dot{A}}&=-\sqrt{2}\phi^{-2}\,dq^{\dot{A}},\end{aligned} \qquad (5)$$

2 See, for example, Hacyan and Plebañski [3]

is a null tetrad[3], where $Q^{\dot{A}\dot{B}}$ is a symmetric object and ϕ is a solution of

$$l^B \nabla_{A\dot{C}} l_B = (l^B \partial_{B\dot{C}} \ln \phi) l_A . \tag{6}$$

Using the Ricci rotation coefficients for the tetrad (5)[4], one finds that Eq. (1) amounts to

$$\begin{aligned}(2k+1-j)\,\partial^{\dot{S}}[\phi^{3k-2j}L_{(j)}] + jD^{\dot{S}}[\phi^{3k-2(j-1)}L_{(j-1)}]\\ -j(k+1-j)(\partial_{\dot{B}}Q^{\dot{B}\dot{S}})\phi^{3k-2(j-1)}L_{(j-1)}\\ -j(j-1)(D_{\dot{B}}Q^{\dot{B}\dot{S}})\phi^{3k-2(j-2)}L_{(j-2)}=0, \quad j=0,1,\dots,2k+1,\end{aligned} \tag{7}$$

where

$$\begin{aligned}\partial_{\dot{A}} &\equiv \partial/\partial p^{\dot{A}},\\ D_{\dot{A}} &\equiv \partial/\partial q^{\dot{A}} - Q^{\dot{B}}_{\dot{A}}\,\partial/\partial p^{\dot{B}}.\end{aligned} \tag{8}$$

Since for the tetrad (5) $C^{(5)}=C^{(4)}=0$, the integrability conditions (3) reduce to

$$\begin{aligned}6(2k+1-j)(j-k)C^{(3)}L_{(j)} + 2j(2j-3k-2)C^{(2)}L_{(j-1)}\\ -j(j-1)C^{(1)}L_{(j-2)}=0, \quad j=0,1,\dots,2k+1.\end{aligned} \tag{9}$$

3. Integration of the Equations

When the "left" conformal curvature, C_{ABCD}, is of type N, one has $C^{(3)}=C^{(2)}=0$. Then from (9) it follows that $L_{(0)}=L_{(1)}=\dots=L_{(2k-1)}=0$. Substituting in (7) one finds that $\phi^{-k}L_{(2k)}=\zeta^k$, where ζ is a function of $q^{\dot{A}}$ only, which has to satisfy the condition[5]

$$\partial^{\dot{B}}Q_{\dot{B}\dot{S}} = \partial \ln\zeta/\partial q^{\dot{S}}. \tag{10}$$

By an appropriate change of coordinates one obtains[6] $\partial'^{\dot{B}}Q'_{\dot{B}\dot{S}}=0$, where the primed quantities refer to the new set of coordinates, which implies that[7] $Q'_{\dot{A}\dot{B}} = -\partial'_{\dot{A}}\partial'_{\dot{B}}\Xi$ for some function Ξ. With respect to the basis induced by the primed coordinates, the only nonvanishing component of the Killing spinor is given by $\phi'^{-k}L'_{(2k)}=\text{const}$.

Assuming now that $C^{(3)}$ does not vanish, from (9) one obtains that if k is an integer then $L_{(0)}=L_{(1)}=\dots=L_{(k-1)}=0$ and, in order to have a nontrivial Killing spinor, $L_{(k)}$ must be different from zero; in contrast, for the values $k=1/2, 3/2, \dots$, the only solution of (9) is $L_{AB\dots D}=0$. Hence, when the left conformal curvature is of type D there can only exist nontrivial $D(k,0)$ Killing spinors for integer values of k. Then the set of Eqs. (7) tells that

$$\begin{aligned}\partial^{\dot{S}}[\phi^k L_{(k)}]=0,\\ k\partial^{\dot{S}}[\phi^{k-2}L_{(k+1)}] + (k+1)D^{\dot{S}}[\phi^k L_{(k)}]=0,\\ (k-1)\partial^{\dot{S}}[\phi^{k-4}L_{(k+2)}] + (k+2)D^{\dot{S}}[\phi^{k-2}L_{(k+1)}]\\ +(k+2)(\partial_{\dot{B}}Q^{\dot{B}\dot{S}})\phi^{k-2}L_{(k+1)} - (k+2)(k+1)(D_{\dot{B}}Q^{\dot{B}\dot{S}})\phi^k L_{(k)}=0, \quad \text{etc.}\end{aligned} \tag{11}$$

3 With $g^{A\dot{B}}\cdot g^{C\dot{D}} = -2\varepsilon^{AC}\varepsilon^{\dot{B}\dot{D}}$

4 See, Finley and Plebański [8] and Torres del Castillo [7]

5 In the case where $l^B\nabla_{A\dot{C}}l_B=0$ (called case I in [8]) it follows that the spinor given by $(\phi\zeta)^{1/2}\delta^2_A$, with respect to the tetrad (5), is covariantly constant

6 See Appendix

7 See Finley and Plebański [8]

The first two of these equations are easily integrated, giving

$$\phi^k L_{(k)} = \delta, \phi^{k-2} L_{(k+1)} = \frac{k+1}{k}(\varepsilon - p^{\dot{R}}\, \partial\delta/\partial q^{\dot{R}}), \tag{12}$$

where δ and ε are functions of $q^{\dot{A}}$ only. Instead of substituting these expressions into the third equation and trying to determine $L_{(k+2)}$ and so on, it is convenient to use the existing freedom in the choice of coordinates in order to simplify these equations.

Indeed, one can find a set of coordinates $q^{\dot{A}}$, $p^{\dot{A}}$ such that[8] $L_{(k+1)} = 0$. Taking $j = k+1$ in (9) and recalling that $L_{(k-1)} = 0$, one gets the condition

$$6kC^{(3)} L_{(k+1)} - 2k(k+1) C^{(2)} L_{(k)} = 0 .$$

Therefore if $L_{(k+1)} = 0$, then $C^{(2)}$ must also vanish. On the other hand, in a tetrad such that $C^{(5)}$ and $C^{(4)}$ are zero, the Weyl spinor C_{ABCD} is of type D if and only if

$$2[C^{(2)}]^2 = 3C^{(1)} C^{(3)}, C^{(3)} \neq 0 .$$

Hence, in a tetrad such that $L_{(k+1)} = 0$, the components $C^{(2)}$ and $C^{(1)}$ must vanish and, as a consequence of (9), it follows that $L_{(k)}$ is the only nonvanishing component of the Killing spinor. Then the set of Eqs. (7) gives $\phi^k L_{(k)} = \text{const}$ and requires $D_{\dot{B}} Q^{\dot{B}\dot{S}} = 0$.

4. Induced Killing Vectors

If L_{AB} is a $D(1,0)$ Killing spinor, then the vector $K_{A\dot{B}} \equiv \frac{2}{3} \nabla^C_{\dot{B}} L_{CA}$ satisfies [9, 10]

$$\nabla^{\dot{R}}_A K^{\dot{S}}_B = 4C^{N\dot{R}\dot{S}}_{(A} L_{B)N} + \varepsilon_{AB} l^{\dot{R}\dot{S}} + \varepsilon^{\dot{R}\dot{S}} l_{AB} , \tag{13}$$

where l_{AB} and $l_{\dot{R}\dot{S}}$ are symmetric spinors. Thus, when $C^{N\dot{R}\dot{S}}_{(A} L_{B)N} = 0$, $K = -\frac{1}{2} K^{A\dot{B}} \partial_{A\dot{B}}$ is a Killing vector. In general, K is a complex vector field. Therefore, due to the linearity of the Killing equations, the real and imaginary parts of K are Killing vectors.

Assuming that the left conformal curvature is of type N and that there exists a set of coordinates such that $\partial_{\dot{B}} Q^{\dot{B}\dot{S}} = 0$, $L_{AB} = \phi \delta^2_A \delta^2_B$ is a $D(1,0)$ Killing spinor. Thus, from (13) it follows that $K = -\frac{1}{3}(\nabla^C_{\dot{B}} L_{CA}) \partial^{A\dot{B}}$ is a Killing vector provided that $C_{11\dot{A}\dot{B}} = C_{12\dot{A}\dot{B}} = 0$. By a direct computation one gets

$$K = -2(\partial^{\dot{A}} \phi) \partial_{\dot{A}} . \tag{14}$$

Similarly, when the left conformal curvature is of type D and $D_{\dot{B}} Q^{\dot{B}\dot{S}} = 0$ in some set of coordinates, $L_{AB} = \phi^{-1} \delta^1_{(A} \delta^2_{B)}$ is a $D(1,0)$ Killing spinor. If $C_{11\dot{A}\dot{B}} = C_{22\dot{A}\dot{B}} = 0$, then $K = -\frac{1}{3}(\nabla^C_{\dot{B}} L_{CA}) \partial^{A\dot{B}}$ is a Killing vector. In this case one obtains

$$K = (\partial^{\dot{A}} \phi) \frac{\partial}{\partial q^{\dot{A}}} - \frac{\partial \phi}{\partial q^{\dot{A}}} \partial^{\dot{A}} . \tag{15}$$

Notice that in both cases the Killing vector K is tangent to the hypersurfaces $\phi = \text{const}$.

8 See Appendix

5. Conclusions

The results derived here show that, in the cases under consideration, the $D(k,0)$ Killing spinors are symmetrized outer products of a single Killing spinor with itself. Therefore, there exists essentially one $D(k,0)$ Killing spinor [of type $D(1/2,0)$ or $D(1,0)$ if C_{ABCD} is of type N or D, respectively] provided, of course, that the corresponding existence conditions are satisfied.

The integration of the equations for the $D(k,0)$ Killing spinors presented here is somewhat simpler than that in the case of $\mathscr{H}$ spaces [4] due to the fact that the integrability conditions are very restrictive when the conformal curvature does not vanish.

The condition for the existence of a $D(k,0)$ Killing spinor when the left conformal curvature is of type N [Eq. (10)] has been integrated giving the form of the metric which admits such spinor field. However, the corresponding condition in the case where the left conformal curvature is of type D ($D_{\dot{B}}Q^{\dot{B}\dot{S}}=0$) has not been integrated here.

Appendix

The coordinates $q^{\dot{A}}$ are two independent functions which are constant on the null strings, i.e., $l^A\partial_{A\dot{B}}q^{\dot{C}}=0$. Therefore one can use in place of $q^{\dot{A}}$ any other pair of independent functions $q'^{\dot{A}}=q'^{\dot{A}}(q^{\dot{R}})$. On the other hand, the function ϕ is not uniquely defined by Eq. (6). If ϕ' is another solution of Eq. (6), then $l^A\partial_{AB}\ln(\phi/\phi')=0$, which means that $\varrho\equiv\phi^2/\phi'^2$ is a function of $q^{\dot{R}}$ only. The new "longitudinal" coordinates $p'^{\dot{A}}$ are then given by

$$p'^{\dot{A}}=-\varrho^{-1}T^{-1\dot{A}}_{\dot{B}}p^{\dot{B}}+\sigma^{\dot{A}}, \tag{A1}$$

where $(T^{-1\dot{A}}_{\dot{B}})$ is the inverse of $(T^{\dot{A}}_{\dot{B}})\equiv(\partial q'^{\dot{A}}/\partial q^{\dot{B}})$ and $\sigma^{\dot{A}}=\sigma^{\dot{A}}(q^{\dot{R}})$.

From (5) it follows that $dq^{\dot{A}}\cdot dq^{\dot{B}}=0$, $dq^{\dot{A}}\cdot dp^{\dot{B}}=\phi^2\varepsilon^{\dot{A}\dot{B}}$, and $dp^{\dot{A}}\cdot dp^{\dot{B}}=-2\phi^2Q^{\dot{A}\dot{B}}$. Hence writing $dp'^{\dot{A}}\cdot dp'^{\dot{B}}=-2\phi'^2Q'^{\dot{A}\dot{B}}$ and using (A1) one finds

$$Q'^{\dot{A}\dot{B}}=\varrho^{-1}T^{-1\dot{A}}_{\dot{C}}T^{-1\dot{B}}_{\dot{D}}Q^{\dot{C}\dot{D}}-T^{-1\dot{C}(\dot{A}}\frac{\partial p'^{\dot{B})}}{\partial q^{\dot{C}}}. \tag{A2}$$

The null tetrad $g'^{A\dot{B}}$, induced by the coordinates $q'^{\dot{A}}$, $p'^{\dot{A}}$, is obtained by replacing the objects which appear in (5) by their primed versions. The result can be written in the form

$$g'^{A\dot{B}}=m^A_C m^{\dot{B}}_{\dot{D}}g^{C\dot{D}}, \tag{A3}$$

with the SL(2, $\mathbb{C}$) matrices (m^A_C) and $(m^{\dot{B}}_{\dot{D}})$ given by

$$(m^A_B)=\begin{pmatrix}\varrho^{-1}T^{-1/2} & \eta T^{1/2}\\ 0 & \varrho T^{1/2}\end{pmatrix},\qquad m^{\dot{C}}_{\dot{D}}=T^{-1/2}T^{\dot{C}}_{\dot{D}}, \tag{A4}$$

where

$$T\equiv\det(T^{\dot{R}}_{\dot{S}}),$$

$$\eta=\frac{1}{2}\phi^2T^{-1\dot{B}}_{\dot{A}}\frac{\partial p'^{\dot{A}}}{\partial q^{\dot{B}}}=\frac{1}{2}\phi^2(\partial\sigma^{\dot{A}}/\partial q'^{\dot{A}}+T^{-1}p^{\dot{A}}\partial\varrho^{-1}/\partial q^{\dot{A}}).$$

Using (A1) and (A2) one gets

$$\partial'^{\dot{A}} Q'_{\dot{A}\dot{B}} = T_{\dot{B}}^{-1\dot{D}}(\partial^{\dot{C}} Q_{\dot{C}\dot{D}} - \partial \ln \varrho^{3/2} T / \partial q^{\dot{D}}),$$

where $\partial'_{\dot{A}}$ denotes $\partial/\partial p'^{\dot{A}}$. Therefore, if $Q_{\dot{C}\dot{D}}$ satisfies (10), then by a coordinate transformation such that $\varrho^{3/2} T = \zeta$, one obtains $\partial'^{\dot{A}} Q'_{\dot{A}\dot{B}} = 0$.

In the case where C_{ABCD} is of type D, denoting by $L'_{(j)}$ the components of the Killing spinor with respect to the basis induced by the coordinates $q'^{\dot{A}}$, $p'^{\dot{A}}$, and since $L_{(0)} = L_{(1)} = \ldots = L_{(k-1)} = 0$, from (A3), (A4), and (12), one gets

$$\begin{aligned} L'_{(k+1)} &= (m_1^1)^k (m_2^2)^{k-1} \{m_1^1 L_{(k+1)} - (k+1) m_2^1 L_{(k)}\} \\ &= (k+1)\varrho^{-1}\phi^{2-k}\{\varrho^{-1} T^{-1}(\varepsilon - p^{\dot{R}}\partial\delta/\partial q^{\dot{R}})/k - \tfrac{1}{2}\delta(\partial\sigma^{\dot{R}}/\partial q'^{\dot{R}} \\ &\quad + T^{-1} p^{\dot{R}} \partial \varrho^{-1}/\partial q^{\dot{R}})\}. \end{aligned}$$

Thus, choosing $q'^{\dot{R}} = q^{\dot{R}}$, $\varrho = \delta^{2/k}$, and $\sigma^{\dot{R}}$ such that $\partial\sigma^{\dot{R}}/\partial q^{\dot{R}} = (2\varepsilon/k)\delta^{-1-2/k}$, one obtains $L'_{(k+1)} = 0$.

References

1. Walker, M., Penrose, R.: On quadratic first integrals of the geodesic equations for type {22} spacetimes. Commun. Math. Phys. **18**, 265–274 (1970)
2. Hughston, L.P., Penrose, R., Sommers, P., Walker, M.: On a quadratic first integral for the charged particle orbits in the charged Kerr solution. Commun. Math. Phys. **27**, 303–308 (1972)
3. Hacyan, S., Plebañski, J.F.: Some basic properties of Killing spinors. J. Math. Phys. **17**, 2203–2206 (1976)
4. Finley III., J.D., Plebañski, J.F.: Killing vectors in plane *HH* spaces. J. Math. Phys. **19**, 760–766 (1978)
5. Plebañski, J.F.: Some solutions of complex Einstein equations. J. Math. Phys. **16**, 2395–2402 (1975)
6. Plebañski, J.F., Hacyan, S.: Null geodesic surfaces and Goldberg-Sachs theorem in complex Riemannian spaces. J. Math. Phys. **16**, 2403–2407 (1975)
7. Torres del Castillo, G.F.: Null strings and Bianchi identities. J. Math. Phys. **24**, 590–596 (1983)
8. Finley III., J.D., Plebañski, J.F.: The intrinsic spinorial structure of hyperheavens. J. Math. Phys. **17**, 2207–2214 (1976)
9. Hughston, L.P., Sommers, P.: The symmetries of Kerr black holes. Commun. Math. Phys. **33**, 129–133 (1973)
10. Hacyan, S., Plebañski, J.F.: $D(1,0)$ Killing structures and $\mathscr{E}$ potentials. J. Math. Phys. **18**, 1517–1519 (1977)

Communicated by S.-T. Yau

Received April 4, 1983